6 Connectivity

Connectivity is a key property of graphs. The central result on connectivity of graphs is the theorem of Menger, a result of min–max type with several connections in other areas of combinatorics and of combinatorial optimization, besides its relevance in graph theory itself. Some structural results related to connectivity are also presented in this chapter, including a theorem of Tutte on 3-connected graphs. The close notion of edge-connectivity is also discussed at the end of the chapter.

6.1 Vertex Connectivity

A graph is **connected** if there is a path connecting any pair of vertices. A **connected component** of a graph is a connected subgraph which cannot be extended (by adding edges or vertices). Every graph is the disjoint union of its connected components. For a subset $X \subset V(\Gamma)$, we denote by $\Gamma[X]$ the subgraph of Γ induced by the vertices in X.

A **tree** is a connected acyclic graph. The following are equivalent definitions of a tree. The proof is a simple exercise.

Proposition 6.1 *For a graph T, the following statements are equivalent:*

i. T is a tree.
ii. T is an edge-maximal acyclic graph: the addition of any edge to T results in a graph which is no longer acyclic.
iii. T is an edge-minimal connected graph: the suppression of any edge of T results in a graph which is no longer connected.
iv. For every pair of vertices in T there is a unique path joining them.
v. $|E(T)| = |V(T)| - 1$ and T is acyclic.
vi. $|E(T)| = |V(T)| - 1$ and T is connected.

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_6

A subgraph T of a graph Γ is a spanning tree of Γ if it is a tree and $V(T) = V(\Gamma)$. A simple characterization of connected graphs is the following one.

Lemma 6.2 *A graph Γ is connected if and only if there is an ordering $\{v_1, \dots, v_n\}$ of the vertices such that $\Gamma[v_1, \dots, v_i]$ is connected for each $i = 1, \dots, n$. In particular, Γ is connected if and only if it contains a spanning tree.*

Proof The first part is a direct consequence of the definition: if Γ is not connected then the condition fails for $i = n$ and every ordering.

Reciprocally, if Γ is connected one can start in any vertex v_1 and define v_{i+1} as the first vertex not in $\{v_1, \dots, v_i\}$ in a path connecting v_1 with some vertex not in that initial segment.

For the second part, we can choose, for every i, one edge joining v_i with some vertex in $\{v_1, \dots, v_{i-1}\}$. In this way we obtain a spanning subgraph (a graph with vertex set $V(\Gamma)$) which has $|V(\Gamma)| - 1$ edges, and it is therefore a tree. □

A natural measure of connectivity of a graph is given by the minimum number of vertices whose deletion disconnects the graph. A subset $S \subset V(\Gamma)$ is a **separator** of Γ if $\Gamma[V \setminus S]$ is not connected. A graph is k**-connected** if $|V(\Gamma)| \geqslant k + 1$ and every separator of G has at least k vertices. For example a tree is 1-connected but not 2-connected. A cycle is 1-connected and also 2-connected but not 3-connected. For the complete graph, which has no separators, the definition is to be seen as a convention: the complete graph K_n is k-connected for every $k \leqslant n - 1$.

6.2 Structure of k-Connected Graphs for Small k

A **cut vertex** v of a connected graph Γ is a vertex such that $\Gamma[V(\Gamma) \setminus \{v\}]$ is not connected, i.e. $S = \{v\}$ is a separator of size one.

A **block** of Γ is a connected subgraph of Γ which contains no cut vertices and cannot be extended to a larger subgraph which contains no cut vertices.

Thus, a block is either an isolated vertex, an edge with its two end vertices or a maximal 2-connected subgraph. By maximality, if two blocks intersect then they have a unique common vertex, which is a cut vertex of the graph. Connected graphs can be structured in a tree of blocks.

Lemma 6.3 *Let Γ be a connected graph and let A be its set of cut vertices. Let $B(\Gamma)$ be the bipartite graph with bipartition $V_1 = A$ and $V_2 = \{B \subset \Gamma :$ B is a block of $\Gamma\}$ where there is an edge joining a cut vertex $a \in A$ with a block $B \in V_2$ if and only if $a \in B$. Then $B(\Gamma)$ is a tree.*

Proof The block graph is connected since Γ is connected. If it has a cycle then this cycle contains $r \geqslant 2$ blocks of Γ with r cut points, which together form a block, contradicting the maximality of the existing blocks. □

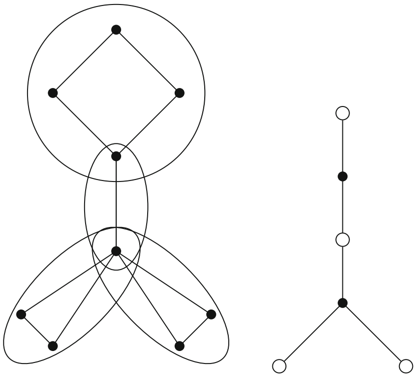


Fig. 6.1 A graph (left) and its block graph (right)

Figure 6.1 shows an example of a graph and its block graph.

As for a block, its structure can be described as follows.

Lemma 6.4 *A graph is* 2*-connected if and only if it can be recursively constructed starting from a cycle by successively adding a path between two vertices previously constructed.*

Proof Suppose that Γ has been recursively constructed starting from a cycle by successively adding a path between two vertices previously constructed. Then every vertex is contained in a cycle, so Γ has no cut vertices. Hence, it is 2-connected.

Suppose that Γ is 2-connected. Let Γ' be a maximal subgraph of Γ constructed as stated. Then Γ' is an induced subgraph of Γ, since we can always add an edge between two vertices of Γ' under the recursion rule, if that edge is an edge of Γ.

If there is a vertex $v \in V(\Gamma) \setminus V(\Gamma')$ then there is a path from v to some vertex in Γ'. Suppose that w is the first vertex of Γ' on such a path. Since Γ is 2-connected, there is another path from v to another vertex $w' \neq w$ in Γ' sharing no other vertices than v with the above path. Thus, v lies on a path joining two previously constructed vertices, which contradicts the maximality of Γ'. □

Thus, if a graph is 2-connected, then there is a sequence

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$$

such that Γ_0 is a cycle and Γ_i is obtained from Γ_{i-1} by adding a path (possibly with internal vertices not in Γ_{i-1}) joining two vertices in Γ_{i-1}.

We next discuss the more substantial structural characterisation of 3-connected graphs.

The **contraction** of an edge $e = xy \in E(\Gamma)$ consists in identifying its two endpoints and the possible multiple edges which may be created by this identification, see Fig. 6.2. The resulting graph is denoted by Γ/e. Contraction is an important notion in the theory of graphs.

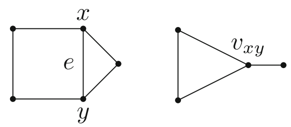


Fig. 6.2 A graph G on the left and the contraction G/e on the right

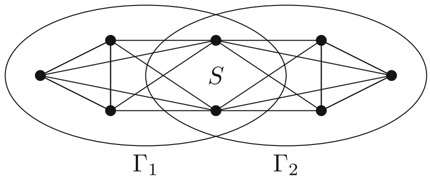


Fig. 6.3 An example of a separation defined by the separator S

We will often use the following view on separators of a graph. If S is a separator of Γ and C is a connected component of $\Gamma[V \setminus S]$ then Γ can be written as $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \Gamma[C \cup S]$ and $\Gamma_2 = \Gamma[V \setminus C]$ are two graphs whose vertex sets intersect in S with the property that there are no edges in Γ connecting vertices in C with vertices in $V \setminus (C \cup S)$. The pair $\{\Gamma_1, \Gamma_2\}$ is a separation of Γ defined by S and C. Figure 6.3 shows an example of such a separation.

The following simple lemma will be useful.

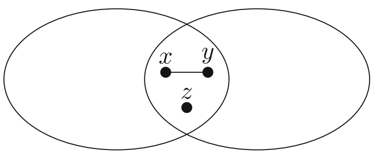
Lemma 6.5 *Let S be a minimum separating set of Γ. Then, every vertex in S is adjacent to a vertex in each connected component of $\Gamma - S$.*

Proof Suppose that $x \in S$ is not adjacent to a component C of $\Gamma - S$. Then, with $S' = S \setminus \{x\}$, C is still a connected connected component of $\Gamma - S'$, contradicting the minimality of $|S|$. □

Lemma 6.6 *Let Γ be a 3-connected graph, $\Gamma \neq K_4$. There is an edge $e \in \Gamma$ such that Γ/e is still 3-connected.*

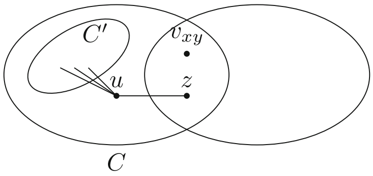
Proof Suppose that Γ/e is not 3-connected for every edge $e = xy \in E(\Gamma)$.

Let v_{xy} be the vertex of Γ/e resulting from the contraction of e. Every separator of Γ/e not containing v_{xy} is also a separator of Γ. Moreover, for every minimum separator $\{v_{xy}, z\}$ of Γ/e, the set $S = \{x, y, z\}$ is a minimum separator of Γ. Therefore, every separator of Γ/e with cardinality less than three must contain v_{xy} and, once this vertex is split, it corresponds to a minimal separator of Γ. It follows that Γ/e is 2-connected. Moreover, for every minimal separator $\{v_{xy}, z\}$ of Γ/e the set $S = \{x, y, z\}$ is a minimal separator of Γ.



For every edge $e = xy$ choose $z \in V(\Gamma)$ such that $\{v_{xy}, z\}$ is a separator of Γ/e and choose the smallest component C of $(\Gamma/xy) - \{v_{xy}, z\}$. From all such possibilities of $e = xy$, z and C, choose one in which C has the smallest possible cardinality.

Since $\{v_{xy}, z\}$ is a minimal cut in Γ/xy, z is adjacent to a vertex $u \in C$. We will show that the choice of $e' = uz$ and some z' results in a separator $\{v_{uz}, z'\}$ of Γ/uz with a component C' with $|C'| < |C|$, contradicting the minimality of $|C|$.



Since $\{x, y, z\}$ is a separator of Γ and C is one of its components, all neighbours of $u \in C$ different from x, y and z belong to C.

There is some vertex z' such that $\{v_{uz}, z'\}$ is a separator of Γ/uz and, as discussed before, $\{u, z, z'\}$ is a separator of Γ. Since x and y are adjacent, they belong to the same connected component of $\Gamma - \{u, z, z'\}$. By Lemma 6.5, u is adjacent to all other connected components. Let C' be such a connected component. Since all neighbours of u different from x and y are contained in C, it follows that $C' \subset C \setminus \{u\}$, giving the claimed contradiction in our choice of C and hence, to the initial assumption.

□

We are now in a position to prove a structural characterisation of 3-connected graphs.

Theorem 6.7 (Tutte) *Every* 3-*Connected Graph* Γ *contains a sequence*

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$$

such that

1. $\Gamma_0 = K_4$,
2. $\Gamma_i = \Gamma_{i+1}/xy$ *for some* $e = xy \in E(\Gamma_{i+1})$ *such that* $d_{\Gamma_{i+1}}(x), d_{\Gamma_{i+1}}(y) \geqslant 3$.

Proof Suppose that Γ is 3-connected. By Lemma 6.6, there is an edge $e \in E(\Gamma)$ whose contraction Γ/e results in a graph which has one vertex less and is still 3-connected. By iterating this procedure, we obtain a sequence as claimed. Note that the only 3-connected graph with four vertices is K_4.

Reciprocally, a graph containing a sequence as described is 3-connected. To see this it suffices to show that, if Γ_i is 3-connected, then a graph Γ_{i+1}, with the property that $\Gamma_i = \Gamma_{i+1}/xy$ for some edge xy, such that $d_{\Gamma_{i+1}}(x), d_{\Gamma_{i+1}}(y) \geqslant 3$, is also 3-connected. Suppose not and let S be a separator of Γ_{i+1} with two vertices. It cannot

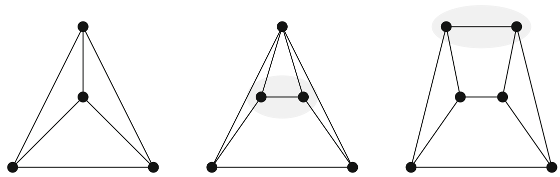


Fig. 6.4 A construction of a 3-connected graph with 6 vertices starting from K_4

be that $S = \{x, y\}$, since otherwise the contracted edge v_{xy} would be a separator of Γ_i. It also cannot be that S is disjoint from $\{x, y\}$, since S otherwise would be a separator of Γ_i. If $S \cap \{x, y\} = \{x\}$ then y is isolated in a singleton component of $\Gamma_{i+1} \setminus S$ since other vertices of that component would be separated by $S \setminus \{x\} \cup \{v_{xy}\}$ in Γ_i. But then this implies that y has degree at most two. □

It follows from Tutte's theorem that every 3-connected graph can be constructed from K_4 by splitting a vertex into two adjacent vertices and connecting them to the old neighborhood distributing the edges among the new two vertices such that each one has degree at least three (Fig. 6.4).

6.3 Menger's Theorem

Menger's theorem connects two dual notions of connectivity: separating sets and number of disjoint paths connecting two sets. Let $A, B \subset V(\Gamma)$ be two sets of vertices. An AB-**separator** is a set S of vertices such that there are no paths connecting A with B in $\Gamma - S$. A vertex in $A \cap B$ is connected by itself by a path of length 0 in this definition, which implies that every AB-separator contains $A \cap B$. An AB-**connector** is a subgraph $\Gamma' \subset \Gamma$ each of its connected components is a path containing precisely one vertex in A and one vertex in B. A graph with no edges can also be an AB-connector, formed by isolated vertices in $A \cap B$.

Theorem 6.8 (Menger, Local Version) *Let A, B be two nonempty subsets of vertices of a graph Γ. The cardinality of a minimum AB-separator equals the maximum number of components (paths) in an AB-connector.*

Proof Let S be a minimal AB-separator and Γ' an AB-connector containing c paths. It is clear that every separator must contain one point of every path in Γ' so $|S| \geqslant c$.

We will prove that there is an AB-connector with $|S|$ paths, by induction on the number of edges of Γ. If Γ is edgeless one can take $A \cap B$ as both, a maximal AB-connector and minimal AB-separator.

Suppose Γ is not edgeless and let s be the cardinality of a minimum AB-separator in Γ.

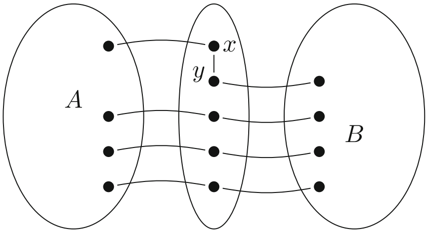


Fig. 6.5 The construction of the AB-connector $\Gamma_1 \cup \Gamma_2 \cup \{xy\}$ in Theorem 6.8

Let $e = xy$ be an edge of Γ. The statement holds in $\Gamma - e$ by induction. If a minimum separator of $\Gamma - e$ has the same cardinality s as in Γ then we are done, as an AB-connector in $\Gamma - e$ is also an AB-connector in Γ.

Suppose that S' is an AB-separator in $\Gamma - e$ with $|S'| < s$. Since $S_1 = S' \cup \{x\}$ and $S_2 = S' \cup \{y\}$ are both AB-separators in Γ, they both have s vertices which implies that $|S'| = s - 1$.

Let S'' be an AS_1-separator in $\Gamma - e$. Observe that S'' is also an AB-separator in Γ, since every path connecting A and B either uses the edge $e = xy$ or intersects a vertex of S'. In particular $|S''| \geqslant s$. By induction, there is an AS_1-connector Γ_1 in $\Gamma - e$ with s paths, thus meeting each point of S_1 precisely once. The same argument applied to an S_2B-separator gives an S_2B-connector Γ_2 with s paths. Now $\Gamma_1 \cup \Gamma_2 \cup \{xy\}$ is an AB-connector with s paths (Fig. 6.5). □

Theorem 6.9 (Menger, Global Version) *A graph Γ with $|V(\Gamma)| > k$ is k-connected if and only if every pair of vertices is joined by k internally disjoint paths.*

Proof Let $x, y \in V(\Gamma)$. Take $A = N(x)$ and $B = N(y)$. Let S be an AB-separator. If $|S| < k$ then we can separate x and y by S contradicting that the graph is k-connected. By Theorem 6.8, there is an AB-connector with k paths. Together with the edges joining A with x and B with y one obtains k internally disjoint paths. □

Menger's theorem is a central result in combinatorics belonging to a family of results called min–max theorems. The theorem of Hall, Theorem 4.3, on the existence distinct representatives of a family of sets, or on the existence of a matching in bipartite graphs, Theorem 5.3, are examples of such results. As an illustration, we show an application of Menger's theorem to prove the following theorem of Ford and Fulkerson.

Let $\{A_1, \dots, A_m\}$ and $\{B_1, \dots, B_m\}$ be two families of subsets of a ground set X. A common system of distinct representatives is a set $\{x_1, \dots, x_m\} \subset X$ such that, for some permutations σ, τ of $\{1, \dots, m\}$, we have $x_i \in A_{\sigma(i)} \cap B_{\tau(i)}$ for each i.

Theorem 6.10 *The families of subsets $\{A_1, \dots, A_m\}$ and $\{B_1, \dots, B_m\}$ have a common system of distinct representatives if and only if for each pair $I, J \subset \{1, \dots, m\}$,*

$$|(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)| \geqslant |I| + |J| - m.$$

Proof Construct the graph Γ with vertex set

$$V(\Gamma) = \{s\} \cup \{A_1, \dots, A_m\} \cup \{v_1, \dots, v_m\} \cup \{B_1, \dots, B_m\} \cup \{t\},$$

and edge set

$$E(\Gamma) = \{sA_i : i \in \{1, \dots, m\}\} \cup \{A_i v_x : i \in \{1, \dots, m\}, x \in A_i\}$$

$$\cup \{B_j v_y : i \in \{1, \dots, m\}, y \in B_i\} \cup \{B_j t : j \in \{1, \dots, m\}\}.$$

See Fig. 6.6 for an example.

We observe that there is a common system of distinct of representatives if and only if there are m internally disjoint paths joining s and t in Γ. Indeed, for any such path with vertices s, A_i, x, B_j, t, the vertex x can be taken to be the common representative of A_i and B_j.

By Menger's theorem, such a set of paths exists if and only if every $\{s, t\}$-separator of Γ has more than m vertices.

Let S be an $\{s, t\}$-separator and set

$$I = \{v_i \in \{v_1, \dots, v_m\} : A_i \notin S\}$$

and

$$J = \{v_j \in \{v_1, \dots, v_m\} : B_j \notin S\}.$$

By the definition of I and J, we have that S contains $\{v_1, \dots, v_m\} \setminus I$ and $\{v_1, \dots, v_m\} \setminus J$.

Moreover, S must contain

$$(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)$$

since if there is a

$$k \in (\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j) \setminus S$$

then there is a path joining s and t which passes through v_k.

Therefore

$$|S| \geqslant |(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)| + (m - |I|) + (m - |J|)$$

which implies

$$|S| \geqslant |I| + |J| - m + (m - |I|) + (m - |J|) = m.$$

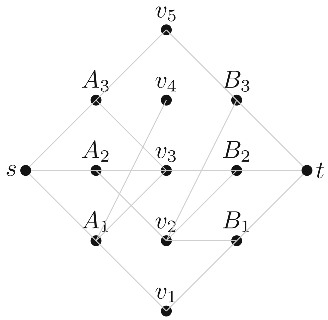


Fig. 6.6 An example of the graph in the proof of Theorem 6.10 for the sets $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 5\}$ and $B_1 = \{1, 2\}$, $B_2 = \{2, 3\}$, $B_3 = \{2, 5\}$

□

The **connectivity** $\kappa(\Gamma)$ of a graph Γ is the largest k such that Γ is k-connected. It follows from Menger's theorem (or from the definition) that $\kappa(\Gamma) \leqslant \delta(\Gamma)$, the minimum degree of Γ. Even if large minimum degree does not ensure high connectivity, the following theorem of Mader gives some connection. Recall that $\Delta(\Gamma)$ indicates graph's maximum degree.

Theorem 6.11 (Mader) *A graph Γ with average degree $\bar{d}(\Gamma) = 4k$ contains a k-connected subgraph Γ' with average degree $\bar{d}(\Gamma') > \bar{d}(\Gamma) - 2k$.*

Proof We observe that

$$n > \Delta(\Gamma) \geqslant \bar{d}(\Gamma) \geqslant 4k$$

and

$$m = \frac{n\bar{d}(\Gamma)}{2} \geqslant 2kn.$$

We will prove, by induction, the stronger statement that if $n \geqslant 2k - 1$ and

$$m \geqslant (2k-3)(n-k+1)+1$$

then Γ has a k-connected subgraph with average degree larger than $\bar{d}(\Gamma) - 2k$.

If $n = 2k - 1$ then $m \geqslant n(n-1)/2$ so that $\Gamma = K_n$ satisfies the claim.

Suppose $n \geqslant 2k$. If Γ is k-connected then there is nothing to prove. Furthermore, if $\delta(\Gamma) \leqslant 2k-3$, we can apply induction on $\Gamma - x$, where x is a vertex of minimum degree in Γ. Therefore, we can suppose that $\delta(\Gamma) \geqslant 2k - 2$.

Let S be a separator in Γ with cardinality $|S| < k$ and let $\Gamma_1, \Gamma_2 \subset \Gamma$ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Gamma[S]$. Let $n_i = |V(\Gamma_i)|$ and $m_i = |E(\Gamma_i)|$. Since $\delta(\Gamma) \geqslant 2k - 2$ and all neighbours of a vertex in $\Gamma_1 \setminus \Gamma_2$ are in Γ_1 we have $n_1 \geqslant 2k-2$ and $n_2 \geqslant 2k-2$ for the analogous reason. Since $n \geqslant n_1+n_2-(k-1)$, one of the two satisfies the induction hypothesis, otherwise

$$m \leqslant m_1 + m_2 < (2k-3)(n_1+n_2-2k+2) \leqslant (2k-3)(n-k+1).$$

□

6.4 Edge Connectivity

The notion of vertex connectivity can be translated to edge-separators. A set $L \subset E(\Gamma)$ is an edge-separator of a graph Γ if $\Gamma - L$ is not connected. A graph Γ is k-**edge-connected** if $\Gamma - L$ is connected for every set $L \subset E(\Gamma)$ with $|L| < k$ edges. If an edge e has the property that $\Gamma - e$ has more connected components than Γ then we say that e is a **bridge**. The minimum k such that Γ is k-edge-connected is the **edge-connectivity** of Γ, which is denoted by $\lambda(\Gamma)$.

The following proposition lists some basic properties of edge connectivity.

Proposition 6.12 *For any graph* Γ,

i. $\kappa(\Gamma) \leqslant \lambda(\Gamma) \leqslant \delta(\Gamma)$.
ii. *every minimal edge-separator of a connected graph separates the graph in two connected components.*
iii. *if* Γ *is* k*-edge-connected then, for every edge* $e \in E(\Gamma)$*, the graph* $\Gamma - e$ *is* $(k-1)$*-edge-connected.*

Proof

i. Suppose L is an edge-separator of Γ. A subset S of vertices which cover the edges of L is a (vertex) separator for Γ and there is some such separator such that $|S| \leqslant |L|$. Hence, $\kappa(\Gamma) \leqslant \lambda(\Gamma)$. If v is a vertex of minimum degree then the set of edges L, incident with v, is an edge-separator of Γ of size $\delta(\Gamma)$. Hence, $\lambda(\Gamma) \leqslant \delta(\Gamma)$.
ii. Suppose L is an edge-separator of Γ. If $\Gamma - L$ is has more than two connected components then $L - e$ is an edge separator for $\Gamma - L$.
iii. This is immediate.

□

We now use Menger's theorem (local version), Theorem 6.8 to prove a similar result for edge-connectivity.

Theorem 6.13 (Menger) *A graph* Γ *is* k*-edge-connected if and only if every pair of vertices can be joined by* k *edge-disjoint paths.*

Proof If every pair of vertices can be joined by k edge-disjoint paths then we must remove at least k edges to disconnect Γ. Hence, Γ is k-edge-connected.

Suppose Γ is k-edge-connected.

Recall that the line graph $L(\Gamma)$ of Γ has the edge set $E(\Gamma)$ as vertex set and two edges are adjacent whenever they are incident in Γ. A set $S \subset E(\Gamma)$ is an edge-separator of Γ if and only if it is a (vertex) separator of $L(\Gamma)$.

Take two vertices $x, y \in V(\Gamma)$ and let A be the set of edges incident with x and let B be the set of edges incident with y. The sets A and B are subsets of *vertices* of $L(\Gamma)$. From the previous paragraph, an AB-separator of $L(\Gamma)$ has size at least k. Thus, by Theorem 6.8, there is an AB-connector with at least k components (paths).

The vertices on these paths in $L(\Gamma)$ describe the edges on disjoint paths in Γ which join a neighbour of x to a neighbour in y. Each of these can then be extended to a path from x to y by adding an edge incident with x and an edge incident with y. □

6.5 Notes and References

Menger's theorem is one of the central theorems in graph theory. The simple proof of the theorem is taken from Goring (2000). The theorem on common distinct representatives was obtained by Ford and Fulkerson (1958), as an application of their max-flow/min-cut theorem, which is one of many min-max theorems equivalent to Menger theorem.

6.6 Exercises

Exercise 6.1 Let Γ be 2-connected. Show (without using Menger's theorem) that every pair of edges is contained in a cycle.

Exercise 6.2 Let Γ be 2-connected different from K_3. Show that, for each edge e, either $\Gamma - e$ or Γ/e is 2-connected.

Exercise 6.3 Let Γ be 3-connected and let xy be an edge of Γ. Show that Γ/xy is 3-connected if and only if $\Gamma - \{x, y\}$ is 2-connected.

Exercise 6.4 Show that if Γ is k-connected, $k \geqslant 2$, then for every k vertices there is a cycle containing them.

Exercise 6.5 Let Γ be k-connected. Show that, for every edge $e \in E(\Gamma)$, $\Gamma - e$ is $(k-1)$-connected.

Exercise 6.6 Let S, S' be distinct minimal separating sets of a graph Γ. Show that, if S intersects at least two connected components of $\Gamma - S'$ then S' intersects each component of $\Gamma - S$ (and S intersects every component of $\Gamma - S'$).

Exercise 6.7 Give an example of a k-edge-connected graph Γ with vertex connectivity $\kappa(\Gamma) = 1$.

Exercise 6.8 Show that a cubic 3-edge connected graph is also 3-connected.

Exercise 6.9 Prove Hall's theorem on the existence of a perfect matching in a bipartite graph by using Menger's theorem.

Exercise 6.10 Show that the n-cube $Q^n = K_2 \times \cdots \times K_2$ (n-times, cartesian product) is n-connected.

Exercise 6.11 A k-**split** of a graph Γ is the graph H obtained from Γ by replacing one vertex x by two adjacent vertices x_1, x_2 such that $N_H(x_1) \cup N_H(x_2) = N_\Gamma(x) \cup \{x_1, x_2\}$ and $d_H(x_1), d_H(x_2) \geqslant k$. Show that, if Γ is k-connected then every k-split of Γ is k-connected.

Exercise 6.12 Let Γ be a k-regular, k-connected graph with an even number of vertices.

For each non-empty subset W of vertices of Γ, let U be the set of odd components of $\Gamma \setminus W$. Consider the bi-partite graph Γ_W with stable sets U and W, where $u_i \in U$ is joined by an edge to $w_j \in W$ if and only if the odd component u_i is joined to w_j in the graph Γ.

i. Prove that if W is separating then $\deg u_i \geqslant k$, $\deg w_j \leqslant k$ and hence $|U| \leqslant |W|$.
ii. Prove that Γ has a perfect matching.
iii. Prove that a 3-connected graph with an even number of vertices which does not have a perfect matching has at least 8 vertices and construct such a graph with 8 vertices.