Graph Colouring

8

Colouring is, alongside planarity, one of the classical topics in graph theory. One of its most celebrated results is the four colour theorem that states that planar graphs can be coloured with just four colours. In this chapter, we first discuss upper bounds on the chromatic number of a graph in terms of the degrees of the vertices, those arising from the greedy colouring algorithm, the Szekeres–Wilf bound and Brooks' theorem. The weaker theorem of Heawood on planar graphs and the characterisation of planar graphs of low chromatic numbers illustrate the framework of the colouring problem for planar graphs. The better bound given by Vizing's theorem on the related edge-chromatic number is also discussed, and the equivalence of the four colour theorem with edge-colourings is considered following on from that. The chapter concludes with the list colouring problem, a proof by Thomasen of the 5-choosability of planar graphs and Galvin's theorem on the edge-choosability of bipartite graphs.

8.1 Vertex Colouring

A **vertex colouring** of a graph $\Gamma = (V, E)$ is a map

$$c : V(\Gamma) \to \{1, \dots, k\}.$$

The colouring is **proper** if no edge is monochromatic, that is adjacent vertices receive distinct colours. The minimum number of colours in a proper vertex colouring of a graph Γ is its **chromatic number**, denoted by $\chi(\Gamma)$.

We denote by $\omega(\Gamma)$ the cardinality of the largest clique (complete subgraph of Γ) and by $\alpha(\Gamma)$ the cardinality of the largest coclique (independent set of Γ). A proper k-colouring of Γ using the k colours induces a partition of its vertex set into k independent sets $c^{-1}(1), \dots, c^{-1}(k)$.

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_8

Lemma 8.1 *For every graph Γ of order n,*

$$\chi(\Gamma) \geqslant \max\{\omega(\Gamma), n/\alpha(\Gamma)\}.$$

Proof Let c be a proper colouring of Γ with $k = \chi(\Gamma)$ colours. Since each vertex of a clique of size $\omega(\Gamma)$ receives a distinct colour, $k \geqslant \omega(\Gamma)$.

On the other hand, $c^{-1}(1), \ldots, c^{-1}(k)$ is a partition of $V(\Gamma)$ into stable sets, so

$$n = \sum_i |c^{-1}(i)| \leqslant k\alpha(\Gamma).$$

□

A graph Γ is k-**critical** if $\chi(\Gamma) = k$ and if by deleting any edge or vertex we obtain a graph which is $(k-1)$-colourable. Observe that, step-by-step removing an edge which does not decrease the chromatic number, it is immediate that every graph with chromatic number k contains a k-critical subgraph.

Lemma 8.2 *If Γ is k-critical then $\delta(\Gamma) \geqslant k-1$.*

Proof By deleting a vertex of minimum degree we obtain a graph which is $(k-1)$-colourable. Thus, if $\delta < k-1$ we can colour the deleted vertex with one of $k-1$ colours, contradicting the fact that $\chi(\Gamma) = k$. □

The following upper bounds are classical results in graph colouring.

Let $\{x_1, \ldots, x_n\}$ be an ordering of the vertices of a graph Γ. The so-called **greedy** colouring algorithm proceeds by giving colour 1 to x_1 and, once x_i is coloured, give to x_{i+1} the smallest available colour among $\{1, 2, \ldots, i+1\}$.

Theorem 8.3 (Szekeres-Wilf) *For every graph Γ*

$$\chi(\Gamma) \leqslant 1 + \max_{\Gamma' \subseteq \Gamma} \delta(\Gamma').$$

Proof Let $d = \max_{\Gamma' \subseteq \Gamma} \delta(\Gamma')$. We define an ordering of the vertices as follows. We choose a vertex x_n with degree at most d in Γ. Once x_{i+1} is defined, we choose a vertex with degree at most d in the subgraph $\Gamma[V \setminus \{x_{i+1}, \ldots, x_n\}]$ of Γ induced by the unchosen vertices. Now the greedy algorithm on $x_1, \ldots, x_n$, which starts colouring x_1 with 1 and colours each x_i with the least available colour to make a proper colouring of $\Gamma[x_1, \ldots, x_i]$, uses at most $d+1$ colours because every x_i is adjacent at most to d previous vertices. □

It follows from the Szekeres–Wilf theorem that $\chi(\Gamma) \leqslant 1+\Delta(\Gamma)$. The following theorem of Brooks states that complete graphs and odd cycles are the only graphs for which the above bound is tight.

Theorem 8.4 (Brooks) *If Γ is a connected graph different from a complete graph or an odd cycle then*

$$\chi(\Gamma) \leqslant \Delta(\Gamma).$$

Proof Suppose that Γ is not a cycle or a complete graph.

If Γ is not regular then Theorem 8.3 implies $\chi(\Gamma) \leqslant \Delta(\Gamma)$, since the minimum degree would be less than Δ.

By Lemma 6.3, since Γ is connected, it is a tree of blocks where we recall that a block is a single vertex, an edge or a 2-connected graph. Observe that the chromatic number of Γ is equal to the maximum of the chromatic numbers of the blocks. Thus, we can assume that Γ is 2-connected, since the statement is trivial if $\chi(\Gamma) = 2$. Moreover, from the previous paragraph, we can assume Γ is regular.

Case 1 Γ is 3-connected. Choose x_n and two non adjacent vertices x_1, x_2 in its neighbourhood (such a choice exists since Γ is not complete). We have that $\Gamma - \{x_1, x_2\}$ is connected, $x_1x_2 \notin E(\Gamma)$ and $x_1x_n, x_2x_n \in E(\Gamma)$. For each i, starting at $i = n-1$, choose $x_i \in V(\Gamma) \setminus \{x_1, x_2, x_{i+1}, \dots, x_n\}$ adjacent to some vertex in $\{x_{i+1}, \dots, x_n\}$, which must exist by connectedness. Now the greedy algorithm allows us to colour x_1, x_2 with 1 and, at each step, x_i is only adjacent to at most $\Delta(\Gamma) - 1$ preceding vertices since it is adjacent to x_j for some $j > i$. In the last step we have to colour x_n, which is adjacent to $\Delta(\Gamma)$ vertices but two of them, x_1 and x_2, have the same colour, leaving one colour available for x_n (Fig. 8.1, left).

Case 2 Γ is 2-connected but not 3-connected. Choose a vertex x_n in a minimal separating set S of Γ, so that $\Gamma' = \Gamma - x_n$ is connected but not 2-connected. By Lemma 6.3, Γ' is a tree of blocks and, by the minimality of $|S|$, the vertex x_n is adjacent to two distinct blocks of this block decomposition of Γ', moreover it is adjacent to vertices x_1 and x_2 which are not articulation vertices of Γ'. Since they belong to distinct blocks of Γ and are not articulation points, x_1 and x_2 are not adjacent in Γ. Moreover, $\Gamma - \{x_1, x_2\}$ is connected as the blocks are 2-connected (Fig. 8.1, left). We can now repeat the argument in Case 1 to produce an ordering of the vertices for which the greedy algorithm uses at most $\Delta(\Gamma)$ colours.

□

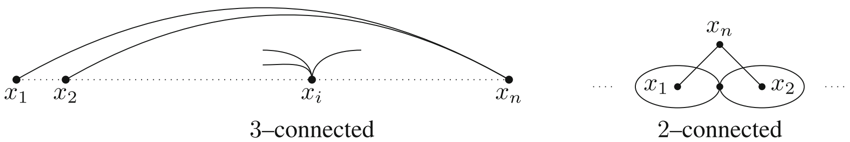

Fig. 8.1 The two cases of the proof of Brooks Theorem

8.2 Planar Graphs

A central result which fostered the development of graph theory is the four colour theorem stating that planar graphs have chromatic number at most four. All known proofs rely on extensive computer checking of hundreds of cases. The following theorem has a much easier proof.

Theorem 8.5 (Heawood) *Every planar graph* Γ *is 5-colourable.*

Proof The proof is by induction on $n = |V(\Gamma)|$, the result being trivial for $n \leqslant 5$. We may assume that Γ is maximal planar, i.e. a graph for which adding any additional edges will give a graph which is not planar. By Corollary 7.7, a planar graph has at most $3n - 6$ edges, which implies that the minimum degree of a planar graph satisfies $\delta(\Gamma) \leqslant 5$.

If Γ has a vertex x such that the degree of x, $d(x) \leqslant 4$ then every 5-colouring of $\Gamma[V \setminus \{x\}]$ can be extended to a 5-colouring of Γ. Suppose that $d(x) = \delta(\Gamma) = 5$ and let $x_1, \ldots, x_5$ be the five neighbours of x listed in clockwise order in a planar embedding of Γ.

If there is a 5-colouring of $\Gamma' = \Gamma[V \setminus \{x\}]$ which does not use the five colours in the neighborhood of x then we can extend the colouring to Γ. We may therefore assume that $\chi(x_i) = i$ for $1 \leqslant i \leqslant 5$ in a 5-colouring of Γ'. Let $\Gamma'[1, 3]$ be the subgraph Γ' induced by the colour classes 1 and 3. This is a graph with maximum degree 2, so that all connected components are either cycles or paths.

If x_1 and x_3 belong to distinct connected components of $\Gamma'[1, 3]$ then we can switch the colours in one of the components and get a proper 5-colouring which uses 4 colours on the neighborhood of x.

Hence, we can assume that x_1 and x_3 belong to the same connected component of $\Gamma'[1, 3]$, see Fig. 8.2. Consider the subgraph $\Gamma'[2, 4]$ of Γ' induced by the colour classes 2 and 4. This time x_2 and x_4 cannot be in the same connected component because every path from x_2 to x_4 must cross a path joining x_1 and x_3, all of whose vertices are not in $\Gamma'[2, 4]$. We again can complete the 5-colouring by switching colours in one of the connected components. □

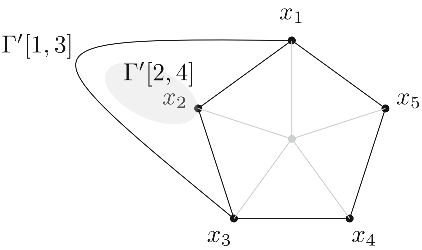

Fig. 8.2 An illustration of the proof of Theorem 8.5

The degree of a face in a planar graph is the number of edges in its boundary. The following theorem gives a characterization of planar 2-connected graph with chromatic number two.

Theorem 8.6 *A planar* 2*-connected graph* Γ *is bipartite if and only if every face of a planar embedding of* Γ *has even degree.*

Proof Suppose Γ is 2-connected and bipartite. By Proposition 7.4, the boundary of every face is a cycle of the graph (a facial cycle), which must have even degree since Γ is bipartite.

To show the reverse implication, note that the edge set of every cycle in a planar 2-connected graph is the symmetric difference of the edge sets of the facial cycles it contains (the boundaries of faces contained in the cycle in a plane embedding of the graph). If all facial cycles have even length, then the same holds for all cycles in Γ and so Γ is bipartite. □

The following theorem gives a characterization of maximal planar graphs with chromatic number three.

Theorem 8.7 (Heawood) *A maximal planar graph has chromatic number* 3 *if and only if every vertex has even degree (the graph is Eulerian).*

Proof If Γ is not Eulerian then a vertex z of odd degree and its neighbours induce an odd wheel in Γ. Three colours are needed to colour the odd cycle of this wheel and z requires a fourth colour. This shows that a maximal planar graph with chromatic number three must be Eulerian.

For the reverse implication we show the stronger statement that a 2-connected near triangulation Γ (all faces but the external one are triangles) in which all internal vertices have even degree has chromatic number three. This we prove by induction on the number f of internal faces. When $f = 1$ then $\Gamma = K_3$. Suppose $f > 1$ and let $e = xy$ be an edge in the external face of Γ. The edge e is in a unique triangle of Γ, let z be the third vertex of this triangle in addition to x and y.

If z is also a vertex on the external face, then one of x and y has degree two, say x (Fig. 8.3, left). Then $\Gamma - x$ is still 2-connected, has one less internal face, and every internal vertex has even degree. By induction, $\Gamma - x$ is 3-colourable. By giving x a colour different from y and z we obtain a 3-colouring of Γ.

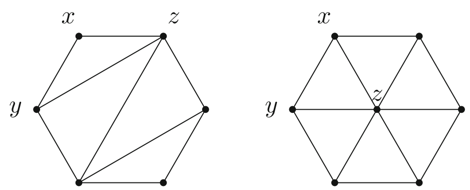

Fig. 8.3 The two cases in the proof of Theorem 8.7

If z is an internal vertex then it has even degree. Consider the even wheel induced by z and its neighbours (Fig. 8.7, right). Now $\Gamma - e$ is still 2-connected, has one less internal face and all internal vertices still have even degree. By induction $\Gamma - e$ is 3-colourable. If z receives colour 1 with a 3-colouring then the rim of the wheel receives colours 2 and 3. Since the vertices x and y are connected by a path of odd length, they receive distinct colours under this 3-colouring of $\Gamma - e$, which is therefore also a 3-colouring of Γ. □

8.3 Edge Colouring

An **edge-colouring** is a map

$$\chi' : E(\Gamma) \to k.$$

An edge-colouring is **proper** if incident edges receive different colours. The minimum number of colours in a proper edge-colouring of Γ is its **edge-chromatic number**, denoted by $\chi'(\Gamma)$. We have

$$\chi'(\Gamma) = \chi(L(\Gamma)),$$

where $L(\Gamma)$ denotes the line graph of Γ. Since all edges incident to a vertex must receive distinct colours under a proper edge-colouring, we clearly have

$$\chi'(\Gamma) \geqslant \Delta(\Gamma).$$

Perhaps surprisingly, this lower bound is never far from the true value of $\chi'(\Gamma)$.

Theorem 8.8 (Vizing) *For every graph* Γ*,*

$$\chi'(\Gamma) \leqslant \Delta(\Gamma) + 1.$$

Proof For every fixed Δ we will prove the bound by induction on m, the number of edges of graphs with maximum degree at most Δ.

If $m = \Delta$ then Γ is a star (and isolated points) which is clearly edge-colourable with Δ colours.

Let $m \geqslant \Delta + 1$, choose a vertex x_0 with degree Δ and remove an edge $x_0 y_0$ from Γ. Let χ_0 be a proper $(\Delta + 1)$-edge colouring of $\Gamma - x_0 y_0$ (which has maximum degree at most Δ). This is a proper edge-colouring of Γ except that one edge, $x_0 y_0$, is still uncoloured.

For every vertex $x \in V(\Gamma)$ denote by $\beta(x)$ the set of colours not used in the edges incident to x. We have $\beta(x) \neq \emptyset$ since we are using $\Delta + 1$ colours.

If $\beta(x_0) \cap \beta(y_0) \neq \emptyset$ then we can use a colour in the intersection to complete the colouring of the edge $x_0 y_0$.

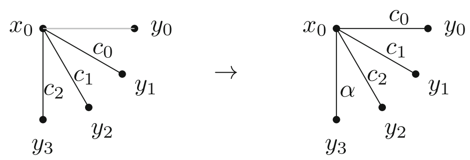

Fig. 8.4 Case 1 of the proof

Suppose that $\beta(x_0) \cap \beta(y_0) = \emptyset$. Choose a colour $c_0 \in \beta(y_0)$ and let x_0y_1 be an edge incident to x_0 coloured with c_0. If $\beta(x_0)\cap\beta(y_1) \neq \emptyset$ we can use a colour in the intersection to recolour x_0y_1 and use c_0 to colour x_0y_0 (which after the recolouring will be available for x_0 and for y_0). Otherwise we construct a maximal sequence $y_0, y_1, \ldots, y_k$ satisfying (i) $\beta(x_0) \cap \beta(y_i) = \emptyset$ and (ii) $c_i \in \beta(y_i)$ is different from $c_1, \ldots, c_{i-1}$ and x_0y_{i+1} has colour c_i.

By the maximality of the length of the chain one of the two cases occur:

Case 1 The chain stopped because we reached a vertex y_k with $\beta(x_0)\cap\beta(y_k) \neq \emptyset$. In this case we use a colour α in the intersection to recolour x_0y_k and use colour c_i for x_0y_i in the vertices $y_0, \ldots, y_{k-1}$, reaching a good edge-colouring for Γ (we push the colours back) (Fig. 8.4).

Case 2 The chain stopped because $\beta(x_0) \cap \beta(y_k) = \emptyset$ but the colours in $\beta(y_k)$ have already appeared in the chain, say $c_{j-1} \in \beta(y_k)$ for some $j < k$. In this case we recolour the edges x_0y_i with c_i for $i = 0, \ldots, j-1$ and leave the edge x_0y_j uncoloured.

Choose a colour $\alpha \in \beta(x_0)$ and consider the subgraph $\Gamma[\alpha, c_{j-1}]$ of Γ induced by the edges coloured α and c_{j-1} after the last recolouring. The graph $\Gamma[\alpha, c_{j-1}]$ has maximum degree two so that the connected components are cycles and paths or isolated vertices. Moreover, the vertices x_0, y_j, y_k have degree one in this subgraph (because $c_{j-1} \notin \beta(x_0)$ implies $d(x_0) = 1$ and $\alpha \notin \beta(y_j) \cup \beta(y_k)$ implies $d(y_j) = d(y_k) = 1$). Therefore, the three vertices can not belong to the same connected component of $\Gamma[\alpha, c_j]$.

Suppose that y_j and x_0 belong to different components. We can exchange the colours of the edges c_{j-1} and α in the connected component containing y_j and the resulting colouring will still be proper. Moreover, after the renaming, α becomes unused at y_j and we can use α to colour the edge x_0y_j completing the colouring of Γ.

Suppose that y_k and x_0 belong to different components. We can exchange the colours of the edges c_{j-1} and α in the connected component containing y_k and the resulting colouring will still be proper. Now we recolour the edges x_0y_i with c_i for $i = j, j+1, \ldots, k-1$ leaving x_0y_k uncoloured, and colour this edge with α. □

By Vizing's theorem, $\chi'(\Gamma) \in \{\Delta(\Gamma), \Delta(\Gamma) + 1\}$. We observe that each colour class is a matching, so Vizing's theorem can be rephrased by saying that every graph admits a partition of its edge set into at most $\Delta(\Gamma) + 1$ edge-disjoint matchings. For instance, for the complete graphs K_{2n} of even order it can easily be seen that $\chi'(\Gamma) = \Delta(\Gamma)$, while the ones of odd order can not be coloured with Δ colours because the largest matching in K_{2n+1} has n edges and the total number of edges is $n(2n+1)$. Thus, for K_{2n+1}, $\chi'(\Gamma) = \Delta(\Gamma) + 1$. The fact that for bipartite graphs, $\chi'(\Gamma) = \Delta(\Gamma)$, is a consequence of Hall's theorem.

Proposition 8.9 *A bipartite graph Γ has edge-chromatic number $\chi'(\Gamma) = \Delta(\Gamma)$.*

Proof If Γ is Δ-regular then, by Theorem 4.3, it contains a perfect matching M_1. Its removal leaves a $(\Delta - 1)$-regular bipartite graph which contains a perfect matching M_2. By iterating this procedure, we decompose the edge set of Γ into Δ edge-disjoint matchings.

If $\Gamma = (A \cup B, E)$ is not Δ-regular we show that there is $\Gamma' \supset \Gamma$ which is bipartite and Δ-regular and $\chi'(\Gamma) \leqslant \chi'(\Gamma') = \Delta$. By adding isolated vertices if needed, we may assume that $|A| = |B|$. If there is a vertex $x \in A$ with degree smaller than Δ then there must be $y \in B$ with the same property and we can add the edge xy to Γ and still get a bipartite graph with maximum degree Δ. By repeating the argument we eventually end up with a bipartite Δ-regular graph $\Gamma' \supset \Gamma$. By the previous paragraph, the edges of Γ' decompose into Δ edge disjoint matchings. □

A final observation on the four colour theorem is the following equivalence.

Theorem 8.10 *The four colour theorem is equivalent to the following statement: every bridgeless cubic planar graph has edge-chromatic number $\chi'(\Gamma) = 3$.*

Proof Every planar graph can be coloured with four colours if and only if every maximal planar graph can be coloured with four colours, so we may restrict ourselves to maximal planar graphs. The dual of a maximal planar graph is a bridgeless cubic graph. Reciprocally, the dual of a bridgeless cubic planar graph is a triangulation, a maximal planar graph.

Let Γ be an embedded cubic planar graph and suppose that there is a 4-colouring χ of its dual Γ^* with elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each $e \in E(\Gamma)$ determines an edge $e^* = xy \in E(\Gamma^*)$, joining the two faces which have the edge e on their boundaries, see Fig. 8.5. We colour e with $\chi'(e) = \chi(x) + \chi(y)$, which is not $(0, 0)$ since x and y (faces of Γ) are coloured with different colours. If e and e' are incident in Γ then the corresponding edges $e^* = xy$, $(e')^* = yz$ are also incident in Γ^* and xyz form a triangle in Γ^*, since Γ^* is a triangulation. Since $\chi(x) \neq \chi(z)$, it follows that

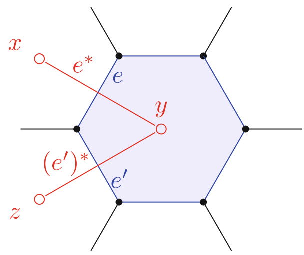

Fig. 8.5 The edge e determines de edge e^* in Theorem 8.10

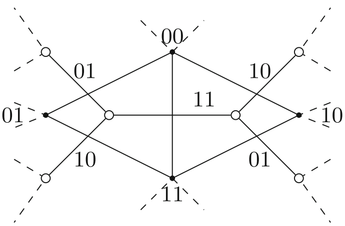

Fig. 8.6 The 4-colouring of a maximal planar graph Γ^* (with black vertices) induced by a 3-edge-colouring of its dual Γ (with white vertices)

$\chi'(e) = \chi(x) + \chi(y) \neq \chi(y) + \chi(z) = \chi'(e')$. Thus we obtain a 3-colouring of Γ with elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$.

Reciprocally, let Γ be a maximal planar graph and suppose that there is a 3-edge-colouring χ^* of its dual Γ^*. Identify the edge colours of Γ^* with the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$ (Fig. 8.6).

Since Γ is a triangulation, every edge e in Γ uniquely defines a dual edge e^* in Γ^* joining the two faces of Γ that have e in common in their boundaries.

Consider a spanning tree of Γ, choose a vertex as a root and colour it with $(0, 0)$. For every pair v, v' of adjacent vertices in the spanning tree define the map $\chi(v) = \chi(v') + \chi^*(e^*)$ where e^* is the dual edge of $e = vv'$ and the sum is in $\mathbb{Z}_2 \times \mathbb{Z}_2$. This is a well defined 4-colouring of Γ. We claim that, for every pair v, v' of adjacent vertices in Γ joined by the edge $e_{vv'}$, we have $\chi(v) = \chi(v') + \chi^*(e^*_{vv'})$, so the colouring is proper.

□

This is the case if $e_{vv'}$ is an edge of the spanning tree, by definition. Otherwise, consider the cycle induced in the tree by this edge e.

Suppose the cycle is facial, namely a triangle $vv`w$. Then e_{vw} and $e_{v'w}$ are edges of the spanning tree and we have that

$$\chi(v) = \chi(w) + \chi^*(e^*_{vw})$$

and

$$\chi(v') = \chi(w) + \chi^*(e^*_{v'w}).$$

Therefore, we have that

$$\chi(v) = \chi(v') + \chi^*(e^*_{vw}) + \chi^*(e^*_{v'w}) = \chi(v') + \chi^*(e^*_{vv'}),$$

since the sum of any elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$ gives the third one.

Finally, if the cycle is not facial then it is the symmetric difference of r facial cycles for some r. Each one of the edges in these faces when added to the spanning tree induces a cycle which is the symmetric difference of less than r facial cycles. By induction on r, we can assume that for these edges $e_{w'w}$

$$\chi(w) = \chi(w') + \chi^*(e^*_{ww'}).$$

Then, since we are summing modulo two, when we take the symmetric difference of r facial cycles, we also conclude that

$$\chi(v) = \chi(v') + \chi^*(e^*_{vv'}).$$

8.4 List Colouring

Let Γ be a graph and let $L(v)$ be a list of colours associated to each vertex $v \in V(\Gamma)$. A **list colouring** of Γ is a proper colouring χ such that

$$\chi(v) \in L(v), \ \forall v \in V(\Gamma).$$

A graph Γ is k**-choosable** if, for every set of lists $\{L(v) : v \in V(\Gamma)\}$ with $|L(v)| \geqslant k$, there is a list colouring with this set of lists. The minimum integer k such that Γ is k-choosable is the **list chromatic number** $\chi_L(\Gamma)$ of Γ. An ordinary k-colouring can be seen as a list colouring where the list of each vertex is $\{1, 2, \dots, k\}$. Therefore,

$$\chi(\Gamma) \leqslant \chi_L(\Gamma).$$

The difference between the two quantities can be arbitrarily large. For example, the list chromatic number of the complete bipartite graph satisfies $\liminf_{n\to\infty} \chi_L(K_{n,n}) = \infty$, even if the graph is bipartite (see Exercise 8.17). The list assignment to $K_{3,3}$ in Fig. 8.7 shows that $\chi_L(K_{3,3}) > 2$.

The arguments using the greedy colouring algorithm, where only the number of colours available at one vertex is significant, show that the statement of the theorems of Szekeres–Wilf and of Brooks still hold for list colourings. The next celebrated theorem by Thomassen shows that list colouring of planar graphs is at most five.

Theorem 8.11 (Thomassen) *For every planar graph* Γ *we have* $\chi_L(\Gamma) \leqslant 5$.

Proof We need to prove that given lists $L(v)$ of size at least 5 for each vertex v, we can choose a colour from $L(v)$ so that the colouring is proper. If we can find a proper list colouring for a graph which contains Γ then we will have found a list colouring for Γ. Thus, we can assume that Γ is a near-triangulation (all faces except the outer one are triangles). □

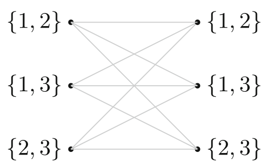

Fig. 8.7 There is no list colouring of $K_{3,3}$ with the displayed list assignment

Claim 8.12 For every list assignment of a near triangulation in which two prescribed adjacent vertices in the outer face have distinct lists of size one, the remaining vertices in the outer face have lists of size three and all inner vertices have lists of size five, there is a list colouring of the graph with these lists.

Proof We will prove this by induction on the number n of vertices. The claim follows for $n = 3$. Suppose $n > 3$.

Let x, y be the vertices with lists of size one. We consider two cases.

Case 1. There is a chord $e = uv$ joining two vertices in the outer face (Fig. 8.8 left).
We consider the two near triangulations Γ_1, Γ_2 which are split by the chord and share this chord in their outer boundary. We may assume that Γ_1 is the near triangulation which contains both of them. We apply induction on Γ_1 and find a list colouring χ_1 of Γ_1. We now apply induction on Γ_2 by redefining the lists of u and v as $L'(u) = \{\chi_1(u)\}$ and $L'(v) = \{\chi_1(v)\}$ to find a list colouring χ_2 with these new lists. The colouring whose restriction to Γ_i is χ_i is a list colouring of Γ.

Case 2. There is no chord joining two vertices in the outer face (Fig. 8.8 right).
Let x, y, u, v be consecutive vertices in the clockwise order in the outer face (it may be that $v = x$). Let $y = u_1, u_2, \ldots, u_k = v$ be the neighbours of u also in clockwise order. Consider the new lists $L'(u) = L(u) \setminus L(y)$ and $L'(u_i) = L(u_i) \setminus L'(u)$, $1 \leqslant i \leqslant k - 1$, which provide the induction hypothesis for the graph $\Gamma - u$. A list colouring χ of $\Gamma - u$ can be completed to a list colouring of Γ with the original lists since none of the u_i's uses the two colours of $L'(u)$ and one can be chosen different from $\chi(v)$.

The statement of the theorem now follows from the Claim 8.12, since an assignment of lists of length five to every vertex fulfils the hypothesis of the claim. □

There are examples of planar graphs whose list chromatic number is five, so the bound in Theorem 8.11 is tight.

Analogous notions for edge-colourings lead to k-edge choosability and edge-list chromatic number $\chi'_L(\Gamma)$. Clearly, $\chi'_L(\Gamma) = \chi_L(L(\Gamma))$. A famous open problem in the area is the list colouring conjecture.

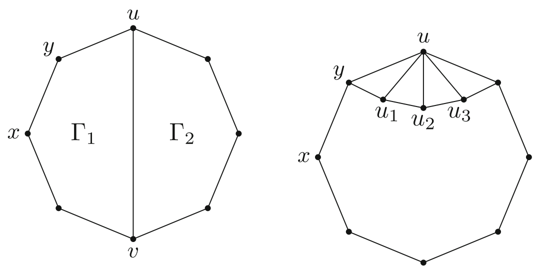

Fig. 8.8 The two cases in the proof of Theorem 8.11

Conjecture 8.13 (List Colouring Conjecture) For every graph Γ we have $\chi'_L(\Gamma) = \chi'(\Gamma)$.

Remarkably, the conjecture has been proved for bipartite graphs.

Theorem 8.14 (Galvin) *If Γ is bipartite then $\chi'_L(\Gamma) = \chi'(\Gamma)$.*

One proof uses the following result. A **kernel** in an oriented graph $\vec{\Gamma} = (V, \vec{E})$ is a nonempty independent set U such that every vertex in $V \setminus U$ has an arc directed to some vertex in U.

Lemma 8.15 *Let $\{L(v) : v \in V\}$ be a set of lists assigned to vertices of a graph $\Gamma = (V, E)$. If there is an orientation $\vec{\Gamma}$ such that every induced subgraph $\vec{\Gamma}'$ of $\vec{\Gamma}$ has a kernel and the out-degree of every vertex satisfies $d^+(v) < |L(v)|$, then Γ can be coloured from the lists.*

Proof The proof is by induction on n. The result is trivial for $n = 1$. Assume $n > 1$. Choose a colour c which occurs in some list and let C be the set of vertices in Γ which have the colour c in their lists. By the hypothesis, $\vec{\Gamma}[C]$ has a kernel U. Colour the vertices of U by c and remove the colour c from all lists in C. The subgraph $\Gamma[V \setminus U]$ satisfies the hypothesis of the lemma, since the out degree of every vertex in $\Gamma[V \setminus U]$ is one less than its out degree in $\Gamma[V]$. whereas $|L(v)|$ has decreased by at most one since we have only removed the colour c from the lists. Hence, by induction, $\Gamma[V \setminus U]$ admits a list colouring with the new lists and this provides a list colouring of Γ with the original lists. □

Proof of Theorem 8.14 Let χ be an edge-colouring of Γ with $\{1, 2, \ldots, k\}$, where $k = \chi'(\Gamma)$. We will prove that the colouring can be used to construct an orientation of the line graph $L(\Gamma)$ of Γ satisfying the hypothesis of Lemma 8.15. The statement then follows from Lemma 8.15.

Let $V(\Gamma) = A \cup B$ be the bipartition of Γ. Let e and e' be two incident edges in Γ with $\chi(e) < \chi(e')$ (since χ is a proper edge-colouring, equality does not hold). We orient the edge $ee' \in E(L(\Gamma))$ from e to e' if $e \cap e' \in A$ and from e' to e if $e \cap e' \in B$ (Fig. 8.9).

The out-degree of an edge e in this orientation is at most $(k-\chi(e))+(\chi(e)-1) < k$. So we only have to show that every subgraph of $L(\Gamma)$ with this orientation has a

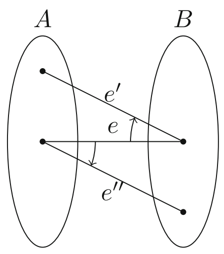

Fig. 8.9 The orientation of $L(\Gamma)$ in the proof of Theorem 8.14 when $\chi(e'') < \chi(e) < \chi(e')$

kernel. We prove that this is the case by induction on $|E'|$, the case $|E'| = 1$ being trivial. Suppose $|E'| > 1$.

Let $E' \subset E(\Gamma)$ be a subset of edges and consider the oriented subgraph Γ' of $L(\Gamma)$ induced by E'. Let $A' \subset A$ be the set of vertices incident with some edge in E'. For each $x \in A'$, let $e_x \in E'$ be the edge incident with x with smallest colour. Let $U = \{e_x : x \in A'\}$. By construction, every edge in $E' \setminus U$ has a directed edge in Γ' to some element in U.

If U is an independent set then we are done.

If not, suppose that two edges e_x and $e_{x'}$ in U meet in some vertex y. Necessarily, by the construction of U, $y \in B$. Suppose $\chi(e_x) < \chi(e_{x'})$, so e_x is directed to $e_{x'}$ in Γ'. By induction, the subgraph $\Gamma' \setminus e_x$ has a kernel U'. If $e_{x'} \in U'$ then U' is a kernel for Γ' and we are done. Otherwise there is an edge e'' incident with $e_{x'}$ such that $e_{x'}$ is directed to e'' in Γ'. Since $e_{x'}$ is the edge incident with x' with minimum colour it must be that e'' is incident with $e_{x'}$ in B, which implies it is incident with y. Thus, $\chi(e'') > \chi(e_{x'})$ which implies $\chi(e_{x''}) > \chi(e_x)$ and so e_x is directed to e'' in Γ'. Hence U' is a kernel for Γ'. □

By Proposition 8.9 and Theorem 8.14, the edge list chromatic number of a bipartite graph Γ is $\chi'_L(\Gamma) = \Delta(\Gamma)$. In particular, $\chi'_L(K_{n,n}) = n$, a statement which had been conjectured by Dinitz in the language of Latin squares. Suppose that each entry of a square $n \times n$ matrix may take one of n distinct values. Dinitz conjectured that one can choose entries such that elements in a row and in a column are pairwise distinct. Observe that when the choices are $\{1, 2, \ldots, n\}$ for each entry we get a Latin square.

8.5 Notes and References

The proof of Brooks' theorem, Theorem 8.4, follows Lovász (1975). A simplified version which avoids the use of 3-connectivity and has further applications is given in Zając (2018). The characterization of 3-chromatic triangulations by Heawood can be complemented by a theorem by Grötzsch that states that every triangle-free planar graph can be 3-coloured, see Thomassen (2003) for a simplified proof. The proof by Thomassen (2004), Theorem 8.11, of the 5-choosability of planar graphs has become a classic in graph theory with wide applications. The proof by Galvin (1995) of the list colouring conjecture also extends to bipartite multigraphs.

8.6 Exercises

Exercise 8.1

i. Show that every graph admits an ordering of the vertices for which the greedy colouring algorithm uses $\chi(\Gamma)$ colours.

ii. Let Γ be the complete bipartite graph $K_{n,n}$ minus a perfect matching. Show that there is an ordering of the vertices such that the greedy colouring algorithm uses n colours.

Exercise 8.2 Let $\Gamma + \Gamma'$ denote the graph resulting from the disjoint union of Γ and Γ' and adding all edges between $V(\Gamma)$ and $V(\Gamma')$. Prove that $\chi(\Gamma + \Gamma') = \chi(\Gamma) + \chi(\Gamma')$.

Exercise 8.3 The cartesian product $\Gamma \Box \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and (x, y) i (x', y') are adjacent if and only if either $x = x'$ and $y \sim y'$ or $y = y'$ and $x \sim x'$. Show that $\chi(\Gamma \Box \Gamma') = \max\{\chi(\Gamma), \chi(\Gamma')\}$.

Exercise 8.4 The direct product $\Gamma \times \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and (x, y) i (x', y') are adjacent if $x \sim x'$ and $y \sim y'$. Show that $\chi(\Gamma \times \Gamma') \leq \min\{\chi(\Gamma), \chi(\Gamma')\}$.

[Hedetniemi conjecture, recently disproved, stated that equality holds.]

Exercise 8.5 A graph Γ is k-**critical** if it has chromatic number k and every proper subgraph of Γ has smaller chromatic number. Show that a k-critical graph is $(k-1)$-edge connected (the graph remains connected after deletion of any set of $k - 1$ edges).

Exercise 8.6 Let Γ be a k-critical graph.

i. Show that, for every pair x, y of non adjacent vertices, there is a k-colouring χ of Γ such that $\chi(x) = \chi(y)$.
ii. Show that Γ must be 2-connected. Moreover, if S, with $|S| = 2$ separates $X \subset V$ from $Y = V \setminus (X \cup S)$, then the induced subgraphs $\Gamma_1 = \Gamma[X \cup S]$ and $\Gamma_2 = \Gamma[Y \cup S]$ have the property that any $(k-1)$-coloring of Γ_1 gives distinct colours to S while any k-colouring of Γ_2 gives the same colour to the vertices in S. Give an example of a 2-connected critical graph with $k = 4$.

Exercise 8.7 (Mycielski construction) Given a graph Γ with vertex set $V = \{v_1, \ldots, v_n\}$, denote by $M(\Gamma)$ the graph with vertex set $V \cup \{u_1, \ldots, u_n\} \cup \{w\}$ where

i. $\{u_1, \ldots, u_n\}$ is an independent set;
ii. For each i, u_i is adjacent to every vertex adjacent to v_i.
iii. w is adjacent to each u_i.

Show that, if Γ is triangle free and $\chi(\Gamma) = k$, then $M(\Gamma)$ is also triangle-free and $\chi(M(\Gamma)) = k + 1$.

Exercise 8.8 Show that $\chi(\Gamma) = k$ if and only if there is an orientation $\vec{\Gamma}$ of Γ whose longer directed path has length k.

Exercise 8.9 For a graph $\Gamma = (V, E)$ and a vertex $x_0 \in V$ let $S_i = \{x \in V(\Gamma) : d(x_0, x) = i\}$ denote the sphere of radius i centered at x_0.

1. Show that

$$\chi(\Gamma) \leqslant \max_i\{\chi(\Gamma[S_i]) + \chi(\Gamma[S_{i+1}])\},$$

where the maximum is taken from $i = 0$ to the eccentricity of x_0 minus one. Give examples showing that the inequality is tight.
2. Show that a graph Γ with chromatic number $\chi(\Gamma) \geqslant 2^t$ has the complete graph K_t as a minor.

Exercise 8.10 Let $f_\Gamma(x)$ be a function such that, fo each positive integer k, $f_\Gamma(k)$ is the number of proper k-colourings of Γ.

i. Show that

$$f_\Gamma(k) = f_{\Gamma-e}(k) - f_{\Gamma/e}(k).$$

ii. Deduce that f_Γ is a polynomial and that $f_\Gamma(x) = x^n - mx^{n-1} +$ terms of lower degree, where $n = |V(\Gamma)|$ and $m = |E(\Gamma)|$.
iii. Compute the polynomial for $\Gamma = K_n$ and for Γ a tree.

Exercise 8.11 Show that a graph Γ with m edges satisfies

$$\chi(\Gamma) \leqslant \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

Exercise 8.12 Show that an outerplanar graph Γ has chromatic number $\chi(\Gamma) \leqslant 3$.

Exercise 8.13 Show that a regular graph Γ with an odd number of vertices satisfies $\chi'(\Gamma) = \Delta(\Gamma) + 1$.

Exercise 8.14 Show that if Γ is a cubic graph with a bridge then $\chi'(\Gamma) = 4$.

Exercise 8.15 Prove that $\chi'(K_{2n}) = 2n - 1$ and $\chi'(K_{2n+1}) = 2n + 1$. Describe the edge-colourings reaching these values.

Exercise 8.16 Let Γ be the graph obtained from the complete bipartite graph $K_{n,n}$ by subdividing one edge by a vertex. Show that $\chi'(\Gamma) = \Delta(\Gamma) + 1$, but $\chi'(\Gamma - e) = \Delta(\Gamma)$ for every edge e.

Exercise 8.17 Show that $\liminf_{n\to\infty} \chi_L(K_{n,n}) = \infty$.

Exercise 8.18 Show that $\chi'_L(\Gamma) \leqslant 2\Delta(\Gamma) - 1$.

Exercise 8.19 Let L be a list assignment to the vertices of a graph Γ such that $d(v) \leqslant |L(v)|$ and the strict inequality holds for at least one vertex. Show that Γ admits a list colouring from these lists.

Exercise 8.20 A graph is k-**degenerated** if every subgraph has a vertex of degree at most k.

i. Prove that the list colouring number of a k-degenerated graph G satisfies $\chi_L(G) \leqslant k + 1$.
ii. Show that a k-degenerated graph G of order $n > k$ with maximal number of edges has $kn - \binom{k+1}{2}$ edges, connectivity $\kappa(G) = k$ and $\chi(G) = k + 1$.
iii. Prove that a non-bipartite outerplanar graph has $\chi_L(G) = 3$

Exercise 8.21 Show that any assignment of lists of length 2 to the vertices of an odd cycle admits a proper list coloring, except when all the lists are the same.