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Simeon Ball
Oriol Serra

A Course in Combinatorics and Graphs

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A Course in Combinatorics and Graphs

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Preface

This book consists of lecture notes given as a fourth-year undergraduate course of the mathematics degree at the Universitat Politècnica de Catalunya. The course consists of four classes of one hour each week during a 14 week term. During this time, roughly two thirds of the classes are dedicated to lectures and one third to solving the exercises. Each exercise is assigned to at least one student, so all exercises are solved in class. Most of the exercises are taken from exams which we have set over the years. They generally fall into the Goldilocks zone; problems which are not too easy whilst being not too difficult or long-winded. In the main part, these exercises are original and, together with the organisation and style, form the most innovative aspect of the book.

The students have studied a first-year course in Discrete Mathematics in which they learn the definitions and some simple results on graphs and basic combinatorial counting, so these are assumed.

The text falls into three parts. The first chapters are about enumeration, the first chapter shows how to count combinatorial objects using generating functions, the second deals with how to count labelled combinatorial objects using exponential generating functions and the third chapter details how to count combinatorial objects up to symmetry. The fourth chapter is a standalone chapter on finite geometries and Latin squares. The third part of the book concerns graphs, where in each chapter a different graph property is investigated, namely matching, connectivity, planarity and colouring. The final chapter is on extremal graph theory, which is the study of graphs which have a critical behaviour with respect to some graph parameter.

We would like to thank all the students who have taken this course at the Universitat Politècnica de Catalunya during the many years we have had the opportunity to teach this material. Their input and enthusiasm has contributed greatly to the quality of the text and the exercises. We would like to thank our colleagues Marc Noy, Lluis Vena and Clement Requilé for their comments and corrections.

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Barcelona, Spain — Oriol Serra
December 2023

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Symbolic Enumeration

1

Generating functions provide a standard tool for enumeration. In this chapter we combine the use of generating functions with the so-called symbolic method which provides a simple systematic way of obtaining the generating function of a class of combinatorial objects by a symbolic description of the class. Generating functions can be thought of as analytic complex functions or can be viewed simply as formal power series, by disregarding convergence issues. Although we will not completely ignore the analytic perspective, we will mostly adopt this latter point of view. Basic definitions and results on formal power series are included, which includes the useful Lagrange inversion formula.

1.1 Formal Power Series

Consider the set of sequences of complex numbers

$$\mathbb{C}^{\mathbb{N}} = \{(a_0, a_1, \ldots) : a_i \in \mathbb{C}\}.$$

Defining addition coordinate by coordinate,

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1, \ldots)$$

and multiplication by convolution,

$$(a_0, a_1, \ldots, a_n, \ldots)(b_0, b_1, \ldots, c_n, \ldots) = (a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, \sum_{k=0}^{n} a_k b_{n-k}, \ldots)$$

one obtains the ring $\mathbb{C}[[z]]$ of formal power series.

This becomes evident if we identify the sequence $(0, 1, 0, \ldots)$ with the symbol z, so that

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$$(a_0, a_1, \ldots) = \sum_{n \geqslant 0} a_n z^n.$$

Proposition 1.1 *$A(z) \in \mathbb{C}[[z]]$ has multiplicative inverse if and only if $a_0 \neq 0$.*

Proof Suppose that there is $B(z) \in \mathbb{C}[[z]]$ such that

$$A(z)B(z) = 1.$$

Then $a_0 b_0 = 1$, so $a_0 \neq 0$.

Reciprocally, if $a_0 \neq 0$ then one can obtain the coefficients of $B(z)$ such that $A(z)B(z) = 1$ from the equations

$$a_0 b_0 = 1 \text{ implies } b_0 = a_0^{-1}, \text{ and}$$
$$\sum_{k=0}^{n} a_k b_{n-k} = 0 \text{ implies } b_n = -a_0^{-1} \sum_{k=1}^{n} a_k b_{n-k},$$

which provide the values of b_n for all $n \geqslant 0$. □

We will use the notation

$$[z^n]A(z)$$

to denote the n-th coefficient a_n of the formal power series $A(z)$.

A brief recap of expansions in power series will be useful. They can be obtained in the context of formal power series, or, in the case of the elementary functions, as a shorthand definition of the series.

$$\left(\frac{1}{1-az}\right)^k = \sum_{n \geqslant 0} \binom{n+k-1}{k-1} a^n z^n, \ k \geqslant 1,$$

$$(1+z)^\alpha = \sum_{n \geqslant 0} \binom{\alpha}{n} z^n, \alpha \in \mathbb{R},$$

$$e^z = \sum_{n \geqslant 0} \frac{z^n}{n!},$$

$$\ln\left(\frac{1}{1-z}\right) = \sum_{n \geqslant 1} \frac{z^n}{n},$$

$$\sin(z) = \sum_{n \geqslant 1} (-1)^{n+1} \frac{z^{2n-1}}{(2n-1)!},$$

$$\cos(z) = \sum_{n \geqslant 0} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Let $A(z), B(z)$ be a formal power series with $b_0 = 0$. We can define the composition of two series by

$$A(B(z)) = \sum_{n \geqslant 0} a_n (B(z))^n.$$

This is well defined since the n-th coefficient of $C = A(B(z))$ is obtained by the finite sum

$$[z^n]C(z) = \sum_{k \geqslant 0} a_k [z^n]((B(z))^k) = \sum_{k=0}^{n} a_k [z^n](B(z)^k).$$

We note that if $b_0 \neq 0$ then the above expression may lead to a series which can be divergent in $\mathbb{C}$. On the other hand, in analysis one can expand in power series the composition $\exp(1+z)$, say, which would not be admitted in the setting of formal power series.

1.2 Combinatorial Classes

A **combinatorial class** $\mathcal{A}$ is a countable family of combinatorial objects equipped with a **size** function

$$|\cdot| : \mathcal{A} \to \mathbb{N},$$

with the condition that the number a_n of objects in $\mathcal{A}$ with size n is finite for each natural n.

Examples are the class of natural numbers $\mathbb{N}$ with $|n| = n$, the class of subsets of $\{1, \ldots, n\}$ with size the cardinality, or the class of permutations,where its size is the size of the ground set on which the permutations are defined.

The **ordinary generating function** of a class $\mathcal{A}$ is defined as

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} = \sum_{n \geqslant 0} a_n z^n,$$

where a_n is the number of objects of size n in $\mathcal{A}$.

The **symbolic method** in enumeration consists of translating formal (symbolic) descriptions of a combinatorial class into algebraic operations of the corresponding generating functions. The symbolic method often provides a simple and clean method to obtain generating functions of combinatorial classes built upon simpler ones.

The **disjoint union** $\mathcal{A} + \mathcal{B}$ of two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ is the class formed by all objects in $\mathcal{A}$ and all objects in $\mathcal{B}$. If $\gamma \in \mathcal{C}$ then it is either in $\mathcal{A}$ and its size is as it is in $\mathcal{A}$ or it is in $\mathcal{B}$ and its size is as it is in $\mathcal{B}$.

Proposition 1.2 *If $\mathcal{C} = \mathcal{A} + \mathcal{B}$ then the ordinary generating function for $\mathcal{C}$ is*

$$C(z) = A(z) + B(z).$$

The **cartesian product** $\mathcal{A} \times \mathcal{B}$ of two combinatorial classes is the class

$$\mathcal{C} = \{(\alpha, \beta), \alpha \in \mathcal{A}, \beta \in \mathcal{B}\},$$

equipped with the size function

$$|(\alpha, \beta)|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}.$$

Proposition 1.3 *If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ then the ordinary generating function for $\mathcal{C}$ is*

$$C(z) = A(z)B(z).$$

Proof We have

$$C(z) = \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} z^{|(\alpha,\beta)|} = \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} z^{|\alpha|+|\beta|} = \left(\sum_{\alpha\in\mathcal{A}} z^{|\alpha|}\right)\left(\sum_{\beta\in\mathcal{B}} z^{|\beta|}\right).$$

□

The **sequence** Seq($\mathcal{A}$) of $\mathcal{A}$ is defined as

$$\text{Seq}(\mathcal{A}) = \{\epsilon\} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \cdots$$

and consists of all k-tuples $(\alpha_1, \ldots, \alpha_k)$ of elements of $\mathcal{A}$ with $k \geqslant 0$. The class $\{\epsilon\}$ is a special class containing a single object ϵ of size zero.

Proposition 1.4 *If $\mathcal{C} = Seq(\mathcal{A})$ then*

$$C(z) = \frac{1}{1 - A(z)}.$$

Proof According to the formal description of Seq($\mathcal{A}$) we have

$$C(z) = 1 + A(z) + A(z)^2 + A^3(z) + \cdots ,$$

which in the ring of formal power series is the multiplicative inverse of $(1 - A(z))$. □

1.3 Examples

The above three operations of classes already provide the means to address a large number of enumerative problems. Some additional operations will be discussed in the list of exercises.

A **composition** of a number n is a k-tuple $(a_1, \ldots, a_k)$ of nonzero integers, for some $k \geqslant 1$, such that

$$n = a_1 + \cdots + a_k.$$

Its size is the sum n of its components.

The class $\mathcal{C}$ of compositions can be constructed from the class $\mathbb{N}$ of integers (not including zero) as

$$\mathcal{C} = \mathrm{Seq}(\mathbb{N}).$$

If $N(z) = \sum_{n \geqslant 1} z^n$ denotes the generating function of $\mathbb{N}$ then

$$C(z) = \frac{1}{1 - N(z)} = \frac{1}{1 - \frac{z}{1-z}} = \frac{1-z}{1-2z}.$$

By expanding the power series we get

$$[z^n]C(z) = 2^{n-1}.$$

Observe that this result can be easily obtained by considering n dots in row. By placing or not placing a separator between two consecutive dots, we obtain a composition of n by letting a_i be the number of dots between the $(i-1)$-th separator and the i-th separator, see Fig. 1.1.

The flexibility of the symbolic method is illustrated by considering variations of an enumeration problem. Consider the class $\mathcal{C}_k$ of compositions of a number into k summands. Then

$$\mathcal{C}_k = (\mathbb{N})^k, \text{ and } C_k(z) = \left(\frac{z}{1-z}\right)^k,$$

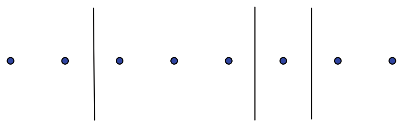


Fig. 1.1 The composition (2, 3, 1, 2) of the number 8

from which

$$[z^n]C_k(z) = \binom{n-1}{k-1}.$$

One can also consider restrictions on the number of parts and on the nature of the parts.

Proposition 1.5 *Let $\mathcal{C}_{A,B}$ be the class of compositions of integers in which the number of parts belongs to $A \subset \mathbb{N}$ and the parts themselves belong to $B \subset \mathbb{N}$. The generating function of $\mathcal{C}_{A,B}$ is*

$$C_{A,B} = \sum_{k \in A} (B(z))^k,$$

where $B(z) = \sum_{b \in B} z^b$ is the generating function of B.

For example, the number of compositions into an odd number of odd parts has generating function

$$\begin{aligned}\sum_{k \geqslant 0} (B(z))^{2k+1} &= B(z) \sum_{k \geqslant 0} (B(z))^{2k} = \frac{B(z)}{1 - B^2(z)} \\ &= \frac{\frac{z}{1-z^2}}{1 - \frac{z^2}{(1-z^2))^2}} = \frac{z(1-z^2)}{(1-z^2)^2 - z^2} \\ &= \frac{z(1-z^2)}{1 - 3z^2 + z^4}.\end{aligned}$$

As another example, the number of compositions of n into parts 1 and 2 is

$$\mathcal{C}_{\{1,2\}} = \mathrm{Seq}(\{1\} + \{2\}),$$

and

$$C_{\{1,2\}}(z) = \frac{1}{1 - z - z^2}.$$

The denominator factorises as

$$1 - z - z^2 = (1 - \phi z)(1 - \bar{\phi} z)$$

where $\phi = \frac{-1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{-1-\sqrt{5}}{2}$. By expanding in power series,

$$C_{\{1,2\}}(z) = \frac{1}{(1-\phi z)(1-\bar{\phi} z)} = \frac{1}{\sqrt{5}}\left(\frac{\phi}{1-\phi z} - \frac{\bar{\phi}}{1-\bar{\phi} z}\right)$$
$$= \frac{1}{\sqrt{5}} \sum_{n \geqslant 0} (\phi^{n+1} - \bar{\phi}^{n+1}) z^n,$$

from which

$$[z^n]C_{\{1,2\}}(z) = \frac{1}{\sqrt{5}}(\phi^{n+1} - \bar{\phi}^{n+1}),$$

is the n-th Fibonacci number.

An **integer partition** is a multi-set $\{a_1, \dots, a_k\}$ where $n = a_1 + \dots + a_k$ is its size. An integer partition can be seen as a sequence of 1, followed by a sequence of 2's and so on, where each sequence can be empty. Therefore,

$$\mathcal{P} = \mathrm{Seq}(\{1\}) \times \mathrm{Seq}(\{2\}) \times \cdots ,$$

where $\{k\}$ is the combinatorial class with 1 object of size k, and so has ordinary generating function z^k.

Hence, the generating function of the class of integer partitions is

$$P(z) = \prod_{k \geqslant 1} \frac{1}{1-z^k}.$$

The Frobenius problem asks for the number of partitions where the parts belong to a set $A \subset \mathbb{N}$. The class $\mathcal{P}_A$ of partitions with parts in $A = \{a_1, a_2, \dots\}$ is

$$\mathcal{P}_A = \mathrm{Seq}(\{a_1\}) \times \mathrm{Seq}(\{a_2\}) \times \cdots ,$$

and its generating function is

$$P_A(z) = \prod_{k \in A} \frac{1}{1-z^k}.$$

For example, if $A = \{1, 2\}$ then

$$P_{\{1,2\}}(z) = \frac{1}{(1-z)} \frac{1}{(1-z^2)} = \frac{1}{4} \frac{1}{(1-z)} + \frac{1}{2} \frac{1}{(1-z)^2} + \frac{1}{4} \frac{1}{(1+z)}.$$

By expanding the above as a power series,

$$P_{\{1,2\}}(z) = \frac{1}{4} \sum_{n \geqslant 0} z^n + \frac{1}{2} \sum_{n \geqslant 0} (n+1) z^n + \frac{1}{4} \sum_{n \geqslant 0} (-1)^n z^n$$

$$= \sum_{n \geqslant 0} \left(\frac{1 + (-1)^n}{4} + \frac{n+1}{2} \right) z^n,$$

which gives

$$[z^n] P_{\{1,2\}}(z) = \begin{cases} (n+2)/2, & n \text{ even} \\ (n+1)/2, & n \text{ odd} \end{cases}$$

again a result which can be obtained by direct analysis.

A **set partition** is a partition of $\{1, \ldots, n\}$ into k parts. The size of a partition is n. The class Π_k denotes the combinatorial class of set partitions into k parts.

A symbolic description of the class of partitions can be obtained with the following encoding of set partitions into k parts. Denote by $\{b_1, \ldots, b_k\}$ the subsets of the partition ordered by its smaller element, so

$$\min b_1 < \min b_2 < \cdots < \min b_k.$$

We construct a word of length n with the letters b_i by placing b_i in the j-th position if and only if $j \in b_i$. The resulting word identifies the partition and it has the property that, for $i < j$, the letter b_j does not appear in the word before b_i. For example,

$$\{\{1, 3, 4\}, \{2, 8\}, \{5, 6, 7\}\} \leftrightarrow (b_1, b_2, b_1, b_1, b_3, b_3, b_3, b_2).$$

The set of such words has the symbolic description

$$\Pi_k = \{b_1\} \times \text{Seq}(\{b_1\}) \times \{b_2\} \times \text{Seq}(\{b_1, b_2\}) \times \cdots \times \{b_k\} \times \text{Seq}(\{b_1, \ldots, b_k\}).$$

Accordingly, its generating function is

$$P_k(z) = \frac{z^k}{(1-z)(1-2z)\cdots(1-kz)}.$$

By decomposing into simple fractions,

$$\frac{1}{(1-z)(1-2z)\cdots(1-kz)} = \sum_{j=1}^{k} \frac{\alpha_j}{(1-jz)},$$

where the value of α_j can be obtained by multiplying both sides by $(1 - jz)$ and setting $z = 1/j$, which gives

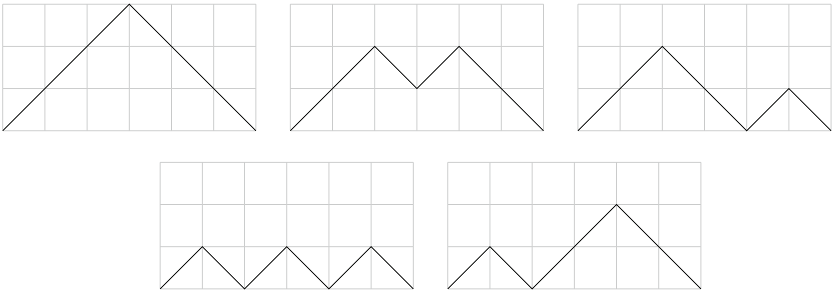


Fig. 1.2 The Dyck paths of length six

$$\alpha_j = \frac{1}{\prod_{i=1,i\neq j}^{k}((j-i)/j)} = (-1)^{k-j}\frac{j^{k-1}}{(j-1)!(k-j)!}.$$

In this way one obtains the formula for the Stirling numbers of the second kind,

$$\left\{ {n \atop k} \right\} = [z^n]P_k(z) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

A **Dyck path** is a sequence of points in the plane integer lattice starting with $(0, 0)$, ending in $(2n, 0)$, and making steps $(1, 1)$ or $(1, -1)$ with the property that the path does not cross the x-axis, i.e. all points have non-negative ordinate (y coordinate). Its size is $2n$.

Figure 1.2 shows the five Dyck paths of length six.

If we represent the steps by $\nearrow$ and $\searrow$ then a Dyck path is a sequence of these two symbols such that the number of $\nearrow$'s is never smaller than the number of $\searrow$'s as we make the path from $(0, 0)$ to $(2n, 0)$. In order to give a symbolic description of this property, we write the following recursive description of the class $\mathcal{D}$ of Dyck paths. By considering the first time the path hits the line $\{y = 0\}$, we can describe (uniquely) a Dyck path as a step $\nearrow$ followed by a Dyck path followed by a step $\searrow$ and another Dyck path. Therefore,

$$\mathcal{D} = \{\epsilon\} + \{\nearrow\} \times \mathcal{D} \times \{\searrow\} \times \mathcal{D}.$$

Note that the use of the class $\{\epsilon\}$ with an only object of size zero is crucial for the correctness of the above description.

This gives, for the generating function, the functional equation

$$D(z) = 1 + z^2(D(z))^2.$$

which gives

$$D(z) = \frac{1 \pm \sqrt{1 - 4z^2}}{2z^2}.$$

The value $D(0) = 1$ indicates the choice of minus sign as the appropriate branch of the solution.

By power expansion we obtain

$$d_{2n} = \frac{1}{n+1}\binom{2n}{n},$$

the n-th Catalan number, for the number of Dyck paths of length $2n$. In Sect. 1.5, we will see a more efficient method to obtain this coefficient directly from the functional equation.

1.4 Rooted Plane Trees

A **rooted plane tree** is a rooted tree drawn in the plane, so that the order of the subtrees pending from the root is taken into account. In other words, a plane tree is a rooted tree in which for every node, there is an ordering of its children. Thus, the two trees in Fig. 1.3 are distinct as plane trees.

Let $\mathcal{T}$ be the class of rooted plane trees, its size being the number of nodes. The class admits a recursive functional description as

$$\mathcal{T} = \mathcal{N} \times \mathrm{Seq}(\mathcal{T}),$$

where $\mathcal{N}$ is a class with a single object of size one. So, a plane tree is a root together with a sequence of plane trees.

This gives the generating function

$$T(z) = z\frac{1}{1 - T(z)}$$

from which we obtain

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2},$$

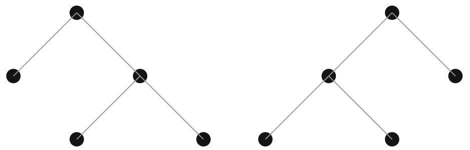


Fig. 1.3 Two distinct rooted plane trees

where the minus sign is chosen so that $T(0) = 0$. By expanding in power series one gets

$$T_n = [z^n]T(z) = \frac{1}{n}\binom{2n-2}{n-1},$$

the $(n-1)$-th Catalan number.

The symbolic method shows its flexibility once more by allowing one to consider restricted classes of plane trees according to the number of children of each node.

Proposition 1.6 *Let $\mathcal{T}_U$ be the class of rooted plane trees in which the number of children of every node is in $U \subset \mathbb{N}$. Its generating function satisfies the functional equation*

$$T_U(z) = z(U(T(z))), \text{ where } U(z) = \sum_{n \in U} z^n.$$

A **binary tree** is a rooted plane tree in which every node has either two or zero children. Let $\mathcal{B}$ be the class of binary trees where the size of a tree is the number of *internal nodes* (not the number of nodes this time). A binary tree is a root from which two binary trees are hanging. The root has now size zero, so

$$\mathcal{B} = \{\epsilon\} + \mathcal{N} \times \mathcal{B} \times \mathcal{B},$$

which gives the functional equation

$$B(z) = 1 + zB^2(z).$$

Therefore,

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z},$$

and

$$b_n = \frac{1}{n+1}\binom{2n}{n},$$

which is again the n-th Catalan number.

1.5 Lagrange Inversion Formula

A natural extension of the ring of formal power series is its quotient field. In order to obtain a field one can use the **formal Laurent series**,

$$\sum_{n\geqslant -k} a_n z^n,$$

where $k \in \mathbb{N}$, so the series has a finite number of negative powers. The sum and product are defined in the same way as before. The field of formal Laurent series is the quotient field of $\mathbb{C}[[z]]$ denoted by $\mathbb{C}((z))$.

If $k \in \mathbb{Z}$ (in particular if $k < 0$) and $A(z) = \sum_{n\geqslant k} a_n z^n$, the multiplicative inverse of $A(z)$, in the set of formal Laurent series, can be obtained by writing

$$A(z) = z^k(\sum_{n\geqslant 0} a_{n+k} z^n) = z^k A_1(z).$$

If $B_1(z) = \sum_{n\geqslant 0} b_n z^n$ is the multiplicative inverse of $A_1(z)$ in $\mathbb{C}[[z]]$ then

$$B(z) = z^{-k} B_1^{-1}(z) = \frac{b_0}{z^k} + \frac{b_1}{z^{k-1}} + \cdots = \sum_{n\geqslant -k} b_{n+k} z^n$$

is the multiplicative inverse of $A(z)$ in $\mathbb{C}((z))$.

The **formal derivation** of a power series $A(z) \sum_{n\geqslant 0} a_n z^n$ is

$$A'(z) = \sum_{n\geqslant 1} n a_n z^{n-1},$$

which can be easily seen to satisfy the usual properties of derivation, for example the product and chain rules hold. The usual rule for derivation of quotients also holds and extends the notion of derivative to the field of formal Laurent series: if $A(z) = \sum_{n\geqslant -k} a_n z^n$, where $k > 0$, then

$$A'(z) = \sum_{n\geqslant -k} n a_n z^{n-1}.$$

The coefficient a_{-1} of z^{-1} in $A(z)$ is called the **residue** of $A(z)$ (notation again coming from analysis). It is denoted $\mathrm{Res}(A(z))$ and it plays a significant role.

Since $[z^{-1}]A'(z) = 0$,

$$\mathrm{Res}(A'(z)) = 0,$$

for every formal Laurent series $A(z)$.

Moreover, if $A(z) = \sum_{n\geqslant -k} a_n z^n$ then

$$\mathrm{Res}(\frac{A'(z)}{A(z)}) = -k. \tag{1.1}$$

To see the last equality one can write

$$A'(z) = z^{-k-1} \sum_{n \geqslant -k} n a_n z^{n+k}$$

and

$$A(z) = z^{-k} \sum_{n \geqslant -k} a_n z^{n+k},$$

so that the quotient

$$\frac{A'(z)}{A(z)} = \frac{1}{z}(c_0 + c_1 z + \cdots),$$

where $-ka_{-k} = c_0 a_{-k}$.

We can now state the main result of this section.

Theorem 1.7 *Let $A(z) = \sum_{n \geqslant 1} a_n z^n$, that is, $a_0 = 0$ and $a_1 \neq 0$. Let B be the functional inverse of A, i.e.*

$$B(A(z)) = z.$$

Then

$$[z^n]B(z) = [z^{-1}]\frac{1}{nA(z)^n}.$$

Proof Let $B(z) = \sum_{n \geqslant 1} b_n z^n$. The formal derivative of $B((A(z)) = z$ leads to

$$\sum_{k \geqslant 1} j b_j A'(z)(A(z))^{j-1} = 1.$$

For a fixed n we divide both sides of the equality by $nA(z)^n$ and take the residue.

For $j \neq n$,

$$(A(z))^{j-1-n} A'(z) = \frac{1}{j-n}(A(z)^{j-n})'$$

is a derivative and therefore the terms for which with $j \neq n$ have residue zero.

For $j = n$ one gets $A'(z)/A(z)$ whose residue is 1 by (1.1) with $k = -1$, which proves the lemma. □

The Lagrange inversion formula is one of the consequences of Theorem 1.7.

Theorem 1.8 (Lagrange inversion) *Let ϕ be an analytic function with $\phi(0) \neq 0$. Let $Y(z)$ satisfy the functional equation*

$$Y(z) = z\phi(Y(z)).$$

Then

$$[z^n]Y(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n.$$

Proof One can write $Y(z)/\phi(Y(z)) = z$ so that $B(w) = w/\phi(w)$ is the functional inverse of $Y(z)$. By Theorem 1.7 we have

$$[z^n]Y(z) = [w^{-1}]\frac{1}{nB(w)^n} = \frac{1}{n}[w^{n-1}](\phi(w))^n,$$

as claimed. □

Example 1.9 For the class $\mathcal{D}$ of Dyck paths we obtained the functional equation

$$D(z) = 1 + z^2 D^2(z).$$

By writing $U(z) = zD(z)$ we get

$$U(z) = z(1 + U^2(z)).$$

By the Lagrange inversion formula,

$$[z^{2m+1}]U(z) = \frac{1}{2m+1}[t^{2m}](1+t^2)^{2m+1} = \frac{1}{2m+1}\binom{2m+1}{m}.$$

Therefore,

$$[z^{2m}]D(z) = [z^{2m+1}]U(z) = \frac{1}{2m+1}\binom{2m+1}{m} = \frac{1}{m+1}\binom{2m}{m}.$$

□

Example 1.10 A **Motzkin path** is a path with steps $(1, 1)$, $(1, -1)$ and $(1, 0)$ which does not cross the x-axis. By considering the first time the path hits the line $\{y = 0\}$, we can describe (uniquely) a Motzkin path either as a step $\nearrow$ followed by a Motzkin path followed by a step $\searrow$ and another Motzkin path, or a horizontal step $\rightarrow$ followed by a Motzkin path. The combinatorial class $\mathcal{M}$ of Motzkin paths is described by the symbolic equation,

$$\mathcal{M} = \{\epsilon\} + \{\rightarrow\} \times \mathcal{M} + \{\nearrow\} \times \mathcal{M} \times \{\searrow\} \times \mathcal{M}.$$

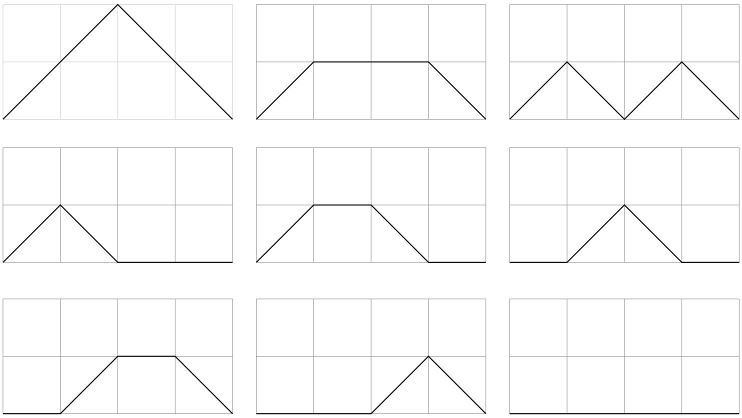


Fig. 1.4 The nine Motzkin path of length four

This leads to the functional equation,

$$M(z) = 1 + zM(z) + z^2M^2(z). \tag{1.2}$$

If we let $U(z) = zM(z)$ then

$$U(z) = z(1 + U(z) + U^2(z)).$$

By Lagrange inversion formula with $\phi(t) = (1 + t + t^2)$ one gets

$$\begin{aligned}[z^n]U(z) &= \frac{1}{n}[t^{n-1}]\phi^n(t) = \frac{1}{n}[t^{n-1}] \sum_{i+j+k=n} \binom{n}{i, j, k} t^{2i+j} \\ &= \frac{1}{n} \sum_{i \geqslant 0} \binom{n}{i} \binom{n-i}{i+1},\end{aligned}$$

and so

$$M_n = [z^{n+1}]U(z) = \frac{1}{n+1} \sum_{i \geqslant 0} \binom{n+1}{i} \binom{n+1-i}{i+1},$$

which gives the sequence 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, . . .

This formula implies that $M_4 = 9$ and Fig. 1.4 shows the 9 Motzkin paths of length four.

□

Example 1.11 A dissection of a convex polygon P is a decomposition of P into polygons by non-crossing diagonals. Let $\mathcal{D}$ be the class of dissections, its size being

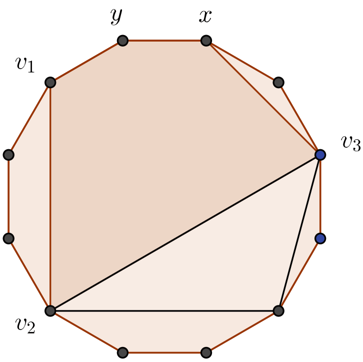


Fig. 1.5 A dissection of the regular 12-gon

the number of vertices of the polygon. We want to find a symbolic description of D and deduce a formula for the number D_n of dissections of a polygon of n sides.

Let δ be a dissection of the polygon. Fix a side xy of the polygon and consider ρ the region containing the edge xy. If the polygon has $r+1$ edges then we can identify the dissection δ with a sequence of r dissections (possibly reduced to a single edge). In Fig. 1.5, the r dissections have sizes 2 (the edge between y and v_1), 4 (from v_1 to v_2), 6 (from v_2 to v_3) and 3 (from v_3 to x). The r-dissections are put together in such a way that $r-1$ of the vertices are counted twice, precisely the vertices v_1, v_2, v_3 in the example. Therefore, a dissection consists of r dissections, in which $r-1$ of the vertices have been counted twice, which gives the D^r/z^{r-1} term in the functional equation for $D(z)$. Observe that a polygon with two vertices is an edge, which has size two and of which there is one dissection.

Therefore, the generating function satisfies the functional equation

$$D = z^2 + \frac{D^2}{z} + \frac{D^3}{z^2} + \cdots + \frac{D^r}{z^{r-1}} + \cdots,$$

which leads to

$$2D^2 - z(1+z)D + z^3 = 0.$$

By writing $D = zU$ we get

$$U = z\frac{U-1}{2U-1}.$$

The coefficients of U can be extracted by the Lagrange inversion formula:

$$[z^n]U = \frac{1}{n}[t^{n-1}]\left(\frac{t-1}{2t-1}\right)^n = \frac{1}{n-1}\sum_{i=0}^{n-2}(-1)^i\binom{n-1}{i}\binom{2n-4-i}{n-2-i}2^{n-2-i}.$$

A useful extension of Lagrange formula is given in the following theorem.

Theorem 1.12 (Bürmann–Lagrange Inversion Formula) *Let ϕ be an analytic function such that $\phi(0) \neq 0$. Let $Y(z)$ satisfy the functional equation*

$$Y(z) = z\phi(Y(z)).$$

Then, for every analytic function g one has

$$[z^n]g(Y(z)) = \frac{1}{n}[t^{n-1}]g'(t)\phi(t)^n.$$

In particular, if $Y(z) = z\phi(Y(z))$ one can obtain a direct expression for the coefficients of $Y^k(z)$ by taking $g(t) = t^k$ in the Bürman–Lagrange formula and get

$$[z^n](Y(z))^k = \frac{1}{n}[t^{n-1}]kt^{k-1}\phi(t)^n = \frac{k}{n}[t^{n-k}](\phi(t))^n.$$

1.6 Notes and References

This symbolic approach to combinatorial enumeration was mostly developed by Flajolet and Sedgewick and their book *Analytic Combinatorics* (Flajolet and Sedgewick, 2009) is an excellent reference on the topic. The Catalan numbers appear in a host of enumeration problems, see for example Stanley (2015). The proof of the Lagrange inversion formula described here is based on Henrici (1964), see also van Lint and Wilson (2001, Appendix).

1.7 Exercises

Exercise 1.1 (Euler) Show that the number of partitions of an integer n into *odd parts* equals the number of partitions in *distinct parts*.

Exercise 1.2 Let $\mathcal{D}_k$ be the class of paths in the plane lattice which start at $(0, k)$ and end at $(n, 0)$ formed by steps $(1, 1)$ or $(1, -1)$ such that the points have non-negative ordinate.

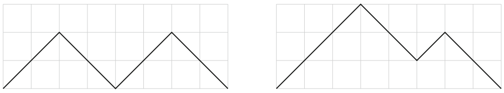
i. Find a symbolic description of $\mathcal{D}_k$ in terms of $\mathcal{D}_0$.
ii. Give an explicit formula for the number of paths of length n in $\mathcal{D}_k$.

Exercise 1.3 (Schroeder paths) Let $\mathcal{S}$ be the class of paths in the plane lattice starting at $(0, 0)$, ending at $(2n, 0)$ with steps $(1, 1), (2, 0), (1, -1)$ such that the ordinate of every point is non-negative. Find the number of such paths with length $2n$.

Exercise 1.4 Let m be a positive integer and let $\mathcal{D}^{\leqslant m}$ be the class of Dyck paths whose height does not exceed m.

i. Show that the generating function of $\mathcal{D}^{\leqslant m}$ is a rational function. Give the generating function for $m = 2$.
ii. Let $\mathcal{S}^{\leqslant \pm m}$ be the class of lattice paths with steps $(1, 1)$ and $(1, -1)$ which start at $(0, 0)$ and end at $(n, 0)$ and that are contained between the lines $y = m$ and $y = -m$. Show that the generating function of $\mathcal{S}^{\leqslant \pm m}$ is a rational function and find an expression for $m = 3$.

Exercise 1.5 Let $\mathcal{D}$ be the class of Dyck paths such that every descent has length exactly two (the two such paths of length 8 are depicted below)



i. Give a symbolic implicit formula for the class and derive a functional equation for the ordinary generating function of the class.
ii. Use the Lagrange inversion to compute D_{4n} the number of paths in the class of length $4n$ in terms of Catalan numbers.
iii. Use the Bürman–Lagrange inversion formula to calculate the number of ordered pairs of paths in $\mathcal{D}$ with total length $4n$.

Exercise 1.6 Let $\mathcal{A}$ be the class of lattice paths starting at $(0, 0)$ and ending at $(3n, 0)$ with steps $(1, 2)$ and $(1, -1)$ which do not cross the x-axis, and let $\mathcal{A}^+$ be the class of paths in $\mathcal{A}$ which do not touch the x-axis except in the initial and last points. The size of a path is its length.

i. Find a symbolic description of $\mathcal{A}$ and deduce a functional equation for its generating function.
ii. By using the Lagrange inversion formula find an explicit formula for a_n. Describe all paths in $\mathcal{A}$ of length six.
iii. Find a symbolic description of $\mathcal{A}^+$ and deduce a functional equation for its generating function. Give an expression for a_n^+ and check it directly for $n = 6$.

Exercise 1.7 Compute the number E_n of rooted plane trees with n nodes such that every node has an even number of children.

Exercise 1.8 Let $\mathcal{W}^{<k}$ be the class of binary words with no k consecutive zeros, the size being their length. Find the generating function of the class $\mathcal{W}^{<k}$.

Exercise 1.9 Let $\mathcal{W}$ be the class of words on the alphabet $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$, the size of a word is its length (including the empty word of size zero).

i. Let $\mathcal{S} \subset \mathcal{W}$ be the class of words which contain no pattern of two repeated consecutive letters and that do not start with a_1.

(i.a) Justify the symbolic identity

$$\mathcal{W} = ((\epsilon + \{a_1\}) \times \mathcal{S}) + (\{a_1\} \times \mathcal{S} \times \mathcal{Z} \times \mathcal{W}) + (\mathcal{S} - \epsilon) \times \mathcal{Z} \times \mathcal{W}).$$

(i.b) Use the above symbolic description to derive the generating function of $\mathcal{S}$.

(i.c) Obtain a formula for S_n. Give a direct combinatorial proof of the latter.

ii Let Π^{nc} be the class of partitions of $[n]$ into k parts so that no part contains two consecutive elements, the size of the partition being the cardinality of $[n]$.

(ii.a) Write a symbolic description of Π^{nc} (recall the symbolic description of the class of all partitions into k parts).

(ii.b) Obtain the generating function $\Pi^{nc}(z)$.

(ii.c) Give an expression for the number of partitions of $[n]$ into blocks not containing consecutive elements. Check your answer for $n = k$ and for $n = k + 1$.

Exercise 1.10 (Euler) Let Π_n be a convex n-gon. A **triangulation** of Π_n is a subdivision of its interior into triangles by noncrossing diagonals.

i. Count the number of diagonals and the number of triangles a triangulation fo Π_n has.
ii. Find the number of triangulations of Π_n.

Exercise 1.11 Let $\binom{[n]}{k}_{\leqslant d}$ denote the set of k-subsets of $[n]$ such that every two consecutive elements are at distance (absolute value of their difference) at most d. For instance $\binom{[4]}{2}_{<2} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Similalry, $\binom{[n]}{k}_{>d}$ denotes the set of k-subsets of $[n]$ such that every two elements are at distance larger than d.

i. Find the generating function $\sum_n \binom{n}{k}_{\leqslant d} z^n$. Deduce that

$$\binom{n}{k}_{\leqslant d} = \sum_j (-1)^j \binom{k-1}{j}\binom{n-jd}{k}.$$

ii. Find the generating function $\sum_n \binom{n}{k}_{>d} z^n$ and an expression for these numbers.

[Hint: It may be useful to start from the formal specification Seq({0}) × Seq(Seq({1} × Seq({0})) of the binary sequences.]

Exercise 1.12 Let k and m be fixed positive integers. Let $\mathcal{T}$ be the class of rooted plane trees in which every node has at most k children, the number of nodes being the size.

i. Give a symbolic implicit formula for the class $\mathcal{T}$ and derive a functional equation for the ordinary generating function.

ii. Use Lagrange inversion to compute T_n, the number of trees in the class with n nodes. Check your answer for $k = 2$ and $n = 3$.
iii. Use the Bürman–Lagrange inversion formula for the number of forests (ordered sequences) of m trees in $\mathcal{T}$ with total number of nodes n.

Exercise 1.13 Let $\mathcal{U}$ be the class of monic polynomials in $\mathbb{F}_q$, the finite field with q elements. Denote by $\mathcal{I}$ the subclass of irreducible polynomials in $\mathcal{U}$. The size of a polynomial is its degree.

i. Give a symbolic relationship among the two classes and deduce a relation among the respective generating functions.
ii. By taking logarithms and expanding in power series, find a relationship between the coefficients of the two generating functions.
iii. By using the Möbius inversion formula find an explicit form of the number of the monic irreducible polynomials of degree n in $\mathbb{F}_q$. How many monic irreducible polynomials of degree three are there in $\mathbb{F}_8$?

Exercise 1.14 A permutation $\sigma = \sigma_1 \dots \sigma_n \in Sym(n)$ contains the pattern 132 if there are subscripts $1 \leqslant i < j < k \leqslant n$ such that $\sigma_i < \sigma_k < \sigma_j$. For example, the permutation 2143 contains the pattern 132 in 243 and in 143, while 4321 does not contain the pattern 132. Let $\mathcal{P}^{132}$ denote the class of permutations avoiding the pattern 132.

i. Justify the equation

$$\mathcal{P}^{132} = \mathcal{E} + \mathcal{P}^{132} \times \mathcal{N} \times \mathcal{P}^{132}.$$

ii. By using the Lagrange inversion formula obtain the number of permutations of n symbols avoiding the pattern 132.

Labelled Enumeration

2

The symbolic method discussed in Chap. 1 may not always be suitable in addressing enumeration problems in combinatorial classes where some natural way of distinguishing objects by its labels appears, for example in the class of permutations. In this chapter, the notion of labelled classes is introduced and the power and flexibility of the symbolic method will again be demonstrated in applications to count classes of permutations, set partitions, labelled trees, words and other combinatorial objects.

2.1 Exponential Generating Functions

The **exponential generating function** of a sequence $a = (a_1, a_2, \cdots)$ of complex numbers is

$$A(z) = \sum_{n \geqslant 0} \frac{a_n}{n!} z^n.$$

We note that, for exponential generating functions, we have

$$a_n = n![z^n]A(z).$$

Exponential generating functions turn out to be a more convenient type of generating functions for labelled classes. The main reason for this is their behaviour with respect to the product. If $A(z)$, $B(z)$ are exponential generating functions of sequences a_n and b_n respectively, then their product

$$A(z)B(z) = \sum_{n \geqslant 0} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) z^n$$

is the exponential generating function of the sequence

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_2

$$c_n = n![z^n]A(z)B(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

The above expression is sometimes called the binomial convolution of the sequences a_n and b_n.

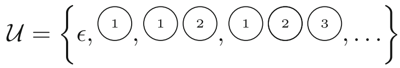
2.2 Labelled Classes

A combinatorial class $\mathcal{A}$ is *labelled* if the objects in $\mathcal{A}$ are labelled graphs, the size of an object being the number of vertices in the graph, and the labels of an object $\alpha \in \mathcal{A}$ of size n are distinct labels in $\{1, \ldots, n\}$ attached to the vertices of the graph.

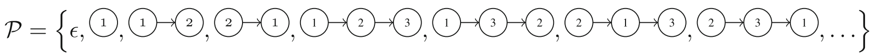
For convenience we also consider the null class ϵ which has a single object of size zero and no labels. We also denote by $\mathcal{N}$ the class with a sole object of size one with label 1.

We first consider some important examples of labelled classes.

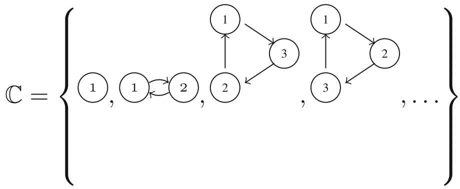
The Class $\mathcal{U}$ of Urns The objects of $\mathcal{U}$ are edgeless labelled graphs. There is a unique way of labelling the n vertices, so there is a unique object of each size.



The Class $\mathcal{P}$ of Permutations The objects of $\mathcal{P}$ are labelled directed paths. There are $n!$ different labelings of a directed path with n vertices.



The Class $\mathcal{C}$ of Cyclic Permutations The objects of $\mathcal{C}$ are labelled directed cycles. There are $(n-1)!$ different labelings of a directed cycle with $n \geqslant 1$ vertices.



In an analogue of the previous section, the **exponential generating function** of a combinatorial class $\mathcal{A}$ is

$$A(z) = \sum_{\alpha} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n \geq 0} A_n \frac{z^n}{n!},$$

where A_n is the number of objects in $\mathcal{A}$ of size n.

As we shall see, the exponential generating function is a more convenient generating function for labelled classes.

For the three examples discussed above, the exponential generating functions are

$$U(z) = \sum_{n \geqslant 0} \frac{z^n}{n!} = e^z,$$

$$P(z) = \sum_{n \geqslant 1} z^n = \frac{1}{1-z},$$

and

$$C(z) = \sum_{n \geqslant 1} \frac{z^n}{n} = -\log(1-z).$$

2.3 Labelled Constructions

As for the unlabelled case, the power of the symbolic method relies on the possibility of describing a class in a formal symbolic way by means of elementary operations. The basic ones are described below.

The **sum** $\mathcal{A} + \mathcal{B}$ of two labelled classes $\mathcal{A}$ and $\mathcal{B}$ is simply its disjoint union. Every object in $\mathcal{A} + \mathcal{B}$ inherits its size and labels from its class.

The labelled product is the most interesting operation. We introduce some convenient terminology as follows. Given any sequence of n pairwise distinct natural numbers $a = (a_1, a_2, \ldots, a_n)$, its **reduction** $\rho(a)$ is the sequence $\sigma(1), \ldots, \sigma(n)$, where $\sigma \in \mathrm{Sym}(n)$ is a permutation of $\{1, \ldots, n\}$, with the property that $\sigma(i) < \sigma(j)$ if and only if $a_i < a_j$. In other words, the reduction is an order preserving map onto $\{1, \ldots, n\}$. For example, $\rho(4, 8, 3, 6, 2) = (3, 5, 2, 4, 1)$.

Let $\mathcal{A}, \mathcal{B}$ be labelled classes. In order to define the labelled product we must define a way to label the pairs $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ with labels in $\{1, 2, \ldots, |\alpha| + |\beta|\}$. The idea is to use all possible labels on (α, β) whose reductions on α and β coincide with the original ones. Therefore, we define

$$\alpha * \beta = \{(\alpha', \beta') \in \mathcal{A} \times \mathcal{B} \mid \rho(\alpha') = \alpha,\ \rho(\beta') = \beta\}.$$

For example, if α is labelled $(1, 2)$ and β is labelled $(1, 3, 2)$ then $\alpha * \beta$ consists of the objects (α', β') with labels

$$(1,2,3,5,4),(1,3,2,5,4),(1,4,2,5,3),(1,5,2,4,3),(2,3,1,5,4),$$
$$(2,4,1,5,3),(2,5,1,4,3),(3,4,1,5,2),(3,5,1,4,2),(4,5,1,3,2).$$

Note that

$$|\alpha * \beta| = \binom{|\alpha| + |\beta|}{|\alpha|}.$$

The **labelled product** of two labelled classes $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} * \mathcal{B} = \bigcup_{\substack{\alpha \in \mathcal{A} \\ \beta \in \mathcal{B}}} (\alpha * \beta).$$

If $\mathcal{C} = \mathcal{A} * \mathcal{B}$ then

$$\begin{aligned} C(z) &= \sum_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \sum_{(\alpha,\beta) \in \alpha * \beta} \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!} \\ &= \sum_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \frac{(|\alpha| + |\beta|)!}{|\alpha|!|\beta|!} \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!} \\ &= \left(\sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} \right) \left(\sum_{\beta \in \mathcal{B}} \frac{z^{|\beta|}}{|\beta|!} \right) \\ &= A(z)B(z), \end{aligned}$$

which explains the use of exponential generating functions instead of ordinary ones. The labelled product is the natural product operation of labelled combinatorial classes.

The **sequence** $\mathrm{Seq}(\mathcal{A})$ of a labelled class $\mathcal{A}$ is defined as

$$\mathrm{Seq}(\mathcal{A}) = \{\epsilon\} + \mathcal{A} + (\mathcal{A} * \mathcal{A}) + (\mathcal{A} * \mathcal{A} * \mathcal{A}) + \cdots = \bigcup_{k \geqslant 0} \mathrm{Seq}_k(\mathcal{A}),$$

where

$$\mathrm{Seq}_k(\mathcal{A}) = \underbrace{\mathcal{A} * \cdots * \mathcal{A} * \mathcal{A}}_{k \text{ times}}.$$

By the expression of generating functions of labelled products, if $\mathcal{C} = \mathrm{Seq}(\mathcal{A})$ then

$$C(z) = 1 + A(z) + A^2(z) + \cdots = \frac{1}{1 - A(z)}.$$

The class of k**-sets** $\mathrm{Set}_k(\mathcal{A})$ of a labelled class $\mathcal{A}$ is

$$\mathrm{Seq}_k(\mathcal{A})/\sim$$

where two objects are identified by the equivalence relation $\sim$ if they only differ on the ordering of its components. The class of sets of $\mathcal{A}$ is

$$\mathrm{Set}(\mathcal{A}) = \bigcup_{k \geq 0} \mathrm{Set}_k(\mathcal{A}).$$

We observe that, in the labelled product, the number of n-tuples of objects in each every equivalence class of $\mathrm{Seq}_n(\mathcal{A}))/\sim$ is $n!$. Accordingly, if $\mathcal{D} = \mathrm{Set}(\mathcal{A})$ then

$$D(z) = 1 + A(z) + \frac{1}{2!}A^2(z) + \frac{1}{3!}A^3(z) + \cdots = e^{A(z)}.$$

This is known as the exponential formula in classical enumerative combinatorics, to which the symbolic method gives a natural and simple derivation.

We summarise the above relations between operations in labelled classes and the corresponding exponential generating functions in the following theorem.

Theorem 2.1 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be labelled classes and denote by $A(z), B(z), C(z)$ their exponential generating functions.*

1. *If $\mathcal{A} = \mathcal{B} + \mathcal{C}$, then $A(z) = B(z) + C(z)$.*
2. *If $\mathcal{A} = \mathcal{B} * \mathcal{C}$, then $A(z) = B(z)C(z)$.*
3. *If $\mathcal{A} = \mathrm{Seq}(\mathcal{B})$, then $A(z) = \frac{1}{1-B(z)}$.*
4. *If $\mathcal{A} = \mathrm{Set}(\mathcal{B})$, then $A(z) = e^{B(z)}$.*

2.4 Permutations

Permutations The class of permutations was defined as the class of labelled directed paths. An alternative symbolic description provides a wealth of enumeration possibilities. Recall that a permutation can be expressed as a product of cycles in a unique way.

Let $\mathcal{C}$ denote the class of cyclic permutations. The number C_n of cyclic permutations of $\{1, \ldots, n\}$ of size n in $\mathcal{C}$ is $(n-1)!$. Therefore, the exponential generating function of $\mathcal{C}$ is

$$C(z) = \sum_{n \geqslant 1} (n-1)! \frac{z^n}{n!} = \log \frac{1}{1-z}.$$

A permutation is a set of disjoint cycles. The class $\mathcal{P}$ of permutations has the formal specification

$$\mathcal{P} = \text{Set}(\mathcal{C}).$$

It follows that the exponential generating function is

$$P(z) = \exp\left(\log \frac{1}{1-z}\right) = \frac{1}{1-z},$$

as we have already seen. However, this specification allows for the flexibility of the symbolic method. The following examples illustrate this fact.

Derangements. A **derangement** is a permutation with no fixed points (that is, with no cycles of length one). The formal specification of the class $\mathcal{D}$ of derangements is

$$\mathcal{D} = \text{Set}\left(\sum_{n \geqslant 2} C_n\right),$$

and thus the generating function is

$$D(z) = \exp\left(\log\left(\frac{1}{1-z}\right) - z\right) = \frac{\exp(-z)}{1-z}.$$

Thus, one obtains directly that

$$D_n = n![z^n]e^{-z}\left(\frac{1}{1-z}\right) = n! \sum_{i=0}^{n} (-1)^i \frac{1}{i!} \approx \frac{n!}{e}.$$

Involutions. An **involution** is a permutation σ with the property that σ^2 is the identity. The cycle decomposition of an involution has only cycles of lengths one or two. The class of involutions is

$$\mathcal{I} = \text{Set}(C_1 + C_2),$$

and its exponential generating function is

$$I(z) = \exp(z + \frac{z^2}{2}).$$

Therefore, the number of involutions of size n is

$$i_n = n![z^n]I(z) = n![z^n]e^z e^{z^2/2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!2^k}.$$

More generally, the class $\mathcal{I}_r$ of permutations satisfying $\sigma^r = 1$ is

$$\mathcal{I}_r = \text{Set}(\sum_{j|r} C_j)$$

and its exponential generating function is

$$I_r(z) = \exp\left(\sum_{j|r} \frac{z^j}{j}\right).$$

Number of Cycles. The class $\mathcal{P}^{(k)}$ of permutations with k disjoint cycles is

$$\mathcal{P}^{(k)} = \text{Set}_k(\mathcal{C}).$$

It has exponential generating function

$$P^{(k)}(z) = \frac{1}{k!}\left(\log \frac{1}{1-z}\right)^k.$$

The number of such permutations of n is the (signless) *Stirling number of first kind*, or the Stirling cycle number,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!}[z^n]\left(\log \frac{1}{1-z}\right)^k = \frac{n!}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1 i_2 \cdots i_k}.$$

Some simple values of the Stirling cycle numbers are

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!, \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}.$$

Of course,

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!.$$

Number of Cycles and Cycle Lengths. We may specify the set $\mathcal{P}_{A.B}$ of permutations which have cycles with length in $A \subset \mathbb{N}$ and a number of cycles which is an integer in $B \subset \mathbb{N}$. The formal specification is

$$\mathcal{P}_{A.B} = \prod_{i \in B} (\sum_{j \in A} \mathcal{C}_j),$$

from which

$$P_{A,B}(z) = \beta(\alpha(z)),$$

where

$$\alpha(z) = \sum_{a \in A} \frac{z^a}{a!}, \quad \beta(z) = \sum_{b \in B} \frac{z^b}{b!}.$$

2.5 Set Partitions

A partition is a collection of nonempty sets. We have already seen an approach to enumerate partitions by describing them as an unlabelled combinatorial class. However, its description as a labelled combinatorial class is more natural: a partition is a set of disjoint subsets. The labelled class of partitions can be described as

$$\mathcal{P} = \mathrm{Set}(\mathcal{U}_1),$$

where $\mathcal{U}_1$ is the class of urns (excluding the empty object). Therefore, the exponential generating function of the class of partitions is

$$P(z) = \exp(\exp(z) - 1).$$

The total number of partitions of $\{1, \ldots, n\}$ is the Bell number B_n. The exponential generating function provides an expression for this number

$$B_n = n![z^n]P(z) = \frac{n!}{e}[z^n] \sum_{k \geqslant 0} \frac{e^{kz}}{k!} = \frac{n!}{e}[z^n] \sum_{k \geqslant 0} \sum_{m \geqslant 0} \frac{k^m z^m}{m!k!} = \frac{1}{e} \sum_{k \geqslant 0} \frac{k^n}{k!},$$

from which one can obtain asymptotic expressions.

The class $\mathcal{P}^{(k)}$ of partitions into k parts is

$$\mathcal{P}^{(k)} = \mathrm{Set}_k(\mathcal{U}_1).$$

Hence,

$$P^{(k)}(z) = \frac{1}{k!}(e^z - 1)^k.$$

This gives an alternative derivation of the Stirling numbers of second kind,

$$\begin{aligned}\left\{ {n \atop k} \right\} &= \frac{n!}{k!}[z^n](e^z - 1)^k \\ &= \frac{n!}{k!}[z^n]\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}e^{mz} \\ &= \frac{n!}{k!}[z^n]\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\sum_{\ell\geqslant 0}\frac{m^\ell}{\ell!}z^\ell \\ &= \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}m^n.\end{aligned}$$

Additional specializations can be obtained as in the case of permutations. For instance, the exponential generating function for the class of partitions with no singletons is

$$\exp(e^z - 1 - z).$$

2.6 Words

Words on an alphabet can also be treated from the perspective of labelled combinatorial classes. A word of length n on an alphabet $A = \{a_1, \ldots, a_r\}$ can be seen as a map

$$f : \{1, \ldots, n\} \to A$$

and it can be specified by the sequence

$$(f^{-1}(a_1), \ldots, f^{-1}(a_r)),$$

a sequence of subsets (including the empty set). Therefore, the class $\mathcal{W}_A$ of words on A can be specified as

$$\mathcal{W}_A = (\mathcal{U})^r,$$

where $\mathcal{U}$ is the class of urns, now including the empty set. This gives

$$W_A(z) = e^{rz} \text{ and } w_{A,n} = r^n,$$

as expected.

If the number of occurrences of the letter i is restricted to a set $A_i \subset \mathbb{N}$ then the symbolic specification is

$$\mathcal{W}_{A;A_1,\dots,A_r} = \mathcal{U}_{A_1} * \cdots * \mathcal{U}_{A_r}$$

where

$$\mathcal{U}_{A_i} = \sum_{a \in A_i} (\mathbb{N})^a / \sim,$$

and so

$$U_{A_i} = \sum_{a \in A_i} \frac{z^a}{a!}.$$

For example, the class of words on an alphabet of r letters in which each letter appears at least twice has exponential generating function

$$W_A^{\geqslant 2} = (e^z - 1 - z)^r$$

2.7 Labelled Trees

Let $\mathcal{T}$ be the class of rooted labelled trees. There is a distinguished vertex (the root) and the nodes of the trees are labelled. The size of the tree is its number of nodes.

A tree in $\mathcal{T}$ consists of a node and a set of trees. Therefore,

$$\mathcal{T} = \mathcal{N} * \mathrm{Set}(\mathcal{T}).$$

The generating function satisfies the equation

$$T(z) = ze^{T(z)}.$$

The Lagrange inversion formula provides the classical Cayley formula for the number of labelled rooted trees with n vertices.

$$T_n = n^{n-1}$$

The above is the well-known Cayley formula for the number of labelled trees (dividing by the n possible roots of a labelled tree).

2.8 Notes and References

For a more comprehensive look at labelled enumeration, see Flajolet and Sedgewick (2009, Chapter 2). The general symbolic approach to enumeration problems can be traced back to Joyal (1981) who derived the exponential formula for the set construction. There are many identities involving the Stirling numbers, for partitions and for permutations, which have not been explored here. For the Stirling permutation numbers there is a closed formula which is more involved than the one obtained here for the Stirling partition numbers. Cayley's formula for the number of spanning trees has many beautiful proofs, see for example Matousek and Nesetril (2008). The one given here is probably the simplest one.

2.9 Exercises

Exercise 2.1 Compute the number $f_n(r)$ of permutations which have no cycles of length r ($f_n(1)$ is the number of derangements). Prove that $\lim_{n\to\infty} f_n(r)/n! = e^{-1/r}$.

Exercise 2.2 Compute the exponential generating function of the permutations which decompose into even cycles. Analogously for the ones decomposing into odd cycles.

Exercise 2.3 Compute the exponential generating function of the permutations which decompose into an even number of cycles. Analogously for the ones decomposing into an odd number of cycles.

Exercise 2.4 Show that the number of permutations of $\{1, \ldots, 2n\}$ whose cycle decomposition contains only even cycles is

$$(2n-1)^2(2n-3)^2 \cdots 3^2.$$

Exercise 2.5 Let $\mathcal{W}^{(k,r)}$ denote the class of words over the alphabet $\{a, b\} \cup \{0, 1\}$ in which every letter appears at most k times and each number appears at least r times. The size of a word is its length.

i. Compute $W_n^{(0,r)}$ for $n > 2(r-1)$, $r \geqslant 1$.
ii. Give an expression of the exponential generating function of $\mathcal{W}^{(k,0)}$.
iii. Compute $W_n^{(2,2)}$ for $n > 6$.

Exercise 2.6 Compute the exponential generating function of set partitions with an odd number of blocks.

Exercise 2.7 Compute the exponential generating function of rooted labelled trees such that the root has exactly k descendants. Find the number of such trees with n nodes.

Exercise 2.8 Let $\mathcal{T}_2$ denote the class of rooted binary labelled trees, that is, every node has zero or two childs, the size being the number of nodes. Let $\mathcal{F}_2$ denote the class of forests in which each connected component is an object in $\mathcal{T}_2$.

i. Find the exponential generating function of $\mathcal{T}_2$.
ii. Give an expression of $T_{2,n}$ in terms of the Catalan numbers. What is the value of $T_{2,n}$ for $n = 1, 2, 3, 4, 5$?
iii. Use the Bürman–Lagrange formula to obtain an expression of $F_{2,n}$, and compute the first four values.

Exercise 2.9 A star is a tree where all but at most one vertex is a leaf. Let $\mathcal{S}$ be the class of rooted labelled star forests (a forest is a set of trees). The size of a star forest is the number of vertices it has.

i. Give a symbolic description of $\mathcal{S}$, the exponential generating function of the class and derive the number of rooted labelled star forests with n vertices. Compute the first few values of these numbers.
ii. Use the above to count the number of maps $f : \{1, \dots, n\} \to \{1, \dots, n\}$ which are idempotent: $f(f(x)) = f(x)$ for all x.
iii. Let $\mathcal{I}^{(3)}$ be the class of maps $f : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $f^3 = f$, the size of a map being n. Give a symbolic description, the exponential generating function of the class and an expression for the number of such maps on $\{1, \dots, n\}$. Compute the first few values of these numbers.

Exercise 2.10 Let $\mathcal{P}_{k,2}$ be the class of partitions of a set into k parts, each of them has cardinality at least two.

1. Find the exponential generating function of $\mathcal{P}_{k,2}$.
2. Show that the number $s_{n,k}$ of doubly surjective maps $f : \{1, \dots, n\} \to \{1, \dots, k\}$ (every pre-image has cardinality at least two) is

$$\sum_{i,j,l:i+j+l=k} \binom{n}{j} \frac{k!}{i!l!} (-1)^{i+j} k^{n-j}.$$

3. Give a formula for the number w_n of words of length n on the alphabet $\{a_1, \dots, a_k\}$ such that each symbol appears at least twice.

Enumeration with Symmetries

3

In the previous chapters we saw some powerful tools which allow us to count many objects, tools which were especially useful if we wanted to count labelled objects. But imagine we wanted to count unlabelled objects, the number of graph on n vertices, for example. This is somewhat complicated by the fact that one has to ascertain when two graph are essentially the same. That is, there is a bijective map from the vertices of one to the vertices of the other which induces a bijection between the edges. To be able to count such objects one must be able to account for these isomorphic copies. Similarly, suppose we wanted to colour the faces of the cube with a set of say r colours. We have to account for the fact that many colourings will essentially be the same colourings when we allow for the rotations of the cube. In this chapter, we will see that such colourings can be counted if we know the symmetries of the object concerned. Polya's theorem tells us that we need only construct a certain polynomial to be able to not only count the number of r-colourings but also the number of colourings with a fixed number of elements coloured a certain colour.

3.1 Group Actions

A **group** is a set G with an associative binary operation which has an identity element e ($xe = ex = x$) and for which every element $x \in G$ has an inverse y ($xy = yx = e$).

A subgroup H of G is a subset of G which is itself a group. A **left coset** of H is

$$\sigma H = \{\sigma h \mid h \in H\},$$

for some $\sigma \in G$. Observe that $\sigma \in \sigma H$ and that $|H| = |\sigma H|$ for any coset σH. Thus the cosets of H partition the elements of G and into equal sized parts, from which it follows that $|H|$ divides $|G|$.

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_3

Let G be a group with identity id and let X be a set.

We say that G **acts on** X if for all $\sigma \in G$ there is a map from X to X satisfying

$$\mathrm{id}(x) = x \quad \text{and} \quad \tau(\sigma(x)) = (\tau\sigma)(x),$$

for all $x \in X$ and $\sigma, \tau \in G$. Observe that we use σ both for the element of G and the map from X to X. It should be clear when we are talking about the map since we use $\sigma(x)$, whereas the group element will be denoted simply as σ.

Observe that if $\sigma(x) = \sigma(y)$ then $\sigma^{-1}(\sigma(x)) = \sigma^{-1}(\sigma(y))$ and so $x = y$. Hence, the map $\sigma : X \to X$ is a bijection, i.e. a permutation of X. We denote by $\mathrm{Sym}(X)$ the set of all permutations of the set X, which forms a group under composition.

If X has some additional structure then G must preserve this structure too. For example, if X is a vector space or the cube or, as in the following example, a graph.

Example 3.1 Let Γ be a graph and let G be a group of symmetries of Γ. Then G acts on the vertices of Γ and it also acts on the edges of Γ.

Example 3.2 Let $G = \{\mathrm{id}, \sigma, \sigma^2\}$ be a group with three elements acting on $X = \{a, b\}$. If $\sigma(a) = b$ then $\sigma(b) = a$. Then, applying σ to $\sigma(a) = b$ gives $\sigma^2(a) = a$ which implies (applying σ) that $a = \sigma(a)$, a contradiction. Hence, the only action of G on X is the action $\sigma(x) = x$ for all $\sigma \in G$.

An action is **faithful** if for all $\sigma \in G$, $\sigma \neq \mathrm{id}$, there is an $x \in X$ such that $\sigma(x) \neq x$. In other words, no element of G, other than the identity, fixes all the elements of X. The action in Example 3.2 is not a faithful action.

For $x \in X$, the **orbit** of X under the action of G

$$\mathrm{orb}(x) = \{\sigma(x) \mid \sigma \in G\}.$$

If there is only one orbit then we say that G acts **transitively** on X, or the action of G on X is **transitive**.

Example 3.3 Let Γ be the graph K_4 and let G be the cyclic group generated by the permutation (1234) (Fig. 3.1). The action of G on the vertices of Γ is transitive, but the action of G on the edges of Γ has two orbits

$$\{(12), (23), (34), (14)\} \quad \text{and} \quad \{(13), (24)\}.$$

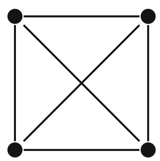


Fig. 3.1 The complete graph with four vertices

Observe that even if we extended the group to the di-hedral group by adding the four reflections, we still only get two orbits for the action on the edges.

For $\sigma \in G$, we denote the fixed elements of X by

$$\text{fix}(\sigma) = \{x \in X \mid \sigma(x) = x\}.$$

For all $x \in X$, we define the **stabiliser subgroup of** x to be

$$G_x = \{\sigma \in G \mid \sigma(x) = x\}.$$

It is clear that G_x is a subgroup of G.

Lemma 3.4 *For $x \in X$,*

$$|G| = |G_x||\text{orb}(x)|$$

and so if G acts transitively on X then $|G| = |G_x||X|$.

Proof For all $y \in \text{orb}(x)$, there is a $\sigma \in G$ such that $\sigma(x) = y$, since G acts transitively on orb(x). Therefore, the coset $\sigma G_x \neq G_x$.

Moreover, if $\tau \in G$ then $\tau(x) = z$ for some $z \in \text{orb}(x)$. If $\sigma G_x = \tau G_x$ then $\tau^{-1}\sigma \in G_x$ which implies $\tau^{-1}\sigma(x) = x$ and so $\sigma(x) = \tau(x)$ and therefore that $z = y$.

Hence, there are $|\text{orb}(x)|$ cosets of G_x and so $|G_x||\text{orb}(x)| = |G|$.

If G acts transitively on X then $\text{orb}(x) = X$. □

The following lemma is called the **orbit-counting lemma**. It will be a direct application of this lemma which will allow us to prove Polya's theorem, Theorem 3.13.

Lemma 3.5 *The number of orbits of G acting on X is*

$$\frac{1}{|G|}\sum_{\sigma \in G} |\text{fix}(\sigma)|.$$

Proof Suppose that G acts transitively on X, i.e. there is only one orbit.

Count pairs (x, σ), where $\sigma(x) = x$. This is

$$\sum_{\sigma \in G} |\text{fix}(\sigma)|.$$

It is also

$$\sum_{x \in X} |G_x|.$$

By Lemma 3.4, $|G_x| = |G|/|X|$.
Hence,

$$\sum_{\sigma \in G} |\text{fix}(\sigma)| = |G|.$$

Suppose that G has orbits $X_1, \ldots, X_t$. Then G acts transitively on X_i. Let

$$\text{fix}_i(\sigma) = \{x \in X_i \mid \sigma(x) = x\}.$$

From above, for all $i \in \{1, \ldots, t\}$,

$$\sum_{\sigma \in G} |\text{fix}_i(\sigma)| = |G|.$$

Clearly,

$$\sum_{i=1}^{t} |\text{fix}_i(\sigma)| = |\text{fix}(\sigma)|.$$

Hence,

$$t|G| = \sum_{i=1}^{t} \sum_{\sigma \in G} |\text{fix}_i(\sigma)| = \sum_{\sigma \in G} |\text{fix}(\sigma)|.$$

□

Example 3.6 As in Example 3.3, let Γ be the graph K_4.

Let G be the cyclic group of generated by the permutation (1234) and consider the action of G on the edges of Γ. The group

$$G = \{\text{id}, (1234), (13)(24), (1432)\}.$$

The permutation id fixes all 6 edges. The permutations (1234) and (1432) fix none. The permutation (13)(24) fixes 2 edges. According to Lemma 3.5, the number of orbits is $(6 + 2)/4 = 2$, which coincides with our observation in Example 3.3.

Consider what happens when we extend G to the di-hedral group by adding the four reflections. Each of these reflections fixes two edges, so the number of fixed elements increases to 16. The size of the group is now 8, so Lemma 3.5 verifies that there are two orbits.

Example 3.7 Let Γ be the graph K_6, let G be the cyclic group of generated by the permutation (123456) and consider the action of G on the edges of Γ. The group

$$G = \{\text{id}, (123456), (135)(246), (14)(25)(36), (153)(264), (165432)\}.$$

The permutation id fixes all 15 edges. The permutations (123456), (135)(246), (153)(264) and (165432) fix none. The permutation (14)(25)(36) fixes 3 edges. Thus, there are 18 elements fixed by the permutations in G. Lemma 3.5 implies that G has three orbits in its action on the edges.

3.2 Group Action on Functions

Given an action of G on a set X, in this section we will define an action of G on the functions defined on the set X. It will be this latter action to which we will apply the orbit counting lemma, Lemma 3.5 and so prove Polya's theorem, Theorem 3.13.

We begin with an example, which will be a function defined on the vertices of a graph and which assigns one of two colours to each vertex.

Example 3.8 Let Γ be the 6-cycle graph and G be a group of automorphisms (symmetries) of Γ. Let $B_i(G)$ denote the number of different ways (under the action of a group G) to 2-colour the vertices so that there are i vertices coloured blue. If C_6 denotes the cyclic group of size 6 then it is fairly easy to calculate $B_i(C_6)$. The four different colourings with three blue vertices are given in Fig. 3.2.

i	0	1	2	3	4	5	6
$B_i(C_6)$	1	1	3	4	3	1	1

Now, consider the different ways to 2-colour the vertices under the action of the di-hedral group D_6. The only i for which $B_i(C_6) \neq B_i(D_6)$ is $i = 3$ when the two central colourings in Fig. 3.2 are equivalent. This gives the following values.

i	0	1	2	3	4	5	6
$B_i(D_6)$	1	1	3	3	3	1	1

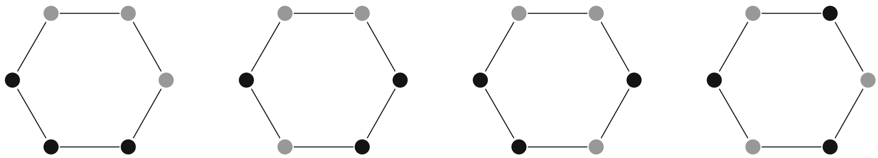


Fig. 3.2 2-colourings of the vertices of the cyclic graph with six vertices

An r-colouring is a map from X to a set C of r colours. Let C^X denote the set of maps from X to a set C of r colours.

Let G be a group acting on a set X. For any element $\sigma \in G$ and $f \in C^X$, define $\sigma(f)$ to be the element of C^X defined by

$$(\sigma(f))(x) = f(\sigma^{-1}(x)).$$

Lemma 3.9 *Let G be a group acting on a set X. The map $f \mapsto \sigma(f)$ defined as above, defines an action of G on C^X.*

Proof Let $\sigma, \tau \in G$. Then

$$\sigma(\tau(f))(x) = \tau(f)(\sigma^{-1}(x)) = f(\tau^{-1}\sigma^{-1}(x)),$$

and

$$(\sigma\tau)(f)(x) = f((\sigma\tau)^{-1})(x) = f(\tau^{-1}\sigma^{-1}(x)).$$

Hence, $\sigma(\tau(f)) = (\sigma\tau)(f)$. □

Example 3.10 As in Example 3.8, let G be the di-hedral group acting on Γ, the 6-cycle graph. Consider the reflective symmetry in the vertical axis, given by the permutation $(12)(36)(45)$. If we label the vertices from the top-left with 1 and continue labelling cyclically clockwise, then this permitation interchanges the two central colourings in Fig. 3.2. The colouring in the left central colouring is given by the function f, where

$$f(1) = f(2) = f(5) = \text{grey}$$

and

$$f(3) = f(4) = f(6) = \text{black}.$$

By definition, the function $\sigma(f)$ is the colouring given by $f(\sigma^{-1}(x))$, which gives

$$\sigma(f)(1) = \sigma(f)(4) = \sigma(f)(6) = \text{grey}$$

and

$$\sigma(f)(2) = \sigma(f)(3) = \sigma(f)(5) = \text{black},$$

which is the colouring in the right central copy of Γ.

One should check that the image of the permutation $\tau = (135)(246)$ on f in the following figure is as claimed (Fig. 3.3).

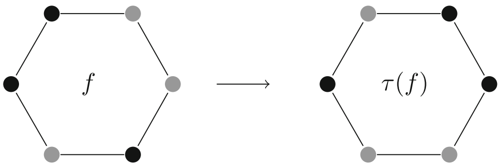


Fig. 3.3 An example of a permutation acting on a function

3.3 The Cycle-Index Polynomial

Suppose G acts on a set X. Equivalently, there is a homomorphism from G to $\mathrm{Sym}(X)$. Every element of G, as an element of $\mathrm{Sym}(X)$, is the disjoint union of cycles. For each $\sigma \in G$, let $c_i^X(\sigma)$ be the number of cycles of length i in the cyclic decomposition of σ. Observe that $c_i^X(\sigma)$ depends on the action of G on X. Furthermore, define

$$c^X(\sigma) = \sum_{i=1}^{n} c_i^X(\sigma),$$

where $n = |X|$.

The **cycle-index polynomial** is defined as

$$Z_G^X(X_1, \ldots, X_n) = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{i=1}^{n} X_i^{c_i^X(\sigma)}.$$

Example 3.11 Let C_6 be the cyclic group acting on $X = \{1, 2, 3, 4, 5, 6\}$. The cyclic group is the set of permutations

$$\{\mathrm{id}, (123456), (135)(246), (14)(25)(36), (153)(264), (165432)\}.$$

Hence, the cycle-index polynomial for this action is

$$Z_{\mathrm{C}_6}^X(X_1, \ldots, X_6) = \frac{1}{6}(X_1^6 + X_2^3 + 2X_3^2 + 2X_6).$$

Now consider what changes when we extend the group to D_6, the di-hedral group acting on $X = \{1, 2, 3, 4, 5, 6\}$. The di-hedral group is the set of permutations of the cyclic group together with the permutations

$$\{(16)(25)(34), (12)(45)(36), (14)(23)(56), (2)(5)(13)(46),$$

$$(1)(4)(26)(35), (3)(6)(15)(24)\}.$$

Thus, we should add $3X_2^3 + 3X_1^2X_2^2$ to the polynomial above and divide by 2, since the size of the group has doubled. This implies that the cycle-index polynomial for action of the di-hedral group is

$$Z^X_{\mathrm{D}_6}(X_1, \ldots, X_6) = \frac{1}{12}(X_1^6 + 4X_2^3 + 2X_3^2 + 2X_6 + 3X_2^2X_1^2).$$

Example 3.12 Let $G = C_4$ be the cyclic group with 4 elements acting on X, the edges of $\Gamma = K_4$, the complete graph with four vertices.

No. of elements	Cyclic decomposition on vertices	Cyclic decomposition on edges
1	(.)(.)(.)(.)	(.)(.)(.)(.)(.)(.)
2	(. . . .)	(. . . .)(..)
1	(..)(..)	(..)(..)(.)(.)

The cycle-index polynomial for this action is

$$Z^X_{\mathrm{C}_4}(X_1, \ldots, X_4) = \frac{1}{4}(X_1^6 + 2X_2X_4 + X_2^2X_1^2).$$

Recall that C^X denotes the set of functions from X to C.

Theorem 3.13 (Polya) *Suppose that G acts on X and let C be a set of r colours. The number of orbits of G acting on C^X, the number of r-colourings of X distinct under G, is*

$$Z^X_G(r, \ldots, r) = \frac{1}{|G|}\sum_{\sigma \in G} r^{c^X(\sigma)},$$

where Z^X_G is the cycle-index polynomial for the action of G on X.

Proof A colouring is fixed by σ if all the elements of X in a cycle of the cyclic decomposition of σ receive the same colour. The cyclic decomposition of σ has $c(\sigma)$ cycles and we can choose a colour for each one in $r^{c(\sigma)}$ ways. Therefore, in the action of G on C^X, $|\mathrm{fix}(\sigma)| = r^{c(\sigma)}$. Two colourings are equivalent under the action of G if and only if they lie in the same orbit of the action of G on C^X. The theorem follows by applying Lemma 3.5. □

In the following example, we check that Theorem 3.13 coincides with our previous calculation for the number of 2-colourings in Example 3.8.

Example 3.14 Regarding the action in Example 3.11 of C_6 acting on X, the vertices of Γ, the cyclic graph with six vertices, we have

$$Z^X_{\mathrm{C}_6}(2, \ldots, 2) = \frac{1}{6}(2^6 + 2^3 + (2 \times 2^2) + (2 \times 2)) = 14$$

different 2-colourings.

On the other hand, for the action of the di-hedral group D_6 acting on X, the vertices of Γ, the cyclic graph with six vertices, we obtain

$$Z_{D_6}^X(2, \ldots, 2) = \frac{1}{12}(2^6 + (4 \times 2^3) + (2 \times 2^2) + (2 \times 2) + (3 \times 2^4)) = 13$$

distinct 2-colourings.

Example 3.15 Consider the action in Example 3.12 of C_4 acting on X, the edges of Γ, the complete graph with four vertices. Theorem 3.13 implies that there are

$$Z_{C_4}^X(r, \ldots, r) = \frac{1}{4}(r^6 + r^4 + 2r^2)$$

distinct r-colourings.

Substituting $r = 2$, we deduce that there are 22 two colourings of the edges of K_4, distinct under the cyclic group with four elements.

If we define E_i to be the number of distinct 2-colourings of the edges with exactly i blue edges then we can verify that there are 22 distinct 2-colourings by calculating manually the value of E_i. Observe that $E_i = E_{6-i}$, so with Figs. 3.4 and 3.5 we get the following array.

i	0	1	2	3	4	5	6
$E_i(C_4)$	1	2	5	6	5	2	1

We will see in Theorem 3.19 that the cyclic index polynomial not only allows us to count the number of distinct r-colourings but also the values of the number of distinct r-colourings with a specified number of edges of each colour.

If we extend the cyclic group to the dihedral group then we see that the first two colourings in Fig. 3.4 coincide as do the central two colourings in Fig. 3.5. These are

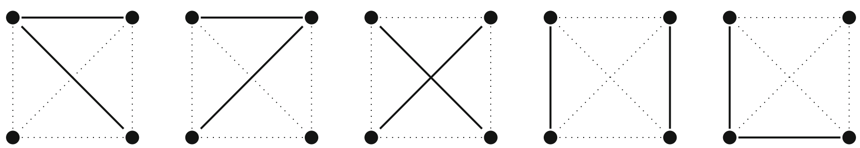


Fig. 3.4 The 2-colourings with 2 black edges distinct under the cyclic group

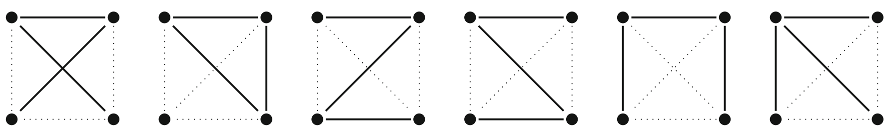


Fig. 3.5 The 2-colourings with 3 black edges distinct under the cyclic group

the only coincidences which occur when we extend the cyclic group to the di-hedral group, so we deduce that the number of 2 colourings with exactly i black edges is given by the following array.

i	0	1	2	3	4	5	6
$E_i(D_4)$	1	2	4	5	4	2	1

Hence, there are 19 different 2 colourings of the edges of K_4 distinct under the action of the di-hedral group.

3.4 The Rotations of the Cube

The cube has 24 rotational symmetries which are summarised in the following array, see Fig. 3.6.

Type	Axis	Order	No. of elements
1	.	1	1
2	Face	2	3
3	Face	4	6
4	Edge	2	6
5	Vertex	3	8

There are no further rotational symmetries that fix a face. So we can use Lemma 3.5 to check that we have not missed any further symmetries. Observe that G is transitive on the faces, Type 1 fixes 6 faces, Type 2 and 3 fix 2 faces and Type 4 and 5 fix none, so

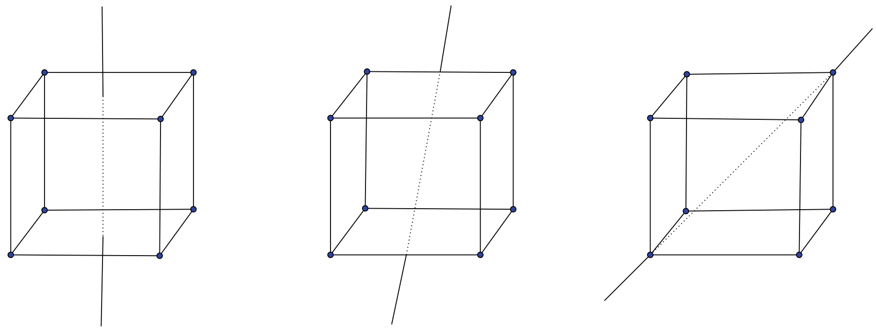


Fig. 3.6 The cube rotations with axis on a face, edge and vertex respectively

$$\sum_{\sigma \in G} |\text{fix}(\sigma)| = 24,$$

and Lemma 3.5 implies that $|G| = 24$.

We could have arrived at the same conclusion by applying Lemma 3.4. The only rotational symmetries which fix a face are the face rotations of which there are 4 (including the identity). Thus, in Lemma 3.4, we have $|G_x| = 4$ and $|X| = 6$, which implies $|G| = 24$.

We can consider G acting on the faces, the edges or the vertices.

Consider first the action of G on the faces.

Type	$c(\sigma)$	Cyclic decomposition	No. of elements
1	6	(.)(.)(.)(.)(.)(.)	1
2	4	(.)(.)(..)(..)	3
3	3	(....)(.)(.)	6
4	3	(..)(..)(..)	6
5	2	(...)(...)	8

Let F_i is the number distinct 2-colourings of the faces with i faces coloured blue. It is a fairly straight-forward task to verify the following array.

i	0	1	2	3	4	5	6
$F_i(G)$	1	1	2	2	2	1	1

This implies that there are 10 distinct 2-colourings of the faces of the cube and Theorem 3.13 verifies this,

$$Z_G^{\text{faces}}(2, \ldots, 2) = (2^6 + (3 \times 2^4) + (12 \times 2^3) + (8 \times 2^2))/24 = 10.$$

Now consider the action of G on the edges.

Type	$c(\sigma)$	Cyclic decomposition	No. of elements
1	12	(.)(.)(.)(.)(.)(.)(.)(.)(.)(.)(.)(.)	1
2	6	(..)(..)(..)(..)(..)(..)	3
3	3	(....)(....)(....)	6
4	7	(.)(.)(..)(..)(..)(..)(..)	6
5	4	(...)(...)(...)(...)	8

Theorem 3.13 implies that the number of distinct 2-colourings of the edges is

$$Z_G^{\text{edges}}(2, \ldots, 2) = (2^{12} + (6 \times 2^7) + (3 \times 2^6) + (6 \times 2^3) + (8 \times 2^4))/24 = 218.$$

3.5 The Number of Non-Isomorphic Graphs

We can use Polya's theorem, Theorem 3.13, to calculate the number of graphs with n vertices.

By way of example, let us count the number of distinct (unlabelled) graphs on four vertices. Let E_i denote the number of graphs with four vertices and i edges. Then we quickly verify the following array.

i	0	1	2	3	4	5	6
E_i	1	1	2	3	2	1	1

Let G be the group Sym(4) and consider the action of G on X, the edges (i.e. the pairs of $\{1, 2, 3, 4\}$).

No. of elements	Cyclic decomposition on vertices	Cyclic decomposition on edges
1	(.)(.)(.)(.)	(.)(.)(.)(.)(.)(.)
6	(.)(.)(..)	(.)(.)(..)(..)
8	(.)(. . .)	(. . . .)(. . .)
6	(. . . .)	(. . . .)(..)
3	(..)(..)	(..)(..)(.)(.)

Therefore, the cyclic decomposition polynomial for G acting on the edges is

$$Z_G^X(X_1, \ldots, X_4) = (X_1^6 + 6X_2^2X_1^2 + 8X_3^2 + 6X_4X_2 + 3X_2^2X_1^2)/24.$$

Applying Theorem 3.13, we deduce that there are

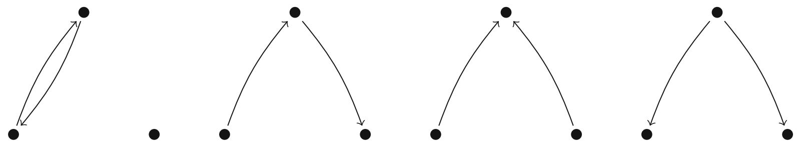
$$Z_G^X(2, \ldots, 2) = (2^6 + (6 \times 2^4) + (8 \times 2^2) + (6 \times 2^2) + (3 \times 2^4))/24 = 11$$

non-isomorphic graphs with four vertices.

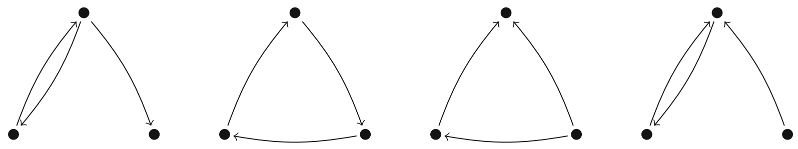
The number of graphs on n vertices can be calculated in the same way by considering the action of Sym(n) on the unordered pairs of $\{1, \ldots, n\}$. Once the cyclic decomposition is obtained the number of non-isomorphic graphs is equal to the number of 2-colouring of the edges of K_n, which can be determined from Theorem 3.13.

To be able to count the number of directed graphs on n vertices, we consider the action of Sym(n) on the ordered pairs of $\{1, \ldots, n\}$.

We can count the number of directed graphs on three vertices by simply writing them down. If we let D_i be the number of directed graphs on three vertices with i directed edges then clearly $D_0 = 1$ and $D_1 = 1$. The distinct directed graphs with two and three edges are drawn below.



The directed graphs with three vertices and two edges.



The directed graphs with three vertices and three edges.

The observation that $D_i = D_{6-i}$ (by taking complements) allows us to fill in the rest of the table.

i	0	1	2	3	4	5	6
D_i	1	1	4	4	4	1	1

To verify that there are 16 directed graphs with three vertices, we first work out the cyclic decomposition of the elements of Sym(3) acting on the directed edges. This is detailed in the following table.

No. of elements	Cyclic decomposition on vertices	Cyclic decomposition on unordered pairs	Cyclic decomposition on ordered pairs
1	(.)(.)(.)	(.)(.)(.)	(.)(.)(.)(.)(.)(.)
3	(.)(..)	(.)(..)	(..)(..)(..)
2	(...)	(...)	(...)(...)

The cyclic decomposition polynomial for G acting on the directed edges is therefore

$$Z_G^X(X_1, \ldots, X_3) = (X_1^6 + 3X_2^3 + 2X_3^2)/6.$$

Applying Theorem 3.13, we deduce that there are

$$Z_G^X(2, \ldots, 2) = (2^6 + (3 \times 2^3) + (2 \times 2^2))/6 = 16$$

non-isomorphic directed graphs with three vertices.

3.6 General Version of Polya's Theorem

We can prove a more general version of Polya's theorem which allows us not only to deduce the number of r-colourings, but also the number of r-colourings in which i of the objects are coloured blue, and even more generally the number of r-colourings in which k_j of the objects are coloured with the j-th colour.

Let X be a set and let C be a set of colours.

We define a bijective map h from C to the set of indeterminates $\{t_1, \ldots, t_r\}$.

We define a function w from C^X to $\mathbb{Z}[t_1, \ldots, t_r]$, by

$$w(f) = \prod_{i=1}^{r} t_i^{m_i(f)} = \prod_{x \in X} h(f(x)),$$

where f is a function from X to C, and $m_i(f)$ is the number of elements of X that f maps to the i-th colour.

Before we state and prove the general version of Polya's theorem, consider the following example.

Example 3.16 Consider again the action of D_6 on the vertices of the cyclic graph with 6 vertices, as in Example 3.11. Suppose that $C = \{b, r\}$ and that $h(b) = t_1$ and $h(r) = t_2$, Let $\tau = (124)(356)$. There are four functions fixed by τ and these are given in the following array, along with their corresponding weights.

f	1	2	3	4	5	6	$w(f)$
f_1	b	b	b	b	b	b	t_1^6
f_2	b	b	r	b	r	r	$t_1^3 t_2^3$
f_3	r	r	b	r	b	b	$t_1^3 t_2^3$
f_4	r	r	r	r	r	r	t_2^6

Therefore,

$$\sum_{f \in \text{fix}(\tau)} w(f) = t_1^6 + 2t_1^3 t_2^3 + t_2^6 = (t_1^3 + t_2^3)^2.$$

Example 3.17 Consider the action of Sym(6) acting on $\{1, 2, 3, 4, 5, 6\}$ and let $C = \{b, r, w\}$. Let $\tau = (123)(45)(6)$. There are 27 functions fixed by τ. For example,

f	1	2	3	4	5	6	$w(f)$
f_1	b	b	b	b	b	b	t_1^6
f_2	b	b	b	b	b	r	$t_1^5t_2$
f_3	b	b	b	b	b	w	$t_1^5t_3$
f_4	b	b	b	r	r	b	$t_1^4t_2^4$
f_5	b	b	b	r	r	r	$t_1^3t_2^3$
f_5	b	b	b	r	r	w	$t_1^3t_2^2t_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
f_{27}	w	w	w	w	w	w	t_3^6

To be fixed by τ, we must assign the same colour to 1, 2 and 3, which will contribute either t_1^3, t_2^3 or t_3^3 to $w(f)$. Then we must assign the same colour to 4 and 5, which will contribute either t_1^2, t_2^2 or t_3^2 to $w(f)$ and finally we can assign a colour to 6, which will contribute either t_1, t_2 or t_3 to $w(f)$. Thus,

$$\sum_{f \in \text{fix}(\tau)} w(f) = (t_1^3 + t_2^3 + t_3^3)(t_1^2 + t_2^2 + t_3^2)(t_1 + t_2 + t_3).$$

Example 3.18 Consider again the action of Sym(4) on the edges of K_4 as in Examples 3.12 and 3.15. Suppose that τ is a permutation whose cyclic decomposition on the edges is of the form (..)(..)(.)(.). Then,

$$\sum_{f \in \text{fix}(\tau)} w(f) = (t_1^2 + t_2^2)^2 (t_1 + t_2)^2.$$

Observe that in the following theorem, if we substitute $t_i = 1$ for all $i = 1, \ldots, n$ then we obtain

$$Z_G^X(r, r, \ldots, r) = |R|,$$

since $w(f) = 1$ for all f. In this way, we can recover Theorem 3.13, since $|R|$ is the number of r-colourings of X distinct under G.

Theorem 3.19 (General Version of Polya) *Suppose that G acts on X and let C be a set of r colours. Let R be a set of representatives of the orbits of the action of G on C^X. Then*

$$Z_G^X(t_1 + \cdots + t_r, t_1^2 + \cdots + t_r^2, \ldots, t_1^n + \cdots + t_r^n) = \sum_{f \in R} w(f).$$

Proof Recall that h is the function which maps a colour to its indeterminate and that

$$\prod_{x \in X} h(f(x)) = w(f).$$

Suppose that $\sigma \in G$, acting on C^X fixes the function f. Then f maps all the elements of X in the same cycle of the cyclic decomposition of σ acting on X, to the same element in C.

Therefore, on an m-cycle we have to choose a colour for f, if f maps an element x in the m-cycle to the i-th colour then $h(f(x)) = t_i$ and the contribution that the m-cycle makes to

$$\prod_{x \in X} h(f(x))$$

will be t_i^m. As we sum over Fix(σ), the set of all functions f that are fixed by σ we get

$$\sum_{f \in \mathrm{Fix}(\sigma)} w(f) = (t_1 + \cdots + t_r)^{c_1^X(\sigma)} (t_1^2 + \cdots + t_r^2)^{c_2^X(\sigma)} \cdots (t_1^n + \cdots + t_n^n)^{c_n^X(\sigma)}.$$

Hence,

$$\sum_{\sigma \in G} \sum_{f \in \mathrm{Fix}(\sigma)} w(f) = |G| Z_G(t_1 + \cdots + t_r, t_1^2 + \cdots + t_r^2, \ldots, t_1^n + \cdots + t_r^n). \tag{3.1}$$

Suppose that g and f are in the same orbit of G acting on C^X. Then, for some $\sigma \in G$, $g = \sigma(f)$, and

$$\begin{aligned} w(g) = \prod_{x \in X} h(g(x)) &= \prod_{x \in X} h((\sigma(f))(x)) = \prod_{x \in X} h(f(\sigma^{-1}(x))) \\ &= \prod_{x \in X} h(f(x)) = w(f). \end{aligned}$$

Switching the order of the sums gives,

$$\sum_{\sigma \in G} \sum_{f \in \mathrm{Fix}(\sigma)} w(f) = \sum_{f \in C^X} \sum_{\sigma \in S_f} w(f) = \sum_{f \in C^X} w(f)|S_f|,$$

where S_f is the stabiliser group of f in G. Let orb(f) denote the orbit of f under G. Since G acts transitively on orb(f), by Lemma 3.4, $|G| = |S_f||\mathrm{orb}(f)|$

$$\sum_{f \in C^X} \sum_{\sigma \in S_f} w(f) = \sum_{f \in C^X} \frac{w(f)|G|}{|\mathrm{orb}(f)|} = |G| \sum_{f \in R} w(f),$$

where the last equality follows since we observed that for functions g in the same orbit as f, we proved that $w(g) = w(f)$.

The theorem follows by equating the above equalities with (3.1). □

Example 3.20 demonstrates what Theorem 3.19 claims.

Example 3.20 Make the substitution $X_i = t_1^i + t_2^i$ in the cyclic decomposition polynomial for the action of D_6 on the vertices of the cyclic graph with 6 vertices from Example 3.11. This gives

$$Z^X_{D_6}(t_1 + t_2, t_1^2 + t_2^2, \dots, t_1^6 + t_2^6)$$

$$= \tfrac{1}{12}((t_1 + t_2)^6 + 4(t_1^2 + t_2^2)^3 + 2(t_1^3 + t_2^3)^2 + 2(t_1^6 + t_2^6) + 3(t_1^2 + t_2^2)^2(t_1 + t_2)^2).$$

$$= t_1^6 + t_1^5 t_2 + 3t_1^4 t_2^2 + 3t_1^3 t_2^3 + +3t_1^2 t_2^4 + t_1 t_2^5 + t_2^6.$$

The coefficient of $t_1^i t_2^{6-i}$ is equal to the number of 2-colouring of the vertices with i blue vertices, which we calculated in Example 3.8.

Example 3.21 If we substitute $X_i = t_1^i + t_2^i$ in the cyclic decomposition polynomial for the action of Sym(4) on the edges of the complete graph with 4 vertices from Sect. 3.5 then we obtain

$$Z^X_G(t_1 + t_2, \dots, t_1^4 + t_2^4)$$

$$= \tfrac{1}{24}((t_1 + t_2)^6 + 6(t_1^2 + t_2^2)^2(t_1 + t_2)^2 + 8(t_1^3 + t_2^3)^2 + 6(t_1^4 + t_2^4)(t_1^2 + t_2^2)$$
$$+ 3(t_1^2 + t_2^2)^2(t_1 + t_2)^2).$$

$$= t_1^6 + t_1^5 t_2 + 2t_1^4 t_2^2 + 3t_1^3 t_2^3 + 2t_1^2 t_2^4 + t_1 t_2^5 + t_2^6.$$

The coefficient of $t_1^i t_2^{6-i}$ is equal to the number of graphs with 4 vertices and i edges.

Example 3.22 Consider the substitution $X_i = t_1^i + t_2^i + t_3^i$ in the cyclic index polynomial for the action of the cyclic group with four elements acting on the edges of the complete graph with 4 vertices, as in Example 3.12. This gives

$$Z^X_{C_4}(t_1 + t_2 + t_3, \dots, t_1^6 + t_2^6 + t_3^6)$$

$$= \tfrac{1}{4}((t_1 + t_2 + t_3)^6 + 2(t_1^2 + t_2^2 + t_3^2)(t_1^4 + t_2^4 + t_3^4) + (t_1^2 + t_2^2 + t_3^2)^2(t_1 + t_2 + t_3)^2).$$

The coefficient of $t_1^i t_2^j t_3^{6-i-r}$ is equal to the number of 3-colourings with i blue edges and j red edges, distinct under the action of the cyclic group.

To count the number of 3-colourings with i blue edges distinct under the action of the cyclic group, we need to sum the coefficient of $t_1^i t_2^j t_3^{6-i-r}$ from $j = 0, \ldots, 6$. This we can do by making the substitution $t_1 = t$, $t_2 = t_3 = 1$. The coefficient of t^i is equal to the number of 3-colouring with i blue edges distinct under the action of the cyclic group.

3.7 Notes and References

Theorem 3.19 was obtained by Pólya (1937) in a celebrated paper, although the result had been obtained by Redfield (1927) some ten years earlier. The result was to have a profound influence in combinatorics providing a unified tool to solve involved problems in counting under symmetries. One of the first applications was the counting of unlabelled graphs.

3.8 Exercises

Exercise 3.1

i. Count the number of colourings of the vertices of the cube with r colours.
ii. Count the number of colourings of faces of the regular octahedron with r colours..

Exercise 3.2 Let G be the automorphism group of the graph Γ, see Fig. 3.7. We say that two colourings are equivalent if they are in the same orbit under the action of G.

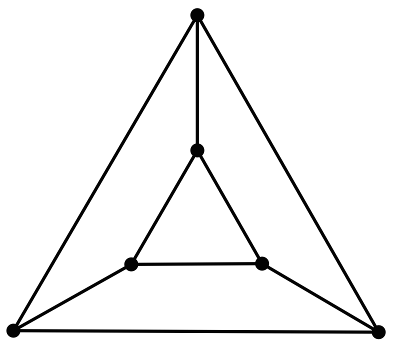


Fig. 3.7 The graph Γ from Exercise 3.2

i. Using the orbit-counting lemma, or otherwise, prove that G has 12 elements.
ii. Calculate the number of distinct r-colourings of the vertices of Γ.
iii. Calculate the number of distinct 2-colourings of the edges of Γ.

Exercise 3.3 Prove that the number of permutations of $\mathrm{Sym}(n)$ with k_i cycles of length i is

$$\frac{n!}{1^{k_1}2^{k_2}\cdots n^{n_k}k_1!k_2!\cdots k_n!}$$

Calculate the cycle-index polynomial of $\mathrm{Sym}(n)$ acting on $\{1, \ldots, n\}$.

Exercise 3.4 Calculate the number of non-isomorphic graphs with 5 vertices.

Exercise 3.5

i. Show that there are 4 ways to 2-colour the edges of C_7 with 3 blue edges, distinct under the action of the dihedral group D_7.
ii. Let p be a prime. Prove that the number of ways to 2-colour the edges of C_p with $j \notin \{0, p\}$ blue edges, distinct under the action of the dihedral group D_p, is

$$\frac{1}{2p}\binom{p}{j} + \frac{1}{2}\binom{(p-1)/2}{\lfloor j/2 \rfloor}.$$

Exercise 3.6 Let W_n denote the wheel graph, the graph with a central vertex joined to the each vertex of a cycle of length n.

i. Calculate the number of distinct two-colourings of the edges of W_6, distinct under the action of the cyclic group .
ii. Calculate the number of distinct two-colourings of the edges of W_6, distinct under the action of the dihedral group.

Exercise 3.7 Calculate the number of directed graphs on 4 vertices.

Exercise 3.8 Suppose that G acts on A and H acts on B, where A and B are disjoint sets of size n and m respectively and $n \geqslant m$. Then there is a natural action of $G \times H$ on $A \cup B$.

Prove that for this action,

$$Z_{G\times H}(X_1, \ldots, X_n) = Z_G(X_1, \ldots, X_n)Z_H(X_1, \ldots, X_m).$$

Exercise 3.9

i. The vertices of the Petersen graph are the 2-subsets of $\{1, 2, 3, 4, 5\}$ where vertices u and v are joined by an edge if and only if $u \cap v = \emptyset$. Draw the Petersen graph and label the vertices with 2-subsets of $\{1, 2, 3, 4, 5\}$.
ii. Prove that the Petersen graph has a group G of symmetries isomorphic to Sym(5).
iii. Calculate the number of 2-colourings of the edges of the Petersen graph, distinct under the action of G.
iv. Is the answer to iii. equal to the number of non-isomorphic subgraphs of the Petersen graph with 10 vertices?

Exercise 3.10 Let T be a triangular prism as in Fig. 3.8.

i. Prove that T has 6 symmetries and let G denote this group of symmetries.
ii. Calculate the cycle-index polynomial for the group G acting on the edges.
iii. Calculate the number of distinct ways to 2-colour the edges.
iv. Calculate the cycle-index polynomial for the group G acting on the faces.
v. Calculate the number of distinct ways to 3-colour the faces.

Exercise 3.11

i. Calculate the cycle-index polynomial of G, the cyclic group of 5 elements, acting on the edges of Γ, the complete graph with 5 vertices.
ii. Calculate the number of r-colourings of the edges of Γ, distinct under G.
iii. Calculate the number of 2-colourings of the edges of Γ with exactly i blue edges, distinct under G.
iv. Calculate the number of 3-colourings of the edges of Γ with exactly i blue edges, distinct under G.

Exercise 3.12 Consider a carbon atom located in the centre of a regular tetrahedron joined to four radicals located at the vertices, which can be either $HOCH_2$, C_2H_5, Cl or H.

i. Prove that there are 36 possible molecules.

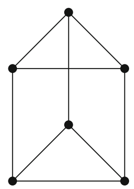


Fig. 3.8 The triangular prism

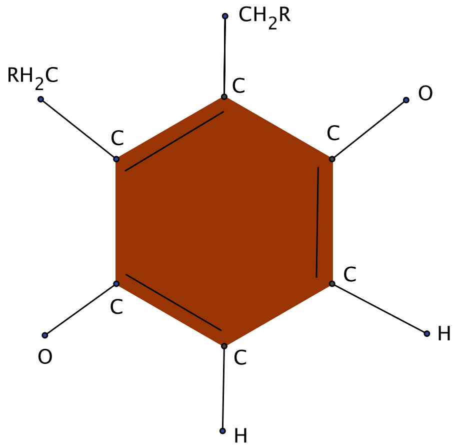


Fig. 3.9 An example of a benzoic acid

ii. Find c_i, the number of molecules that contain exactly i radicals Cl.
iii. Find h_i, the number of molecules that contain exactly i atoms H.

Exercise 3.13 The benzoic acid group consists of a benzene ring with a radical attached to each carbon which can be any radical in the set {O, H, CH_2R, RH_2C}. An example of such a molecule is given in Fig. 3.9.

i. Calculate the total number of possible benzoic acids.
ii. Calculate the total number of possible benzoic acids with i oxygen atoms.
iii. Calculate the total number of possible benzoic acids with i hydrogen atoms.

In all cases, express the answer as the coefficient of t^i in some polynomial.

4 Finite Geometries and Latin Squares

The study of latin squares stretches far back into history and our fascination with them appears undiminished today, evidenced by its appearance in the popular Soduku puzzles. As we shall prove in this chapter, there are a very large number of $n \times n$ latin squares. There are however, very few sets of mutually orthogonal latin squares. The problem of finding two mutually orthogonal latin squares can be rephrased in natural terms as the problem of lining up n regiments of n different ranking officers, on parade in an $n \times n$ grid, such that in each row and column we find exactly one officer from each regiment and one officer of each of the n ranks. We will see that there is a solution to this problem for every $n \neq 2, 6$. Finding larger sets of mutually orthogonal latin squares will lead us to consider finite geometries, incidence structures of points and lines in which the set of points and the set of lines are finite. We will consider properties of geometries defined from a finite vector space, focussing on affine and projective planes, as well as higher-dimensional projective spaces.

4.1 Systems of Distinct Representatives

Let X be a set and suppose that $A_1, \ldots, A_n$ are non-empty subsets of X.

A **system of distinct representatives** (SDR) for $A_1, \ldots, A_n$ is a subset $\{a_1, \ldots, a_n\}$ of X of size n with the property that $a_i \in A_i$.

Example 4.1 Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

The subsets $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_3, x_4\}$, $A_3 = \{x_2, x_5\}$, $A_4 = \{x_4, x_6\}$ and $A_5 = \{x_1, x_3\}$ have a system of distinct representatives, for example

$$a_1 = x_1,\ a_2 = x_4,\ a_3 = x_2,\ a_4 = x_6,\ \text{and } a_5 = x_3.$$

Example 4.2 Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$.

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_4

The subsets $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_3, x_4\}$, $A_3 = \{x_1, x_2, x_5, x_7\}$, $A_4 = \{x_1, x_4, x_6, x_7\}$, $A_5 = \{x_1, x_3\}$, $A_6 = \{x_2, x_3, x_4\}$ and $A_7 = \{x_1, x_4\}$ have no system of distinct representatives, since

$$A_1 \cup A_2 \cup A_5 \cup A_6 \cup A_7 = \{x_1, x_2, x_3, x_4\}$$

and we cannot choose 5 distinct representatives from only four elements.

For any subset J of $\{1, \ldots, n\}$ define

$$A(J) = \cup_{i \in J} A_i.$$

Theorem 4.3 (Hall's Marriage Theorem) *An SDR exists for $A_1, \ldots, A_n$ if and only if*

$$|A(J)| \geqslant |J|,$$

for all subsets J of $\{1, \ldots, n\}$ (we call this condition Hall's condition).

Proof

($\Rightarrow$) Suppose J is a subset of $\{1, \ldots, n\}$ and there is an SDR for $A_1, \ldots, A_n$. Then there is a distinct $x_i \in A$ for each $i \in J$, so

$$|A(J)| \geqslant |J|.$$

($\Leftarrow$) By induction on n.
Suppose $n = 1$. The hypothesis with $J = \{1\}$ implies that A_1 is non-empty, so we can choose $x_1 \in A_1$ as a representative.

Case 1 There are no subsets $\emptyset \neq J \subset \{1, \ldots, n\}$ such that $|A(J)| = |J|$.
The condition implies that the sets A_i are non-empty, so we can choose an $a_n \in A_n$.
Define $A'_i = A_i \setminus \{a_n\}$ for $i = 1, \ldots, n-1$.
For $J \subseteq \{1, \ldots, n-1\}$,

$$|A'(J)| \geqslant |A(J)| - 1 \geqslant |J| + 1 - 1 = |J|,$$

where

$$A'(J) = \bigcup_{i \in J} A'_i.$$

Hence, by induction, there is an SDR for $A'_1, \ldots, A'_{n-1}$ (which are subsets of $X \setminus \{a_n\}$).

Case 2 There is a subset J for which $\emptyset \neq J \subset \{1, \dots, n\}$ and $|A(J)| = |J|$.
Choose J so that $|J|$ is minimal with the property that $|A(J)| = |J|$.
For each $i \in \{1, \dots, n\} \setminus J$, define $A_i^* = A_i \setminus A(J)$.
Then

$$\{A_i^* \mid i \in \{1, \dots, n\} \setminus J\}$$

is a set of subsets of $X \setminus A(J)$.
For $K \subseteq \{1, \dots, n\} \setminus J$,

$$A^*(K) = \bigcup_{i \in K} A_i^* = \bigcup_{i \in K} (A_i \setminus A(J))$$

$$= (\bigcup_{i \in K} A_i) \setminus (\bigcup_{j \in J} A_j) = (\bigcup_{i \in K \cup J} A_i) \setminus (\bigcup_{j \in J} A_j) = A(K \cup J) \setminus A(J).$$

Now,

$$|A^*(K)| = |A(K \cup J)| - |A(J)| \geqslant |K \cup J| - |J| = |K|,$$

so by induction there is an SDR for A_i^* $(i \in \{1, \dots, n\} \setminus J)$.
Also, by induction, there is an SDR for the sets A_j, where $j \in J$, which are all subsets of $A(J)$. Observe that choosing J so that $|J|$ is minimal with the property that $|A(J)| = |J|$ implies that we are in Case 1 when we apply the induction to the set of subsets (of $A(J)$) $\{A_j \mid j \in J\}$.
Putting these two SDR's together we get an SDR for $A_1, \dots, A_n$.

□

We will see an algorithmic proof of Theorem 4.3 when we study matchings in graphs, see Theorem 5.3. An algorithmic proof has the advantage over an inductive proof in that it actually tells us how to find the SDR (not only that it exists). However, the advantage of the inductive proof is that it allows us to prove the following theorem.

Theorem 4.4 (Hall's Extended) *Suppose that $A_1, \dots, A_n$ are subsets of a set X, satisfy Hall's condition and $|A_i| \geqslant r$, for all $i = 1, \dots, n$. If $r \leqslant n$ then there are at least $r!$ SDR's.*

Proof If $r = 1$ and it follows that there is an SDR by Theorem 4.3. If $n = r$ then $|A_i| \geqslant n$ for all i and so there are at least $n!$ SDR's. We shall prove that the (n, r) case reduces to the $(n-1, r-1)$ case or the $(n-1, r)$ case. This suffices to prove the statement by induction since it will then eventually reduce to the case $r = 1$ or $n = r$.

If we are in Case 1 (of the proof of Theorem 4.3) then there are r choices for a_n. For each choice, $A'_1, \ldots, A'_{n-1}$ consists of $n-1$ sets of size at least $r-1$, so by induction there are at least $r(r-1)!$ SDR's.

If we are in Case 2 (of the proof of Theorem 4.3) then for the J such that $|A(J)| = |J|$ we have that $r \leqslant |J| < n$. By induction, $\{A_j \mid j \in J\}$, has at least $r!$ SDR's. □

Theorem 4.5 *Suppose that* $|A_i| = r$*, for each* $i = 1, \ldots, n$*, and each element* $x \in X$ *is contained in exactly* r *of the sets* $A_1, \ldots, A_n$*. Then* $A_1, \ldots, A_n$ *satisfy Hall's condition.*

Proof Note by counting pairs (x, A_i), where $x \in A_i$, we have that $|X| = n$.

Now count (x, j) where $x \in A_j$ and $j \in J$.

We have $|J|$ choices for j and r choices for x, which gives $r|J|$.

Choosing $x \in A(J)$ first, we have $|A(J)|$ choices for x. The element x is in r sets (not all of which may have an indice in J) so there are at most r choices for j, i.e. counting in this way we have at most $|A(J)|r$ pairs.

Hence, $|A(J)| \geqslant |J|$. □

4.2 Latin Squares

A **latin square of order** n is an $n \times n$ array with entries taken from a set X of size n, such that each element of X appears exactly once in each row and column.

Example 4.6 Let $X = \{g_1, \ldots, g_n\}$ be a group with operation $\circ$. Recall that a group satisfies three axioms, that there is an element e such that $a \circ e = e \circ a = a$ for all $a \in X$, that for all $a \in X$ there is a a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = e$ and for all $a, b, c \in X$, $a \circ (b \circ c) = (a \circ b) \circ c$. From this, we deduce that if $a \circ b = a \circ c$ then $a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$ and so $(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c$ which implies $b = c$. Similarly $b \circ a = c \circ a$ implies $b = c$. Thus, the array whose (i, j)-th entry is $g_i \circ g_j$ is a latin square.

A latin square defines a binary operation $\circ$ on the set X by labelling the rows and columns with the elements of X and then defining $a \circ b$ to the entry in the latin square of the row labelled with a and the column labelled with b. The pair $(X, \circ)$ is called a **quasigroup**. The two concepts are interchangeable. Sometimes it is more convenient to think of the latin square as a quasigroup and vice-versa.

Example 4.7 The following array is a latin square of order four which is not a group. There's no identity element and it's also not associative. To see that there is no identity element, observe that no element commutes with all the others. To prove that it is not associative, it is enough to note that $3 = 1 \circ (1 \circ 1) \neq (1 \circ 1) \circ 1 = 0$.

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	3	0	1	2
3	2	3	0	1

A **latin rectangle of size** $r \times n$ is an $r \times n$ array with entries taken from a set X of size n, such that each element of X appears exactly once in each row and at most once in each column.

Lemma 4.8 *A latin rectangle of size $r \times n$ $(r < n)$ can be extended to a latin rectangle of size $(r+1) \times n$ in at least $(n-r)!$ ways.*

Proof Let A_i be the subset of X of elements not appearing in the i-th column of the $r \times n$ latin rectangle.

Then $|A_i| = n-r$ and each element of X appears in $n-r$ of the sets $A_1, \ldots, A_n$. Theorem 4.5 implies that $A_1, \ldots, A_n$ satisfy Hall's condition. Theorem 4.4 implies that there are at least $(n-r)!$ SDR's. Each SDR allows us to extend to a latin rectangle of size $(r+1) \times n$. □

Theorem 4.9 *There are at least $\prod_{k=1}^{n} k!$ latin squares of order n.*

We can ask ourselves how good is the bound in Theorem 4.9.

For $n = 2$ we have just two latin squares of order two, so the bound is tight.

1	2
2	1

2	1
1	2

Also for $n = 3$, once we fix the first row with one of the six possible permutations, the second row has exactly two possibilities, both of which complete uniquely to a latin square. Thus, we have exactly twelve latin square of order three, which coincides with the bound in Theorem 4.9.

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

However, for $n \geqslant 4$ it is no longer the case that there are $(n-1)!$ ways of extending a latin rectangle of order $1 \times n$ to a latin rectangle of order $2 \times n$. Again, we can fix the first row with a given permutation, so we choose σ_1 to be the permutation $\sigma_1(x) = x$. We can then extend the latin rectangle with any permutation σ_2 with the property that $\sigma_2(x) \neq x$, for all $x \in X$. As we saw in Sect. 2.4, such a permutation is called a derangement. We deduced that

$$d_n = n!\left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right).$$

This gives $d_n > (n-1)!$ for $n \geqslant 4$, so there are more than $(n-1)!$ ways of extending a latin rectangle of order $1 \times n$ to a latin rectangle of order $2 \times n$. Observe that

$$\left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right) \to \frac{1}{e},$$

so $d_n \sim n!/e$.

Given a $r \times n$ latin rectangle L with entries from the set $X = \{1, \ldots, n\}$, let A_i be the subset of X of elements **not** appearing in the i-th column of the $r \times n$ latin rectangle.

Let $\mathrm{M} = (m_{ij})$ be the matrix where $m_{ij} = 1$, if $i \in A_j$ and zero otherwise, $i, j \in \{1, \ldots, n\}$.

For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \to \mathrm{M} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Another example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \to \mathrm{M} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The **permanent** of an $n \times n$ matrix M is

$$\mathrm{Perm(M)} = \sum_{\sigma \in \mathrm{Sym}(n)} \prod_{i=1}^{n} m_{i\sigma(i)}.$$

Observe that the permanent differs from the determinant since there is no $\mathrm{sign}(\sigma)$ in the summand, where $\mathrm{sign}(\sigma)$ denotes the sign of the permutation σ.

Theorem 4.10 *The latin rectangle* L *of size* $r \times n$ $(r < n)$ *can be extended to a latin rectangle of size* $(r+1) \times n$ *in* Perm(M) *ways.*

Proof Suppose $\sigma \in \mathrm{Sym}(n)$ is such that $\prod_{i=1}^{n} m_{i\sigma(i)}$ is non-zero. Now,

$$\prod_{i=1}^{n} m_{i\sigma(i)} \neq 0$$

if and only if

$$i \in A_{\sigma(i)}$$

for all i, which is if and only if

$$\sigma^{-1}(j) \in A_j$$

for all j. Hence, $\{\sigma^{-1}(j) \mid j = 1, \ldots, n\}$ is an SDR, which gives an extension of L to a latin rectangle of size $(r+1) \times n$ and vice-versa. □

For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \rightarrow \mathrm{M} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

There are nine permutations (written as $(\sigma(1), \sigma(2), \sigma(3), \sigma(4)))$ $(2, 1, 4, 3)$, $(2, 3, 4, 1)$, $(2, 4, 1, 3)$, $(3, 1, 4, 2)$, $(3, 4, 1, 2)$, $(3, 4, 2, 1)$, $(4, 1, 2, 3)$, $(4, 3, 1, 2)$ and $(4, 3, 2, 1)$ which all contribute a 1 to the sum, so $\mathrm{Perm(M)} = 9$. Note that we have simply listed

$$(\sigma(1), \sigma(2), \sigma(3), \sigma(4)),$$

where σ is a derangement.

The proof of the following theorem falls outside the scope of this book. It follows from Bregman's theorem, which was conjectured by Henryk Mink in 1963 and proved by Bregman 10 years later.

Theorem 4.11 *If* M *is a matrix of zeros and ones with* r *ones in each column then*

$$\mathrm{Perm(M)} \leqslant (r!)^{n/r}.$$

Theorem 4.10 and Theorem 4.11 imply that an $r \times n$ latin rectangle can be extended to a $(r+1) \times n$ latin rectangle in at most $((n-r)!)^{n/(n-r)}$ ways. Hence, it follows that

$$\prod_{k=1}^{n} (k!)^{n/k} \geqslant L(n) \geqslant \prod_{k=1}^{n} k!.$$

We can improve on the lower bound by using Theorem 4.12, a conjecture of Van der Waerden, proved by G. P. Egorychev.

Theorem 4.12 *If* M *is an* $n \times n$ *matrix of non-negative real numbers whose columns and row each sum to* 1 *then*

$$\text{Perm(M)} \geqslant n!/n^n.$$

An $n \times n$ matrix of non-negative real numbers whose columns and rows each sum to 1 is called a **doubly stochastic matrix**. Note that the bound in Theorem 4.12 is tight. It is a simple matter to check that the permanent of

$$\frac{1}{n}J,$$

where J denotes the $n \times n$ matrix all of whose entries are 1, has permanent $n!/n^n$.

Theorem 4.13 *The number of latin squares of order n satisfies*

$$\prod_{k=1}^{n} (k!)^{n/k} \geqslant L(n) \geqslant \frac{(n!)^{2n}}{(n^n)^n}.$$

Proof We have already observed that the upper bound follows from Theorem 4.11.

Given a latin rectangle L of size $r \times n$, construct the matrix M as before. The matrix $\text{M}/(n-r)$ is a doubly stochastic matrix. Therefore, by Theorem 4.12,

$$\text{Perm}(\text{M}/(n-r)) \geqslant n!/n^n,$$

which implies

$$\text{Perm(M)} \geqslant n!(n-r)^n/n^n.$$

By Theorem 4.10, L extends to a $(r+1) \times n$ latin rectangle in Perm(M) ways. Therefore,

$$L(n) \geqslant \prod_{r=0}^{n-1} n! \left(\frac{(n-r)}{n} \right)^n = \frac{(n!)^n (n!)^n}{(n^n)^n}.$$

□

To get a feel for what these exponential bounds imply, we take logarithms. Observe that the total number of $n \times n$ matrices over a set of size n is n^{n^2}, whose logarithm is $n^2 \log n$.

Corollary 4.14 *Up to order* n^2*,*

$$\log L(n) \sim (\log n - 2\log e)n^2.$$

Proof Taking logarithms of the lower bound in Theorem 4.13 and we have

$$\log \frac{(n!)^n (n!)^n}{(n^n)^n} = 2n \log n! - n^2 \log n.$$

Using the approximation $n! \sim \sqrt{(2\pi)} n^{n+\frac{1}{2}}/e^n$, we have that

$$\log n! \sim n \log n - n \log e + \tfrac{1}{2} \log n + \tfrac{1}{2} \log(2\pi),$$

which gives, up to order n^2,

$$\log \frac{(n!)^n (n!)^n}{(n^n)^n} \sim n^2 \log n - 2n^2 \log e$$

for an asymptotic approximation of the lower bound.

Meanwhile,

$$\log \prod_{k=1}^{n} (k!)^{n/k} = \sum_{k=1}^{n} \frac{n}{k} \log k!,$$

which, applying the same approximation for $\log k!$, gives

$$\log \prod_{k=1}^{n} (k!)^{n/k} \sim \sum_{k=1}^{n} \frac{n}{k} (k \log k - k \log e + \tfrac{1}{2} \log k + \tfrac{1}{2} \log(2\pi))$$

$$\sim n \sum_{k=1}^{n} \log k - (\log e) n^2,$$

up to order n^2.

Note that

$$\sum_{k=1}^{n} \log k = \log n!$$

so, up to order n^2,

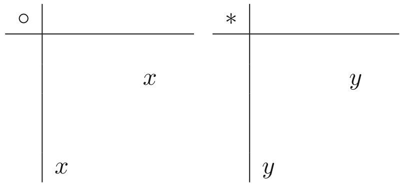
$$\log \prod_{k=1}^{n} (k!)^{n/k} \sim n^2 \log n - 2n^2 \log e$$

for an asymptotic approximation of the upper bound. □

4.3 Mutually Orthogonal Latin Squares

Let G and H be two finite sets of the same size.

Two latin squares (or quasigroups) $(G, \circ)$ and $(H, *)$ are **orthogonal** if the following does not occur.



In other words, two latin squares are orthogonal then when we superimpose one on the other we see every pair of elements in $G \times H$ exactly once.

Example 4.15 The following arrays are orthogonal latin squares of order three.

$\circ$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

$*$	A	B	C
A	A	C	B
B	C	B	A
C	B	A	C

Observe that permuting the elements in either of the latin squares, does not affect the orthogonality property or indeed the latin square property.

Example 4.16 The following array shows two orthogonal latin squares of order four superimposed one on top of the other. Here, $G = \{J, Q, K, A\}$ and $H = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$.

$J\clubsuit$	$Q\diamondsuit$	$K\heartsuit$	$A\spadesuit$
$Q\heartsuit$	$J\spadesuit$	$A\clubsuit$	$K\diamondsuit$
$K\spadesuit$	$A\heartsuit$	$J\diamondsuit$	$Q\clubsuit$
$A\diamondsuit$	$K\clubsuit$	$Q\spadesuit$	$J\heartsuit$

Presenting the problem of finding two orthogonal latin squares of order four, as in Example 4.16, can be generalised to the problem of finding two orthogonal latin squares of order n by supposing we have n^2 cards from n suits and n different numbers. Of course, since a standard deck of cards only has 4 suits, this is only feasible with a standard pack of cards for $n \leqslant 4$. However, Euler phrased this problem in terms of officers and ranks. Given a group of n^2 officers which are from n different regiments and within each regiment there is an officer from each of n ranks he asked "is it possible to line-up the officers on a $n \times n$ grid in such a way that each row and column contains exactly one officer from each regiment and of each rank?". We will return to the problem of finding two mutually orthogonal latin squares of order n later in this section.

Let $\mathbb{F}_q$ denote the finite field with $q = p^h$ elements, where p is a prime and h is a positive integer. Recall, that if $q = p$ then $\mathbb{F}_q$ is isomorphic to $\mathbb{Z}/q\mathbb{Z}$. In general, $\mathbb{F}_q$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})[X]/(f)$, where f is an irreducible polynomial of $(\mathbb{Z}/p\mathbb{Z})[X]$ of degree h and (f) is the ideal generated by f.

Let G be the set of elements of $\mathbb{F}_q$ and define the quasigroup $(G, *_m)$, for each non-zero $m \in \mathbb{F}_q$, by

$$g *_m h = mg + h.$$

Example 4.17 The following arrays are latin squares of order four defined on $\mathbb{F}_4 = \{0, 1, e, e^2\}$, where $e^2 = e + 1$. Any two of these latin squares are orthogonal.

$*_1$	0	1	e	e^2
0	0	1	e	e^2
1	1	0	e^2	e
e	e	e^2	0	1
e^2	e^2	e	1	0

$*_e$	0	1	e	e^2
0	0	1	e	e^2
1	e	e^2	0	1
e	e^2	e	1	0
e^2	1	0	e^2	e

$*_{e^2}$	0	1	e	e^2
0	0	1	e	e^2
1	e^2	e	1	0
e	1	0	e^2	e
e^2	e	e^2	0	1

Compare these squares with those in Example 4.16. One readily sees that the first two squares are isomorphic to Example 4.16, but now we have another latin square, orthogonal to both the previous ones. We can make the three mutually orthogonal latin squares into a problem of cards by selecting the cards from four different packs. For example, $\{J♣, K\diamondsuit, A\heartsuit, Q♠\}$ come from the same pack since they are in the position where a "0" appears in the third latin square. Curiously, presenting the problem in this way, and insisting that in each row and column we see a card from each pack, actually makes the problem easier.

Lemma 4.18 *The quasigroups* $(G, *_m)$ *and* $(G, *_j)$ *are orthogonal.*

Proof Observe that two quasigroups $(G, *)$ and $(G, \circ)$ are orthogonal if for all $(x, y) \in G \times G$, there is a unique $(g, g') \in G \times G$ such that $x = g * g'$ and $y = g \circ g'$.

Suppose that there exists a pair $(x, y) \in G \times G$ for which there exists a $(g, h) \in G \times G$ and a $(g', h') \in G \times G$ such that

$$x = mg + h = mg' + h' \text{ and } \quad y = jg + h = jg' + h'.$$

Then $x - y = (m - j)g = (m - j)g'$, and so $g = g'$ and then $h = h'$. □

A set of latin squares are **mutually orthogonal latin squares** if they are pairwise orthogonal.

Theorem 4.19 *If $n = p^h$ for some prime p then there are $n-1$ mutually orthogonal latin squares of order n.*

Proof By Lemma 4.18, and the existence of a finite field with p^h elements. □

Now that we have established that, at least for some n, there are $n - 1$ mutually orthogonal latin squares of order n, we will now prove that one cannot find n mutually orthogonal latin squares of order n.

Theorem 4.20 *There are at most $n - 1$ mutually orthogonal latin squares of order n.*

Proof Observe that permuting the symbols in a latin square does not affect the fact that it is a latin square, nor its orthogonality with another latin square.

Suppose we have a set of N mutually orthogonal latin squares of order n. We can assume that the latin squares are defined over the same set G. In each latin square, permute the symbols so that the element x appears in the $(1, 1)$-cell in the arrays. There are $n - 1$ other columns and rows (i.e. $n - 1$ columns which are not the first column and $n-1$ rows that are not the first row). In each of the $(n-1)^2$ cells of these rows and columns, x appears in a different position in each of the latin squares since when we take any two of the latin squares, the pair (x, x) comes from the $(1, 1)$-cell. There are $n - 1$ entries of x in these $(n - 1)^2$ cells, so $N(n - 1) \leqslant (n - 1)^2$. □

Suppose that $(G, \circ)$ and $(H, \bullet)$ are latin squares. Define $(G \times H, \circ\!\bullet)$ as $(g_1, h_1) \circ\!\bullet (g_2, h_2) = (g_1 \circ g_2, h_1 \bullet h_2)$.

Lemma 4.21 *The pair $(G \times H, \circ\!\bullet)$ is a latin square.*

Proof Suppose that for some row, Labelled by (g_1, h_1) we have

$$(g_1, h_1) \circ\!\bullet (g_2, h_2) = (g_1, h_1) \circ\!\bullet (g_3, h_3).$$

Then, by definition. $g_1 \circ g_2 = g_1 \circ g_3$ and since $(G, \circ)$ is a quasigroup, $g_2 = g_3$. Similarly, $h_2 = h_3$. The same argument works for the columns. □

Theorem 4.22 *Suppose that $(G, \triangle)$ and $(G, \circ)$ are mutually orthogonal latin squares and $(H, \blacktriangle)$ and $(H, \bullet)$ are mutually orthogonal latin squares. Then $(G \times H, \triangle\!\blacktriangle)$ and $(G \times H, \circ\!\bullet)$ are mutually orthogonal latin squares.*

Proof Suppose they are not orthogonal. Then there is a

$$((x, y), (x', y')) \in (G \times H) \times (G \times H)$$

with the property that there exist

$$((g_1, h_1), (g_2, h_2)) \in (G \times H) \times (G \times H)$$

and

$$((g_1', h_1'), (g_2', h_2')) \in (G \times H) \times (G \times H)$$

such that

$$(g_1, h_1) \circ\!\bullet (g_2, h_2) = (g_1', h_1') \circ\!\bullet (g_2', h_2') = (x, y)$$

and

$$(g_1, h_1) \vartriangle\!\blacktriangle (g_2, h_2) = (g_1', h_1') \vartriangle\!\blacktriangle (g_2', h_2') = (x', y').$$

By definition, this implies $g_1 \circ g_2 = g_1' \circ g_2' = x$ and $g_1 \vartriangle g_2 = g_1' \vartriangle g_2' = x'$, which contradicts the orthogonality of $(G, \circ)$ and $(G, \vartriangle)$. □

Theorem 4.23 *If $n \not\equiv 2$ modulo 4 then there exist two mutually orthogonal latin squares of order n.*

Proof If $n \not\equiv 2$ modulo 4 then $n = 2^h p_1 \cdots p_r$, for some $h \geqslant 2$ and odd primes $p_1, \ldots, p_r$ or $n = p_1 \cdots p_r$, for some odd primes $p_1, \ldots, p_r$. Since there are $2^h - 1$ MOLS of order 2^h and two MOLS of order p for all odd primes p, the theorem follows by repeated applying Theorem 4.22. □

Conjecture 4.24 (Euler) If $n \equiv 2$ modulo 4 then there do not exist two orthogonal latin squares of order n.

It would take about 200 years before Euler's conjecture was shown to be false. It was shown to be true by Gaston Tarry in 1901 for $n = 6$, by exhaustive search. There is still no known proof that there are no two orthogonal latin squares of order six that does not involve some exhaustive search. It was Ernie Parker, together with R.C. Bose and S. S. Shrikhande, who refuted the conjecture for all $n \notin \{2, 6\}$. It is still not known whether there are three mutually orthogonal latin squares of order 10.

Theorem 4.25 *Conjecture 4.24 is false for all $n \neq 6$.*

Proof (for $n = 10$.) Consider the two partial latin squares of order 10 below. By moving the coloured diagonals to the coloured dots above and to the side, and replacing the moved entries by 7, 8 and 9 depending on whether they are yellow, blue or red, we can construct two mutually orthogonal latin squares of order 10. Observe that we can consider the first array as containing the quasigroup $(\mathbb{Z}/7\mathbb{Z}, *_1)$ and the second array containing the quasigroup $(\mathbb{Z}/7\mathbb{Z}, *_4)$, where we label the columns with $x \in \mathbb{Z}/7\mathbb{Z}$ and the rows by $y \in \mathbb{Z}/7\mathbb{Z}$. The yellow entries in the first square are $4 + 2y$ and in the second square are $5 - 2y$. Likewise the blue entries in the first square are $2 + 2y$ and in the second square are $6 - 2y$ and the red entries in the first square are $1 + 2y$ and in the second square are $-2y$. When we move the entries to the columns on the right we line up these columns so that the sums of the elements in the same column are the same, and in different columns they are different. We repeat the process by moving the same elements to the upper rows. Thus, we obtain in the first square rows $6 + 2x$, $5 + 2x$ and $3 + 2x$ and the second square $2 - 2x$, $4 - 2x$ and $1 - 2x$. Clearly, every pair of elements from $\{7, 8, 9\} \times \{0, 1, 2, 3, 4, 5, 6\}$ and $\{0, 1, 2, 3, 4, 5, 6\} \times \{7, 8, 9\}$ occur exactly once. Since in the top-right corner we have two orthogonal latin squares of order 3, every pair of elements from $\{7, 8, 9\} \times \{7, 8, 9\}$ occurs once. Finally, by lining up the top rows and the right-hand columns so the sums are distinct, we can ensure that every pair of elements from $\{0, 1, 2, 3, 4, 5, 6\} \times \{0, 1, 2, 3, 4, 5, 6\}$ occurs exactly once.

·	·	·	·	·	·	·	7	8	9
·	·	·	·	·	·	·	9	7	8
·	·	·	·	·	·	·	8	9	7
6	0	1	2	3	4	5	·	·	·
5	6	0	1	2	3	4	·	·	·
4	5	6	0	1	2	3	·	·	·
3	4	5	6	0	1	2	·	·	·
2	3	4	5	6	0	1	·	·	·
1	2	3	4	5	6	0	·	·	·
0	1	2	3	4	5	6	·	·	·

·	·	·	·	·	·	·	7	8	9
·	·	·	·	·	·	·	8	9	7
·	·	·	·	·	·	·	9	7	8
3	4	5	6	0	1	2	·	·	·
6	0	1	2	3	4	5	·	·	·
2	3	4	5	6	0	1	·	·	·
5	6	0	1	2	3	4	·	·	·
1	2	3	4	5	6	0	·	·	·
4	5	6	0	1	2	3	·	·	·
0	1	2	3	4	5	6	·	·	·

□

4.4 Linear Spaces

An **incidence structure** Γ is a pair (P, L) where P is a set (of points) and L is a multi-set of non-empty subsets of P (called **lines**).

We do not rule out the possibility that there are two lines which are incident with the same set of points. This is necessary since we want the following definition to give an incidence structure.

The dual Γ^* of $\Gamma = (P, L)$ is the incidence structure (L, M) where for all $x \in P$ we have a line $m \in M$ such that $x \in \ell$ if and only if $\ell \in m$. This reflexive relation ensures that $(\Gamma^*)^* = \Gamma$.

In Fig. 4.1 we see that the dual of four intersecting lines, each incident with three points and no three concurrent, is the complete graph on four vertices.

Observe that the dual of a single line with r points in a single point with r lines all of which contain just the single point.

A **linear space** is an incidence structure with the property that any two points are incident with a unique line. We implicitly assume that a linear space has at least two points.

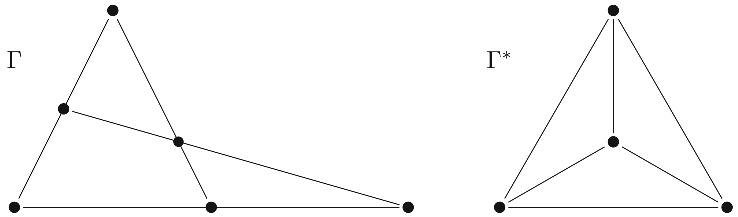


Fig. 4.1 The dual of an incidence structure

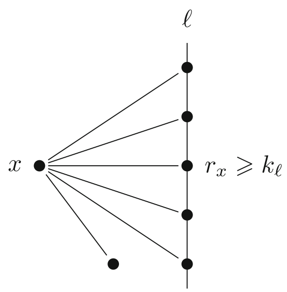


Fig. 4.2 The number of lines incident with a point is at least the number of points incident with a line

Theorem 4.26 (Erdős-de Bruijn) *If there is no line containing all the points of a finite linear space* (P, L) *then* $|L| \geqslant |P|$.

Proof For any point x, let r_x denote the number of lines incident with x.

For any line ℓ, let k_ℓ denote the number of points incident with ℓ.

Suppose

$$|P| \geqslant |L|. \tag{4.1}$$

Since there is no line containing all the points, there is a non-incident pair (x, ℓ), where ℓ is a line and x is a point.

For each pair (x, ℓ), where ℓ is a line and x is a point not incident with ℓ,

$$r_x \geqslant k_\ell, \tag{4.2}$$

since any two points are joined by a unique line, see Fig. 4.2.

Hence,

$$|P|r_x \geqslant |L|k_\ell,$$

Then

$$(|P| - k_\ell)|L| \geqslant (|L| - r_x)|P|.$$

Since not all of the points are contained in a line $|P| \neq k_\ell$ and since not all lines are incident with the same point $|L| \neq r_x$. Hence,

$$\frac{1}{(|L| - r_x)|P|} \geqslant \frac{1}{(|P| - k_\ell)|L|}$$

Now sum both sides of this inequality over all such pairs (x, ℓ).

There are exactly $|L| - r_x$ lines not incident with x, so the left-hand side of the inequality gives

$$\sum_{x \in P} \sum_{\ell \not\ni x} \frac{1}{(|L| - r_x)|P|} = \sum_{x \in P} \frac{1}{|P|} = 1.$$

There are exactly $|P| - k_\ell$ points not incident with ℓ, so the right-hand side of the inequality gives

$$\sum_{\ell \in L} \sum_{x \notin \ell} \frac{1}{(|P| - k_\ell)|L|} = \sum_{\ell \in L} \frac{1}{|L|} = 1.$$

Therefore, the inequalities (4.1) and (4.2) must be equalities.

In particular, we have that $|P| = |L|$. □

In the next theorem we analyse more carefully the case in which $|P| = |L|$.

Theorem 4.27 *In a finite linear space $\Gamma = (P, L)$ in which $|P| = |L|$ any two lines are concurrent. Equivalently, Γ^* is also a finite linear space. Moreover, if there are four points, no three collinear, then there is an integer n, such that every line is incident with $n + 1$ points and every point is incident with $n + 1$ lines.*

Proof Since $|P| = |L|$ implies $|P| \geqslant |L|$, we can repeat the proof of Theorem 4.26 and once again conclude that the inequality (4.2) is an equality. Thus, for every non-incident pair (x, ℓ), we have that $r_x = k_\ell$, .

Case 1

Suppose there are points x and y for which $r_x \neq r_y$. Then every line is either incident with x or incident with y, since if there were a line ℓ incident with neither then $r_x = k_\ell = r_y$. Let z be a point not incident with the line joining x and y. Since $r_x \neq r_y$, r_z must differ from either r_x or r_y, so, without loss of generality, let us assume that $r_x \neq r_z$. Then, every line is either incident with x or z. Therefore, there is a unique line ℓ, not incident with x, which joins y and z, see Fig. 4.3.

Case 2

Suppose that $r_x = r_y = n + 1$ for all points x and y. Then, since ℓ is not incident with some point, $k_\ell = n + 1$ for all $\ell \in L$. Counting the number of points on the

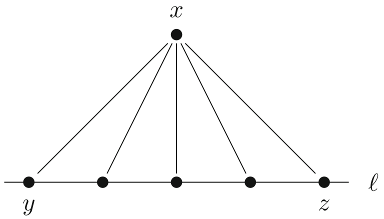


Fig. 4.3 Case 1 of the proof of Theorem 4.27: a degenerate projective plane

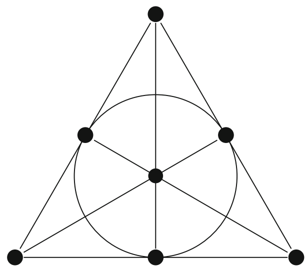


Fig. 4.4 The projective plane of order two

lines incident with x, we have that $|P| = 1 + (n+1)n = n^2 + n + 1$ and so $|L| = n^2 + n + 1$.
Let ℓ be a line and count the number of lines intersecting ℓ. There are $n+1$ points incident with ℓ and each of these points is incident with n other lines. Therefore, ℓ is incident with $(n+1)n$ lines, which is all the other lines.

□

4.5 Projective Planes

A linear space in which any two lines are concurrent is called a **projective plane**. Thus, Theorem 4.27 implies that a linear space (P, L) for which $|P| = |L|$ is a projective plane. A projective plane is **non-denegerate** if it contains four points, no three of which are collinear.

Theorem 4.28 *A non-degenerate finite projective plane has an order n, with the property that every line is incident with* $n+1$ *points and every point is incident with* $n+1$ *lines.*

Proof This follows directly from Theorem 4.27. □

For example, in Fig. 4.4, we have a projective plane of order 2. It is known as the **Fano plane**.

This example is the first case of the following examples, which are denoted PG$(2, q)$.

Example 4.29 Let $V_3(\mathbb{F}_q)$ denote the three-dimensional vector space over the field with q elements. Let P be the set of one-dimensional subspaces of $V_3(\mathbb{F}_q)$ and let L be the set of two-dimensional subspaces of $V_3(\mathbb{F}_q)$ and let incidence be containment. i.e. $x \in P$ is incident with $\ell \in L$ if as subspace of $V_3(\mathbb{F}_q)$, $x \subset \ell$. Then (P, L) is a projective plane since any two elements of P span an element of L and any two elements of L intersect in an element of P.

The number of points incident with a line is the number of 1-dimensional subspaces contained in a 2-dimensional subspace, which is $(q^2-1)/(q-1) = q+1$. We will return to this example in Sect. 4.7.

Example 4.29 implies that there is a projective plane of order n whenever n is the power of a prime. It is not known if there are projective planes of non-prime power order. The following is the **prime-power conjecture**.

Conjecture 4.30 If there exists a non-degenerate projective plane of order n then n is the power of a prime.

The only evidence we have for the conjecture is Theorem 4.31, which rules out many possibilities for n including 6, 14, 21 and 22, and that there is no projective plane of order 10, which was proven with the aid of a computer.

Theorem 4.31 (Bruck–Ryser) *If $n \equiv 1$ or 2 modulo 4 and there exists a non-degenerate projective plane of order n then n is the sum of two squares.*

Proof Suppose that (P, L) is a projective plane of order n and let $m = n^2 + n + 1$.

Let A be the $m \times m$ matrix whose rows are indexed by elements of P and whose columns are indexed by elements of L and where the (x, ℓ) entry is 1 if and only if x is incident with ℓ. Then

$$\mathrm{AA}^t = J + nI,$$

where J is the all one matrix and I is the $m \times m$ identity matrix.

For indeterminates $x_1, \dots, x_m$, define

$$z_j = \sum_{i=1}^{m} a_{ji} x_i.$$

Then, pre and post multiplying the above equality by $x = (x_1, \dots, x_m)$ we get

$$z_1^2 + \cdots + z_m^2 = w^2 + n(x_1^2 + \cdots + x_m^2), \tag{4.3}$$

where $w = x_1 + \cdots + x_m$.

By Lagrange's four-square theorem, n is the sum of four squares. Moreover, for any integers a_i, b_i, one can write

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2) = c_1^2 + c_2^2 + c_3^2 + c_4^2 \tag{4.4}$$

where,

$$\begin{pmatrix} -a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{4.5}$$

Therefore we can write the above as

$$z_1^2 + \cdots + z_m^2 + nx_{m+1}^2 = w^2 + c_1^2 + \cdots + c_{m+1}^2, \tag{4.6}$$

where we have added nx_{m+1}^2 to both sides, since $m + 1 \equiv 0$ modulo 4.

Now we can solve $z_j^2 = c_j^2$ with either $z_j = c_j$ or $z_j = -c_j$, for $j = 1, \dots, m$. We eliminate x_j by solving either $z_j = c_j$ or $z_j = -c_j$, where solving involves substituting x_j as a rational linear combination of the remaining x_i's.

Having solved this system of equations for all x_i, $i \neq m + 1$, we will have $z_i = a_i x_{m+1}$ and $c_i = \pm a_i x_{m+1}$ for $i = 1, \dots, m$, for some rational numbers a_i. Putting $x_{m+1} = 1$, we get

$$n = w^2 + c_{m+1}^2,$$

where c_{m+1} and w are some rational numbers.

An elementary number-theoretic argument implies that if an integer n is the sum of two rational squares then it is the sum of two integer squares, which is what we wanted to prove.

The only possible problem is that at some stage it may be that x_i no longer appears in any expression for the remaining z_j's or c_j's. However, the matrix in (4.5) is non-singular so if we substitute $x_1, \dots, x_{j-1}$ with rational number linear combinations of $x_j, \dots, x_m$, the expression for c_j will still have a x_j term. □

4.6 Affine Planes

An **affine plane** is a linear space (P, L) with the property that for all $x \in P$ and $\ell \in L$, where $x \notin \ell$, there is a unique line m such that $x \in m$ and $m \cap \ell = \emptyset$ (Fig. 4.5).

Lemma 4.32 *Define a relation* $\sim$ *on the lines, by* $m \sim \ell$ *if* $m = \ell$ *or* $m \cap \ell = \emptyset$. *Then* $\sim$ *defines an equivalence relation.*

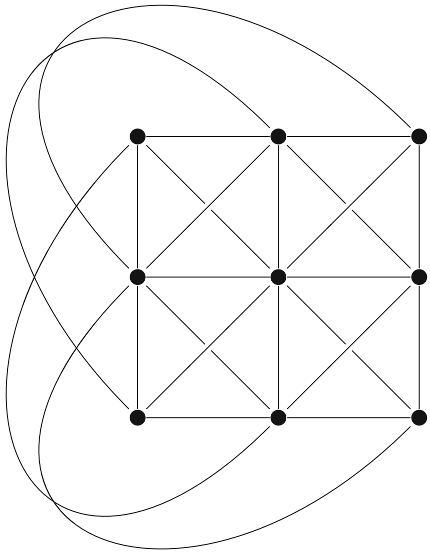


Fig. 4.5 The affine plane of order three

Proof Clearly $\ell \sim \ell$ and $\ell \sim m$ if and only if $m \sim \ell$, so we only have to show transitivity. If $\ell \sim m$ amd $\ell \sim m'$ but $m \not\sim m'$ then there is a point $x \in m \cap m'$. But then $x \notin \ell$, which contradicts the uniqueness of m. □

Let E denote the set of equivalence classes of L. For $\ell \in L$, define $\ell^* = \ell \cup \{e\}$, where e is the equivalence class containing ℓ. Let $\ell_\infty = E$. Define $L^* = \{\ell^* \mid \ell \in L\} \cup \{\ell_\infty\}$.

Theorem 4.33 *Given an affine plane* (P, L)*, the incidence structure* $(P \cup E, L^*)$ *is a projective plane.*

Proof First we prove that a point of P and a point of E are joined by a line. By the parallel axiom, the lines of a parallel class cover all the points in P. Thus, if $x \in P$ and $e \in E$ then there exists an $m \in L$ with the property that $x \in m$ and $m \in e$. This implies $x, e \in m^*$. Since an affine plane is a linear space, any two points of P are joined by a line of L^*. Any two points of E are joined by the line ℓ_∞. Hence, $(P \cup E, L^*)$ is a linear space.

For $\ell, m \in L$ either $\ell \cap m \neq \emptyset$, in which case they intersect in a point, or $\ell \cap m = \emptyset$. In the latter case, this implies that there is an $e \in E$ such that $\ell, m \in e$ and so ℓ^* and m^* intersect in e. Hence, $(P \cup E, L^*)$ is a dual linear space.

Thus, by Theorem 4.27, $(P \cup E, L^*)$ is a projective plane. □

Corollary 4.34 *If* (P, L) *is a finite affine plane then there is an* n *such that every point is incident with* $n + 1$ *lines and every line is incident with* n *points.*

Proof Since the projective plane we obtain from Theorem 4.33 is not a degenerate projective plane, the projective plane has an order n in which every point is incident with $n + 1$ lines and every line is incident with $n + 1$ points. The corollary follows by observing that lines in L have one less point than the lines in L^*. □

Theorem 4.35 *The linear space obtained from a projective plane π by deleting a line ℓ_∞ and all the points on the line ℓ_∞, is an affine plane.*

Proof Suppose that x is a point and ℓ is a line such that $x \not\in \ell$. Let e be the intersection of ℓ and ℓ_∞. Then the line m joining x to e in π is the unique line incident with x and skew to ℓ. □

Example 4.36 Consider the 2 mutually orthogonal latin squares $A_m = (a_{ij}^m)$ of order 3 below.

We construct an affine plane (P, L) of order 3, where $P = \{(i, j) \mid i, j = 1, 2, 3\}$.

There are two parallel classes of lines given by

$$\ell_{mk} = \{(i, j) \mid a_{ij}^m = k\},$$

for $k = 1, 2, 3$ and $m = 1, 2$.

$$A_1 = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \quad \begin{matrix} \ell_{11} = \{(1, 1), (2, 3), (3, 2)\} \\ \ell_{12} = \{(1, 2), (2, 1), (3, 3)\} \\ \ell_{13} = \{(1, 3), (2, 2), (3, 1)\} \end{matrix}$$

$$A_2 = \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \quad \begin{matrix} \ell_{21} = \{(1, 1), (2, 2), (3, 3)\} \\ \ell_{22} = \{(1, 2), (2, 3), (3, 1)\} \\ \ell_{23} = \{(1, 3), (2, 1), (3, 2)\} \end{matrix}$$

And two further sets of parallel lines, the horizontal and vertical lines.

$$\begin{matrix} h_1 = \{(1, 1), (1, 2), (1, 3)\} & v_1 = \{(1, 1), (2, 1), (3, 1)\} \\ h_2 = \{(2, 1), (2, 2), (2, 3)\} & v_2 = \{(1, 2), (2, 2), (3, 2)\} \\ h_3 = \{(3, 1), (3, 2), (3, 3)\} & v_3 = \{(1, 3), (2, 3), (3, 3)\} \end{matrix}$$

Theorem 4.37 *Given $n - 1$ mutually orthogonal latin squares of order n one can construct an affine plane of order n.*

Proof Let $P = \{(i, j) \mid i, j = 1, \dots, n\}$.

Define (horizontal) lines $h_j = \{(j, x) \mid x = 1, \dots, n\}$, for $j = 1, \dots, n$ and (vertical) lines $v_j = \{(x, j) \mid x = 1, \dots, n\}$, for $j = 1, \dots, n$.

Let $\{1, \dots, n\}$ be the set over which the latin squares are defined.

For each latin square A_m and for each $k \in \{1, \dots, n\}$, define a line

$$\ell_{mk} = \{(i, j) \mid (A_m)_{ij} = k\}.$$

Let

$$L = \{\ell_{mk} \mid m{=}1, \dots, n{-}1, k{=}1, \dots, n\} \cup \{h_j \mid j{=}1, \dots, n\} \cup \{v_j \mid j = 1, \dots, n\}.$$

Suppose that (i, j) and (i', j') are joined by both the lines $\ell_{m,k}$ and $\ell_{m',k'}$. Then $(A_m)_{ij} = (A_m)_{i'j'} = k$ and $(A_{m'})_{ij} = (A_{m'})_{i'j'} = k'$, contradicting the orthogonality of the latin squares A_m and $A_{m'}$. This implies that any two points are joined by at most one line.

There are $n^2 + n$ lines, each incident with n points and n^2 points in P. Counting triples (x, y, ℓ) where x and y are points incident with the line ℓ, we have that

$$N = (n^2 + n)\binom{n}{2}$$

where N is the number of pairs of points joined by a line. Since $(n^2 + n)\binom{n}{2} = \binom{n^2}{2}$ it follows that any two points are joined by a line.

Hence, (P, L) is a linear space.

If $m \neq m'$ then $\ell_{m,k}$ and $\ell_{m',k'}$ intersect in a unique point, since orthogonality implies there exists a unique (i, j) such that $(A_m)_{ij} = k$ and $(A_{m'})_{ij} = k'$.

Suppose that x is a point not incident with the line $\ell_{m,k}$. Since x is a point, it is a cell of the latin square A_m and there is some k' such that $\ell_{m,k'}$ contains x, precisely the one in which A_m has entry k' in the cell x. This proves the parallel axiom for the non-horizontal and non-vertical lines. The parallel axiom for the horizontal and vertical lines is immediate. □

Theorem 4.38 *Given an affine plane of order n one can construct $n - 1$ mutually orthogonal latin squares of order n.*

Proof An affine plane of order n has n^2 points and $n + 1$ parallel classes of lines. Select two of these classes $\{H_1, \dots, H_n\}$ and $\{V_1, \dots, V_n\}$. Any point is incident with one horizontal line H_i and one vertical line V_j. Give this point the coordinates (i, j).

Let $\{L_1, \dots, L_n\}$ be a further parallel class of lines. Define a matrix $A = (a_{ij})$ by the rule $a_{ij} = k$ if and only if $(i, j) \in L_k$. Then A is a latin square, since each line L_k meets each horizontal line and each vertical line exactly once. Moreover, A and A′, where A′ is the latin square we obtain from the parallel class of lines $\{L'_1, \dots, L'_n\}$, are orthogonal since each line of $\{L_1, \dots, L_n\}$ and $\{L'_1, \dots, L'_n\}$ meet in a unique point. □

There are affine planes of order $n = p^h$ where $h \geqslant 2$ and $p^h \geqslant 8$, which are not isomorphic to $AG(2, q)$, the affine plane obtained by deleting a line from the projective plane $PG(2, q)$. These planes will give $n - 1$ mutually orthogonal latin squares of order n which are not isomorphic to the set of mutually orthogonal latin squares,

$$\{(\mathbb{F}_n, *_m) \mid m \in \mathbb{F}_n,\ m \neq 0\}.$$

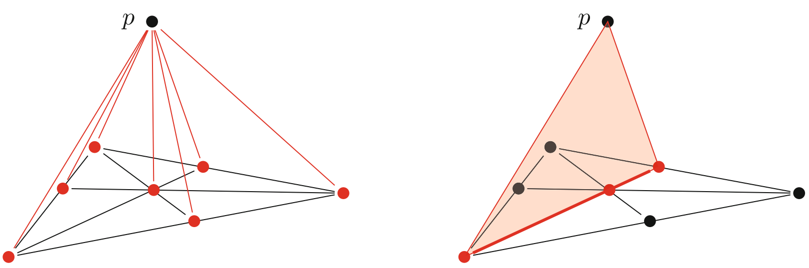
We shall see some examples of such affine planes in Theorem 4.43.

4.7 Projective Spaces

Let $V_n(\mathbb{F})$ denote the n-dimensional vector space over the field $\mathbb{F}$.

Consider the geometry we get which has as points the set of one-dimensional subspaces of $V_n(\mathbb{F}_q)$, as lines the set of two-dimensional subspaces of $V_n(\mathbb{F}_q)$, as planes the set of three-dimensional subspaces of $V_n(\mathbb{F}_q)$, etc. This structure is denoted $PG(n-1, q)$. It has a large group of symmetries. We can multiply the vectors of $V_n(\mathbb{F}_q)$ by any $n \times n$ non-singular matrix with entries from $\mathbb{F}_q$ and preserve the subspace structure of the vector space and therefore preserve the subspace structure of $PG(n-1, q)$.

We define a symmetry of an incidence structure (P, L) to be a bijection of P which induces a bijection on L. Lemma 4.39 implies that the Fano plane in Fig. 4.4 has 168 symmetries, since any 3-tuple of linearly independent vectors in $V_n(\mathbb{F}_2)$ will give a 3×3 non-singular matrix with entries from $\mathbb{F}_2$.



(a) In $PG(3,2)$, there are seven lines through each point p.

(b) A non-incident point-line pair span a plane in $PG(3,2)$.

In the above figure, we see some substructures contained in the geometry $PG(3, 2)$. The points of this geometry can be labelled by the non-zero vectors of $V_4(\mathbb{F}_2)$ of which there are 15. Since any two points are joined by a unique line, which is incident with three points, there are $15.14/3.2 = 35$ lines. In Theorem 4.40, we will deduce a formula which allows us to count the number of subspaces of a finite vector space, and thereby the number of subspaces of $PG(n-1, q)$.

Lemma 4.39 *The number of k-tuples of linearly independent vectors in $V_n(\mathbb{F}_q)$ is $(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$.*

Proof Consider a k-tuple $(v_1, \ldots, v_k)$ of linearly independent vectors.

There are $q^n - 1$ non-zero vectors, so $q^n - 1$ choices for v_1.

There are $q^n - q$ vectors not in the subspace $\langle v_1 \rangle$, so $q^n - q$ choices for v_2.

In the same way, there are $q^n - q^j$ vectors not in the subspace $\langle v_1, \ldots, v_j \rangle$, so $q^n - q^j$ choices for v_{j+1}. □

Theorem 4.40 *The number of k-dimensional subspaces of* $\mathrm{V}_n(\mathbb{F}_q)$ *is*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})}.$$

Proof Each k-tuple of linearly independent vectors in $\mathrm{V}_n(\mathbb{F}_q)$ spans a k-dimensional subspace. By Lemma 4.39, there are $(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$ such k-tuples. Again by Lemma 4.39, there are $(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$ k-tuples which span the same k-dimensional subspace. □

In the following we use the notation $x \oplus y$ to denote the line joining x and y and $\mathrm{PG}_{k-1}(\mathbb{F})$ denotes the projective space obtained from the vector space $\mathrm{V}_n(\mathbb{F})$.

Theorem 4.41 *Suppose that x_1, x_2, x_3 and y_1, y_2, y_3 are two sets of three non-collinear points of* $\mathrm{PG}_{k-1}(\mathbb{F})$*, $k \geqslant 3$, where there is a point z such that z, x_i, y_i are collinear for $i = 1, 2, 3$, see Fig. 4.6.*

Then there are points $z_{ij} = (x_i \oplus x_j) \cap (y_i \oplus y_j)$, for all $i \neq j$, and z_{12}, z_{13} and z_{23} are collinear.

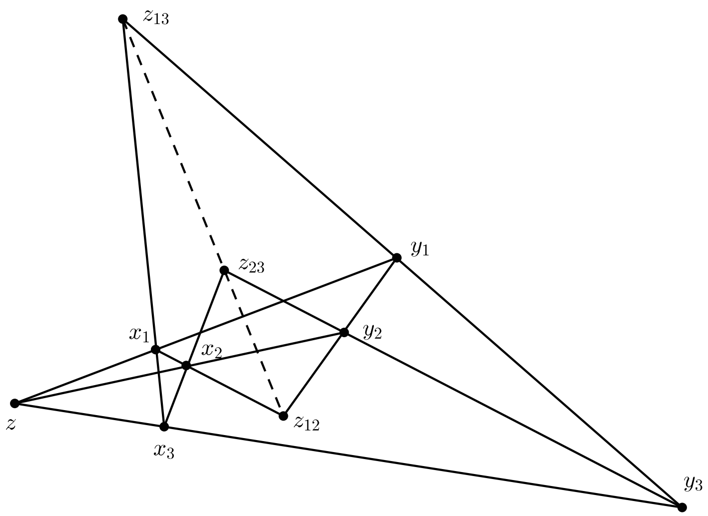


Fig. 4.6 Desargues configuration

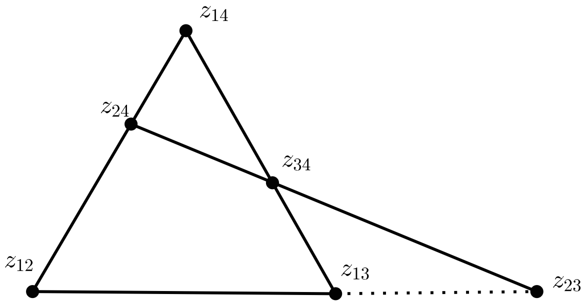


Fig. 4.7 The points z_{12}, z_{13}, z_{23} and z_{14} are co-planar

Proof Suppose that $k \geqslant 4$.

Since the lines $x_i \oplus y_i$ contain the point z, the points x_i, x_j, y_i, y_j are contained in a plane. Thus the lines $x_i \oplus x_j$ and $y_i \oplus y_j$ have a point of intersection, which we define as z_{ij}, see Fig. 4.6. Furthermore, the whole configuration is contained in a three dimensional subspace.

Suppose the configuration is not contained in a plane of $\mathrm{PG}_{k-1}(\mathbb{F})$. Then $\pi_x = x_1 \oplus x_2 \oplus x_3$ and $\pi_y = y_1 \oplus y_2 \oplus y_3$ are planes of $\mathrm{PG}_{k-1}(\mathbb{F})$ which intersect in a line ℓ of $\mathrm{PG}_{k-1}(\mathbb{F})$. Furthermore, ℓ contains z_{12}, z_{13} and z_{23}, so these three points are collinear.

Suppose the configuration is contained in a plane π of $\mathrm{PG}_{k-1}(\mathbb{F})$. Let x_4 and y_4 be points of $\mathrm{PG}_{k-1}(\mathbb{F}) \setminus \pi$ such that z, x_4 and y_4 are collinear. By the previous paragraph, z_{12}, z_{14}, z_{24} are collinear, z_{13}, z_{14}, z_{34} are collinear, and z_{23}, z_{24}, z_{34} are collinear. Therefore, z_{12}, z_{13}, z_{23} and z_{14} are co-planar, see Fig. 4.7. Let π_4 denote the plane containing these points. Then $\pi_4 \cap \pi$ is a line of $\mathrm{PG}_{k-1}(\mathbb{F})$ containing z_{12}, z_{13} and z_{23}.

The same proof works for $k = 3$ since we can embed $\mathrm{PG}_2(\mathbb{F})$ in $\mathrm{PG}_3(\mathbb{F})$. $\square$

A **spread** of $\mathrm{V}_{2d}(\mathbb{F}_q)$ is a set of d-dimensional subspaces which partition the non-zero vectors of $\mathrm{V}_{2d}(\mathbb{F}_q)$.

Lemma 4.42 *Let $q = p^h$, where $h \geqslant 2$ and p is an odd prime p. Let η be an element of $\mathbb{F}_q$, such that $\eta \neq c^2$, for all $c \in \mathbb{F}_q$. For each $a, b \in \mathbb{F}_q$, let*

$$U_{ab} = \langle (1, 0, a, b), (0, 1, b, \eta a^p) \rangle$$

and let

$$U_\infty = \langle (0, 0, 1, 0), (0, 0, 0, 1) \rangle.$$

Then

$$\{U_{ab} \mid a, b \in \mathbb{F}_q\} \cup \{U_\infty\}$$

is a spread of $\mathrm{V}_4(\mathbb{F}_q)$.

Proof To prove this is a spread, we have to show that each non-zero vector of $\mathbb{F}_q^4$ is incident with exactly one of the subspaces

$$\{U_{ab} \mid a, b \in \mathbb{F}_q\} \cup \{U_\infty\}.$$

It suffices to prove that for all $(x, y) \in \mathbb{F}_q^2$, $(x, y) \neq (0, 0)$, the set of vectors

$$\{x(1, 0, a, b) + y(0, 1, b, \eta a^p) \mid a, b \in \mathbb{F}_q\}$$

is a set of q^2 (distinct) vectors. Note that this implies that all of the vectors of $V_4(\mathbb{F}_q)$ which are not in U_∞ are contained in some element of the spread.

If not then there are a, b and a', b' such that

$$\begin{pmatrix} a & b \\ b & \eta a^p \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a' & b' \\ b' & \eta(a')^p \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

which implies

$$\begin{pmatrix} a - a' & b - b' \\ b - b' & \eta a^p - (a')^p \end{pmatrix}$$

is a singular matrix. But $a^p - (a')^p = (a - a')^p$, so this implies

$$\eta(a - a')^{p+1} = (b - b')^2,$$

which contradicts the condition on η. □

Theorem 4.43 *Let P be the set of vectors of* $\mathrm{V}_{2d}(\mathbb{F}_q)$ *and let L be the set of cosets of the subspaces of a spread of* $\mathrm{V}_{2d}(\mathbb{F}_q)$*. Then (P, L) is an affine plane of order q^d.*

Proof Let u and v be two vectors of $\mathrm{V}_{2d}(\mathbb{F}_q)$. Then $u - v \in U$, for some U in the spread. This implies $v \in U + v$ (since $0 \in U$) and $u \in U + v$ (since $u - v \in U$) and so (P, L) is a linear space. The set of lines which are cosets of a fixed subspace of the spread are parallel lines, so the parallel axiom holds. □

Theorem 4.41 gives us a geometric way of checking if a projective plane is isomorphic to $\mathrm{PG}(2, q)$ or not. If one can find two triangles in perspective for which the points z_{ij} are not collinear then the projective plane is not isomorphic to $\mathrm{PG}(2, q)$; it is called a **non-Desarguesian** plane.

The spread in Lemma 4.42, allows us to define an affine plane by applying Theorem 4.43, which can be completed to a projective plane of order q^2 according to Theorem 4.33. This projective plane is not isomorphic to $\mathrm{PG}(2, q^2)$, if we can find elements e and η, so that $\overline{a} \neq a$ or a^p. The value of a is the (unique) solution to the equation

$$a - \eta a^p = e.$$

whereas $\overline{a}$ is the (unique) solution to

$$-\eta\overline{a} + \overline{a}^p = e^p.$$

If $q = p^h$, for some $h \geqslant 2$, then we can choose $e \neq 1 - \eta$ then $a \neq a^p$ and therefore $\overline{a}$ is not equal to at least one of a and a^p. Thus, we have constructed non-Desarguesian planes of order p^{2h} for all $h \geqslant 2$ and odd primes p (Fig. 4.8).

In fact, there are non-Desarguesian planes of order q known, for all prime powers q where q is not a prime and $q \geqslant 9$. This has led to the following conjecture, which is known to be true only for $p \leqslant 7$.

Conjecture 4.44 A projective plane of prime order p is isomorphic to PG(2, p).

Observe that by Theorem 4.35, we can obtain affine planes by removing any line from a projective plane, so the non-Desarguesian projective plane, will give non-Desarguesian affine planes. These affine planes, by Theorem 4.38, will give us sets of $n - 1$ mutually orthogonal latin squares of order n, which are not isomorphic to the sets of $n - 1$ mutually orthogonal latin squares of order n constructed in Lemma 4.18.

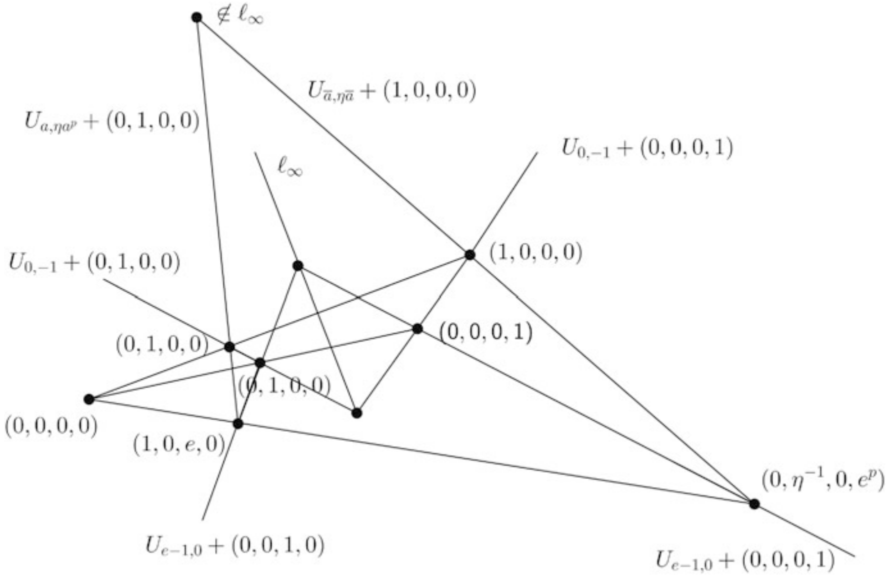


Fig. 4.8 The two triangles in perspective do not complete to Desargues configuration

4.8 Notes and References

Bregman's theorem, mentioned in the text, which was conjectured by Henryk Mink in 1963, a proof of which appeared in Bregman (1973). G. P. Egorychev gave the first proof of van der Waerden's conjecture in Egorychev (1981). The falsity of Euler's conjecture is from Bose et al. (1960) and earned the authors a mention the front page of the New York Times on April 26, 1959 and a detailed article with photos. The proof of Theorem 4.26, the Erdős-de Bruijn theorem, is due to J. H. Conway. The fact that there is no projective plane of order 10 was proven by Lam et al. (1989), with the aid of a computer, in 1989.

4.9 Exercises

Exercise 4.1

i. Prove that a quasi-group of order 4 with an identity element is a group.
ii. Find a quasi-group of order 5 with an identity element which is not a group.

Exercise 4.2

i. Prove that if a latin square of order m contains a latin square of order n then $m \geqslant 2n$.
ii. Construct a latin square of order $2n$ that contains Q, a latin square of order n.

Exercise 4.3

i. Let L be a latin square of order $2m$ that contains a latin square of order m. Prove that if m is odd then L has no orthogonal mate (i.e. there is no latin square orthogonal to L).
ii. Prove that if $m = 2^k$ then there exists a latin square of order $2m$ which contains a sub-latin square of order m and which belongs to a set of $2m - 1$ MOLS. (Recall that over $\mathbb{F}_q$, $L_a(x, y) = ax + y$, $a \neq 0$, forms a set of $q - 1$ MOLS.)

Exercise 4.4 Suppose G is an abelian group of order n and L is a latin square of order n (indexed by the elements of G) with $\mathrm{L}(x, y) = x + y$. Prove that there is a latin square orthogonal to L if and only if there is a permutation σ of the elements of G such that the map $\tau : G \to G$ defined by $\tau(x) = \sigma(x) - x$ is also bijective.

Exercise 4.5 Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, \ldots, B_m\}$ be two partitions of a set S. Let $\mathrm{M} = (m_{ij})$ be the matrix where $m_{ij} = |A_i \cap B_j|$. Prove that the number of common systems of distinct representatives of $\mathcal{A}$ and $\mathcal{B}$ is equal to the permanent of M.

Exercise 4.6 A latin square L on the ordered set $\{x_1, \ldots, x_n\}$ is **idempotent** if its (i, i)-th entry is x_i, for all $i = 1, \ldots, n$.

1. Given a set of m mutually orthogonal latin squares of order n, construct a set of $m - 1$ mutually orthogonal idempotent latin squares of order n.
2. Given a set of n mutually orthogonal idempotent latin squares of order $s + 1$ and a linear space $\Gamma = (P, L)$ in which for all lines $\ell \in L$, ℓ is incident with $s + 1$ points, construct a set of n mutually orthogonal idempotent latin squares of order $|P|$.
   [Hint: Let f be a bijective map from the elements of P to the set $\{1, \ldots, |P|\}$. Then for each line ℓ of Γ, there is a set of n mutually orthogonal idempotent latin squares $L_1, \ldots, L_n$ of order $s+1$ on the set $S_\ell = \{f(x) \mid x \in \ell\}$ (so the rows and columns are Labelled by elements of the set S_ℓ). Construct a (partial) latin square L_k^* of order $|P|$, whose (i, j)-th entry is the (i, j)-th entry in the latin square L_k for the line ℓ joining the points $f^{-1}(i)$ and $f^{-1}(j)$ of Γ.]
3. Construct three mutually orthogonal idempotent latin squares of order 21.
4. By deleting three points from PG(2, 4), construct two mutually orthogonal latin squares of order 18.

Exercise 4.7 Let A be a finite set.

i. Suppose that $C \subseteq A^n$ has the property that any two elements of C agree on at most 1 coordinate.
   Prove that $|C| \leqslant |A|^2$ and that in the case of equality for all $(i, j) \in A^2$, there exists a unique element of C such that $(u_1, \ldots, u_{n-2}, i, j) \in C$.
ii. Suppose that $C \subseteq A^n$ has the property that any two elements of C agree on at most 1 coordinate and $|C| = |A|^2$. For each $m \in \{1, \ldots, n-2\}$, let L_k be the $|A| \times |A|$ array whose (i, j) entry is u_m, where $(u_1, \ldots, u_{n-2}, i, j) \in C$.
   Prove that $\{L_m \mid m \in \{1, \ldots, n-2\}\}$ is a set of $n - 2$ mutually orthogonal Latin squares of order $|A|$.
iii. Suppose that $C \subseteq A^n$ has the property that any two elements of C agree on at most $k - 1$ coordinates and that $|C| = |A|^k$.
   Prove that $n \leqslant k + m$, where m is the maximum number of mutually orthogonal Latin squares of order $|A|$.

Exercise 4.8 Prove that a set of $n - 2$ mutually orthogonal latin squares of order n can be extended to a set of $n - 1$ mutually orthogonal latin squares of order n. Hence, prove that there is no set of four mutually orthogonal latin squares of order 6.

Exercise 4.9 Prove that a linear space with $k^2 + k + 1$ points in which each line is incident with $k + 1$ points is a projective plane.

Exercise 4.10 Prove that there is only one projective plane of order 2 and only one projective plane of order 3.

Exercise 4.11 Suppose that π is a projective plane of order n with a subplane π' of order m.

i. Prove that if $m^2 < n$ then $m^2 + m \leqslant n$.
ii. Prove that if $m^2 = n$ then every line of π is incident with 1 or $m + 1$ points of π'.
iii. Let S be a set of points of π with the property that every line of π is incident with 1 or $t + 1$ points of S. Prove that t divides n.

Exercise 4.12 A difference set of an abelian group G is a subset D with the property that every non-zero element of G can be expressed uniquely as the difference of two elements of D.

i. Construct difference sets of $\mathbb{Z}/7\mathbb{Z}$ and $\mathbb{Z}/13\mathbb{Z}$.
ii. Construct a projective plane from a difference set of $\mathbb{Z}/(n^2 + n + 1)\mathbb{Z}$.

Exercise 4.13 Let G be an abelian group of size $n^2 - 1$ with a subgroup N of size $n - 1$.

A relative difference set is a subset D of G in which every element of $G \setminus N$ can be expressed uniquely as the difference of two elements of D and no non-zero element of N can be expressed as the difference of two elements of D.

i. Find a relative difference set of $\mathbb{Z}/8\mathbb{Z}$.

Suppose that D is a relative difference set of G.

Let $\ell_g = \{g + d \mid d \in D\}$.

Let R be a set of coset representatives of N and let $m_r = \{r + x \mid x \in N\} \cup \{\infty\}$, for each $r \in R$.

Let $P = G \cup \{\infty\}$ and let $L = \{\ell_g \mid g \in G\} \cup \{m_r \mid r \in R\}$

ii. Prove that (P, L) is an affine plane of order n.

Exercise 4.14 Let A be a randomly chosen $n \times n$ matrix with entries from $\mathbb{F}_q$. Prove that the probability that A is non-singular tends to some positive constant $c(q)$ as n tends to infinity.

Exercise 4.15 Let $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denote the number of k-dimensional subspaces of the n-dimensional vector space over $\mathbb{F}_q$.

Let $F_q(n)$ denote the total number of subspaces of the n-dimensional vector space over $\mathbb{F}_q$.

i. Prove that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

ii. Prove that $F_q(n)$ satisfies the relation

$$F_q(n+1) = 2F_q(n) + (q^n - 1)F_q(n-1),$$

where $F_q(0) = 1$ and $F_q(1) = 2$.

iii. Prove that

$$F_q(n) \geqslant q^{\lfloor n^2/4 \rfloor}.$$

Exercise 4.16 An inversive plane is an incidence structure (P, L) with the property that every three points are incident with exactly one $\ell \in L$ (the elements of L are called *circles* for an inversive plane) and that if x and y are two points and ℓ is a circle incident with x and not incident with y then there is a unique circle m incident with y with the property that $\ell \cap m = \{x\}$.

i. Let x be a point of an inversive plane (P, L) and let

$$L^* = \{\ell \setminus \{x\} \mid \ell \in L,\ \ell \ni x\}.$$

Prove that $(P \setminus \{x\}, L^*)$ is an affine plane.

ii. Conclude that a finite inversive plane has an order n in which every circle contains $n+1$ points and any two points are incident with $n+1$ circles.

iii. Prove that if (P, L) is a finite inversive plane of order n then $|P| = n^2 + 1$ and $|L| = n^3 + n$.

iv. Construct an inversive plane of order q from an elliptic quadric $\mathcal{O}$ of PG(3, q).

[An elliptic quadric is a set of points

$$\mathcal{O} = \{(1 : x : y : f(x, y)) \mid x, y \in \mathbb{F}_q\} \cup \{(0 : 0 : 0 : 1)\},$$

where $f(x, y)$ is an irreducible polynomial of degree 2. In an analogous way to an elliptic quadric in real space, you may assume that a planar section of $\mathcal{O}$ is either a conic or a tangent plane. Thus, a planar section contains either 1 or $q+1$ points of $\mathcal{O}$ and each point of $\mathcal{O}$ is incident with a unique tangent plane.]

Exercise 4.17

i. Let η be a non-square element of $\mathbb{F}_q$ and define

$$\ell_{bc} = \langle (1, 0, c, b), (0, 1, b, \eta c) \rangle,$$

and

$$\ell_\infty = \langle (0, 0, 1, 0), (0, 0, 0, 1) \rangle.$$

Prove that

$$\{\ell_{bc} \mid b, c \in \mathbb{F}_q\} \cup \{\ell_\infty\},$$

is a spread of $\mathbb{F}_q^4$.

ii. Define

$$\ell_a = \langle (1, a, 0, 0), (0, 0, a, 1) \rangle,$$

and

$$\ell'_\infty = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle.$$

Prove that the subspaces

$$\{\ell_a \mid a \in \mathbb{F}_q\} \cup \{\ell'_\infty\},$$

contain the same vectors as the subspaces

$$\{\ell_{b0} \mid b \in \mathbb{F}_q\} \cup \{\ell_\infty\}.$$

iii. For every $n = p^{2h}$, where p is prime, construct two sets of $n - 1$ mutually orthogonal latin squares of order n, which share $n - \sqrt{n} - 2$ latin squares.

Matchings

5

Imagine that one is tasked with assigning hotel rooms, which are all for two occupants, to n guests, but where there is a list of couples who are incompatible and cannot, for some reason, share a hotel room. Is it possible to find a solution to this problem? In terms of graphs, this is the matching problem, asking if there is a perfect matching of a given graph. In this chapter, we shall study matchings and prove Tutte's theorem, which proves that the existence of a perfect matching is equivalent to the connected structure of its subgraphs. We will also consider stable matchings for graphs where each vertex has a preference order for its neighbours. One can think of this as the real-life situation of singles having an order preference for the others singles that they know. Given a set of singles and their preferences, one can ask if it is possible to match up the singles in couples in such a way that there is no pair, not coupled with each other, who both prefer each other to their own partner. We will investigate this problem and give an algorithm to solve this in the case of bi-partite graphs.

5.1 König's Theorem

A **matching** of a graph Γ is a set of disjoint edges, edges which share no end-vertex.

A matching is **maximal** if it cannot be extended and **perfect** if it covers all the vertices.

A **vertex cover** of a graph Γ is a subset U of the vertices of Γ such that each edge has an end-vertex in U.

Figure 5.1 is a bipartite graph whose largest matching and smallest vertex cover have size three.

Since a vertex cover must cover all the edges of a graph, it is immediate that any vertex cover is at least the size of any matching. Thus, if we define $\mathrm{m}(\Gamma)$ to be the maximum size of a matching of Γ and $\mathrm{vc}(\Gamma)$ to be the minimum size of a vertex cover of Γ then it is immediate that

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_5

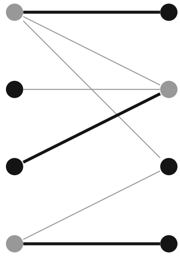


Fig. 5.1 The black edges are a matching and the grey vertices are a vertex cover

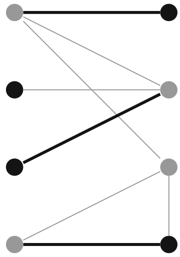


Fig. 5.2 The black edges are a matching and the grey vertices are a vertex cover

$$\mathrm{m}(\Gamma) \leqslant \mathrm{vc}(\Gamma). \tag{5.1}$$

It is, however, possible to find graphs for which this inequality is strict. The graph in Fig. 5.2 has a matching of maximum size three, whereas the smallest vertex cover has size four.

Nevertheless, Kőnig's theorem maintains that if the graph is bipartite then we do have equality in the inequality (5.1).

Theorem 5.1 (Kőnig) *If* Γ *is a bipartite graph then* $\mathrm{m}(\Gamma) = \mathrm{vc}(\Gamma)$.

Proof Suppose that the statement is not true and let Γ be the bi-partite graph with the least number of vertices, and amongst the graphs with the least number of vertices, the least number of edges, which is a counterexample to the statement.

If Γ is a path or a even cycle then $\mathrm{m}(\Gamma) = \mathrm{vc}(\Gamma)$, so Γ must have a vertex of degree at least three. By minimality, Γ must be connected.

Let u be a vertex of degree at least three and let v be a neighbour of u.

Suppose that all matchings of size $\mathrm{m}(\Gamma)$ cover v. This implies that

$$\mathrm{m}(\Gamma \setminus \{v\}) = \mathrm{m}(\Gamma) - 1.$$

By the minimality of Γ, the graph $\Gamma \setminus \{v\}$ has a vertex cover of size $\mathrm{m}(\Gamma) - 1$, which implies Γ has a vertex cover of size $\mathrm{m}(\Gamma)$), which is a contradiction. Therefore, there is a matching M of Γ which does not cover v.

Since u has degree at least three, there is an edge e of Γ which is incident with u, is not incident with v, and is not an edge of M. Let W be a minimal vertex cover of

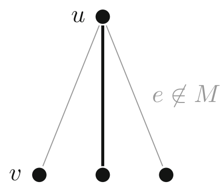


Fig. 5.3 The vertex u has at least three neighbours and there is a matching of size $\mathrm{m}(\Gamma)$ which does not cover v

$\Gamma \setminus \{e\}$. Since M is also a matching of $\Gamma \setminus \{e\}$, and the statement holds for $\Gamma \setminus \{e\}$, by the minimality of Γ, we have that $|W| = \mathrm{m}(\Gamma)$ (Fig. 5.3).

The vertex cover W must cover all the edges of M and has size $|M|$, so cannot contain v. Thus, since v is not covered by an edge of M, W must contain u and cover e too. Hence, W is a vertex cover of Γ of size $\mathrm{m}(\Gamma)$. □

5.2 Hall's Marriage Theorem

Recall that a **path** in Γ is a set of edges which connect a sequence of vertices which are all distinct from one another.

Let E be the set of edges of Γ. An **alternating path** with respect to a matching M is a path which starts at an unmatched vertex and whose edges contain alternately edges from $E \setminus M$ and M.

An alternating path is **augmenting** if it ends in an unmatched vertex, i.e. a vertex not covered by an edge of the matching.

Lemma 5.2 *If there exists an augmenting path with respect to a matching M then M is not of maximum size.*

Proof Suppose that the alternating path is $e_1 m_1 e_2 m_2 \ldots e_t m_t e_{t+1}$. Then the set of edges $M \setminus \{m_1, \ldots, m_t\} \cup \{e_1, \ldots, e_{t+1}\}$ is a matching with more edges than M has. □

We have already proved Hall's theorem, Theorem 4.3, in Chap. 4. Here, we give an algorithmic proof of Hall's theorem by finding a matching of a bipartite graph which satisfies Hall's condition. This is very useful because it actually details an algorithm which allows us to find a system of distinct representatives in a number of operations which is polynomial in the number of vertices.

Let X be a set and suppose that $A_1, \ldots, A_n$ are non-empty subsets of X.

Define a bipartite graph Γ with vertex partition $X \cup A$, where $A = \{A_1, \ldots, A_n\}$, and where $x \in X$ is joined to a vertex $A_i \in A$ if and only if $x \in A_i$.

Recall that we defined a system of distinct representatives (SDR) for $A_1, \ldots, A_n$ as a subset $\{x_1, \ldots, x_n\}$ of X with the property that $x_i \in A_i$. Finding an SDR for $A_1, \ldots, A_n$ is equivalent to finding a matching in the graph Γ which covers the vertices of A. Note that if the subset $\{x_1, \ldots, x_n\}$ of X has the property that $x_i \in A_i$

(i.e. is an SDR) then the set of edges $M = \{x_i A_i \mid i = 1, \ldots, n\}$ is a matching covering the vertices in A.

For any subset J of $\{1, \ldots, n\}$, define

$$A(J) = \bigcup_{i \in J} A_i,$$

so the union of the neighbours of $\{a_j \mid j \in J\}$.

Recall that Hall's condition is

$$|A(J)| \geqslant |J|,$$

for all subsets J of $\{1, \ldots, n\}$.

In terms of bipartite graphs Hall's theorem, Theorem 4.3, is the following theorem.

Theorem 5.3 *The bipartite graph Γ with vertex partition $A \cup X$ has a matching covering the vertices of A if and only if for each subset J of A, the number of vertices in X which are neighbour to some vertex in J is at least $|J|$. (Again, we call this condition Hall's condition.)*

Proof Start with any matching M, which could be empty.

We will prove that if M does not cover all the vertices in A then there is a larger matching than M. This implies that we will find a matching which covers all the vertices of A.

We will assume the vertices in A are $A_1, \ldots, A_n$, where we are identifying the subset A_i of X as a set of edges joining A_i to the elements in A_i.

Suppose that, after a suitable relabelling of the indices, A_1 is not covered by an edge of M. Then Hall's condition implies that there is an $x_1 \in X$ such that $x_1 A_1$ is an edge of Γ.

If x_1 is covered by an edge of M then there is an $A_2 \in A$ such that $x_1 A_2$ is an edge of M. Hall's condition implies that there is an $x_2 \in X \setminus \{x_1\}$ such that at least one of the vertices A_1 or A_2 is joined by x_2 by an edge of Γ.

We then repeat this process with x_2, and iteratively x_j. i.e. if x_j is covered by an edge of M then there is an $A_{j+1} \in A$ such that $x_j A_{j+1}$ is an edge of M. Hall's condition implies there is an $x_{j+1} \in X \setminus \{x_1, \ldots, x_j\}$ such that x_{j+1} is a neighbour of one of the vertices $A_1, \ldots, A_{j+1}$. We can then repeat the process again using x_{j+1}.

Since A_1 is not covered by an edge of M and by Hall's condition $|X| \geqslant n$, we will eventually find x_k which is not covered by an edge of M. Furthermore, by construction, there is an edge (necessarily not in M) from x_k to some A_i, where $i \in \{1, \ldots, k\}$. We construct an augmenting path P from x_k starting on this edge, then adjoining the edge $x_{i-1} A_i$ from M we then choose an edge from x_{i-1} to A_j, for some $j \in \{1, \ldots, i-1\}$, which we know to exist since x_{i-1} is neighbour to some

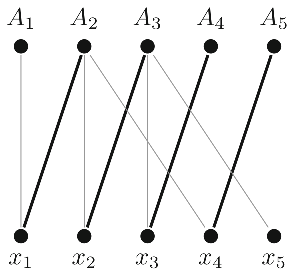


Fig. 5.4 An augmenting path $x_5A_3x_2A_2x_1A_1$, with the black edges from M

vertex in $\{A_1, \ldots, A_{i-1}\}$. In this way we find P, an alternating path with respect to M, leading back to A_1, see Fig. 5.4.

Since P is an alternating path which leads from one unmatched vertex to another unmatched vertex, P is an augmenting path for M, and Lemma 5.2 implies that M is not of maximum size.

□

A k-**regular** graph is a graph in which every vertex has degree k.

The equivalent theorem to Theorem 4.5 is the following theorem.

Theorem 5.4 *An k-regular bipartite graph Γ has a perfect matching.*

Proof Suppose that $A \cup X$ is the vertex partition of Γ. Observe that $|A| = |X|$. Let J be a subset of A and count (x, j) where $x \in X$ is a neighbour of $j \in J$. Considering x this is at most $k|A(J)|$ and considering j, this is $k|J|$. Therefore, $|J| \leqslant |A(J)|$ and Hall's condition holds. Theorem 5.3 implies that Γ has a perfect matching. □

A k-**factor** of Γ is a k-regular subgraph of Γ whose vertex set is that of Γ. A k-**factorisation** partitions the edges of the graph into disjoint k-factors. In particular, a 1-factor is a perfect matching and a 2-factor is the union of disjoint cycles which cover all the vertices.

Corollary 5.5 *A k-regular bipartite graph Γ has a 1-factorisation.*

Proof Theorem 5.4 implies that Γ has a 1-factor. Removing the 1-factor we obtain a $(k-1)$-regular bipartite graph. Theorem 5.4 implies that this subgraph has a 1-factor. We continue applying Theorem 5.4 k times in all to obtain a 1-factorisation.

□

5.3 Stable Matchings

Suppose that each vertex in a graph Γ has an order of preference towards its neighbours.

An edge xy is an **unstable edge** (with respect to the matching and the preferences) if $xy \notin M$ and one of following occurs.

i. x and y are covered by M, x prefers y to the vertex it is matched with in M and y prefers x to the vertex it is matched with in M.
ii. x is covered by M, y is not covered by M and x prefers y to the vertex it is matched with in M.
iii. y is covered by M, x is not covered by M and y prefers x to the vertex it is matched with in M.
iv. neither x nor y is covered by M.

We call a matching is **stable** if there are no unstable edges. Observe that by iv, a stable matching is maximal.

The following is an algorithmic proof that not only proves that a stable matching exists, it also tells us how to find one.

Theorem 5.6 *A bipartite graph* Γ *has a stable matching.*

Proof Let $A \cup B$ be the vertex partition of the graph Γ. For each $a_i \in A$, let A_i be the subset of B consisting of the neighbours of a_i.

In Round j, each vertex in A proposes to its most preferred vertex in B. If $b \in B$ receives more than one proposal then it rejects all but the most preferable. Thus if $b \in A_i$ and b rejects a_i's proposal then b is removed from A_i. Continue until all vertices in B receive at most one proposal. Let M be the matching consisting of these proposals.

Suppose that xy is an unstable edge, $x \in A$, $y \in B$. We consider four cases as in the definition of an unstable edge.

i. Suppose that x is on the edge $xy' \in M$ and y is on the edge of $x'y \in M$ (Fig. 5.5).

   This case does not occur since x proposed to y before y', at which point y rejects the proposal from x'.
ii. Suppose that x is on the edge $xy' \in M$ and y is unmatched. Since y is unmatched it never gets proposed to. However, since xy is unstable x prefers y to y' so will propose to y before it proposes to y', so this case does not occur (Fig. 5.6).
iii. Suppose that y is on the edge of $x'y \in M$ and x is unmatched.

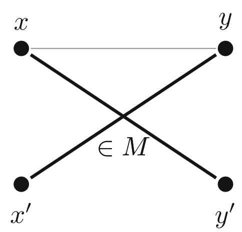


Fig. 5.5 x and y are covered by an edge of the matching

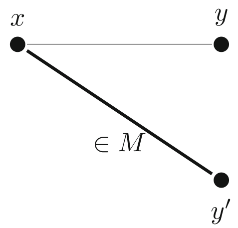


Fig. 5.6 x is covered by an edge of the matching

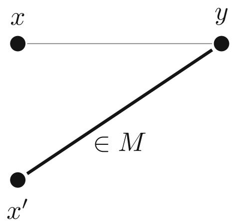


Fig. 5.7 y is covered by an edge of the matching

Figure 5.7 x is unmatched, his proposals must have been rejected, so some vertex in B must have received more than one proposal and so the algorithm will continue with another round. Therefore, at some point x will propose to y and when he does, y will reject x' proposal. Therefore, this case does not occur either.

iv. Suppose that neither x nor y is covered by M. Since x is connected to y by an edge it will propose to y at some point, so y will be matched. Hence, this case does not occur.

Hence, there are no unstable edges and M is a stable matching. □

Corollary 5.7 *The complete bipartite graph $K_{n,n}$ has a perfect stable matching.*

Proof Any stable matching is maximal and a maximal matching of $K_{n,n}$ is perfect. □

Example 5.8 Consider the graph in Fig. 5.8, with the following preferences.

x_1	$y_3 > y_1 > y_2$	y_1	$x_1 > x_2$
x_2	y_1	y_2	x_1
x_3	y_3	y_3	$x_3 > x_1$

Applying the algorithm from the proof of Theorem 5.6, we end up with a stable matching, which is not of maximum size.

Observe that the graph does have a perfect matching but that matching is not stable. The petitions in the algorithm are given in the following array.

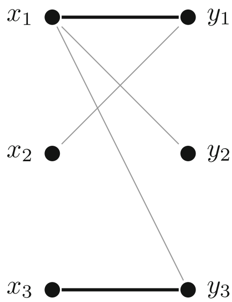


Fig. 5.8 A stable matching indicated by the black edges

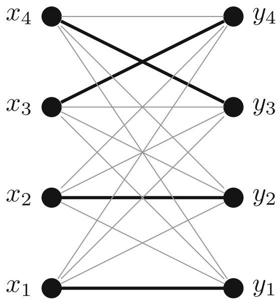


Fig. 5.9 The black edges are a stable matching with respect to the given preferences

Round	1	2	3
x_1	y_3	y_1	y_1
x_2	y_1	y_1	.
x_3	y_3	y_3	y_3

Example 5.9 Consider the complete bipartite graph $K_{4,4}$ with the following preferences.

x_1	$y_1 > y_2 > y_3 > y_4$	y_1	$x_1 > x_2 > x_4 > x_3$
x_2	$y_1 > y_3 > y_2 > y_4$	y_2	$x_2 > x_1 > x_3 > x_4$
x_3	$y_2 > y_4 > y_1 > y_3$	y_3	$x_3 > x_1 > x_4 > x_2$
x_4	$y_2 > y_1 > y_3 > y_4$	y_4	$x_2 > x_1 > x_4 > x_3$

The algorithm from the proof of Theorem 5.6 gives a stable matching after five rounds (Fig. 5.9).

Round	1	2	3	4	5
x_1	y_1	y_1	y_1	y_1	y_1
x_2	y_1	y_3	y_3	y_2	y_2
x_3	y_2	y_2	y_2	y_2	y_4
x_4	y_2	y_1	y_3	y_3	y_3

5.4 Tutte's Theorem

A graph is **connected** if there is a path between any two of its vertices.

A subgraph of Γ is **induced** if it can be obtained by taking a subset V' of the vertices of Γ and has the same edges as Γ, restricted to the vertices V'.

Lemma 5.10 *A connected graph $\Gamma \neq K_n$ with at least 3 vertices has an induced subgraph isomorphic to $K_{1,2}$.*

Proof We have to prove that there is a vertex with two neighbours who are not neighbours of each other. If not then the vertices of a vertex v induce a complete graph. If the neighbourhood of every vertex is a complete graph then Γ is a complete graph. □

Connectedness is an equivalence relation on the vertices of a graph Γ. A subgraph induced by the vertices of an equivalence class is called a **component** of Γ.

An **odd component** is a component which has an odd number of vertices. Similarly, an **even component** is a component which has an even number of vertices.

Let $\mathrm{oc}(\Gamma)$ denote the number of odd components of a graph Γ.

Let S be a subset of the vertices of Γ. The graph $\Gamma \setminus S$ is the graph obtained from Γ by deleting the vertices of S (and necessarily any edges which have an end vertex in S).

Theorem 5.11 (Tutte) *A graph Γ has a perfect matching if and only if for all subsets S of the vertices of Γ, the inequality $\mathrm{oc}(\Gamma \setminus S) \leqslant |S|$ holds.*

Proof

($\Leftarrow$) Suppose that Γ has no perfect matching.

The condition with $S = \emptyset$ implies that Γ has an even number of vertices. Let Γ^* be a graph obtained from Γ by adding edges until adding any other edge will give a graph with a perfect matching. Adding edges to Γ does not change the property that the number of odd components of $\Gamma \setminus S$ is at most $|S|$. Indeed, adding edges may join two odd components into a larger even component, but this does not affect the property. Hence, for all subsets S of the vertices of Γ^*, $\mathrm{oc}(\Gamma^* \setminus S) \leqslant |S|$.

Let K be the set of vertices of Γ^* which are adjacent to all the other vertices.

Suppose $\Gamma^* \setminus K$ has a non-complete component. By Lemma 5.10, there exists three vertices a, b, c of $\Gamma^* \setminus K$ with the property that ab and bc are edges but ac is a non-edge. Since b is not in K, there is a vertex d of Γ^* which is not adjacent to b.

Since we added edges so that adding any other edge will give a graph with a perfect matching, both the graphs $\Gamma^* + \{ac\}$ and $\Gamma^* + \{bd\}$ have perfect matchings. Call these perfect matching M_{ac} and M_{bd} respectively. Observe that ac is an edge of M_{ac} and bd is an edge of M_{bd}.

Now, construct the path P which starts at d with the edge of M_{ac} and alternates between an edge of M_{bd} and an edge of M_{ac}. Note that the edge bd is not an edge of $\Gamma^* + \{ac\}$ and therefore not an edge of the matching M_{ac}. Now as the edges alternate between an edge of M_{bd} and an edge of M_{ac}, we cannot return to a previous vertex along the path since this vertex is already covered by an edge of both matchings M_{ac} and M_{bd}.

Eventually, either P gets to b or one of a or c. Since P, if completed to a circuit, would return to d on an edge of M_{bd} which must necessarily be the edge bd, it must get to b at some point.

Suppose that P gets to the vertex b before a or c, see Fig. 5.10.

Since bd is an edge of M_{bd}, the path P arrives at b on an edge of M_{ac}. Adjust, M_{bd} removing the edges of M_{bd} in P and replacing them with the edges of M_{ac} in P and removing the edge bd. Then this set of edges is a perfect matching of Γ^*, since it does not contain the edge bd (nor the edge ac, since M_{bd} is a matching of $\Gamma^* + \{bd\}$).

Suppose that P gets to one of the vertices a or c before b and without loss of generality assume that it is a, see Fig. 5.11.

The path P arrives at a on an edge of M_{bd} since the edge containing a in M_{ac} is the edge ac. Adjust, M_{bd} removing the edges of M_{bd} in P and replacing them with the edges of M_{ac} in P, removing the edge bd and add the edge ab. Then, as in the previous paragraph, this set of edges is a perfect matching of Γ^*.

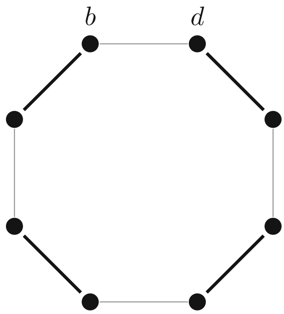


Fig. 5.10 The path P arrives at b before a or c, the grey edges are from M_{bd} and the black edges from M_{ac}

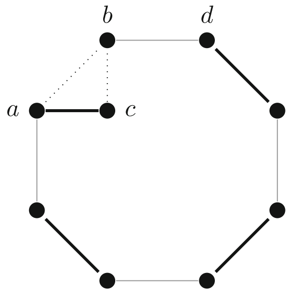


Fig. 5.11 The path P starting at d arrives at a before b or c, the grey edges are from M_{bd} and the black edges from M_{ac}. The edge ab is not in either M_{bd} nor M_{ac}

Suppose $\Gamma^* \setminus K$ has only complete components. The number of odd components of $\Gamma^* \setminus K$ is at most $|K|$, so we can pair up the vertices in the even components and then pair up all but one of the vertices in the odd components. The remaining vertices can then be paired to the vertices in K, of which there are sufficient since the number of odd components of $\Gamma^* \setminus K$ is at most $|K|$. Finally, the remaining vertices in K can be paired up since, as we observed the beginning of the proof, the total number of vertices in Γ^* is even. Thus, Γ^* has a perfect matching, a contradiction. Hence, Γ has a perfect matching.

($\Rightarrow$) Suppose that Γ has a perfect matching M. An odd component of $\Gamma \setminus S$ must (in Γ) have an edge of M joining it to a vertex of S. Therefore $|S|$ is at least as large as the number of odd components of $\Gamma \setminus S$.

□

The following corollary uses Tutte's theorem to give a lower bound on the number of vertices which are covered by the largest matching.

Corollary 5.12 (Tutte-Berge) *Let $\Gamma = (V, E)$ be a graph and for every subset S of the vertices of Γ, define*

$$d(S) = \mathrm{oc}(\Gamma \setminus S) - |S|.$$

Let

$$d = \max\{d(S) \mid S \subseteq V(\Gamma)\}.$$

If Γ has no perfect matching then there is a matching of Γ which covers at least $|V| - d$ of the vertices.

Proof By Theorem 5.11, Γ has a perfect matching if and only if $d \leqslant 0$.

Suppose $d \geqslant 1$. Consider the graph Γ^* which consists of appending a complete graph K on d vertices to Γ and connecting every vertices in Γ to every vertex in K (Fig. 5.12).

Let S^* be a subset of Γ^*. We will prove that Γ^* satisfies the hypothesis of Theorem 5.11.

If $S^* = \emptyset$ then the number of vertices in Γ^* is $d + |V|$. By definition,

$$d + |V| = \mathrm{oc}(\Gamma \setminus S) - |S| + |V|$$

for some subset S of V. Observe that

$$\mathrm{oc}(\Gamma \setminus S) = |V \setminus S| \pmod 2,$$

so $d + |V|$ is even. Hence, the number of vertices in Γ^* is even. Since Γ^* is connected, it only has one component, so this implies that

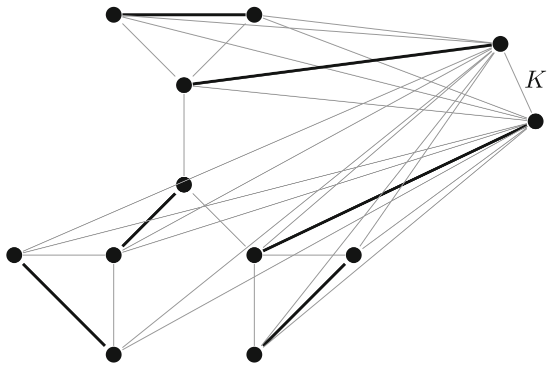


Fig. 5.12 The graph Γ^* appending the vertices K to Γ with the black edges giving a perfect matching of Γ^*

$$\mathrm{oc}(\Gamma^* \setminus S^*) = 0 = |S^*|.$$

If S^* is non-empty and does not contain all the vertices of K then $\Gamma^* \setminus S^*$ is connected, so has at most one odd component. Therefore,

$$\mathrm{oc}(\Gamma^* \setminus S^*) \leqslant 1 \leqslant |S^*|.$$

If S^* contains all the vertices of K then $\Gamma^* \setminus S^*$ is $\Gamma \setminus S$ for some subset S of V. Thus,

$$\mathrm{oc}(\Gamma^* \setminus S^*) = \mathrm{oc}(\Gamma \setminus S) = |S| + d(S) \leqslant |S| + d = |S^*|,$$

where the inequality follows by the definition of d.

Theorem 5.11 implies that Γ^* has a perfect matching and this matching restricted to Γ covers at least $|V| - d$ of the vertices. □

A graph is **cubic** if every vertex has degree three. An edge in a graph is a **bridge** if deleting it increases the number of components.

Corollary 5.13 *Every bridgeless cubic graph has a perfect matching.*

Proof It is sufficient to prove this for a connected bridgeless cubic graph.

Suppose that S is a subset of the vertices of a bridgeless cubic graph Γ and that D is an odd component of $\Gamma \setminus S$ (Fig. 5.13).

The sum of the degrees of the vertices in D, as vertices in Γ, sum to an odd number, so there are an odd number of edges between D and $\Gamma \setminus D$. Moreover, S disconnects D from the other components of $\Gamma \setminus S$ so these edges must be edges from D to S. If there is only one such edge, then this edge would be a bridge in Γ, so there are three edges from a vertex in D to a vertex in S. Therefore, the number of edges between S and $\Gamma \setminus S$ is at least $3\mathrm{oc}(\Gamma \setminus S)$. Clearly, since Γ is a cubic

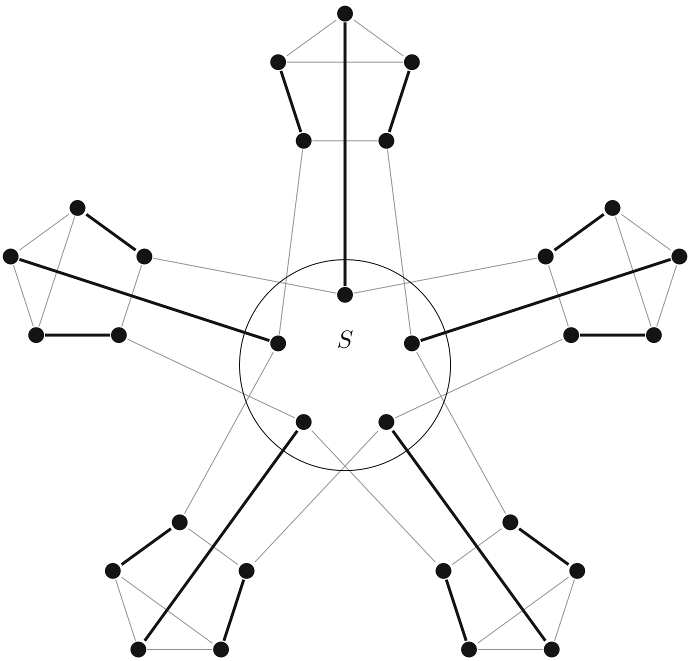


Fig. 5.13 There are at least three edges joining an odd component of $\Gamma \setminus S$ to S

graph, the number of edges between S and $\Gamma \setminus S$ is at most $3|S|$, so we have that $|S| \geqslant \mathrm{oc}(\Gamma \setminus S)$. Thus, the hypothesis in Theorem 5.11 is satisfied. □

5.5 Coverings and Independent Sets

Let $\Gamma = (V, E)$ be a graph. An **independent set** of Γ is a subset of the vertices with the property that no two of the vertices are joined by an edge. This is also known as a **co-clique**.

Let $\mathrm{is}(\Gamma)$ be the maximum size of an independent set. Let $\mathrm{vc}(\Gamma)$ be the minimum size of a vertex cover.

Lemma 5.14 *For any graph* $\Gamma = (V, E)$*,*

$$\mathrm{is}(\Gamma) + \mathrm{vc}(\Gamma) = |V|.$$

Proof Suppose that U is a vertex cover of minimum size. Then $V \setminus U$ is an independent set, so $|V \setminus U| \leqslant \mathrm{is}(\Gamma)$. Hence,

$$|V| - \mathrm{vc}(\Gamma) \leqslant \mathrm{is}(\Gamma).$$

Suppose that I is an independent set of maximum size. Since, there is no edge between two vertices in I, $V \setminus I$ is a vertex cover, $|V \setminus I| \geqslant \mathrm{vc}(\Gamma)$. Hence,

$$|V| - \mathrm{is}(\Gamma) \geqslant \mathrm{vc}(\Gamma).$$

□

An **edge-covering** of a graph $\Gamma = (V, E)$ is a subset of the edges that cover all the vertices.

Let $\mathrm{m}(\Gamma)$ be the maximum size of a matching. Let $\mathrm{ec}(\Gamma)$ be the minimum size of an edge covering.

Lemma 5.15 *For any graph* $\Gamma = (V, E)$ *without isolated vertices,*

$$\mathrm{m}(\Gamma) + \mathrm{ec}(\Gamma) = |V|.$$

Proof Suppose that L is an edge covering of minimum size. Since L is minimal, it contains no paths of length three, since we could delete the middle edge and still have an edge covering. Therefore, L is the union of stars, where a star is a $K_{1,s}$ for some s. Let ℓ_i count the number of edges in each of the k components of L. A component with ℓ_i edges has $\ell_i + 1$ vertices, so

$$\sum_{i=1}^{k} (\ell_i + 1) = |V|.$$

Hence, $|L| = |V| - k$. Taking an edge from each component of L, we get a matching of size k, so $k \leqslant \mathrm{m}(\Gamma)$, which gives the inequality

$$\mathrm{m}(\Gamma) + \mathrm{ec}(\Gamma) \geqslant |V|.$$

Suppose that M is a matching of maximum size. There are $|V| - 2|M|$ vertices not covered by M, so we can make an edge covering of size $|M| + |V| - 2|M|$. Therefore,

$$|V| - |M| = |V| - \mathrm{m}(\Gamma) \geqslant \mathrm{ec}(\Gamma).$$

□

5.6 Notes and References

Kőnig claimed a proof of Theorem 5.1 in 1914, later publishing the article Kőnig (1916). Note that the proof in Theorem 5.3 of Hall's theorem is somewhat better than the proof of Theroem 4.3, since it is a constructive proof and if the algorithm given in the proof is followed then the desired matching will be found. A proof of Hall's theorem first appeared in Hall (1935).

Stable matchings date back to the 1962 article Gale and Shapley (1962). The Tutte–Berge formula from Corollary 5.12 dates back to Berge (1958).

5.7 Exercises

Exercise 5.1 Prove that a tree admits at most one perfect matching.

Exercise 5.2 Prove that if M is not of maximum size then there is an augmenting path with respect to M.

Exercise 5.3 Prove Hall's theorem using Kőnig's theorem.

Exercise 5.4 Prove that if Γ is $2k$-regular then Γ has a 2-factor.

Exercise 5.5 Find a bipartite graph and a set of preferences such that no matching of maximum size is stable and no stable matching is of maximum size.

Exercise 5.6 Find a non-bipartite graph and a set of preferences which has no stable matching.

Exercise 5.7 Prove the the maximum number of rounds that the algorithm in Theorem 5.6 has to perform to find a stable matching for $K_{n,n}$ is $n(n-2)+2$ and find a set of preferences which requires this number of rounds for the algorithm to complete.

Exercise 5.8 Find a 1-factor of the Petersen graph and prove that the Petersen graph has no 1-factorisation.

Exercise 5.9 Find a cubic graph which has no perfect matching.

Exercise 5.10 Prove Hall's theorem using Tutte's theorem, assuming Γ is a bipartite graph with vertex partitions of equal size.

Exercise 5.11 Recall that $\text{oc}(\Gamma)$ denotes the number of odd components of a graph Γ.

i. Let S be a subset of the vertices of a graph Γ. Prove that

$$|\text{oc}(\Gamma \setminus S)| - |S| = n \pmod 2,$$

where n is the number of vertices of the graph Γ.

ii. Prove that for a cubic graph with no perfect matching there is a subset S of the vertices such that

$$|\mathrm{oc}(\Gamma \setminus S)| - |S| \geqslant 2.$$

iii. Prove that a cubic graph Γ with at most 2 bridges has a perfect matching.

Exercise 5.12 Let Γ_0 be a simple connected graph. Let Γ be the line graph of Γ_0, i.e. the graph whose vertices are the edges of Γ_0 and where two vertices of Γ are adjacent if and only if the corresponding edges in Γ_0 share a vertex. Use Tutte's theorem to prove that if Γ has an even number of vertices then Γ has a perfect matching.

Exercise 5.13 A connected component C of a multipartite graph $\Gamma = (V_1 \cup \cdots \cup V_k, E)$ is unbalanced if $|C \cap V_i| > \sum_{j=1, j \neq i}^{k} |C \cap V_j|$ for some i. Let $\mathrm{uc}(\Gamma)$, $\mathrm{euc}(\Gamma)$ and $\mathrm{oc}(\Gamma)$ be the number of unbalanced, even unbalanced and odd components of Γ respectively.

i. Prove that a multipartite graph Γ of even order has a perfect matching if and only if for all subsets S of $V(\Gamma)$,

$$\mathrm{oc}(\Gamma \setminus S) + 2\mathrm{euc}(\Gamma \setminus S) \leqslant |S|.$$

ii. If Γ is bipartite, show that it has a perfect matching if and only if for all subsets S of $V(\Gamma)$,

$$\mathrm{uc}(\Gamma \setminus S) \leqslant |S|.$$

iii. Find a tripartite graph Γ which has a perfect matching but for which there is an S such that

$$\mathrm{oc}(\Gamma \setminus S) + \mathrm{uc}(\Gamma \setminus S) > |S|,$$

and a tripartite graph Γ' with no perfect matching but for all subsets S of $V(\Gamma')$

$$\mathrm{uc}(\Gamma' \setminus S) \leqslant |S|.$$

Connectivity

6

Connectivity is a key property of graphs. The central result on connectivity of graphs is the theorem of Menger, a result of min–max type with several connections in other areas of combinatorics and of combinatorial optimization, besides its relevance in graph theory itself. Some structural results related to connectivity are also presented in this chapter, including a theorem of Tutte on 3-connected graphs. The close notion of edge-connectivity is also discussed at the end of the chapter.

6.1 Vertex Connectivity

A graph is **connected** if there is a path connecting any pair of vertices. A **connected component** of a graph is a connected subgraph which cannot be extended (by adding edges or vertices). Every graph is the disjoint union of its connected components. For a subset $X \subset V(\Gamma)$, we denote by $\Gamma[X]$ the subgraph of Γ induced by the vertices in X.

A **tree** is a connected acyclic graph. The following are equivalent definitions of a tree. The proof is a simple exercise.

Proposition 6.1 *For a graph T, the following statements are equivalent:*

i. *T is a tree.*
ii. *T is an edge-maximal acyclic graph: the addition of any edge to T results in a graph which is no longer acyclic.*
iii. *T is an edge-minimal connected graph: the suppression of any edge of T results in a graph which is no longer connected.*
iv. *For every pair of vertices in T there is a unique path joining them.*
v. $|E(T)| = |V(T)| - 1$ *and T is acyclic.*
vi. $|E(T)| = |V(T)| - 1$ *and T is connected.*

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_6

A subgraph T of a graph Γ is a spanning tree of Γ if it is a tree and $V(T) = V(\Gamma)$. A simple characterization of connected graphs is the following one.

Lemma 6.2 *A graph Γ is connected if and only if there is an ordering $\{v_1, \ldots, v_n\}$ of the vertices such that $\Gamma[v_1, \ldots, v_i]$ is connected for each $i = 1, \ldots, n$. In particular, Γ is connected if and only if it contains a spanning tree.*

Proof The first part is a direct consequence of the definition: if Γ is not connected then the condition fails for $i = n$ and every ordering.

Reciprocally, if Γ is connected one can start in any vertex v_1 and define v_{i+1} as the first vertex not in $\{v_1, \ldots, v_i\}$ in a path connecting v_1 with some vertex not in that initial segment.

For the second part, we can choose, for every i, one edge joining v_i with some vertex in $\{v_1, \ldots, v_{i-1}\}$. In this way we obtain a spanning subgraph (a graph with vertex set $V(\Gamma)$) which has $|V(\Gamma)| - 1$ edges, and it is therefore a tree. □

A natural measure of connectivity of a graph is given by the minimum number of vertices whose deletion disconnects the graph. A subset $S \subset V(\Gamma)$ is a **separator** of Γ if $\Gamma[V \setminus S]$ is not connected. A graph is k**-connected** if $|V(\Gamma)| \geqslant k + 1$ and every separator of G has at least k vertices. For example a tree is 1-connected but not 2-connected. A cycle is 1-connected and also 2-connected but not 3-connected. For the complete graph, which has no separators, the definition is to be seen as a convention: the complete graph K_n is k-connected for every $k \leqslant n - 1$.

6.2 Structure of *k*-Connected Graphs for Small *k*

A **cut vertex** v of a connected graph Γ is a vertex such that $\Gamma[V(\Gamma) \setminus \{v\}]$ is not connected, i.e. $S = \{v\}$ is a separator of size one.

A **block** of Γ is a connected subgraph of Γ which contains no cut vertices and cannot be extended to a larger subgraph which contains no cut vertices.

Thus, a block is either an isolated vertex, an edge with its two end vertices or a maximal 2-connected subgraph. By maximality, if two blocks intersect then they have a unique common vertex, which is a cut vertex of the graph. Connected graphs can be structured in a tree of blocks.

Lemma 6.3 *Let Γ be a connected graph and let A be its set of cut vertices. Let $B(\Gamma)$ be the bipartite graph with bipartition $V_1 = A$ and $V_2 = \{B \subset \Gamma :$ B is a block of $\Gamma\}$ where there is an edge joining a cut vertex $a \in A$ with a block $B \in V_2$ if and only if $a \in B$. Then $B(\Gamma)$ is a tree.*

Proof The block graph is connected since Γ is connected. If it has a cycle then this cycle contains $r \geqslant 2$ blocks of Γ with r cut points, which together form a block, contradicting the maximality of the existing blocks. □

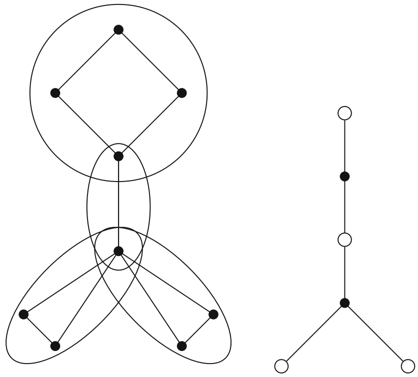


Fig. 6.1 A graph (left) and its block graph (right)

Figure 6.1 shows an example of a graph and its block graph.

As for a block, its structure can be described as follows.

Lemma 6.4 *A graph is* 2*-connected if and only if it can be recursively constructed starting from a cycle by successively adding a path between two vertices previously constructed.*

Proof Suppose that Γ has been recursively constructed starting from a cycle by successively adding a path between two vertices previously constructed. Then every vertex is contained in a cycle, so Γ has no cut vertices. Hence, it is 2-connected.

Suppose that Γ is 2-connected. Let Γ' be a maximal subgraph of Γ constructed as stated. Then Γ' is an induced subgraph of Γ, since we can always add an edge between two vertices of Γ' under the recursion rule, if that edge is an edge of Γ.

If there is a vertex $v \in V(\Gamma) \setminus V(\Gamma')$ then there is a path from v to some vertex in Γ'. Suppose that w is the first vertex of Γ' on such a path. Since Γ is 2-connected, there is another path from v to another vertex $w' \neq w$ in Γ' sharing no other vertices than v with the above path. Thus, v lies on a path joining two previously constructed vertices, which contradicts the maximality of Γ'. □

Thus, if a graph is 2-connected, then there is a sequence

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$$

such that Γ_0 is a cycle and Γ_i is obtained from Γ_{i-1} by adding a path (possibly with internal vertices not in Γ_{i-1}) joining two vertices in Γ_{i-1}.

We next discuss the more substantial structural characterisation of 3-connected graphs.

The **contraction** of an edge $e = xy \in E(\Gamma)$ consists in identifying its two endpoints and the possible multiple edges which may be created by this identification, see Fig. 6.2. The resulting graph is denoted by Γ/e. Contraction is an important notion in the theory of graphs.

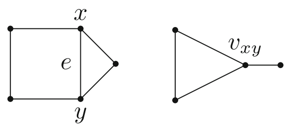


Fig. 6.2 A graph G on the left and the contraction G/e on the right

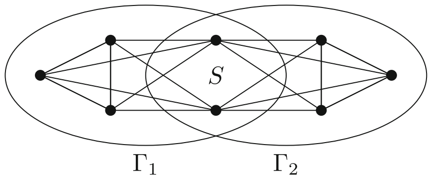


Fig. 6.3 An example of a separation defined by the separator S

We will often use the following view on separators of a graph. If S is a separator of Γ and C is a connected component of $\Gamma[V \setminus S]$ then Γ can be written as $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \Gamma[C \cup S]$ and $\Gamma_2 = \Gamma[V \setminus C]$ are two graphs whose vertex sets intersect in S with the property that there are no edges in Γ connecting vertices in C with vertices in $V \setminus (C \cup S)$. The pair $\{\Gamma_1, \Gamma_2\}$ is a separation of Γ defined by S and C. Figure 6.3 shows an example of such a separation.

The following simple lemma will be useful.

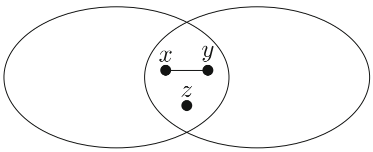
Lemma 6.5 *Let S be a minimum separating set of Γ. Then, every vertex in S is adjacent to a vertex in each connected component of $\Gamma - S$.*

Proof Suppose that $x \in S$ is not adjacent to a component C of $\Gamma - S$. Then, with $S' = S \setminus \{x\}$, C is still a connected connected component of $\Gamma - S'$, contradicting the minimality of $|S|$. □

Lemma 6.6 *Let Γ be a 3-connected graph, $\Gamma \neq K_4$. There is an edge $e \in \Gamma$ such that Γ/e is still 3-connected.*

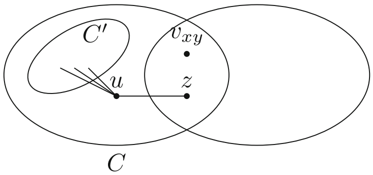
Proof Suppose that Γ/e is not 3-connected for every edge $e = xy \in E(\Gamma)$.

Let v_{xy} be the vertex of Γ/e resulting from the contraction of e. Every separator of Γ/e not containing v_{xy} is also a separator of Γ. Moreover, for every minimum separator $\{v_{xy}, z\}$ of Γ/e, the set $S = \{x, y, z\}$ is a minimum separator of Γ. Therefore, every separator of Γ/e with cardinality less than three must contain v_{xy} and, once this vertex is split, it corresponds to a minimal separator of Γ. It follows that Γ/e is 2-connected. Moreover, for every minimal separator $\{v_{xy}, z\}$ of Γ/e the set $S = \{x, y, z\}$ is a minimal separator of Γ.



For every edge $e = xy$ choose $z \in V(\Gamma)$ such that $\{v_{xy}, z\}$ is a separator of Γ/e and choose the smallest component C of $(\Gamma/xy) - \{v_{xy}, z\}$. From all such possibilities of $e = xy$, z and C, choose one in which C has the smallest possible cardinality.

Since $\{v_{xy}, z\}$ is a minimal cut in Γ/xy, z is adjacent to a vertex $u \in C$. We will show that the choice of $e' = uz$ and some z' results in a separator $\{v_{uz}, z'\}$ of Γ/uz with a component C' with $|C'| < |C|$, contradicting the minimality of $|C|$.



Since $\{x, y, z\}$ is a separator of Γ and C is one of its components, all neighbours of $u \in C$ different from x, y and z belong to C.

There is some vertex z' such that $\{v_{uz}, z'\}$ is a separator of Γ/uz and, as discussed before, $\{u, z, z'\}$ is a separator of Γ. Since x and y are adjacent, they belong to the same connected component of $\Gamma - \{u, z, z'\}$. By Lemma 6.5, u is adjacent to all other connected components. Let C' be such a connected component. Since all neighbours of u different from x and y are contained in C, it follows that $C' \subset C \setminus \{u\}$, giving the claimed contradiction in our choice of C and hence, to the initial assumption.

□

We are now in a position to prove a structural characterisation of 3-connected graphs.

Theorem 6.7 (Tutte) *Every* 3-*Connected Graph* Γ *contains a sequence*

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$$

such that

1. $\Gamma_0 = K_4$,
2. $\Gamma_i = \Gamma_{i+1}/xy$ *for some* $e = xy \in E(\Gamma_{i+1})$ *such that* $d_{\Gamma_{i+1}}(x), d_{\Gamma_{i+1}}(y) \geqslant 3$.

Proof Suppose that Γ is 3-connected. By Lemma 6.6, there is an edge $e \in E(\Gamma)$ whose contraction Γ/e results in a graph which has one vertex less and is still 3-connected. By iterating this procedure, we obtain a sequence as claimed. Note that the only 3-connected graph with four vertices is K_4.

Reciprocally, a graph containing a sequence as described is 3-connected. To see this it suffices to show that, if Γ_i is 3-connected, then a graph Γ_{i+1}, with the property that $\Gamma_i = \Gamma_{i+1}/xy$ for some edge xy, such that $d_{\Gamma_{i+1}}(x), d_{\Gamma_{i+1}}(y) \geqslant 3$, is also 3-connected. Suppose not and let S be a separator of Γ_{i+1} with two vertices. It cannot

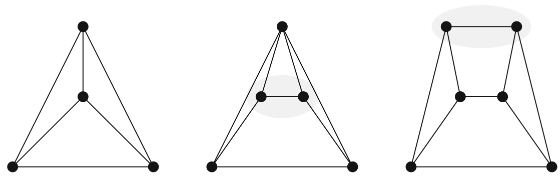


Fig. 6.4 A construction of a 3-connected graph with 6 vertices starting from K_4

be that $S = \{x, y\}$, since otherwise the contracted edge v_{xy} would be a separator of Γ_i. It also cannot be that S is disjoint from $\{x, y\}$, since S otherwise would be a separator of Γ_i. If $S \cap \{x, y\} = \{x\}$ then y is isolated in a singleton component of $\Gamma_{i+1} \setminus S$ since other vertices of that component would be separated by $S \setminus \{x\} \cup \{v_{xy}\}$ in Γ_i. But then this implies that y has degree at most two. □

It follows from Tutte's theorem that every 3-connected graph can be constructed from K_4 by splitting a vertex into two adjacent vertices and connecting them to the old neighborhood distributing the edges among the new two vertices such that each one has degree at least three (Fig. 6.4).

6.3 Menger's Theorem

Menger's theorem connects two dual notions of connectivity: separating sets and number of disjoint paths connecting two sets. Let $A, B \subset V(\Gamma)$ be two sets of vertices. An AB-**separator** is a set S of vertices such that there are no paths connecting A with B in $\Gamma - S$. A vertex in $A \cap B$ is connected by itself by a path of length 0 in this definition, which implies that every AB-separator contains $A \cap B$. An AB-**connector** is a subgraph $\Gamma' \subset \Gamma$ each of its connected components is a path containing precisely one vertex in A and one vertex in B. A graph with no edges can also be an AB-connector, formed by isolated vertices in $A \cap B$.

Theorem 6.8 (Menger, Local Version) *Let A, B be two nonempty subsets of vertices of a graph Γ. The cardinality of a minimum AB-separator equals the maximum number of components (paths) in an AB-connector.*

Proof Let S be a minimal AB-separator and Γ' an AB-connector containing c paths. It is clear that every separator must contain one point of every path in Γ' so $|S| \geqslant c$.

We will prove that there is an AB-connector with $|S|$ paths, by induction on the number of edges of Γ. If Γ is edgeless one can take $A \cap B$ as both, a maximal AB-connector and minimal AB-separator.

Suppose Γ is not edgeless and let s be the cardinality of a minimum AB-separator in Γ.

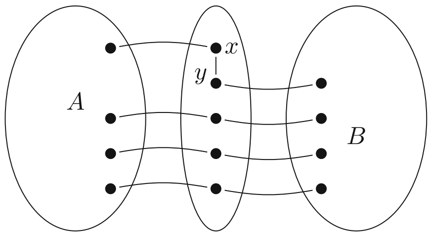


Fig. 6.5 The construction of the AB-connector $\Gamma_1 \cup \Gamma_2 \cup \{xy\}$ in Theorem 6.8

Let $e = xy$ be an edge of Γ. The statement holds in $\Gamma - e$ by induction. If a minimum separator of $\Gamma - e$ has the same cardinality s as in Γ then we are done, as an AB-connector in $\Gamma - e$ is also an AB-connector in Γ.

Suppose that S' is an AB-separator in $\Gamma - e$ with $|S'| < s$. Since $S_1 = S' \cup \{x\}$ and $S_2 = S' \cup \{y\}$ are both AB-separators in Γ, they both have s vertices which implies that $|S'| = s - 1$.

Let S'' be an AS_1-separator in $\Gamma - e$. Observe that S'' is also an AB-separator in Γ, since every path connecting A and B either uses the edge $e = xy$ or intersects a vertex of S'. In particular $|S''| \geqslant s$. By induction, there is an AS_1-connector Γ_1 in $\Gamma - e$ with s paths, thus meeting each point of S_1 precisely once. The same argument applied to an S_2B-separator gives an S_2B-connector Γ_2 with s paths. Now $\Gamma_1 \cup \Gamma_2 \cup \{xy\}$ is an AB-connector with s paths (Fig. 6.5). □

Theorem 6.9 (Menger, Global Version) *A graph Γ with $|V(\Gamma)| > k$ is k-connected if and only if every pair of vertices is joined by k internally disjoint paths.*

Proof Let $x, y \in V(\Gamma)$. Take $A = N(x)$ and $B = N(y)$. Let S be an AB-separator. If $|S| < k$ then we can separate x and y by S contradicting that the graph is k-connected. By Theorem 6.8, there is an AB-connector with k paths. Together with the edges joining A with x and B with y one obtains k internally disjoint paths. □

Menger's theorem is a central result in combinatorics belonging to a family of results called min–max theorems. The theorem of Hall, Theorem 4.3, on the existence distinct representatives of a family of sets, or on the existence of a matching in bipartite graphs, Theorem 5.3, are examples of such results. As an illustration, we show an application of Menger's theorem to prove the following theorem of Ford and Fulkerson.

Let $\{A_1, \dots, A_m\}$ and $\{B_1, \dots, B_m\}$ be two families of subsets of a ground set X. A common system of distinct representatives is a set $\{x_1, \dots, x_m\} \subset X$ such that, for some permutations σ, τ of $\{1, \dots, m\}$, we have $x_i \in A_{\sigma(i)} \cap B_{\tau(i)}$ for each i.

Theorem 6.10 *The families of subsets $\{A_1, \dots, A_m\}$ and $\{B_1, \dots, B_m\}$ have a common system of distinct representatives if and only if for each pair $I, J \subset \{1, \dots, m\}$,*

$$|(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)| \geqslant |I| + |J| - m.$$

Proof Construct the graph Γ with vertex set

$$V(\Gamma) = \{s\} \cup \{A_1, \dots, A_m\} \cup \{v_1, \dots, v_m\} \cup \{B_1, \dots, B_m\} \cup \{t\},$$

and edge set

$$E(\Gamma) = \{sA_i : i \in \{1, \dots, m\}\} \cup \{A_i v_x : i \in \{1, \dots, m\}, x \in A_i\}$$

$$\cup \{B_j v_y : i \in \{1, \dots, m\}, y \in B_i\} \cup \{B_j t : j \in \{1, \dots, m\}\}.$$

See Fig. 6.6 for an example.

We observe that there is a common system of distinct of representatives if and only if there are m internally disjoint paths joining s and t in Γ. Indeed, for any such path with vertices s, A_i, x, B_j, t, the vertex x can be taken to be the common representative of A_i and B_j.

By Menger's theorem, such a set of paths exists if and only if every $\{s, t\}$-separator of Γ has more than m vertices.

Let S be an $\{s, t\}$-separator and set

$$I = \{v_i \in \{v_1, \dots, v_m\} : A_i \notin S\}$$

and

$$J = \{v_j \in \{v_1, \dots, v_m\} : B_j \notin S\}.$$

By the definition of I and J, we have that S contains $\{v_1, \dots, v_m\} \setminus I$ and $\{v_1, \dots, v_m\} \setminus J$.

Moreover, S must contain

$$(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)$$

since if there is a

$$k \in (\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j) \setminus S$$

then there is a path joining s and t which passes through v_k.

Therefore

$$|S| \geqslant |(\cup_{i\in I} A_i) \cap (\cup_{j\in J} B_j)| + (m - |I|) + (m - |J|)$$

which implies

$$|S| \geqslant |I| + |J| - m + (m - |I|) + (m - |J|) = m.$$

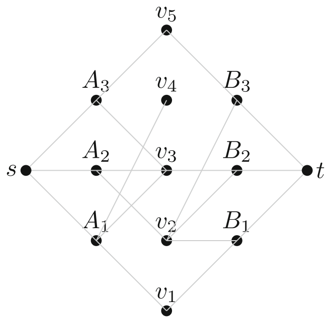


Fig. 6.6 An example of the graph in the proof of Theorem 6.10 for the sets $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 5\}$ and $B_1 = \{1, 2\}$, $B_2 = \{2, 3\}$, $B_3 = \{2, 5\}$

□

The **connectivity** $\kappa(\Gamma)$ of a graph Γ is the largest k such that Γ is k-connected. It follows from Menger's theorem (or from the definition) that $\kappa(\Gamma) \leqslant \delta(\Gamma)$, the minimum degree of Γ. Even if large minimum degree does not ensure high connectivity, the following theorem of Mader gives some connection. Recall that $\Delta(\Gamma)$ indicates graph's maximum degree.

Theorem 6.11 (Mader) *A graph Γ with average degree $\bar{d}(\Gamma) = 4k$ contains a k-connected subgraph Γ' with average degree $\bar{d}(\Gamma') > \bar{d}(\Gamma) - 2k$.*

Proof We observe that

$$n > \Delta(\Gamma) \geqslant \bar{d}(\Gamma) \geqslant 4k$$

and

$$m = \frac{n\bar{d}(\Gamma)}{2} \geqslant 2kn.$$

We will prove, by induction, the stronger statement that if $n \geqslant 2k - 1$ and

$$m \geqslant (2k - 3)(n - k + 1) + 1$$

then Γ has a k-connected subgraph with average degree larger than $\bar{d}(\Gamma) - 2k$.

If $n = 2k - 1$ then $m \geqslant n(n - 1)/2$ so that $\Gamma = K_n$ satisfies the claim.

Suppose $n \geqslant 2k$. If Γ is k-connected then there is nothing to prove. Furthermore, if $\delta(\Gamma) \leqslant 2k - 3$, we can apply induction on $\Gamma - x$, where x is a vertex of minimum degree in Γ. Therefore, we can suppose that $\delta(\Gamma) \geqslant 2k - 2$.

Let S be a separator in Γ with cardinality $|S| < k$ and let $\Gamma_1, \Gamma_2 \subset \Gamma$ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Gamma[S]$. Let $n_i = |V(\Gamma_i)|$ and $m_i = |E(\Gamma_i)|$. Since $\delta(\Gamma) \geqslant 2k - 2$ and all neighbours of a vertex in $\Gamma_1 \setminus \Gamma_2$ are in Γ_1 we have $n_1 \geqslant 2k - 2$ and $n_2 \geqslant 2k - 2$ for the analogous reason. Since $n \geqslant n_1 + n_2 - (k - 1)$, one of the two satisfies the induction hypothesis, otherwise

$$m \leqslant m_1 + m_2 < (2k-3)(n_1 + n_2 - 2k + 2) \leqslant (2k-3)(n-k+1).$$

□

6.4 Edge Connectivity

The notion of vertex connectivity can be translated to edge-separators. A set $L \subset E(\Gamma)$ is an edge-separator of a graph Γ if $\Gamma - L$ is not connected. A graph Γ is k-**edge-connected** if $\Gamma - L$ is connected for every set $L \subset E(\Gamma)$ with $|L| < k$ edges. If an edge e has the property that $\Gamma - e$ has more connected components than Γ then we say that e is a **bridge**. The minimum k such that Γ is k-edge-connected is the **edge-connectivity** of Γ, which is denoted by $\lambda(\Gamma)$.

The following proposition lists some basic properties of edge connectivity.

Proposition 6.12 *For any graph* Γ*,*

i. $\kappa(\Gamma) \leqslant \lambda(\Gamma) \leqslant \delta(\Gamma)$.
ii. every minimal edge-separator of a connected graph separates the graph in two connected components.
iii. if Γ *is* k*-edge-connected then, for every edge* $e \in E(\Gamma)$*, the graph* $\Gamma - e$ *is* $(k-1)$*-edge-connected.*

Proof

i. Suppose L is an edge-separator of Γ. A subset S of vertices which cover the edges of L is a (vertex) separator for Γ and there is some such separator such that $|S| \leqslant |L|$. Hence, $\kappa(\Gamma) \leqslant \lambda(\Gamma)$. If v is a vertex of minimum degree then the set of edges L, incident with v, is an edge-separator of Γ of size $\delta(\Gamma)$. Hence, $\lambda(\Gamma) \leqslant \delta(\Gamma)$.
ii. Suppose L is an edge-separator of Γ. If $\Gamma - L$ is has more than two connected components then $L - e$ is an edge separator for $\Gamma - L$.
iii. This is immediate.

□

We now use Menger's theorem (local version), Theorem 6.8 to prove a similar result for edge-connectivity.

Theorem 6.13 (Menger) *A graph* Γ *is* k*-edge-connected if and only if every pair of vertices can be joined by* k *edge-disjoint paths.*

Proof If every pair of vertices can be joined by k edge-disjoint paths then we must remove at least k edges to disconnect Γ. Hence, Γ is k-edge-connected.

Suppose Γ is k-edge-connected.

Recall that the line graph $L(\Gamma)$ of Γ has the edge set $E(\Gamma)$ as vertex set and two edges are adjacent whenever they are incident in Γ. A set $S \subset E(\Gamma)$ is an edge-separator of Γ if and only if it is a (vertex) separator of $L(\Gamma)$.

Take two vertices $x, y \in V(\Gamma)$ and let A be the set of edges incident with x and let B be the set of edges incident with y. The sets A and B are subsets of *vertices* of $L(\Gamma)$. From the previous paragraph, an AB-separator of $L(\Gamma)$ has size at least k. Thus, by Theorem 6.8, there is an AB-connector with at least k components (paths).

The vertices on these paths in $L(\Gamma)$ describe the edges on disjoint paths in Γ which join a neighbour of x to a neighbour in y. Each of these can then be extended to a path from x to y by adding an edge incident with x and an edge incident with y. □

6.5 Notes and References

Menger's theorem is one of the central theorems in graph theory. The simple proof of the theorem is taken from Goring (2000). The theorem on common distinct representatives was obtained by Ford and Fulkerson (1958), as an application of their max-flow/min-cut theorem, which is one of many min-max theorems equivalent to Menger theorem.

6.6 Exercises

Exercise 6.1 Let Γ be 2-connected. Show (without using Menger's theorem) that every pair of edges is contained in a cycle.

Exercise 6.2 Let Γ be 2-connected different from K_3. Show that, for each edge e, either $\Gamma - e$ or Γ/e is 2-connected.

Exercise 6.3 Let Γ be 3-connected and let xy be an edge of Γ. Show that Γ/xy is 3-connected if and only if $\Gamma - \{x, y\}$ is 2-connected.

Exercise 6.4 Show that if Γ is k-connected, $k \geqslant 2$, then for every k vertices there is a cycle containing them.

Exercise 6.5 Let Γ be k-connected. Show that, for every edge $e \in E(\Gamma)$, $\Gamma - e$ is $(k-1)$-connected.

Exercise 6.6 Let S, S' be distinct minimal separating sets of a graph Γ. Show that, if S intersects at least two connected components of $\Gamma - S'$ then S' intersects each component of $\Gamma - S$ (and S intersects every component of $\Gamma - S'$).

Exercise 6.7 Give an example of a k-edge-connected graph Γ with vertex connectivity $\kappa(\Gamma) = 1$.

Exercise 6.8 Show that a cubic 3-edge connected graph is also 3-connected.

Exercise 6.9 Prove Hall's theorem on the existence of a perfect matching in a bipartite graph by using Menger's theorem.

Exercise 6.10 Show that the n-cube $Q^n = K_2 \times \cdots \times K_2$ (n-times, cartesian product) is n-connected.

Exercise 6.11 A k-**split** of a graph Γ is the graph H obtained from Γ by replacing one vertex x by two adjacent vertices x_1, x_2 such that $N_H(x_1) \cup N_H(x_2) = N_\Gamma(x) \cup \{x_1, x_2\}$ and $d_H(x_1), d_H(x_2) \geqslant k$. Show that, if Γ is k-connected then every k-split of Γ is k-connected.

Exercise 6.12 Let Γ be a k-regular, k-connected graph with an even number of vertices.

For each non-empty subset W of vertices of Γ, let U be the set of odd components of $\Gamma \setminus W$. Consider the bi-partite graph Γ_W with stable sets U and W, where $u_i \in U$ is joined by an edge to $w_j \in W$ if and only if the odd component u_i is joined to w_j in the graph Γ.

i. Prove that if W is separating then $\deg u_i \geqslant k$, $\deg w_j \leqslant k$ and hence $|U| \leqslant |W|$.
ii. Prove that Γ has a perfect matching.
iii. Prove that a 3-connected graph with an even number of vertices which does not have a perfect matching has at least 8 vertices and construct such a graph with 8 vertices.

7 Planarity

Planarity is one of the classical topics in graph theory, partly due to the celebrated 4-colour theorem which largely fostered the development of graph theory. This theorem will be discussed in the next chapter, whilst this chapter will focus solely on the planarity property. As a blending of combinatorics and topology, there are some topological preliminaries which are necessary, but these will not be discussed too deeply. The central result in this chapter is Kuratowski's theorem, which characterises planar graphs in terms of forbidden minors. This is a result with deep extensions in graph theory.

7.1 Plane Graphs

A **plane** graph $\Gamma = (V, E)$ is a graph where V is a finite set of points of $\mathbb{R}^2$ and E is a set of simple arcs (continuous functions $f : [0, 1] \to \mathbb{R}^2$ such that $\{f(0), f(1)\}$ an edge of Γ) which are internally pairwise disjoint (only meet at incident vertices of edges). A graph is **planar** if it is isomorphic to a plane graph. Such an isomorphism is an **embedding** of Γ in the plane. A **face** of a plane graph Γ is an arc-connected component of $\mathbb{R}^2 \setminus E(\Gamma)$, i.e. any two points in the face are connected by a continuous, not necessarily straight, line.

Lemma 7.1 *Let Γ be a planar graph. There is an embedding of Γ in the plane such that edges are polygonal arcs.*

Proof Consider a plane embedding of the graph. For every vertex u let $\epsilon_u > 0$ be such that the closed disc $D(u, \epsilon_u)$ centred at u with radius ϵ_u intersects only the edges incident with u and no other edge in the embedding. For every edge $e = uv$ let u_e and v_e be the intersections of the simple arc joining u and v with the boundary of the discs $D(u, \epsilon_u)$ and $D(v, \epsilon_v)$ respectively. Since, Γ is planar, for every point x in the restriction of this edge joining u_e with v_e, there is an open disc centered

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_7

at x which does not intersect any other edge of the graph. Since this restriction is a compact set in $\mathbb{R}^2$, there is a finite number of these discs which cover it. Now we can replace the arc within each such disc by a straight line joining its two intersections with the boundary of the disc. Similarly we can replace the arc in $D(u, \epsilon_u)$ with a straight line joining u with u_e. In this way every edge can be replaced by a polygonal line and no two of them intersect except at the incident vertices of edges, and we obtain a plane embedding of Γ in which edges are polygonal arcs. □

A basic topological ingredient in what follows is the Jordan Curve theorem, a version of which we now prove for polygonal curves. This is sufficient for many applications to planar graphs.

Theorem 7.2 (Jordan Curve Theorem) *A closed simple polygonal arc C in the plane splits $\mathbb{R}^2$ into precisely two arc-connected components, one of which is unbounded, having C as a common boundary.*

Proof Let $P_1, \ldots, P_n$ be the straight segments of C. We may assume that none of the segments P_i is horizontal. To each point $z \in \mathbb{R}^2 \setminus C$ we define $\pi(z)$ to be the number of points of C that a horizontal ray starting at z to the right intersects, with the convention that vertices of C are counted as intersections only if the two segments meeting in that point lie in different sides of the ray. Let $\bar{\pi}(z) = \pi(z) \pmod 2$.

We observe that two points on a segment in $\mathbb{R}^2 \setminus C$ have the same value of $\bar{\pi}$, since the value of π changes along the segment only when a vertex of C is met and the two segments incident to that vertex lie on the same side of the ray, thus not changing the parity of π (see Fig. 7.1 (left)). It follows that a polygonal line joining two points with different values of $\bar{\pi}$ must intersect C and these two points lie in different arc-connected components of $\mathbb{R}^2 \setminus C$.

It remains to show that two points z_1, z_2 with $\bar{\pi}(z_1) = \bar{\pi}(z_2)$ can be connected by a polygonal line not intersecting C. If the segment z_1z_2 does not intersect C then we are done. Otherwise let z_1' and z_2' be the first and last intersections of the segment z_1z_2 with C. Consider a sufficiently small $\epsilon > 0$ such that the set $U = \{x \in \mathbb{R}^2 : d(x, C) = \epsilon\}$ does not intersect C. Let $z_1'\prime$ be the point in U at distance ϵ from z_1'

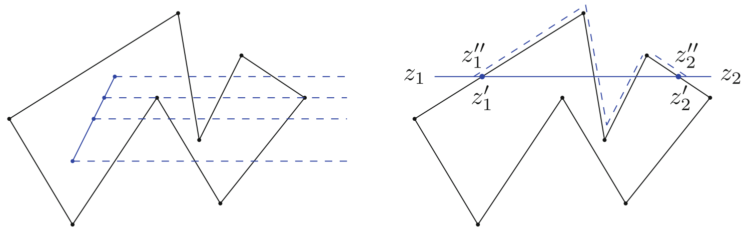


Fig. 7.1 Two points in a segment not intersecting C have the same value of $\bar{\pi}$ (left) and two points with the same value of $\bar{\pi}$ can be connected by a (dotted) arc in $\mathbb{R}^2 \setminus C$ (right)

on the segment z_1z_1', and follow the polygonal line along U from $z_1'{}'$ until we reach the point z''_2 in U at distance ϵ to z_2' (see Fig. 7.1 (right)), an arc which does not intersect C. Now the segment z''_2z_2 does not intersect C either as, if it did, then z''_2 and z_2 would be in different sides of the segment of C containing z_2' and would have different value of $\bar{\pi}$. □

It is clear that a graph is planar if and only if each of its connected components is planar. For connected graphs the following three results describe properties of the faces of Γ.

Proposition 7.3 *Every tree T is planar and a plane embedding of T has only one face.*

Proof By induction on the number n of vertices. The result is clear for $n = 1, 2$. If $n > 2$, let x be a leaf of T and y be its only neighbour in T. By induction $T' = T - x$ is planar and every plane embedding of T' has an only face. For every point in $\mathbb{R}^2 \setminus T'$ there is an arc joining it with the vertex y, giving a plane embedding of T. Moreover such an arc cannot create a new arc-connected component of $\mathbb{R}^2 - T$. □

Proposition 7.4 *Let Γ be a 2-connected planar graph. Every face of Γ is bounded by a cycle.*

Proof By induction on the number of cycles of Γ. If Γ is a cycle then the statement holds. Otherwise, by Lemma 6.4, Γ can be obtained from some subgraph Γ' of Γ by adding a path P with its end points in Γ' and no other vertex in Γ'. By induction, every face of Γ' is bounded by a cycle. The interior of P lies in a face f' of Γ' which, by induction, is bounded by a cycle C. Hence, every face of Γ is bounded by a cycle. □

One can say more if Γ is 3-connected.

Proposition 7.5 (Tutte) *Let Γ be a 3-connected planar graph. A cycle C is the boundary of a face in a plane embedding of Γ if and only if C is induced (it has no chords) and nonseparating ($\Gamma - C$ is connected).*

Proof The reverse implication follows from the Jordan curve theorem, Theorem 7.2. If C is an induced cycle then one of the two faces of C contains no vertices of Γ. Hence, C is the boundary of a face of Γ.

Suppose now that C is a boundary of a face f in a plane embedding of Γ. Suppose that C has a chord xy joining two nonconsecutive vertices along C. Then C must be the boundary of the external face of the plane embedding. Let C' be the cycle formed by a path from x to y in C and the edge xy. Let u, v be two vertices in C in different segments of $C \setminus \{x, y\}$, with u on C', see Fig. 7.2. Every path joining u with v in Γ not using the two edges of C incident to u must cross the cycle C' and

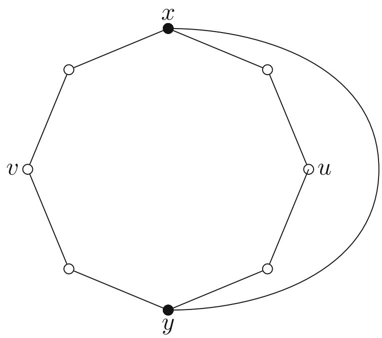


Fig. 7.2 If the boundary cycle C of a face is not induced then u and v are only connected by two disjoint paths: G is not 3-connected

therefore the edge xy, contradicting the assumption that Γ is planar. Therefore there are only 2 disjoint paths from u to v in Γ which, by Theorem 6.9, contradicts the assumption that Γ is 3-connected. Hence, C is induced.

To show that C is nonseparating, let x, y be two vertices not in C. Since Γ is 3-connected, Theorem 6.9 implies that there are three internally disjoint paths P_1, P_2, P_3 joining x and y. The union of these three paths separate the plane into three arc-connected components, and every face of Γ is contained in one of them. Therefore, the removal of C from Γ does not intersect one of the three paths and so $\Gamma - C$ is connected. □

We now prove Euler's formula for planar graphs.

Theorem 7.6 (Euler Formula) *Let Γ be a connected plane graph with n vertices, m edges and f faces. Then*

$$n - m + f = 2.$$

Proof For each fixed n we prove the result by induction on m. When $m = n - 1$ then Γ is a tree and $\mathbb{R}^2 \setminus E(\Gamma)$ has only one face. For $m > n - 1$, the graph has at least one cycle. Let e be an edge in this cycle, which is in the boundary of two faces of Γ. The removal of this edge in Γ results in a graph with one less face and one less edge. The formula follows by induction. □

As a consequence of Euler's formula we obtain the following crucial statements about planar graphs. In the following statement a **maximal planar graph** is a graph which has a planar embedding but for which adding an edge gives a graph which does not.

Corollary 7.7 *A planar graph Γ has at most $3n - 6$ edges with equality if and only if Γ is a triangulation (every face is a triangle). Every planar graph is a spanning subgraph of a maximal planar graph, which is a triangulation.*

If Γ is bipartite then it has at most $2n - 4$ edges with equality if and only if Γ is a quadrangulation (every face is a quadrangle). Every bipartite planar graph is a spanning subgraph of a maximal bipartite planar graph, which is a quadrangulation.

In particular, K_5 and $K_{3,3}$ are not planar.

Proof Every edge is in two faces and every face is defined by at least three edges, so $2m \geqslant 3f$. By substituting in Euler's formula, we have

$$3n - m \geqslant 3n - 3m + 3f = 6.$$

Planarity is preserved by the addition of a chord in a cycle of length at least four in a planar graph Γ. By successively adding such chords we eventually obtain a planar graph Γ' all of whose faces are triangles. Thus, it has $3n - 6$ edges, and has Γ as a spanning subgraph.

If Γ is bipartite then it is triangle-free, so a similar double counting argument gives $2m \geqslant 4f$. Substituting in Euler's formula gives $m \leqslant 2n - 4$. Again, adding chords to even cycles of length at least 6 in a bipartite planar graph Γ while preserving bipartiteness leads to a bipartite graph Γ' all of all whose faces are quadrangles, so it has $2m - 4$ edges, and contains Γ as a spanning subgraph.

Since K_5 has $10 > 3 \cdot 5 - 6$ edges, it can not be planar.

Similarly, $K_{3,3}$ has $9 > 2 \cdot 6 - 4$ edges, so it is not planar. □

7.2 Kuratowski's Theorem

A **subdivision** of an edge $e = xy$ in a graph Γ is the substitution of e by a path joining x and y (and its internal vertices have degree two). A subdivision of a graph Γ is the graph obtained from Γ by subdividing some of its edges. We say that a graph Γ is a **topological minor** of a graph Γ' if Γ' contains a subgraph which is a subdivision of Γ. We write $\Gamma \preceq_T \Gamma'$ if Γ is a topological minor of Γ'. The relation $\preceq_T$ is a partial order on the class of all graphs.

It is clear from the definition that if Γ is a planar graph and $\Gamma' \preceq_T \Gamma$ then Γ' is also planar. In other words, planarity is an hereditary property of the topological minor relation. In particular, a planar graph can not contain K_5 nor $K_{3,3}$ as topological minors. The celebrated theorem of Kuratowski characterizes planarity in terms of forbidden topological minors.

Theorem 7.8 (Kuratowski) *A graph is planar if and only if it contains no K_5 and no $K_{3,3}$ as topological minors.*

We first show 3-connected graphs free of K_5, $K_{3,3}$ topological minors satisfy a stronger property. A plane embedding of a graph Γ is *convex* if every edge is a straight line and every face is a convex polygon (the complement of the outer face is also a convex polygon).

Lemma 7.9 *Let* Γ *be a* 3*-connected graph. If* $K_5, K_{3,3} \not\leq_T \Gamma$ *then* Γ *admits a plane convex embedding.*

Proof The proof is by induction on n. For $n = 4$ a 3-connected graph is K_4, which admits a convex plane embedding.

Suppose $n \geqslant 5$. By Lemma 6.6, there is an edge $e = xy$ in Γ such that Γ/e is 3-connected. We note that Γ/e is free of subdivisions of K_5 or $K_{3,3}$ (otherwise these subdivisions would occur in Γ as well). By the induction hypothesis, Γ/e admits a convex embedding in the plane.

Let v_{xy} be the vertex of Γ/e obtained by contracting the edge e. Since Γ/e is 3-connected, $(\Gamma/e) - v_{xy}$ is 2-connected. Therefore, in the embedding of Γ/e, when we remove the vertex v_{xy}, it follows that the face which contains the neighbours of v_{xy} is a cycle C. The cycle C contains all neighbours of x and of y in $\Gamma - e$. Let $x_1, \dots, x_k$ be the neighbours of x in this cycle listed in anti-clockwise order. If all neighbours of y are in the segment of the cycle between x_i and x_{i+1} for some i (modulo k), then the convex embedding of Γ/e can be extended to one of Γ by placing x in the position of v_{xy} and placing y within the triangle xx_ix_{i+1} (here we use the property that the embedding of Γ/e is convex). We show that, if this is not the case, then we reach a contradiction. We consider three cases.

Since Γ is 3-connected, the vertex y has at least two neighbours in C.

Suppose all neighbours of of y are neighbours of x.

Case (a). The vertex y has three neighbours z, z', z'' in common with x. Then the five vertices x, y, x_0, x_1, x_2 form a subdivision of K_5 (see Fig. 7.3a).

Case (b). The vertex y has two nonconsecutive neighbours x_i, x_j in common with x. Then, the vertices y, x_{i-1}, x_{i+1} and x, x_i, x_j form a subdivision of $K_{3,3}$ (see

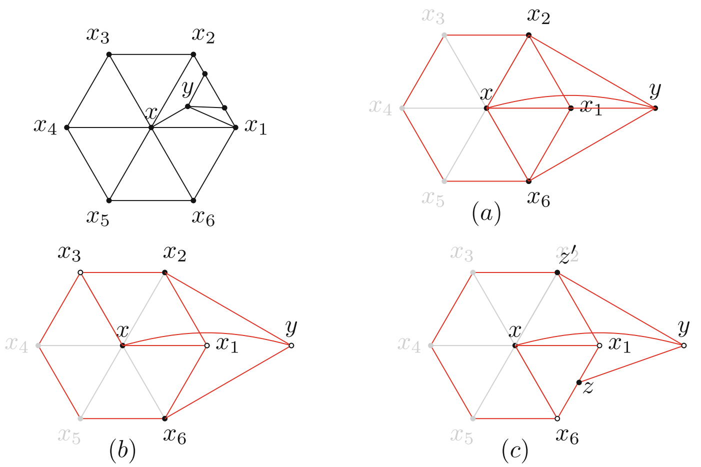


Fig. 7.3 An illustration of the three cases in the proof of Lemma 7.9

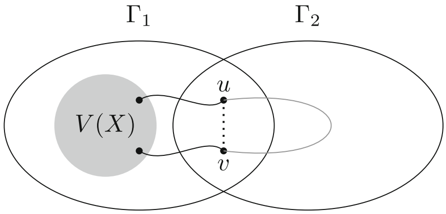


Fig. 7.4 The case that $\{u, v\}$ is a separating set of uv is not an edge

Fig. 7.3b). Note that x_j is joined by disjoint paths (contained in C) to x_i and x_{i+1} and these paths do not contain the edges $x_i x_{i+1}$ or $x_i x_{i-1}$.

In the final case we suppose y has a neighbour which is not a neighbour of x.

Case (c). The vertex y has a neighbour z in the interior of some segment $x_i x_{i+1}$ and another one z' in a different segment (here we use that Γ is 3-connected: y has at least two neighbours in the cycle C). Then the vertices y, x_i, x_{i+1} and x, z, z' form a subdivision of $K_{3,3}$ (see Fig. 7.3c).

□

The second step in the proof of Kuratowski's theorem is to show that 3-connectedness can be assumed.

Lemma 7.10 *Let Γ be a graph such that $K_5, K_{3,3} \not\preceq_T \Gamma$ and Γ is edge-maximal with this property. Then Γ is either K_1, K_2, K_3 or 3-connected.*

Proof This is proven by induction on n again, the cases with $n \leqslant 4$ being clear. Let Γ be an edge maximal graph with $n > 4$ vertices not containing $X \in \{K_{3,3}, K_5\}$ as a topological minor. Let S be a minimum separating set of Γ and let $\Gamma_1, \Gamma_2 \subset \Gamma$ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $V(\Gamma_1) \cap V(\Gamma_2) = S$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$. We note that each of Γ_1 and Γ_2 contain no $X \in \{K_{3,3}, K_5\}$ as topological minor, since otherwise it would be present in Γ.

If $S = \emptyset$ then clearly Γ is not edge-maximal since any edge xy joining Γ_1 with Γ_2 cannot produce a subdivision of X in $\Gamma + xy$ that was not present in Γ.

Suppose that $S = \{u\}$. Let $x \in V(\Gamma_1) \cap N(u)$ and $y \in V(\Gamma_2) \cap N(u)$. Suppose there is a subdivision of $X \in \{K_{3,3}, K_5\}$ in $\Gamma + xy$. Since X is 3-connected, $V(X)$ must be contained in one of Γ_1 or Γ_2, say in Γ_1. A path P in $\Gamma + xy$ which is a subdivided edge of X using xy must contain a sub-path P_{yu} from y to u (possibly the edge yu), since it has to come back to Γ_1. Thus, the edge xy and the sub-path P_{yu} in P can be replaced by xu, so that $X \preceq_T \Gamma_1 \subset \Gamma$, a contradiction.

Suppose that $S = \{u, v\}$, see Fig. 7.4. If the edge uv is not in Γ then there is a subdivision of $X \in \{K_{3,3}, K_5\}$ in $\Gamma + uv$. As in the previous case when $|S| = 1$, since X is 3-connected, we can assume that $V(X)$ is contained in Γ_1. A path in $\Gamma + uv$ which is a subdivided edge of X using uv can be replaced by a path in Γ_2 joining u and v and such a path must exist by the minimality of $|S|$. This would imply a subdivision of X in Γ. Therefore, we can assume $uv \in E(\Gamma)$.

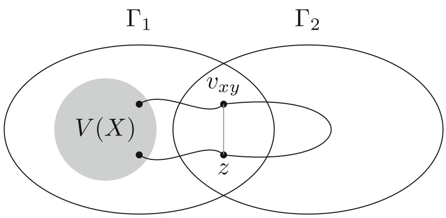


Fig. 7.5 The case that $\{u, v\}$ is a separating set of uv is an edge

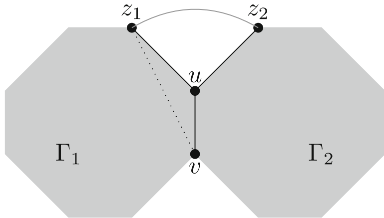


Fig. 7.6 Plane embedding of Γ in Lemma 7.10

Suppose $uv \in E(\Gamma)$, see Fig. 7.5. Each of Γ_1 and Γ_2 is edge-maximal, since the addition of an edge xy to Γ_1 produces a subdivision of $X \in \{K_{3,3}, K_5\}$ in $\Gamma + xy$ which, since X is 3-connected, must have its vertex set contained in Γ_1. Every path which is a subdivided edge of X using edges in Γ_2 can be replaced by a path using the edge uv, so that $X \preceq_T \Gamma_1$. By the induction hypothesis, each of Γ_1 and Γ_2 are 3-connected or K_3 and, by Lemma 7.9, they are planar.

Consider convex embeddings of Γ_1 and Γ_2 such that uv lies in the outer face in each embedding, and identify the edge uv in both embeddings. Let $z_i \in V(\Gamma_i)$ be a vertex adjacent to u different from v in the outer face for each $i = 1, 2$ (see Fig. 7.6). Then $\Gamma + z_1z_2$ is planar and so it does not contain $X \in \{K_{3,3}, K_5\}$ as a topological minor, contradicting that Γ is edge-maximal with this property.

□

We are now in a position to prove Kuratowski's theorem.

Proof of Theorem 7.8 One implication is clear by Corollary 7.7: K_5 or $K_{3,3}$ are not planar and planarity is a hereditary property for the topological minor relation.

For the reciprocal, we can assume that Γ is edge-maximal, does not contain K_5 and $K_{3,3}$ as topological minors and that $n \geqslant 4$. By Lemma 7.10, Γ is 3-connected, and by Lemma 7.9, it is planar. □

7.3 Wagner's Theorem

A **minor** Γ' of a graph Γ is a subgraph of a graph obtained from Γ by contracting some edges. We denote by $\Gamma' \preceq \Gamma$ the relation of being a minor. If Γ' is a minor of Γ then there is as a subgraph $\Gamma' \subset \Gamma$ which admits a partition

$$V(\Gamma') = X_1 \cup \cdots \cup X_m$$

such that each $\Gamma'[X_i]$ is connected and, by contracting all its edges, we obtain Γ'. We say that this partition is **associated** to Γ'.

We observe that $\Gamma' \preceq_T \Gamma$ implies $\Gamma' \preceq \Gamma$. The relation $\preceq$ is a partial order on the family of all graphs. The deep significance of this partial order in graph theory grew partly from Kuratowski's theorem, or rather by the following equivalent version of it.

Theorem 7.11 (Wagner) *A graph is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of* Γ.

The proof of Theorem 7.11 uses the following lemmas relating minors and topological minors. Recall that $\Delta(\Gamma)$ indicates the maximum degree of Γ.

Lemma 7.12 *Let K be a graph with maximum degree $\Delta(K) \leqslant 3$. A graph Γ containing K as a minor also contains K as a topological minor.*

Proof Let Γ' be a minimal subgraph of Γ containing a minor of K, and let $V(\Gamma') = X_1, \ldots, X_m$, $m = |V(K)|$, be a partition of Γ' such that the contraction of each X_i in Γ' gives rise to K. By the minimality of Γ', each induced subgraph $\Gamma'[X_i]$ is a tree. Since $\Delta(K) \leqslant 3$, the edges of K attached to $\Gamma'[X_i]$ use at most three vertices of this tree. Again by the minimality of Γ', $\Gamma'[X_i]$ has at most three leaves, namely, it is a subdivision of K_1, K_2 or $K_{1,3}$. Hence, Γ' (and Γ) contains a subdivision of K. □

Lemma 7.13 *If a graph Γ contains K_5 as a minor, it also contains $K_{3,3}$ or K_5 as a topological minor.*

Proof Suppose that $K_5 \preceq \Gamma$ and let $\Gamma' \leqslant \Gamma$ again be a minimal subgraph which contains K_5 as a minor. Let $X_1, \ldots, X_5 \subset V(\Gamma')$ be a partition of Γ' such that the contraction of each X_i in Γ' results in K_5. By minimality of Γ', each $\Gamma'[X_i]$ is a tree and it is connected by a single edge with each $\Gamma'[X_j]$, $j \neq i$. In particular, it has at most four leaves. If each $\Gamma'[X_i]$ is either one vertex or isomorphic to $K_{1,4}$ then Γ' contains a subdivision of K_5. Suppose that $\Gamma'[X_1]$ is a tree different from a single vertex or $K_{1,4}$. Then $\Gamma'[X_1]$ together with the four edges joining X_1 with X_2, X_3, X_4, X_5 and their end vertices in these copies forms a tree with four leaves and maximum degree at most three. Therefore, it contains two vertices $x, y \in X_1$ with degree three. By contracting $\Gamma'[X_1]$ on these two vertices and every other $H[X_i]$, $i > 1$ to a single vertex, we get $K_{3,3} \preceq \Gamma'$, see the illustration in Fig. 7.7. By Lemma 7.12, $K_{3,3} \preceq_T \Gamma'$ and hence, $K_{3,3} \preceq_T \Gamma$. □

Proof of Theorem 7.11 If Γ is not planar then by Kuratowski's theorem, Theorem 7.8, it contains a subdivision of K_5 or $K_{3,3}$, which, in particular, is a minor of Γ.

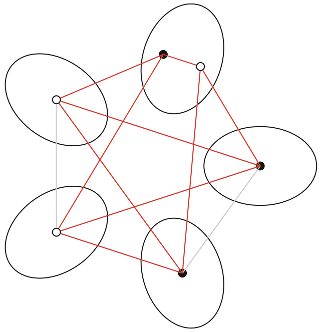


Fig. 7.7 The subdivision of $K_{3,3}$ in the proof of Lemma 7.13

Conversely, if $K_{3,3} \preceq \Gamma$ or $K_5 \preceq \Gamma$ then, by the Lemmas 7.12 and 7.13, Γ contains $K_{3,3}$ or K_5 as a topological minor, and by Kuratowski's theorem, Theorem 7.8, Γ is not planar. □

7.4 Whitney Theorem

Our last result concerns uniqueness of embeddings. Let Γ be a planar graph and let Γ', Γ'' be two plane graphs isomorphic to Γ: we say that the two plane embeddings are **equivalent** if there is a graph isomorphism $\phi : \Gamma' \to \Gamma''$ which can be naturally extended to faces. In other words, ϕ preserves the incidence of vertices, edges and faces. There is the stronger notion of topological incidence which requires that ϕ is additionally an homeomorphism of the sphere S^2.

Theorem 7.14 (Whitney) *Let Γ be a 3-connected graph. Then all plane embeddings of Γ are equivalent.*

Proof By Proposition 7.5, C is the boundary of a face in a plane embedding of Γ if and only if C is induced and non-separating. Thus, every two plane embeddings of Γ have the same set of faces, the induced non-separating cycles. Every graph isomorphism between two plane embeddings defines a bijection between induced non-separating cycles. This bijection preserves the incidence of faces, so the two plane embeddings are equivalent. □

7.5 Notes and References

The Jordan curve theorem is a classical fundamental result in topology. The proof of the simpler polygonal version in Theorem 7.2 given here follows Courant and Robbins (1979, page 267). Some additional topological preliminaries have been omitted here, good references are Diestel (2018, Chapter 4) and the excellent monograph by Mohar and Thomassen (2001).

Kuratowski's theorem, Theorem 7.8 and its minor version by Wagner are fundamental results which form the starting point of graph minor theory, leading to deep results in structural and algorithmic graph theory. From the topological perspective, Kuratowski's theorem is a special case of the general result that the class of graphs which can be embedded in a compact surface of given Euler characteristic are characterised by a finite list of excluded minors. The relatively simple proof of Kuratowski's theorem given here follows Tutte (1963), see also Thomassen (1981).

The fact that the skeleton of a convex polyhedra is a planar 3-connected graph is one implication of a theorem by Steinitz showing that the reciprocal implication also holds. The fact that maximal planar graphs have the maximum number of edges among graphs free from K_5 as a minor is a special case of a theorem by Mader that states that every graph with $3n-5$ edges contains K_5 as a minor. The decomposition of a planar graph into three edge disjoint trees follows from a more general theorem of Schynder (1989), which gives a characterisation of planar graphs in terms of associated posets.

7.6 Exercises

Exercise 7.1 Let Γ be a planar graph. Show that every subgraph $\Gamma' \subset \Gamma$ has minimum degree $\delta(\Gamma') \leqslant 5$. If Γ is triangle-free then $\delta(\Gamma') \leqslant 4$ for every subgraph $\Gamma' \subset \Gamma$.

Exercise 7.2 Show that a plane 2-connected graph is bipartite if and only if the boundary of every face is an even cycle.

Exercise 7.3 Find a planar graph with an exponential number of distinct plane embeddings.

Exercise 7.4 A graph is **outerplanar** if it admits a plane embedding with all vertices on the outer face. Show that a 2-connected outerplanar graph consists of a cycle with nonintersecting chords and it can not be 3-connected.

Exercise 7.5 Show that K_4 and $K_{2,3}$ are not outerplanar graphs. Show that a graph is outerplanar if and only if it does not have subdivisions of K_4 or $K_{2,3}$.

Exercise 7.6 A triangulation is a plane graph with all faces triangles. Let Γ be a triangulation with $n \geqslant 4$ vertices.

i. Show that Γ is maximal planar.
ii. Show that there are at least n edges $e \in E(G)$ such that the contraction Γ/e is also maximal planar.

iii. Show that Γ is 3-connected.
iv. Show that, if S is a vertex cutset with $|S| = 3$ then the subgraph $\Gamma[S]$ induced by the vertices in S is a triangle.

Exercise 7.7 Let Γ be a 3-connected graph with $n > 5$ vertices. Show that if Γ contains a subdivision of K_5 then it also contains a subdivision of $K_{3,3}$.

Exercise 7.8 Show that the skeleton of a bounded polyhedra (3-dimensional polytope) is a planar 3-connected graph.

Exercise 7.9 Show that the addition of one edge to a maximal planar graph with at least 6 vertices produces both a K_5 and a $K_{3,3}$ minor.

Exercise 7.10 Show that a maximal planar graph can be decomposed into three trees. Show that a maximal planar bipartite graph can be decomposed into two trees.

Graph Colouring

8

Colouring is, alongside planarity, one of the classical topics in graph theory. One of its most celebrated results is the four colour theorem that states that planar graphs can be coloured with just four colours. In this chapter, we first discuss upper bounds on the chromatic number of a graph in terms of the degrees of the vertices, those arising from the greedy colouring algorithm, the Szekeres–Wilf bound and Brooks' theorem. The weaker theorem of Heawood on planar graphs and the characterisation of planar graphs of low chromatic numbers illustrate the framework of the colouring problem for planar graphs. The better bound given by Vizing's theorem on the related edge-chromatic number is also discussed, and the equivalence of the four colour theorem with edge-colourings is considered following on from that. The chapter concludes with the list colouring problem, a proof by Thomasen of the 5-choosability of planar graphs and Galvin's theorem on the edge-choosability of bipartite graphs.

8.1 Vertex Colouring

A **vertex colouring** of a graph $\Gamma = (V, E)$ is a map

$$c : V(\Gamma) \to \{1, \ldots, k\}.$$

The colouring is **proper** if no edge is monochromatic, that is adjacent vertices receive distinct colours. The minimum number of colours in a proper vertex colouring of a graph Γ is its **chromatic number**, denoted by $\chi(\Gamma)$.

We denote by $\omega(\Gamma)$ the cardinality of the largest clique (complete subgraph of Γ) and by $\alpha(\Gamma)$ the cardinality of the largest coclique (independent set of Γ). A proper k-colouring of Γ using the k colours induces a partition of its vertex set into k independent sets $c^{-1}(1), \ldots, c^{-1}(k)$.

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_8

Lemma 8.1 *For every graph* Γ *of order* n*,*

$$\chi(\Gamma) \geqslant \max\{\omega(\Gamma), n/\alpha(\Gamma)\}.$$

Proof Let c be a proper colouring of Γ with $k = \chi(\Gamma)$ colours. Since each vertex of a clique of size $\omega(\Gamma)$ receives a distinct colour, $k \geqslant \omega(\Gamma)$.

On the other hand, $c^{-1}(1), \ldots, c^{-1}(k)$ is a partition of $V(\Gamma)$ into stable sets, so

$$n = \sum_i |c^{-1}(i)| \leqslant k\alpha(\Gamma).$$

□

A graph Γ is k-**critical** if $\chi(\Gamma) = k$ and if by deleting any edge or vertex we obtain a graph which is $(k-1)$-colourable. Observe that, step-by-step removing an edge which does not decrease the chromatic number, it is immediate that every graph with chromatic number k contains a k-critical subgraph.

Lemma 8.2 *If* Γ *is* k*-critical then* $\delta(\Gamma) \geqslant k - 1$.

Proof By deleting a vertex of minimum degree we obtain a graph which is $(k-1)$-colourable. Thus, if $\delta < k - 1$ we can colour the deleted vertex with one of $k - 1$ colours, contradicting the fact that $\chi(\Gamma) = k$. □

The following upper bounds are classical results in graph colouring.

Let $\{x_1, \ldots, x_n\}$ be an ordering of the vertices of a graph Γ. The so-called **greedy** colouring algorithm proceeds by giving colour 1 to x_1 and, once x_i is coloured, give to x_{i+1} the smallest available colour among $\{1, 2, \ldots, i + 1\}$.

Theorem 8.3 (Szekeres-Wilf) *For every graph* Γ

$$\chi(\Gamma) \leqslant 1 + \max_{\Gamma' \subseteq \Gamma} \delta(\Gamma').$$

Proof Let $d = \max_{\Gamma' \subseteq \Gamma} \delta(\Gamma')$. We define an ordering of the vertices as follows. We choose a vertex x_n with degree at most d in Γ. Once x_{i+1} is defined, we choose a vertex with degree at most d in the subgraph $\Gamma[V \setminus \{x_{i+1}, \ldots, x_n\}]$ of Γ induced by the unchosen vertices. Now the greedy algorithm on $x_1, \ldots, x_n$, which starts colouring x_1 with 1 and colours each x_i with the least available colour to make a proper colouring of $\Gamma[x_1, \ldots, x_i]$, uses at most $d + 1$ colours because every x_i is adjacent at most to d previous vertices. □

It follows from the Szekeres–Wilf theorem that $\chi(\Gamma) \leqslant 1+\Delta(\Gamma)$. The following theorem of Brooks states that complete graphs and odd cycles are the only graphs for which the above bound is tight.

Theorem 8.4 (Brooks) *If Γ is a connected graph different from a complete graph or an odd cycle then*

$$\chi(\Gamma) \leqslant \Delta(\Gamma).$$

Proof Suppose that Γ is not a cycle or a complete graph.

If Γ is not regular then Theorem 8.3 implies $\chi(\Gamma) \leqslant \Delta(\Gamma)$, since the minimum degree would be less than Δ.

By Lemma 6.3, since Γ is connected, it is a tree of blocks where we recall that a block is a single vertex, an edge or a 2-connected graph. Observe that the chromatic number of Γ is equal to the maximum of the chromatic numbers of the blocks. Thus, we can assume that Γ is 2-connected, since the statement is trivial if $\chi(\Gamma) = 2$. Moreover, from the previous paragraph, we can assume Γ is regular.

Case 1 Γ is 3-connected. Choose x_n and two non adjacent vertices x_1, x_2 in its neighbourhood (such a choice exists since Γ is not complete). We have that $\Gamma - \{x_1, x_2\}$ is connected, $x_1x_2 \notin E(\Gamma)$ and $x_1x_n, x_2x_n \in E(\Gamma)$. For each i, starting at $i = n-1$, choose $x_i \in V(\Gamma) \setminus \{x_1, x_2, x_{i+1}, \dots, x_n\}$ adjacent to some vertex in $\{x_{i+1}, \dots, x_n\}$, which must exist by connectedness. Now the greedy algorithm allows us to colour x_1, x_2 with 1 and, at each step, x_i is only adjacent to at most $\Delta(\Gamma) - 1$ preceding vertices since it is adjacent to x_j for some $j > i$. In the last step we have to colour x_n, which is adjacent to $\Delta(\Gamma)$ vertices but two of them, x_1 and x_2, have the same colour, leaving one colour available for x_n (Fig. 8.1, left).

Case 2 Γ is 2-connected but not 3-connected. Choose a vertex x_n in a minimal separating set S of Γ, so that $\Gamma' = \Gamma - x_n$ is connected but not 2-connected. By Lemma 6.3, Γ' is a tree of blocks and, by the minimality of $|S|$, the vertex x_n is adjacent to two distinct blocks of this block decomposition of Γ', moreover it is adjacent to vertices x_1 and x_2 which are not articulation vertices of Γ'. Since they belong to distinct blocks of Γ and are not articulation points, x_1 and x_2 are not adjacent in Γ. Moreover, $\Gamma - \{x_1, x_2\}$ is connected as the blocks are 2-connected (Fig. 8.1, left). We can now repeat the argument in Case 1 to produce an ordering of the vertices for which the greedy algorithm uses at most $\Delta(\Gamma)$ colours.

□

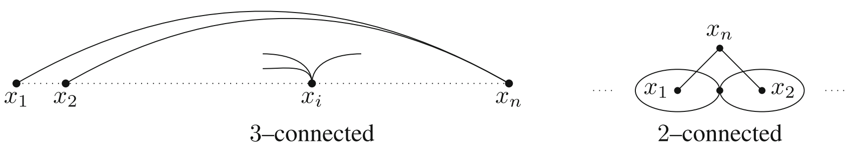


Fig. 8.1 The two cases of the proof of Brooks Theorem

8.2 Planar Graphs

A central result which fostered the development of graph theory is the four colour theorem stating that planar graphs have chromatic number at most four. All known proofs rely on extensive computer checking of hundreds of cases. The following theorem has a much easier proof.

Theorem 8.5 (Heawood) *Every planar graph Γ is 5-colourable.*

Proof The proof is by induction on $n = |V(\Gamma)|$, the result being trivial for $n \leqslant 5$. We may assume that Γ is maximal planar, i.e. a graph for which adding any additional edges will give a graph which is not planar. By Corollary 7.7, a planar graph has at most $3n-6$ edges, which implies that the minimum degree of a planar graph satisfies $\delta(\Gamma) \leqslant 5$.

If Γ has a vertex x such that the degree of x, $d(x) \leqslant 4$ then every 5-colouring of $\Gamma[V \setminus \{x\}]$ can be extended to a 5-colouring of Γ. Suppose that $d(x) = \delta(\Gamma) = 5$ and let $x_1, \ldots, x_5$ be the five neighbours of x listed in clockwise order in a planar embedding of Γ.

If there is a 5-colouring of $\Gamma' = \Gamma[V \setminus \{x\}]$ which does not use the five colours in the neighborhood of x then we can extend the colouring to Γ. We may therefore assume that $\chi(x_i) = i$ for $1 \leqslant i \leqslant 5$ in a 5-colouring of Γ'. Let $\Gamma'[1,3]$ be the subgraph Γ' induced by the colour classes 1 and 3. This is a graph with maximum degree 2, so that all connected components are either cycles or paths.

If x_1 and x_3 belong to distinct connected components of $\Gamma'[1,3]$ then we can switch the colours in one of the components and get a proper 5-colouring which uses 4 colours on the neighborhood of x.

Hence, we can assume that x_1 and x_3 belong to the same connected component of $\Gamma'[1,3]$, see Fig. 8.2. Consider the subgraph $\Gamma'[2,4]$ of Γ' induced by the colour classes 2 and 4. This time x_2 and x_4 cannot be in the same connected component because every path from x_2 to x_4 must cross a path joining x_1 and x_3, all of whose vertices are not in $\Gamma'[2,4]$. We again can complete the 5-colouring by switching colours in one of the connected components. □

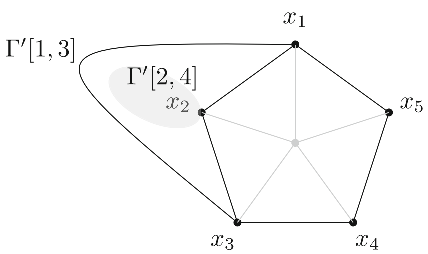


Fig. 8.2 An illustration of the proof of Theorem 8.5

The degree of a face in a planar graph is the number of edges in its boundary. The following theorem gives a characterization of planar 2-connected graph with chromatic number two.

Theorem 8.6 *A planar* 2*-connected graph* Γ *is bipartite if and only if every face of a planar embedding of* Γ *has even degree.*

Proof Suppose Γ is 2-connected and bipartite. By Proposition 7.4, the boundary of every face is a cycle of the graph (a facial cycle), which must have even degree since Γ is bipartite.

To show the reverse implication, note that the edge set of every cycle in a planar 2-connected graph is the symmetric difference of the edge sets of the facial cycles it contains (the boundaries of faces contained in the cycle in a plane embedding of the graph). If all facial cycles have even length, then the same holds for all cycles in Γ and so Γ is bipartite. □

The following theorem gives a characterization of maximal planar graphs with chromatic number three.

Theorem 8.7 (Heawood) *A maximal planar graph has chromatic number* 3 *if and only if every vertex has even degree (the graph is Eulerian).*

Proof If Γ is not Eulerian then a vertex z of odd degree and its neighbours induce an odd wheel in Γ. Three colours are needed to colour the odd cycle of this wheel and z requires a fourth colour. This shows that a maximal planar graph with chromatic number three must be Eulerian.

For the reverse implication we show the stronger statement that a 2-connected near triangulation Γ (all faces but the external one are triangles) in which all internal vertices have even degree has chromatic number three. This we prove by induction on the number f of internal faces. When $f = 1$ then $\Gamma = K_3$. Suppose $f > 1$ and let $e = xy$ be an edge in the external face of Γ. The edge e is in a unique triangle of Γ, let z be the third vertex of this triangle in addition to x and y.

If z is also a vertex on the external face, then one of x and y has degree two, say x (Fig. 8.3, left). Then $\Gamma - x$ is still 2-connected, has one less internal face, and every internal vertex has even degree. By induction, $\Gamma - x$ is 3-colourable. By giving x a colour different from y and z we obtain a 3-colouring of Γ.

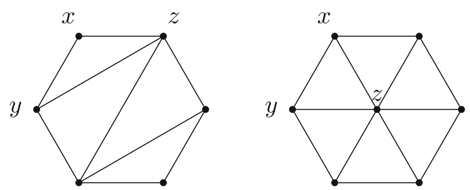


Fig. 8.3 The two cases in the proof of Theorem 8.7

If z is an internal vertex then it has even degree. Consider the even wheel induced by z and its neighbours (Fig. 8.7, right). Now $\Gamma - e$ is still 2-connected, has one less internal face and all internal vertices still have even degree. By induction $\Gamma - e$ is 3-colourable. If z receives colour 1 with a 3-colouring then the rim of the wheel receives colours 2 and 3. Since the vertices x and y are connected by a path of odd length, they receive distinct colours under this 3-colouring of $\Gamma - e$, which is therefore also a 3-colouring of Γ. □

8.3 Edge Colouring

An **edge-colouring** is a map

$$\chi' : E(\Gamma) \to k.$$

An edge-colouring is **proper** if incident edges receive different colours. The minimum number of colours in a proper edge-colouring of Γ is its **edge-chromatic number**, denoted by $\chi'(\Gamma)$. We have

$$\chi'(\Gamma) = \chi(L(\Gamma)),$$

where $L(\Gamma)$ denotes the line graph of Γ. Since all edges incident to a vertex must receive distinct colours under a proper edge-colouring, we clearly have

$$\chi'(\Gamma) \geqslant \Delta(\Gamma).$$

Perhaps surprisingly, this lower bound is never far from the true value of $\chi'(\Gamma)$.

Theorem 8.8 (Vizing) *For every graph* Γ*,*

$$\chi'(\Gamma) \leqslant \Delta(\Gamma) + 1.$$

Proof For every fixed Δ we will prove the bound by induction on m, the number of edges of graphs with maximum degree at most Δ.

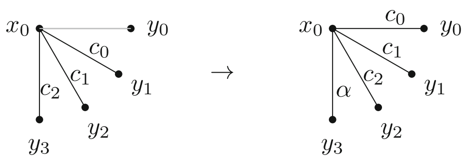
If $m = \Delta$ then Γ is a star (and isolated points) which is clearly edge-colourable with Δ colours.

Let $m \geqslant \Delta + 1$, choose a vertex x_0 with degree Δ and remove an edge $x_0 y_0$ from Γ. Let χ_0 be a proper $(\Delta + 1)$-edge colouring of $\Gamma - x_0 y_0$ (which has maximum degree at most Δ). This is a proper edge-colouring of Γ except that one edge, $x_0 y_0$, is still uncoloured.

For every vertex $x \in V(\Gamma)$ denote by $\beta(x)$ the set of colours not used in the edges incident to x. We have $\beta(x) \neq \emptyset$ since we are using $\Delta + 1$ colours.

If $\beta(x_0) \cap \beta(y_0) \neq \emptyset$ then we can use a colour in the intersection to complete the colouring of the edge $x_0 y_0$.

Fig. 8.4 Case 1 of the proof



Suppose that $\beta(x_0) \cap \beta(y_0) = \emptyset$. Choose a colour $c_0 \in \beta(y_0)$ and let x_0y_1 be an edge incident to x_0 coloured with c_0. If $\beta(x_0) \cap \beta(y_1) \neq \emptyset$ we can use a colour in the intersection to recolour x_0y_1 and use c_0 to colour x_0y_0 (which after the recolouring will be available for x_0 and for y_0). Otherwise we construct a maximal sequence $y_0, y_1, \ldots, y_k$ satisfying (i) $\beta(x_0) \cap \beta(y_i) = \emptyset$ and (ii) $c_i \in \beta(y_i)$ is different from $c_1, \ldots, c_{i-1}$ and x_0y_{i+1} has colour c_i.

By the maximality of the length of the chain one of the two cases occur:

Case 1 The chain stopped because we reached a vertex y_k with $\beta(x_0) \cap \beta(y_k) \neq \emptyset$. In this case we use a colour α in the intersection to recolour x_0y_k and use colour c_i for x_0y_i in the vertices $y_0, \ldots, y_{k-1}$, reaching a good edge-colouring for Γ (we push the colours back) (Fig. 8.4).

Case 2 The chain stopped because $\beta(x_0) \cap \beta(y_k) = \emptyset$ but the colours in $\beta(y_k)$ have already appeared in the chain, say $c_{j-1} \in \beta(y_k)$ for some $j < k$. In this case we recolour the edges x_0y_i with c_i for $i = 0, \ldots, j-1$ and leave the edge x_0y_j uncoloured.

Choose a colour $\alpha \in \beta(x_0)$ and consider the subgraph $\Gamma[\alpha, c_{j-1}]$ of Γ induced by the edges coloured α and c_{j-1} after the last recolouring. The graph $\Gamma[\alpha, c_{j-1}]$ has maximum degree two so that the connected components are cycles and paths or isolated vertices. Moreover, the vertices x_0, y_j, y_k have degree one in this subgraph (because $c_{j-1} \notin \beta(x_0)$ implies $d(x_0) = 1$ and $\alpha \notin \beta(y_j) \cup \beta(y_k)$ implies $d(y_j) = d(y_k) = 1$). Therefore, the three vertices can not belong to the same connected component of $\Gamma[\alpha, c_j]$.

Suppose that y_j and x_0 belong to different components. We can exchange the colours of the edges c_{j-1} and α in the connected component containing y_j and the resulting colouring will still be proper. Moreover, after the renaming, α becomes unused at y_j and we can use α to colour the edge x_0y_j completing the colouring of Γ.

Suppose that y_k and x_0 belong to different components. We can exchange the colours of the edges c_{j-1} and α in the connected component containing y_k and the resulting colouring will still be proper. Now we recolour the edges x_0y_i with c_i for $i = j, j+1, \ldots, k-1$ leaving x_0y_k uncoloured, and colour this edge with α. □

By Vizing's theorem, $\chi'(\Gamma) \in \{\Delta(\Gamma), \Delta(\Gamma) + 1\}$. We observe that each colour class is a matching, so Vizing's theorem can be rephrased by saying that every graph admits a partition of its edge set into at most $\Delta(\Gamma) + 1$ edge-disjoint matchings. For instance, for the complete graphs K_{2n} of even order it can easily be seen that $\chi'(\Gamma) = \Delta(\Gamma)$, while the ones of odd order can not be coloured with Δ colours because the largest matching in K_{2n+1} has n edges and the total number of edges is $n(2n+1)$. Thus, for K_{2n+1}, $\chi'(\Gamma) = \Delta(\Gamma) + 1$. The fact that for bipartite graphs, $\chi'(\Gamma) = \Delta(\Gamma)$, is a consequence of Hall's theorem.

Proposition 8.9 *A bipartite graph Γ has edge-chromatic number $\chi'(\Gamma) = \Delta(\Gamma)$.*

Proof If Γ is Δ-regular then, by Theorem 4.3, it contains a perfect matching M_1. Its removal leaves a $(\Delta - 1)$-regular bipartite graph which contains a perfect matching M_2. By iterating this procedure, we decompose the edge set of Γ into Δ edge-disjoint matchings.

If $\Gamma = (A \cup B, E)$ is not Δ-regular we show that there is $\Gamma' \supset \Gamma$ which is bipartite and Δ-regular and $\chi'(\Gamma) \leqslant \chi'(\Gamma') = \Delta$. By adding isolated vertices if needed, we may assume that $|A| = |B|$. If there is a vertex $x \in A$ with degree smaller than Δ then there must be $y \in B$ with the same property and we can add the edge xy to Γ and still get a bipartite graph with maximum degree Δ. By repeating the argument we eventually end up with a bipartite Δ-regular graph $\Gamma' \supset \Gamma$. By the previous paragraph, the edges of Γ' decompose into Δ edge disjoint matchings. □

A final observation on the four colour theorem is the following equivalence.

Theorem 8.10 *The four colour theorem is equivalent to the following statement: every bridgeless cubic planar graph has edge-chromatic number $\chi'(\Gamma) = 3$.*

Proof Every planar graph can be coloured with four colours if and only if every maximal planar graph can be coloured with four colours, so we may restrict ourselves to maximal planar graphs. The dual of a maximal planar graph is a bridgeless cubic graph. Reciprocally, the dual of a bridgeless cubic planar graph is a triangulation, a maximal planar graph.

Let Γ be an embedded cubic planar graph and suppose that there is a 4-colouring χ of its dual Γ^* with elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each $e \in E(\Gamma)$ determines an edge $e^* = xy \in E(\Gamma^*)$, joining the two faces which have the edge e on their boundaries, see Fig. 8.5. We colour e with $\chi'(e) = \chi(x) + \chi(y)$, which is not $(0, 0)$ since x and y (faces of Γ) are coloured with different colours. If e and e' are incident in Γ then the corresponding edges $e^* = xy$, $(e')^* = yz$ are also incident in Γ^* and xyz form a triangle in Γ^*, since Γ^* is a triangulation. Since $\chi(x) \neq \chi(z)$, it follows that

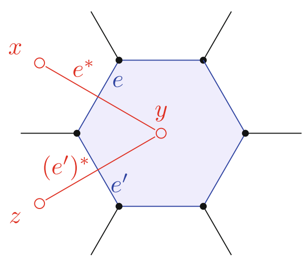


Fig. 8.5 The edge e determines de edge e^* in Theorem 8.10

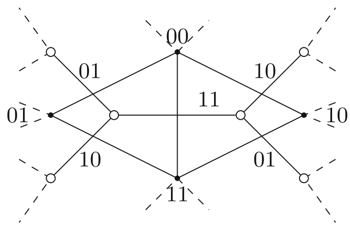


Fig. 8.6 The 4-colouring of a maximal planar graph Γ^* (with black vertices) induced by a 3-edge-colouring of its dual Γ (with white vertices)

$\chi'(e) = \chi(x) + \chi(y) \neq \chi(y) + \chi(z) = \chi'(e')$. Thus we obtain a 3-colouring of Γ with elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$.

Reciprocally, let Γ be a maximal planar graph and suppose that there is a 3-edge-colouring χ^* of its dual Γ^*. Identify the edge colours of Γ^* with the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$ (Fig. 8.6).

Since Γ is a triangulation, every edge e in Γ uniquely defines a dual edge e^* in Γ^* joining the two faces of Γ that have e in common in their boundaries.

Consider a spanning tree of Γ, choose a vertex as a root and colour it with $(0, 0)$. For every pair v, v' of adjacent vertices in the spanning tree define the map $\chi(v) = \chi(v') + \chi^*(e^*)$ where e^* is the dual edge of $e = vv'$ and the sum is in $\mathbb{Z}_2 \times \mathbb{Z}_2$. This is a well defined 4-colouring of Γ. We claim that, for every pair v, v' of adjacent vertices in Γ joined by the edge $e_{vv'}$, we have $\chi(v) = \chi(v') + \chi^*(e^*_{vv'})$, so the colouring is proper.

□

This is the case if $e_{vv'}$ is an edge of the spanning tree, by definition. Otherwise, consider the cycle induced in the tree by this edge e.

Suppose the cycle is facial, namely a triangle $vv`w$. Then e_{vw} and $e_{v'w}$ are edges of the spanning tree and we have that

$$\chi(v) = \chi(w) + \chi^*(e^*_{vw})$$

and

$$\chi(v') = \chi(w) + \chi^*(e^*_{v'w}).$$

Therefore, we have that

$$\chi(v) = \chi(v') + \chi^*(e^*_{vw}) + \chi^*(e^*_{v'w}) = \chi(v') + \chi^*(e^*_{vv'}),$$

since the sum of any elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0, 0)\}$ gives the third one.

Finally, if the cycle is not facial then it is the symmetric difference of r facial cycles for some r. Each one of the edges in these faces when added to the spanning tree induces a cycle which is the symmetric difference of less than r facial cycles. By induction on r, we can assume that for these edges $e_{w'w}$

$$\chi(w) = \chi(w') + \chi^*(e^*_{ww'}).$$

Then, since we are summing modulo two, when we take the symmetric difference of r facial cycles, we also conclude that

$$\chi(v) = \chi(v') + \chi^*(e^*_{vv'}).$$

8.4 List Colouring

Let Γ be a graph and let $L(v)$ be a list of colours associated to each vertex $v \in V(\Gamma)$. A **list colouring** of Γ is a proper colouring χ such that

$$\chi(v) \in L(v), \ \forall v \in V(\Gamma).$$

A graph Γ is k**-choosable** if, for every set of lists $\{L(v) : v \in V(\Gamma)\}$ with $|L(v)| \geqslant k$, there is a list colouring with this set of lists. The minimum integer k such that Γ is k-choosable is the **list chromatic number** $\chi_L(\Gamma)$ of Γ. An ordinary k-colouring can be seen as a list colouring where the list of each vertex is $\{1, 2, \ldots, k\}$. Therefore,

$$\chi(\Gamma) \leqslant \chi_L(\Gamma).$$

The difference between the two quantities can be arbitrarily large. For example, the list chromatic number of the complete bipartite graph satisfies $\liminf_{n\to\infty} \chi_L(K_{n,n}) = \infty$, even if the graph is bipartite (see Exercise 8.17). The list assignment to $K_{3,3}$ in Fig. 8.7 shows that $\chi_L(K_{3,3}) > 2$.

The arguments using the greedy colouring algorithm, where only the number of colours available at one vertex is significant, show that the statement of the theorems of Szekeres–Wilf and of Brooks still hold for list colourings. The next celebrated theorem by Thomassen shows that list colouring of planar graphs is at most five.

Theorem 8.11 (Thomassen) *For every planar graph* Γ *we have* $\chi_L(\Gamma) \leqslant 5$.

Proof We need to prove that given lists $L(v)$ of size at least 5 for each vertex v, we can choose a colour from $L(v)$ so that the colouring is proper. If we can find a proper list colouring for a graph which contains Γ then we will have found a list colouring for Γ. Thus, we can assume that Γ is a near-triangulation (all faces except the outer one are triangles). □

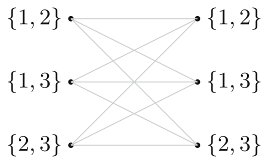


Fig. 8.7 There is no list colouring of $K_{3,3}$ with the displayed list assignment

Claim 8.12 For every list assignment of a near triangulation in which two prescribed adjacent vertices in the outer face have distinct lists of size one, the remaining vertices in the outer face have lists of size three and all inner vertices have lists of size five, there is a list colouring of the graph with these lists.

Proof We will prove this by induction on the number n of vertices. The claim follows for $n = 3$. Suppose $n > 3$.

Let x, y be the vertices with lists of size one. We consider two cases.

Case 1. There is a chord $e = uv$ joining two vertices in the outer face (Fig. 8.8 left).
We consider the two near triangulations Γ_1, Γ_2 which are split by the chord and share this chord in their outer boundary. We may assume that Γ_1 is the near triangulation which contains both of them. We apply induction on Γ_1 and find a list colouring χ_1 of Γ_1. We now apply induction on Γ_2 by redefining the lists of u and v as $L'(u) = \{\chi_1(u)\}$ and $L'(v) = \{\chi_1(v)\}$ to find a list colouring χ_2 with these new lists. The colouring whose restriction to Γ_i is χ_i is a list colouring of Γ.

Case 2. There is no chord joining two vertices in the outer face (Fig. 8.8 right).
Let x, y, u, v be consecutive vertices in the clockwise order in the outer face (it may be that $v = x$). Let $y = u_1, u_2, \ldots, u_k = v$ be the neighbours of u also in clockwise order. Consider the new lists $L'(u) = L(u) \setminus L(y)$ and $L'(u_i) = L(u_i) \setminus L'(u)$, $1 \leqslant i \leqslant k - 1$, which provide the induction hypothesis for the graph $\Gamma - u$. A list colouring χ of $\Gamma - u$ can be completed to a list colouring of Γ with the original lists since none of the u_i's uses the two colours of $L'(u)$ and one can be chosen different from $\chi(v)$.

The statement of the theorem now follows from the Claim 8.12, since an assignment of lists of length five to every vertex fulfils the hypothesis of the claim. □

There are examples of planar graphs whose list chromatic number is five, so the bound in Theorem 8.11 is tight.

Analogous notions for edge-colourings lead to k-edge choosability and edge-list chromatic number $\chi'_L(\Gamma)$. Clearly, $\chi'_L(\Gamma) = \chi_L(L(\Gamma))$. A famous open problem in the area is the list colouring conjecture.

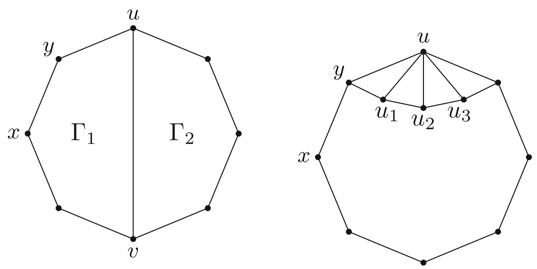


Fig. 8.8 The two cases in the proof of Theorem 8.11

Conjecture 8.13 (List Colouring Conjecture) For every graph Γ we have $\chi'_L(\Gamma) = \chi'(\Gamma)$.

Remarkably, the conjecture has been proved for bipartite graphs.

Theorem 8.14 (Galvin) *If Γ is bipartite then $\chi'_L(\Gamma) = \chi'(\Gamma)$.*

One proof uses the following result. A **kernel** in an oriented graph $\vec{\Gamma} = (V, \vec{E})$ is a nonempty independent set U such that every vertex in $V \setminus U$ has an arc directed to some vertex in U.

Lemma 8.15 *Let $\{L(v) : v \in V\}$ be a set of lists assigned to vertices of a graph $\Gamma = (V, E)$. If there is an orientation $\vec{\Gamma}$ such that every induced subgraph $\vec{\Gamma}'$ of $\vec{\Gamma}$ has a kernel and the out-degree of every vertex satisfies $d^+(v) < |L(v)|$, then Γ can be coloured from the lists.*

Proof The proof is by induction on n. The result is trivial for $n = 1$. Assume $n > 1$. Choose a colour c which occurs in some list and let C be the set of vertices in Γ which have the colour c in their lists. By the hypothesis, $\vec{\Gamma}[C]$ has a kernel U. Colour the vertices of U by c and remove the colour c from all lists in C. The subgraph $\Gamma[V \setminus U]$ satisfies the hypothesis of the lemma, since the out degree of every vertex in $\Gamma[V \setminus U]$ is one less than its out degree in $\Gamma[V]$. whereas $|L(v)|$ has decreased by at most one since we have only removed the colour c from the lists. Hence, by induction, $\Gamma[V \setminus U]$ admits a list colouring with the new lists and this provides a list colouring of Γ with the original lists. □

Proof of Theorem 8.14 Let χ be an edge-colouring of Γ with $\{1, 2, \ldots, k\}$, where $k = \chi'(\Gamma)$. We will prove that the colouring can be used to construct an orientation of the line graph $L(\Gamma)$ of Γ satisfying the hypothesis of Lemma 8.15. The statement then follows from Lemma 8.15.

Let $V(\Gamma) = A \cup B$ be the bipartition of Γ. Let e and e' be two incident edges in Γ with $\chi(e) < \chi(e')$ (since χ is a proper edge-colouring, equality does not hold). We orient the edge $ee' \in E(L(\Gamma))$ from e to e' if $e \cap e' \in A$ and from e' to e if $e \cap e' \in B$ (Fig. 8.9).

The out-degree of an edge e in this orientation is at most $(k-\chi(e))+(\chi(e)-1) < k$. So we only have to show that every subgraph of $L(\Gamma)$ with this orientation has a

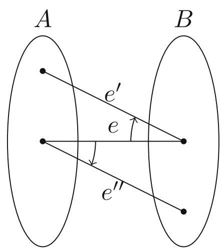


Fig. 8.9 The orientation of $L(\Gamma)$ in the proof of Theorem 8.14 when $\chi(e'') < \chi(e) < \chi(e')$

kernel. We prove that this is the case by induction on $|E'|$, the case $|E'| = 1$ being trivial. Suppose $|E'| > 1$.

Let $E' \subset E(\Gamma)$ be a subset of edges and consider the oriented subgraph Γ' of $L(\Gamma)$ induced by E'. Let $A' \subset A$ be the set of vertices incident with some edge in E'. For each $x \in A'$, let $e_x \in E'$ be the edge incident with x with smallest colour. Let $U = \{e_x : x \in A'\}$. By construction, every edge in $E' \setminus U$ has a directed edge in Γ' to some element in U.

If U is an independent set then we are done.

If not, suppose that two edges e_x and $e_{x'}$ in U meet in some vertex y. Necessarily, by the construction of U, $y \in B$. Suppose $\chi(e_x) < \chi(e_{x'})$, so e_x is directed to $e_{x'}$ in Γ'. By induction, the subgraph $\Gamma' \setminus e_x$ has a kernel U'. If $e_{x'} \in U'$ then U' is a kernel for Γ' and we are done. Otherwise there is an edge e'' incident with $e_{x'}$ such that $e_{x'}$ is directed to e'' in Γ'. Since $e_{x'}$ is the edge incident with x' with minimum colour it must be that e'' is incident with $e_{x'}$ in B, which implies it is incident with y. Thus, $\chi(e'') > \chi(e_{x'})$ which implies $\chi(e_{x''}) > \chi(e_x)$ and so e_x is directed to e'' in Γ'. Hence U' is a kernel for Γ'. □

By Proposition 8.9 and Theorem 8.14, the edge list chromatic number of a bipartite graph Γ is $\chi'_L(\Gamma) = \Delta(\Gamma)$. In particular, $\chi'_L(K_{n,n}) = n$, a statement which had been conjectured by Dinitz in the language of Latin squares. Suppose that each entry of a square $n \times n$ matrix may take one of n distinct values. Dinitz conjectured that one can choose entries such that elements in a row and in a column are pairwise distinct. Observe that when the choices are $\{1, 2, \ldots, n\}$ for each entry we get a Latin square.

8.5 Notes and References

The proof of Brooks' theorem, Theorem 8.4, follows Lovász (1975). A simplified version which avoids the use of 3-connectivity and has further applications is given in Zając (2018). The characterization of 3-chromatic triangulations by Heawood can be complemented by a theorem by Grötzsch that states that every triangle-free planar graph can be 3-coloured, see Thomassen (2003) for a simplified proof. The proof by Thomassen (2004), Theorem 8.11, of the 5-choosability of planar graphs has become a classic in graph theory with wide applications. The proof by Galvin (1995) of the list colouring conjecture also extends to bipartite multigraphs.

8.6 Exercises

Exercise 8.1

i. Show that every graph admits an ordering of the vertices for which the greedy colouring algorithm uses $\chi(\Gamma)$ colours.

ii. Let Γ be the complete bipartite graph $K_{n,n}$ minus a perfect matching. Show that there is an ordering of the vertices such that the greedy colouring algorithm uses n colours.

Exercise 8.2 Let $\Gamma + \Gamma'$ denote the graph resulting from the disjoint union of Γ and Γ' and adding all edges between $V(\Gamma)$ and $V(\Gamma')$. Prove that $\chi(\Gamma + \Gamma') = \chi(\Gamma) + \chi(\Gamma')$.

Exercise 8.3 The cartesian product $\Gamma\Box\Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and (x, y) i (x', y') are adjacent if and only if either $x = x'$ and $y \sim y'$ or $y = y'$ and $x \sim x'$. Show that $\chi(\Gamma\Box\Gamma') = \max\{\chi(\Gamma), \chi(\Gamma')\}$.

Exercise 8.4 The direct product $\Gamma \times \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and (x, y) i (x', y') are adjacent if $x \sim x'$ and $y \sim y'$. Show that $\chi(\Gamma \times \Gamma') \le \min\{\chi(\Gamma), \chi(\Gamma')\}$.

[Hedetniemi conjecture, recently disproved, stated that equality holds.]

Exercise 8.5 A graph Γ is k-**critical** if it has chromatic number k and every proper subgraph of Γ has smaller chromatic number. Show that a k-critical graph is $(k-1)$-edge connected (the graph remains connected after deletion of any set of $k - 1$ edges).

Exercise 8.6 Let Γ be a k-critical graph.

i. Show that, for every pair x, y of non adjacent vertices, there is a k-colouring χ of Γ such that $\chi(x) = \chi(y)$.
ii. Show that Γ must be 2-connected. Moreover, if S, with $|S| = 2$ separates $X \subset V$ from $Y = V \setminus (X \cup S)$, then the induced subgraphs $\Gamma_1 = \Gamma[X \cup S]$ and $\Gamma_2 = \Gamma[Y \cup S]$ have the property that any $(k-1)$-coloring of Γ_1 gives distinct colours to S while any k-colouring of Γ_2 gives the same colour to the vertices in S. Give an example of a 2-connected critical graph with $k = 4$.

Exercise 8.7 (Mycielski construction) Given a graph Γ with vertex set $V = \{v_1, \ldots, v_n\}$, denote by $M(\Gamma)$ the graph with vertex set $V \cup \{u_1, \ldots, u_n\} \cup \{w\}$ where

i. $\{u_1, \ldots, u_n\}$ is an independent set;
ii. For each i, u_i is adjacent to every vertex adjacent to v_i.
iii. w is adjacent to each u_i.

Show that, if Γ is triangle free and $\chi(\Gamma) = k$, then $M(\Gamma)$ is also triangle-free and $\chi(M(\Gamma)) = k + 1$.

Exercise 8.8 Show that $\chi(\Gamma) = k$ if and only if there is an orientation $\vec{\Gamma}$ of Γ whose longer directed path has length k.

Exercise 8.9 For a graph $\Gamma = (V, E)$ and a vertex $x_0 \in V$ let $S_i = \{x \in V(\Gamma) : d(x_0, x) = i\}$ denote the sphere of radius i centered at x_0.

1. Show that

$$\chi(\Gamma) \leqslant \max_i \{\chi(\Gamma[S_i]) + \chi(\Gamma[S_{i+1}])\},$$

   where the maximum is taken from $i = 0$ to the eccentricity of x_0 minus one. Give examples showing that the inequality is tight.
2. Show that a graph Γ with chromatic number $\chi(\Gamma) \geqslant 2^t$ has the complete graph K_t as a minor.

Exercise 8.10 Let $f_\Gamma(x)$ be a function such that, fo each positive integer k, $f_\Gamma(k)$ is the number of proper k-colourings of Γ.

i. Show that

$$f_\Gamma(k) = f_{\Gamma - e}(k) - f_{\Gamma/e}(k).$$

ii. Deduce that f_Γ is a polynomial and that $f_\Gamma(x) = x^n - mx^{n-1} +$ terms of lower degree, where $n = |V(\Gamma)|$ and $m = |E(\Gamma)|$.
iii. Compute the polynomial for $\Gamma = K_n$ and for Γ a tree.

Exercise 8.11 Show that a graph Γ with m edges satisfies

$$\chi(\Gamma) \leqslant \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

Exercise 8.12 Show that an outerplanar graph Γ has chromatic number $\chi(\Gamma) \leqslant 3$.

Exercise 8.13 Show that a regular graph Γ with an odd number of vertices satisfies $\chi'(\Gamma) = \Delta(\Gamma) + 1$.

Exercise 8.14 Show that if Γ is a cubic graph with a bridge then $\chi'(\Gamma) = 4$.

Exercise 8.15 Prove that $\chi'(K_{2n}) = 2n - 1$ and $\chi'(K_{2n+1}) = 2n + 1$. Describe the edge-colourings reaching these values.

Exercise 8.16 Let Γ be the graph obtained from the complete bipartite graph $K_{n,n}$ by subdividing one edge by a vertex. Show that $\chi'(\Gamma) = \Delta(\Gamma) + 1$, but $\chi'(\Gamma - e) = \Delta(\Gamma)$ for every edge e.

Exercise 8.17 Show that $\liminf_{n\to\infty} \chi_L(K_{n,n}) = \infty$.

Exercise 8.18 Show that $\chi'_L(\Gamma) \leqslant 2\Delta(\Gamma) - 1$.

Exercise 8.19 Let L be a list assignment to the vertices of a graph Γ such that $d(v) \leqslant |L(v)|$ and the strict inequality holds for at least one vertex. Show that Γ admits a list colouring from these lists.

Exercise 8.20 A graph is k-**degenerated** if every subgraph has a vertex of degree at most k.

i. Prove that the list colouring number of a k-degenerated graph G satisfies $\chi_L(G) \leqslant k + 1$.
ii. Show that a k-degenerated graph G of order $n > k$ with maximal number of edges has $kn - \binom{k+1}{2}$ edges, connectivity $\kappa(G) = k$ and $\chi(G) = k + 1$.
iii. Prove that a non-bipartite outerplanar graph has $\chi_L(G) = 3$

Exercise 8.21 Show that any assignment of lists of length 2 to the vertices of an odd cycle admits a proper list coloring, except when all the lists are the same.

Extremal Graph Theory

9

Extremal graph theory is the study of graphs which have a critical behaviour with respect to some graph parameter within a certain class characterized by some graph property. The typical example that we will consider in this chapter consists in finding the maximum number of edges that a graph can have within the class of graphs which do not contain a fixed subgraph H. The main result in this area is the Erdős–Stone theorem. This theorem provides an asymptotic expression for the maximum number of edges a graph can have which has no subgraph H. This expression depends only on the chromatic number of the graph H. The theorem is not informative when H is bipartite, and the last part of the chapter is devoted to study this case in which finite geometries will reappear.

9.1 Graphs Without Triangles

Let H be a fixed graph with less than n vertices. Consider the set of graphs which contain no copy of H as a subgraph. Let Γ be a graph in this set with n vertices. The maximum number of edges that Γ can have is a function of n, which we denote by $\mathrm{ex}(n, H)$. Although we may not be able to explicitly give a formula for $\mathrm{ex}(n, H)$, we would at least like to know the asymptotic behaviour of the function.

Example 9.1 In Fig. 9.1, the graph with six vertices and nine edges contains no triangle. As we shall see, this is the maximum number of edges a graph with six vertices can have, which has the property that it contains no triangle, i.e. $\mathrm{ex}(6, K_3) = 9$.

In general, to avoid a triangle we can take Γ to be the graph $K_{\frac{1}{2}n,\frac{1}{2}n}$ if n is even and $K_{\frac{1}{2}(n+1),\frac{1}{2}(n-1)}$ if n is odd. These graphs have $\frac{1}{4}n^2$ edges and $\frac{1}{4}(n^2-1)$ edges respectively, and contain no triangle.

The following theorem proves that this is best possible.

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_9

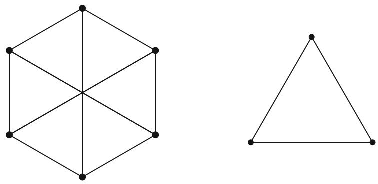


Fig. 9.1 A graph with six vertices containing no triangle

Theorem 9.2 (Mantel) *A graph Γ with n vertices which contains no triangle has at most $\frac{1}{4}n^2$ edges.*

Proof (Inductive Proof of Theorem 9.2) A graph with two vertices has no triangle and a graph with three vertices and two edges contains no triangle.

Let xy be an edge of Γ. Since Γ contains no triangles, the neighbours of x are disjoint from the neighbours of y. Hence,

$$d(x) - 1 + d(y) - 1 \leqslant n - 2, \tag{9.1}$$

where $d(u)$ denotes the degree of the vertex u.

By induction, the graph $H = \Gamma \setminus \{x, y\}$, has at most $\frac{1}{4}(n-2)^2$ edges, since H is a graph with $n - 2$ vertices which contains no triangle. Hence, Γ has at most

$$d(x) - 1 + d(y) - 1 + 1 + \tfrac{1}{4}(n-2)^2$$

edges. □

Proof (Largest Independent Set Proof of Theorem 9.2) Let A be the largest independent set in Γ. Then $d(x) \leqslant |A|$, since the neighbourhood of any vertex is an independent set.

Let B be the set of vertices of $\Gamma \setminus A$. Every edge of Γ must have an end-vertex in B, so

$$m \leqslant \sum_{x \in B} d(x),$$

where m again denotes the number of edges in the graph Γ.

Since $d(x) \leqslant |A|$,

$$\sum_{x \in B} d(x) \leqslant |A||B| = ab \leqslant \left(\frac{a+b}{2}\right)^2 = \tfrac{1}{4}n^2,$$

where $|A| = a$ and $|B| = b$. □

This last proof also classifies the extremal case.

Suppose n is even. Then $a+b=n$ and $ab=\frac{1}{4}n^2$ imply $a=b=\frac{1}{2}n$. Moreover, equality implies that $d(x)=\frac{1}{2}n$ for all $x\in B$. Counting pairs (x,e), where x is a vertex incident with the edge e, we get $\frac{1}{4}n^2$ contribution from the vertices in B, which implies we get a $\frac{1}{4}n^2$ contribution from the vertices in A. Since $a=\frac{1}{2}n$ and $d(x)\leqslant\frac{1}{2}n$ for $x\in A$, we get $d(x)=\frac{1}{2}n$ for $x\in A$ too. Hence Γ is the graph $K_{\frac{1}{2}n,\frac{1}{2}n}$.

Suppose n is odd. Then $a+b=n$ and $ab=\frac{1}{4}(n^2-1)$ implies $\{a,b\}=\{\frac{1}{2}(n-1),\frac{1}{2}(n+1)\}$ and so $m\leqslant\frac{1}{4}(n^2-1)$. Moreover, equality implies that $d(x)=a$ for all $x\in B$. Counting pairs (x,e), where x is a vertex incident with the edge e, we get ab contribution from the vertices in B, which implies we get a $ab=\frac{1}{4}(n^2-1)$ contribution from the vertices in A. Since $d(x)\leqslant b$ for $x\in A$, we get $d(x)=b$ for $x\in A$. But, then these ab edges with end-vertices in A are all the edges, so Γ is the graph $K_{a,b}$.

9.2 Graphs Without Complete Subgraphs

Turán's theorem concerns the natural generalisation of Mantel's theorem and considers graphs which contain no copy of K_{r+1}. The Turán graph is the complete multi-partite graph where the vertices are partitioned into r parts of roughly equal size (the most equal that they can be). The Turań graph has roughly $\frac{1}{2}n(n-n/r)=\frac{1}{2}n^2(1-1/r)$ edges. By the pigeon-hole principle, it contains no copy of K_{r+1}.

Theorem 9.3 (Turán) *A graph with n vertices containing no copy of K_{r+1} has at most $\frac{1}{2}n^2(1-1/r)$ edges.*

Proof (Induction Proof of Theorem 9.3) Assume that Γ has the maximum number of edges containing no copy of K_{r+1}. Since adding an edge to Γ makes a copy of K_{r+1}, the graph Γ contains a copy of K_r.

For $n\in\{r,\dots,2r-1\}$, the $n-r$ vertices not in K_r are incident with at most $r-1$ edges incident with a vertex of K_r. Thus, the number of edges is at most

$$\binom{n}{2}-r(n-r)\leqslant\tfrac{1}{2}n(n-1)-r(n-r)-\frac{1}{2r}(n-r)(n-2r^2)=\tfrac{1}{2}n^2(1-1/r).$$

By induction on n. The graph $\Gamma\setminus K_r$ has at most

$$\tfrac{1}{2}(1-1/r)(n-r)^2$$

edges.

A vertex in $\Gamma\setminus K_r$ has at most $r-1$ neighbours in K_r, since Γ contains no K_{r+1}. Therefore, Γ has at most

$$\tfrac{1}{2}(1-1/r)(n-r)^2+(r-1)(n-r)+\tfrac{1}{2}r(r-1)=\tfrac{1}{2}(1-1/r)n^2.$$

edges. □

The following proof of Theorem 9.3, also classifies the graphs with n vertices which have the maximum number of edges as Turán graphs.

Proof (Non-Adjacency Proof of Theorem 9.3) Suppose that Γ is a graph with n vertices that contains no copy of K_{r+1} and has the maximum number of edges such a graph can have.

We want to prove that non-adjacency is an equivalence relation. Suppose that y is not joined by an edge to neither x nor z. We have to show that x is not adjacent to z. Suppose to the contrary that xz is an edge.

If $d(y) < d(x)$ then let Γ' be the graph where we delete y and add a copy x' of the vertex x (adjacent to the same neighbours as x). Then Γ' has n vertices, more edges than Γ and contains no copy of K_{r+1}, since Γ does not. This contradicts the maximality of the number of edges in Γ.

Therefore, $d(y) \geqslant d(x)$ and similarly $d(y) \geqslant d(z)$. Now, let Γ' be the graph obtained from Γ by deleting the vertices x and z and adding copies y' and y'' of the vertex y. Then Γ has n vertices and

$$m-(d(x)+d(z)-1)+2d(y)\geqslant m+1,$$

edges, where m denotes the number of edges in Γ. Again, Γ contains no copy of K_{r+1} since Γ does not, which contradicts the maximality of the number of edges in Γ.

Therefore, non-adjacency is an equivalence relation and Γ is a complete multipartite graph with at most r parts. Let n_i denote the number of vertices in the i-th part of Γ. Then $n = n_1 + \cdots + n_r$. The number of edges in Γ is

$$\tfrac{1}{2}\sum_{i=1}^{r} n_i(n-n_i)=\tfrac{1}{2}n^2-\sum_{i=1}^{r} n_i^2.$$

The sum

$$\sum_{i=1}^{r} n_i^2$$

is minimised when n_i are chosen to be as close to n/r as possible.

Hence, the number of edges is maximised when the graph Γ is the complete r-partite graph in which each maximum independent subset of vertices has size as close to n/r as possible. □

9.3 Erdős–Stone Theorem

In the previous section, we managed to determine which graphs on n vertices, which contain no copy of K_{r+1}, have the largest number of edges. In this section, we consider graphs containing no copy of an arbitrary graph H.

Recall that $\chi(H)$ denotes the chromatic number of the graph H, the minimum number of colours required to colour the vertices of H in such a way that no two adjacent vertices receive the same colour.

The Turán graph Γ, the complete r-partite graph on n vertices, whose stable sets are of size roughly n/r, provides an example of a graph which contains no copy of any graph H with chromatic number $\chi(H) = r + 1$, since Γ is r-colourable.

Note that, $\chi(K_{r+1}) = r + 1$, so the following theorem is a generalisation of Turán's theorem, Theorem 9.3. The lower bound follows from the existence of the Turán graph.

Theorem 9.4 (Erdős-Stone) *For all $\epsilon > 0$, there is an n_0, such that for all $n > n_0$,*

$$\left(1 - \frac{1}{\chi(H) - 1} - \epsilon\right)\tfrac{1}{2}n^2 \leqslant \mathrm{ex}(n, H) \leqslant \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right)\tfrac{1}{2}n^2.$$

Proof We show that a graph Γ with more edges than the stated upper bound contains some copy of every graph with chromatic number $r + 1 = \chi(H)$. This is achieved by showing that Γ contains a complete $(r+1)$-partite graph with parts of cardinality $t = |V(H)|$. Note that since H is $(r+1)$-colourable, we can place the vertices of H coloured with the same colour in the same stable set of the complete $(r + 1)$-partite subgraph and find a copy of H as a subgraph of Γ. The proof is build upon a series of claims. The first one shows that Γ contains a large subgraph in which every vertex has large degree. □

Claim 9.5 Let $r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1/r$. There is a $\delta = \delta(r, \epsilon)$ and an $n_0 \in \mathbb{N}$ such that for all graphs Γ with $n \geqslant n_0$ vertices and at least $(1 - \frac{1}{r} + \epsilon)\frac{1}{2}n^2$ edges, there is a subgraph Γ' of Γ with at least δn vertices and minimum degree $\delta(\Gamma') \geqslant (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\delta n$.

Proof We repeatedly remove vertices of minimum degree until the resulting graph Γ' has minimum degree $\delta(\Gamma') \geqslant (1 - \frac{1}{r} + \frac{1}{2}\epsilon)n'$, where n' is the number of vertices of Γ'. The number of edges that have been removed to obtain Γ' is at most

$$\sum_{\ell=n'+1}^{n} (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\ell = (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\left(\frac{(n - n')(n + n' + 1)}{2}\right)$$
$$\leqslant (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\left(\frac{n^2 - (n')^2}{2} + \frac{n - n'}{2}\right).$$

Therefore,

$$\begin{aligned} |E(\Gamma)| &\leqslant |E(\Gamma')| + (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\frac{n^2 - (n')^2}{2} + \frac{n - n'}{2} \\ &\leqslant \frac{(n')^2}{2} + (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\frac{n^2 - (n')^2}{2} + \frac{n - n'}{2} \\ &= (1 - \frac{1}{r} + \frac{1}{2}\epsilon)\frac{n^2}{2} + (\frac{1}{r} - \frac{1}{2}\epsilon)\frac{(n')^2}{2} + \frac{n - n'}{2}. \end{aligned}$$

By comparing with the lower bound on $|E(\Gamma)|$ we obtain

$$(\frac{1}{r} - \frac{1}{2}\epsilon)\frac{(n')^2}{2} - \frac{n'}{2} > \epsilon\frac{n^2}{4} - \frac{n}{2},$$

which implies $n' \geqslant \sqrt{r\epsilon}\, n$. □

The second claim shows that in a graph in which every vertex has sufficiently large degree we can find a large blow-up of $K_{1,r}$.

Claim 9.6 Let $r \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1/r$. Let t, s such that $s > 2t/r\epsilon$. Let Γ be a graph with n vertices, where n is sufficiently large and minimum degree $\delta(\Gamma) \geqslant (1 - \frac{1}{r} + \frac{1}{2}\epsilon)n$. For every family $B_1, \ldots, B_r$ of r disjoint subsets of $V(\Gamma)$ of cardinality s there is a subset A_{r+1} of cardinality t disjoint from $B_1, \ldots, B_r$ and subsets $A_1 \subset B_1, \ldots, A_r \subset B_r$ each of cardinality t such that every vertex in A_{r+1} is adjacent to every vertex in each A_i.

Proof Let $B = \cup_{i=1}^{r} B_i$. Let W be the set of vertices in $U = V(G) \setminus B$ which have at least t neighbours in each B_i. Let $e(U, B)$ be the number of edges with one end in U and one end in B. Then,

$$\begin{aligned} |B|\delta(\Gamma) - \binom{|B|}{2} \leqslant e(U, B) &\leqslant |W||B| + (|U| - |W|)(|B| - (s - t)) \\ &= |W|(s - t) + (n - |B|)(|B| - (s - t)), \end{aligned}$$

given that a vertex in $U \setminus W$ has at most $|B| - (s - t)$ neighbours in B, since it has at least $s - t$ non-edges to one of the B_i.

Thus we have that

$$|W| \geqslant \frac{1}{s - t}\left((s - t) - |B|(\frac{1}{r} - \frac{\epsilon}{2})\right)n + |B|((|B| + 1)/2 - (s - t)) = \alpha n + \beta,$$

for some constants α, β which depend only on the parameters s, t, r. From our choice of s we have $\alpha > 0$. By choosing n sufficiently large we may assume

$$|W| > \binom{s}{t}^r (t-1).$$

Every element in W has at least t neighbours in each B_i. There are $\binom{s}{t}^r$ r-tuples $(A_1, \ldots, A_r)$, each $A_i \subset B_i$ of cardinality t. For each $w \in W$, there is such an r-tuple. By the pigeonhole principle, there are (at least) t vertices in W which are adjacent to all vertices of a common r-tuple $(A_1, \ldots, A_r)$, each $A_i \subset B_i$ of cardinality t. We can define A_{r+1} as the set of t such vertices. □

Our last claim unveils the existence of a complete $(r+1)$-partite subgraph $K_{t,\ldots,t}$ in Γ.

Claim 9.7 Let $r, t \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1/r$. Let Γ be a graph with minimum degree $\delta(\Gamma) \geqslant (1 - \frac{1}{r} + \frac{1}{2}\epsilon)n$. If $n = |V(\Gamma)|$ is sufficiently large then Γ contains a copy of the complete $(r+1)$-partite subgraph $K_{t,\ldots,t}$.

Proof The condition on the minimum degree of Γ in Claim 9.6 is satisfied for each $1 \leqslant r' \leqslant r$. For each value of $r' \in \{1, \ldots, r\}$, we successively build a complete $(r'+1)$-partite subgraph $K_{t_{r'},\ldots,t_{r'}}$ with $t_r = t$ and $t_{r'} \geqslant 2t_{r'+1}/r\epsilon$ for $1 \leqslant r' < r$. At each step the stable sets of this subgraph play the role of the B_i's from Claim 9.6 in the next one and the conclusion of Claim 9.6 provides the complete multipartite graph at this step.

The above claims provide a proof of the statement of the theorem. By choosing n large enough we can find, by Claim 9.5, a sufficiently large subgraph Γ' which satisfies the condition on the minimum degree of Claim 9.7 and therefore we find a copy of the complete $(r+1)$-partite graph $K_{t,\ldots,t}$ for every fixed t. By choosing $t = |V(H)|$ this subgraph contains H as a subgraph. □

Theorem 9.4 tells us the asymptotic behaviour of $\text{ex}(n, H)$ for all graphs H for which $\chi(H) \neq 2$. If $\chi(H) = 2$ then H is bipartite and Theorem 9.4 only tells us that we cannot take some positive proportion of all the edges in the graph Γ if we wish to avoid H; there is more work to be done in this case. In the remaining sections we will consider particular classes of bipartite graphs.

9.4 Graphs Without Complete Bipartite Graphs

Consider first the case that H is a star, $H = K_{1,t}$, for some t. If Γ contains no copy of H then the maximum degree of its vertices is $t-1$ and so it has at most $\frac{1}{2}n(t-1)$ edges. This is realizable if $\frac{1}{2}n(t-1)$ is an integer. Keep adding edges to a graph with n vertices until every vertex has degree $t-1$, except possibly one that has degree $t-2$. Then the number of edges is $\frac{1}{2}n(t-1)$ if there is no vertex of degree $t-2$. In the case there is a unique vertex of degree $t-2$ then the number of edges is $\frac{1}{2}((n-1)(t-1)+t-2) = \frac{1}{2}n(t-1) - \frac{1}{2}$.

Therefore, $\mathrm{ex}(n, K_{1,t})$ is linear in n, whereas for non-bipartite graph the Erdős-Stone–Simonovits theorem, Theorem 9.4 tells us that $\mathrm{ex}(n, H)$ is quadratic.

Let us consider the graph $K_{2,2} = C_4$. We wish to determine the asymptotic behaviour of the function $\mathrm{ex}(n, K_{2,2})$.

Example 9.8 Let (P, L) be a linear space and let Γ be the bipartite graph with vertices $P \cup L$ where $x\ell$ is an edge if and only x is incident with ℓ, where $x \in P$ and $\ell \in L$. Since two points x and y are joined by a unique line, the graph Γ contains no $K_{2,2}$.

In the case that (P, L) is a projective plane of order q, see Sect. 4.5, the graph Γ has $n = 2(q^2 + q + 1)$ vertices and $\frac{1}{2}n(q + 1)$ edges which implies

$$\mathrm{ex}(n, K_{2,2}) > (\tfrac{1}{2}n)^{3/2},$$

since $(q + 1)^2 > \frac{1}{2}n$.

The Kőváry–Sós–Turán theorem (Theorem 9.9 below) gives an upper bound on $ex(n, K_{s,t})$, the maximum number of edges in a graph with n vertices and no copy of $K_{s,t}$ as a subgraph. For its proof we will use the fact that, for every positive integer t, the function $f_t(x)$ defined as $\binom{x}{t} = x(x-1)\cdots(x-t+1)/t!$ for $x \geqslant t$ and zero otherwise is a convex function. Thus, for any numbers $d_1, \ldots, d_n$,

$$\frac{1}{n}\sum_{i=1}^{n}\binom{d_i}{t} \geqslant \binom{(\sum_{i=1}^{n} d_i)/n}{t}.$$

Theorem 9.9 (Kőváry–Sós–Turán) *For all $\epsilon > 0$, there is a n_0, such that for all $n > n_0$, a graph with n vertices which contains no $K_{t,s}$ has at most*

$$\tfrac{1}{2}(s-1)^{1/t}(1+\epsilon)n^{2-1/t}$$

edges. In other words,

$$\mathrm{ex}(n.K_{s,t}) < \tfrac{1}{2}(s-1)^{1/t}(1+\epsilon)n^{2-1/t},$$

for n large enough.

Proof Let Γ be a graph with n vertices, m edges and containing no $K_{t,s}$.

Let N be the number of copies of $K_{1,t}$ contained in Γ.

For every subset T of size t, the vertices in T have at most $s - 1$ common neighbours. Hence,

$$N \leqslant \binom{n}{t}(s-1) \leqslant \frac{n^t}{t!}(s-1)(1+\epsilon)^{t-1}.$$

Let $d(v)$ denote the degree of the vertex v and let $\delta = 2m/n$ denote the average degree. By the convexity of the binomial coefficients,

$$N = \sum_v \binom{d(v)}{t} \geqslant n\binom{\delta}{t} > n\frac{\delta^t}{t!} - n\delta^{t-1},$$

for n large enough.

Suppose

$$m > \frac{1}{2}(s-1)^{1/t}(1+\epsilon)n^{2-1/t}.$$

Then,

$$\delta > (s-1)^{1/t}(1+\epsilon)n^{1-1/t}.$$

and comparing the inequalities for N,

$$\frac{n^t}{t!}(s-1)(1+\epsilon)^{t-1} \geqslant \frac{n^t}{t!}(s-1)(1+\epsilon)^t - cn^{t-(t-1)/t},$$

for some constant $c = c(s, t, \epsilon)$. This gives

$$cn^{t-(t-1)/t} \geqslant \frac{n^t}{t!}(s-1)(1+\epsilon),$$

and so

$$ct! \geqslant (s-1)n^{1-1/t}(1+\epsilon),$$

which is a contradiction for n large enough. □

Corollary 9.10 *For all $\epsilon > 0$ and n large enough, a graph containing no $K_{2,2}$ has at most $\frac{1}{2}(1+\epsilon)n^{3/2}$ edges.*

In the construction in Example 9.8, we had a lower bound $ex(n, K_{2,2}) \geqslant (\frac{1}{2}n)^{3/2}$. The next example is a refinement which improves the constant $(\frac{1}{2})^{3/2}$ to $\frac{1}{2}$ and thereby asymptotically meets the upper bound.

Example 9.11 Let (P, L) be a projective plane . A **polarity** is a bijection $\sigma : P \to L$ with the property that $x \in \ell$ if and only if $\sigma^{-1}(\ell) \in \sigma(x)$.

Consider the graph Γ whose vertices are the points P of a projective plane, equipped with a polarity σ. The vertices x and y are joined by an edge if and only if $x \in \sigma(y)$. This defines a simple graph since $x \in \sigma(y)$ if and only if $y \in \sigma(x)$.

A common neighbour of both x and y is a point in the intersection of two lines, $\sigma(x)$ and $\sigma(y)$, which is unique since (P, L) is a projective plane.

Suppose that (P, L) is a projective plane of order q. As we saw in Sect. 4.5, $|L| = |P| = n^2 + n + 1$, so Γ has $n = q^2 + q + 1$ vertices. Since each line is incident with $n + 1$ points, Γ has at least $\frac{1}{2}nq$ edges, which is approximately $\frac{1}{2}n^{3/2}$ edges, which asymptotically is the bound in Theorem 9.9.

It only remains to construct a projective plane with a polarity. Consider $\mathrm{PG}(2, q)$ where σ is defined by

$$\sigma(\langle(x_1, x_2, x_3)\rangle) = \ker(x_1X_1 + x_2X_2 + x_3X_3).$$

Then $y \in \sigma(x)$ if and only if $x_1y_1 + x_2y_2 + x_3y_3 = 0$ which is if and only if $x \in \sigma(y)$.

The proof of the following theorem uses the probabilistic method to prove the existence of graphs with no $K_{t,s}$ and many edges. For $t \geqslant 6$ and $s \leqslant (t-1)!$ there is no better construction known.

Theorem 9.12 *For some constant c and n large enough, there is a graph with n vertices and* $cn^{2-(s+t-2)/(st-1)}$ *edges containing no* $K_{t,s}$.

Proof Construct a graph with n vertices in which each edge is there with probability p.

Let X be the random variable which counts the number of copies of $K_{t,s}$.

Let Y be the random variable which counts the number of edges.

Then, the expectation of Y is

$$E(Y) = \binom{n}{2}p \geqslant c'n^2p,$$

for some constant c'.

The expectation of X is

$$E(X) = \binom{n}{s}\binom{n-s}{t}p^{st} < c''n^{s+t}p^{st},$$

for some constant c'', depending on s and t.

By linearity of expectation,

$$E(Y - X) \geqslant c'pn^2 - c''p^{st}n^{s+t}.$$

Put

$$p = \left(\frac{c'}{2c''}\right)^{1/(st-1)} n^{-(s+t-2)/(st-1)}.$$

Then,

$$E(Y - X) \geqslant \tfrac{1}{2} pc'n^2 = cn^{2-(s+t-2)/(st-1)},$$

for some constant c.

Since the expectation of $Y - X$ is at least $cn^{2-(s+t-2)/(st-1)}$, there must be a graph Γ' for which $Y - X \geqslant cn^{2-(s+t-2)/(st-1)}$. Removing an edge from each copy of $K_{t,s}$ in Γ', we obtain a graph Γ with at least $cn^{2-(s+t-2)/(st-1)}$ edges, containing no copy of $K_{t,s}$. □

The probabilistic construction only gives a lower bound of $cn^{4/3}$ for graphs containing no $K_{2,2}$, whereas we have seen a deterministic construction with $\frac{1}{2}n^{3/2}$ edges. Observe that although our construction only works for $n = q^2 + q + 1$ and q a prime power, the primes are dense enough that, asymptotically, the construction for $n_0 = q^2 + q + 1$ and $n - n_0$ isolated vertices is enough to prove that

$$\mathrm{ex}(n, K_{2,2}) \sim \tfrac{1}{2} n^{3/2}.$$

Deterministic constructions are known which imply that $\mathrm{ex}(n, K_{2,s}) = c(s)n^{3/2} + o(n^{3/2})$ and $\mathrm{ex}(n, K_{3,s}) = c(s)n^{5/3} + o(n^{5/3})$. The asymptotic behaviour of $\mathrm{ex}(n, K_{4,4})$ is unknown, although it is known that

$$\mathrm{ex}(n, K_{t,s}) \sim c(s,t)n^{2-1/t},$$

for $s \geqslant (t-1)! + 1$.

9.5 Graphs Without Even Cycles

The following theorem bounds the number of edges that a graph contatining no C_{2t} can have.

Theorem 9.13 *A graph with n vertices which contains no copy of C_{2t} has at most $cn^{1+(1/t)}$ edges, for some constant c.*

As we shall see in the exercises, the probabilistic construction gives a lower bound of $c'n^{1+1/(2t-1)}$.

There are deterministic constructions known which give a lower bound of $cn^{1+2/(3t-3)}$ edges.

In this section, we will prove Theorem 9.14. In the proof we will construct a **generalised quadrangle**, an incidence structure (P, L) with the property that for a pair $(x, \ell) \in (P, L)$, such that $x \notin \ell$, there is a unique line m incident with x and concurrent with ℓ. Figure 9.2 is an example of a generalised quadrangle with 15 points and 15 lines.

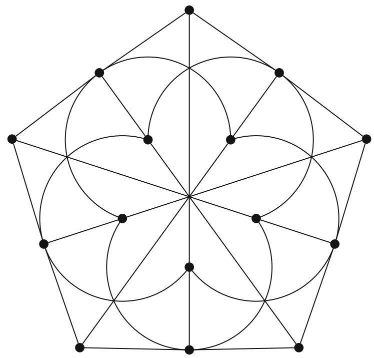


Fig. 9.2 A generalised quadrangle with 15 points and 15 lines

Theorem 9.14 *For all $\epsilon > 0$, there is a n_0 such that for all $n > n_0$, there is a graph with n vertices and at least $((\frac{1}{2})^{4/3} - \epsilon)n^{4/3}$ edges which contains no copy of C_6.*

Proof Let $b(u, v)$ be a non-degenerate alternating form defined on $\mathbb{F}_q^4$, for example

$$b(u, v) = u_1v_2 - v_1u_2 + u_3v_4 - v_3u_4.$$

Consider the incidence structure (P, L) whose points are the 1-dimensional subspace of $\mathbb{F}_q^4$ and whose lines are the 2-dimensional totally isotropic subspaces. In other words, $\ell \in L$ is a 2-dimensional subspace with the property that $b(u, v) = 0$ for all $u, v \in \ell$. We define incidence in (P, L) to be inclusion as subspaces in $\mathbb{F}_q^4$.

Then P has size $q^3 + q^2 + q + 1$. A point $\langle u\rangle$ is incident with the $q + 1$ lines which are the 2-dimensional subspaces contained in the 3-dimensional subspace, $\ker b(u, v)$. There are $q + 1$ such subspaces, so each point is incident with $q + 1$ lines. This implies that there are $q^3 + q^2 + q + 1$ lines in L.

Let $x = \langle u\rangle$ be a point not incident with a line ℓ. The points which are collinear with x are the 1-dimensional subspaces contained in $\ker b(u, v)$. Moreover, the totally isotropic 2-dimensional subspaces contained in $\ker b(u, v)$ all contain x, so ℓ is not contained in $\ker b(u, v)$. Therefore the 3-dimensional subspace $\ker b(u, v)$ meets the 2-dimensional subspace ℓ in a 1-dimensional subspace. In the incidence structure (P, L) this implies that there is a unique point $y \in \ell$ which is collinear with x. Therefore, the incidence structure contains no triangles.

Let Γ be the bipartite graph with vertices $P \cup L$ and where x and ℓ are joined by an edge if and only if $x \in \ell$. Then Γ has roughly $n = 2(q^3+q^2+q+1)$ vertices and $(q^3 + q^2 + q + 1)(q + 1)$ edges. Since (P, L) contains no triangles, Γ contains no 6-cycles and Γ has n vertices and more than $(\frac{1}{2}n)^{4/3}$ edges. The statement follows from density of primes amongst the integers. □

9.6 Notes and References

Mantel's theorem was proven by Mantel in 1907 (Mantel 1907) and later extended to Turán's theorem by Turán in 1941 (Turan 1941) and generalized to the Erdős-Stone theorem by Erdős and Stone in Erdős and Stone (1946).

The smallest complete bipartite graph Γ for which the asymptotic behavious of $\text{ex}(n, \Gamma)$ is not known is $\text{ex}(n, K_{4,4})$. It is known that the asymptotics of the function lies between $cn^{5/3}$ (see Exercise 9.6) and $c'n^{7/4}$ (by Theorem 9.9).

9.7 Exercises

Exercise 9.1 Let Γ be a graph with n vertices in which every vertex has degree d and suppose Γ contains no C_4.

i. Prove that $n \geqslant d^2 + 1$.
ii. Let Γ be the graph whose vertices are the points of a Desargues configuration, see Fig. 9.3, and where two vertices are joined by an edge if they are not collinear in Desargues configuration. Prove that Γ contains no C_4 and meets the bound in i.
iii. Suppose that we can label the 35 lines of PG(3, 2) with a triple from a set X of size 7 in such a way that two lines of PG(3, 2) intersect if and only if the corresponding triples intersect in precisely one element. Let Γ be the graph whose vertices are the points and the lines of PG(3, 2). A point x is joined to a line ℓ in the graph Γ if and only if $x \in \ell$. No two points are joined by an edge. Two lines are joined to an edge if and only if there corresponding triples

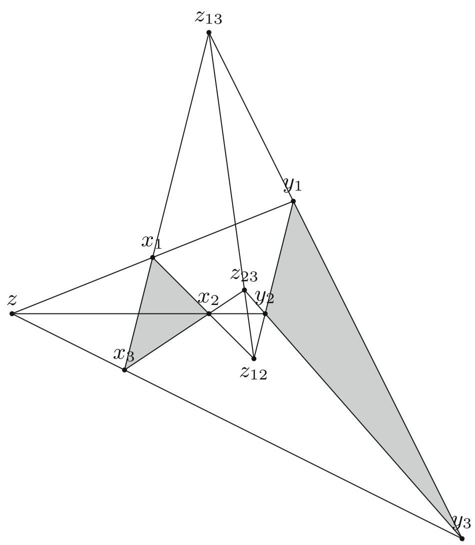
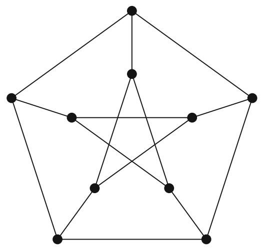


Fig. 9.3 Desargues configuration. It has 10 points, 10 lines, every point is in three lines and every line is incident with 3 points

Fig. 9.4 The Petersen graph



are disjoint. Prove that every vertex of Γ has degree 7, Γ contains no C_4 and meets the bound in i.

The graph constructed in Exercise 9.1 ii. is the Petersen graph (Fig. 9.4). The labelling described in Exercise 9.1 iii. is possible and the graph constructed is the Hoffman-Singleton graph.

Exercise 9.2 Let Γ be a graph with n vertices in which every vertex has degree at least $n/2$. Prove that Γ contains a cycle of length n.

Exercise 9.3 Let E be the set of edges of Γ and let V be the set of vertices of a graph Γ.

i. Summing over the edges xy in Γ, prove that

$$\sum_{xy \in E} (d(x) + d(y)) = \sum_{x \in V} d(x)^2,$$

where $d(x)$ denotes the degree of the vertex x.

ii. Use the Cauchy-Schwarz inequality to prove that

$$(u_1^2 + \cdots + u_n^2)n \geqslant (u_1 + \cdots + u_n)^2.$$

for $(u_1, \ldots, u_n) \in \mathbb{R}^n$.

iii. Use the inequaiities in i. and ii. to prove Theorem 9.2.

Exercise 9.4 Suppose that $ex(n, H) \leqslant \rho\binom{n}{2}$, for $n \geqslant n_0$ and let $V(H)$ denote the set of vertices of H. Prove that if Γ is a graph with n vertices and more than $(\rho + \epsilon)\binom{n}{2}$ edges then it contains at least $cn^{|V(H)|}$ copies of H, for some c not dependent on n.

Exercise 9.5 Let (P, L) be a finite linear space with an injective map σ from P to L with the property that $x \in \sigma(y)$ if and only $y \in \sigma(x)$. Let G be the graph whose vertices are the elements of P and where x is joined to y if and only $x \in \sigma(y)$.

i. Prove that G contains no subgraph isomorphic to C_4.
ii. Using a non-degenerate symmetric bilinear form $b(u, v)$ on $\mathbb{F}_m^3 \times \mathbb{F}_m^3$, prove that $(P, L) = \mathrm{PG}(2, m)$ has such a map σ.
iii. Prove that for $n = m^2 + m + 1$,

$$\mathrm{ex}(n, C_4) \geqslant \frac{1}{2}(n-1)(m+1).$$

Exercise 9.6 Let S be a set of $q^2 + 1$ points of an ellipsoid in $\mathrm{PG}(3, q)$ embedded in the hyperplane $X_5 = 0$ of $\mathrm{PG}(4, q)$. Note that no three points of S are collinear. Let $v = (0, 0, 0, 0, 1)$ and let σ be a bijective map from the points of $S \setminus \{v\}$ to the hyperplanes of the space, given by

$$\sigma((x_1, x_2, x_3, x_4, x_5)) = x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5.$$

For example, the point v is mapped to the hyperplane $X_5 = 0$.

Let Γ be the graph whose vertices are the points on lines joining S to v (not including the point v), where a point x is adjacent to a point y if and only if x is a point incident with the hypreplane $\sigma(y)$.

i. Prove that Γ contains no $K_{3,3}$.
ii. Prove that for all $\epsilon > 0$ and n sufficiently large,

$$\mathrm{ex}(n, K_{3,3}) > \frac{1}{2}(1-\epsilon)n^{5/3}.$$

Exercise 9.7 Let $\mathrm{exb}(n, H)$ be the maximum number of edges a bipartite graph can have which contains no subgraph isomorphic to H.

i. Prove that for all $\epsilon > 0$ and n large enough,

$$\mathrm{exb}(n, K_{t,s}) \leqslant (1+\epsilon)(s-1)^{1/t}(\frac{n}{2})^{2-1/t}.$$

ii. Prove that the bound in (a) is asymptotically tight for $K_{2,2}$.
iii. Prove, using a probabilistic construction, that for all $\epsilon > 0$ and n large enough,

$$\mathrm{exb}(n, K_{t,t}) \geqslant c_t n^{2-2/(t+1)},$$

for some constant c_t depending on t.

Exercise 9.8

i. Prove that a bipartite graph with vertex sets of size t and s which contains no C_4 has less than $t^{1/2}s + t$ edges.
ii. Construct a bipartite graph with vertex sets of size $t = s^{4/3}+s+s^{2/3}$ and s which contains no C_4 and has more than $t^{1/2}s$ edges, for every prime power $s^{1/3}$.

Exercise 9.9 Let H be the cyclic graph C_{2t}. Prove that for all $\epsilon > 0$, there exists an n_0 and $c = c(t)$ such that for all $n \geqslant n_0$,

$$ex(n, H) \geqslant c(1 - \epsilon)n^{1+(1/(2t-1))}.$$

Exercise 9.10 Let D be a finite subset of positive numbers and let S be a set of n points in the real plane.

i. Prove that there are at most $cn^{3/2}$ pairs of points in S that are at distance one from each other, for some constant c that does not depend on n.
ii. Prove that there are at most $cn^{3/2}$ pairs of points in S that are at distance $d \in S$ from each other, for some constant c that does not depend on n.

Hints and Solutions to Selected Exercises

10

1.1
Let $\mathcal{P}_O$ be the class of partitions into odd number of parts. Its symbolic description is

$$\mathcal{P}_O = \text{Seq}(\{1\}) \times \text{Seq}(\{3\}) \times \text{Seq}(\{5\}) \times \cdots$$

On the other hand, the class $\mathcal{P}_d$ of distinct partitions can be written as

$$\mathcal{P}_d = (\epsilon + \{1\}) \times (\epsilon + \{2\}) \times \cdots$$

1.2 A path in $\mathcal{D}_k$ is a Dyck path followed by a $\searrow$ step, followed by a second Dyck path and $\searrow$ step and so on up to k times and finally an additional Dyck path:

$$\mathcal{D}_k = (\mathcal{D}_0 \times \{\searrow\})^k \times \mathcal{D}_0.$$

From the above description one gets

$$D_k(z) = z^k D_0^{k+1}(z),$$

For $k = 2$, the 9 paths of length 6 in $\mathcal{D}_2$ are depicted in Fig. 10.1.
1.3

$$\mathcal{S} = \{\epsilon\} + \{\rightarrow\rightarrow\} \times \mathcal{S} + \{\nearrow\} \times \mathcal{S} \times \{\searrow\} \times \mathcal{S}.$$

The length of the path must be even, so if we declare the size of a path of length $2n$ to be n we get the functional equation

$$S(z) = 1 + z(S(z) + S^2(z)).$$

S. Ball, O. Serra, *A Course in Combinatorics and Graphs*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-031-55384-4_10

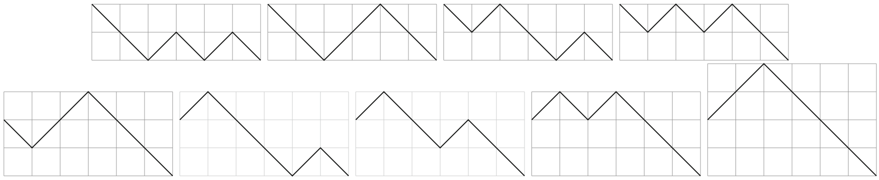


Fig. 10.1 $\mathcal{D}_2$ paths

1.4

As in the case of Dyck paths, in order to find a recursive formula we consider the smallest $i > 0$ such that the path contains the point $(i, 0)$. Every path in $\mathcal{D}^{\leqslant m}$ can be described as a step up followed by a path in $\mathcal{D}^{m-1}$ followed by a step down and a path in $\mathcal{D}^{\leqslant m}$:

$$\mathcal{D}^{\leqslant m} = \epsilon + \{\nearrow\} \times \mathcal{D}^{\leqslant m-1} \times \{\searrow\} \times \mathcal{D}^{m},$$

which gives the functional equation

$$D^{\leqslant m}(z) = 1 + z^2 D^{\leqslant m-1}(z) D^{\leqslant m}(z).$$

1.6

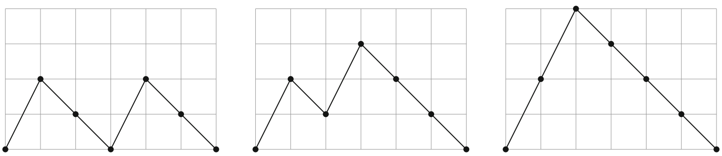
Let U denote a step $(1, 2)$ and D a step $(1, -1)$. The first step must be U. By decomposing the path by the first passage through the line $y = 1$ and then by the first passage through the line $y = 0$, the path can be uniquely described as

$$\mathcal{A} = \epsilon + \{U\} \times \mathcal{A} \times \{D\} \times \mathcal{A} \times \{D\} \times \mathcal{A}.$$

which gives the equation

$$A(z) = 1 + z^3 A^3(z).$$

The three paths of length six are depicted below.



1.7

If $\mathcal{N}$ denotes the class with a single node, then the class of trees such that the number of children at every node belongs to a subset $I \subset \mathbb{N}$ can be written as

$$\mathcal{T}_I = \mathcal{N} \times (1 + \sum_{i \in I} (\mathcal{T}_I)^i).$$

1.10

i. There are $(n-3)$ diagonals and $n-2$ triangles.
ii. Let $\mathcal{T}$ be the class of triangulations. Every triangulation can be described by a sequence of a triangulation, a triangle and a triangulation:

$$\mathcal{T} = \{\epsilon\} + \mathcal{T} \times \{T\} \times \mathcal{T}.$$

2.1

Hint: The class $\mathcal{C}$ of cycles has (exponential) generating function $C(z) = \log \frac{1}{1-z}$ and the class of *even* cycles is its even part

$$\frac{1}{2}(C(z) + C(-z)) = \frac{1}{2}\left(\log \frac{1}{1-z} + \log \frac{1}{1+z}\right) = \log(\frac{1}{\sqrt{1-z^2}}).$$

The class of permutations which have only even cycles in their cycle decomposition is $\mathcal{P}_{\text{even}} = \mathrm{S}et(\mathcal{C}_{\text{even}})$, ans its generating function is

$$P_{\text{even}}(z) = \exp(C_{\text{even}}(z)) = \frac{1}{\sqrt{1-z^2}} = \sum_{n \geqslant 0} \binom{-1/2}{n} z^{2n}.$$

2.5

The class $\mathcal{P}^{\text{even}}$ of permutations which decompose into an even number of cycles can be extracted from the construction $(\mathcal{C})$ by selecting only the even powers:

$$\mathcal{P}^{\text{even}} = \sum_{k \geqslant 1} \frac{\mathcal{C}^{*2k}}{(2k)!}$$

and the generating function is

$$P^{\text{even}}(z) = \cosh\left(\log \frac{1}{1-z}\right) = \frac{1}{2}\left(\frac{1}{1-z} + (1-z)\right).$$

Its number is, for $n > 1$,

$$n![z^n]P^{\text{even}}(z) = n!/2,$$

as one may expect.

2.6

Hint: A word in $\mathcal{W}^{(k,r)}$ can be identified by the 4-tuple $(f^{-1}(a), f^{-1}(b), f^{-1}(\alpha), f^{-1}(\beta))$ where $f : [n] \to \{a, b, \alpha, \beta\}$ gives the positions of the letters in a word of length n, with the condition that $f^{-1}(a)$, $f^{-1}(b)$ have size at most k and $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ have size at least r.

2.9

Hint: A labelled rooted star tree is a root together with a set of nodes which are the leaves. The class of rooted star trees can be described as $\mathcal{Z} * \mathsf{Set}(\mathcal{Z})$. A forest of rooted labelled stars is

$$\mathcal{S} = \mathsf{Set}(\mathcal{Z} * \mathsf{Set}(\mathcal{Z})).$$

2.10

i. Let $\mathcal{U}_2$ be the class of urns of cardinality at least 2. We have

$$\mathcal{P}_{k,2} = \overbrace{\mathcal{U}_2 * \cdots * \mathcal{U}_2}^{k} / \sim_k .$$

where $\sim_k$ denotes the equivalence relation identifying two k-tuples that differ only in a permutation of its entries.

ii. Every doubly surjective map $f : [n] \to [k]$ is identified by a sequence of k sets of cardinality at least two:

$$\mathcal{S}_k = \overbrace{\mathcal{U}_2 * \cdots * \mathcal{U}_2}^{k}.$$

iii. Every word of length n on the alphabet $\{a_1, \ldots, a_k\}$ such that each symbol appears at least twice is identified with a doubly surjective map $f : [n] \to \{a_1, \ldots, a_k\}$.

3.1

i. $\frac{1}{24}(r^8 + 17r^4 + 6r^2)$.
ii. $\frac{1}{24}(r^8 + 17r^4 + 6r^2)$.

3.2

ii. $\frac{1}{12}(r^6 + 3r^4 + 4r^3 + 2r^2 + 2r)$.
iii. 74.

3.4

34.

3.6

i. 700.
ii. 414.

3.7
218.
3.9

iii. 396.
iv. No.

3.10

iii. 104.

3.12

i. 36.
ii. The coefficient of t^j in $15 + 11t + 6t^2 + 3t^3 + t^4$.
iii. The coefficient of t^j in

$$\tfrac{1}{12}((1+t+t^3+t^5)^4+3(1+t^2+t^6+t^{10})^2+8(1+t^3+t^9+t^{15})(1+t+t^3+t^5)).$$

3.13

i. The number of 4-colourings is

$$Z(4, 4, 4, 4, 4, 4)$$

where

$$Z(X_1, \dots, X_6) = \frac{1}{12}(X_1^6 + 4X_2^3 + 2X_3^2 + 2X_6 + 3X_2^2X_1^2).$$

ii. The coefficient of t^j in

$$Z(t + 3, t^2 + 3, t^3 + 3, t^4 + 3, t^5 + 3, t^6 + 3).$$

iii. The coefficient of t^j in

$$Z(t+2t^2+1, t^2+2t^4+1, t^3+2t^6+1, t^4+2t^8+1, t^5+2t^{10}+1, t^6+2t^{12}+1).$$

4.1

i. Prove that they are either $\mathbb{Z}/4\mathbb{Z}$ with addition and the second is $\mathbb{Z}/5\mathbb{Z}$ with multiplication.

ii.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	0	4	1	3
3	3	4	0	2	1
4	4	3	1	0	2

4.2

iii. Let $q = 2m$ and consider the latin square L_a defined on the elements of $\mathbb{F}_q$ with binary operation $x \circ y = ax + y$, where a is a non-zero element of $\mathbb{F}_q$.
Recall that, by construction, $\mathbb{F}_q = \mathbb{F}_2[X]/(f)$, where f is an irreducible polynomial over $\mathbb{F}_2$ of degree $k + 1$.
Order the elements of $\mathbb{F}_q$ so that the first m are represented by polynomials of degree at most $k - 1$. The top-left $m \times m$ array in L_1 will contain only these elements, since summing two polynomials of degree at most $k - 1$ gives a polynomial of degree at most $k - 1$. By Theorem 4.19, L_1 belongs to a set of $2m - 1$ mutually orthogonal latin squares of order $2m$.

4.9
Count triples (x, y, ℓ) where x and y are points incident with the line ℓ in two ways and apply the Erdős-De Bruijn theorem.
4.12

i. $\{0, 1, 3\}$ is a difference set of $\mathbb{Z}/7\mathbb{Z}$ and $\{0, 1, 3, 9\}$ is a difference set of $\mathbb{Z}/13\mathbb{Z}$.

4.13

i. $D = \{0, 1, 3\}$ is a relative difference set of $\mathbb{Z}/8\mathbb{Z}$.

5.1
Hint: Removing the leaves from the tree and their fathers, we get a smaller tree.
5.3
Hint: Let $U = A' \cup B'$ be a vertex cover, of minimum size where $A' \subseteq A$ and $B' \subseteq B$. Since U is a vertex cover, there are no edges between $A \setminus A'$ and $B \setminus B'$. Show that Hall's condition fails for $A \setminus A'$.
5.4
Hint: Consider an Eulerian circuit C, substituting each vertex v by two vertices v^- and v^+ and every edge $v_i v_{i+1}$ of C by the edge $v_i^+ v_{i+1}^-$. The resulting graph is bipartite k-regular.
5.7
Consider the set of preferences in which, for $i \neq n$

$$x_i \mid y_i > y_{i-1} > \cdots y_1 > y_{n-1} > \cdots y_{i+1} > y_n$$

and for x_n

$$x_n \mid y_{n-1} > y_{n-2} > \cdots > y_1 > y_n$$

and for $i \neq n$,

$$y_i \mid x_{i-1} > x_{i-2} > \cdots x_1 > x_n > x_{n-1} > \cdots > x_i.$$

5.8

A 1-factorisation of the Petersen graph is equivalent to a proper 3-colouring of the edges. Up to change of colour there is a unique way to colour the outer edges, which leaves no choice for the colours of the outer to inner edges. But then the inner edges cannot be properly 3-coloured.

5.10

Let $X \cup A$ be the vertex partition of a bipartite graph Γ, where $|X| = |A|$.

Suppose that Γ does not have a perfect matching. Then we will find a subset J of the vertices of A with the property that $N(J)$, the union of the neighbours of J, is smaller than J. This suffices to prove that if $|N(J)| \geqslant |J|$ for all subsets J of A then Γ has a perfect matching.

Adding all edges between any two vertices of X, does not alter the property that Γ does not have a perfect matching, since a perfect matching is a set of edges which have one end-vertex in A and the other in X. Hence, we can assume that the vertices X in Γ form a complete subgraph.

Theorem 5.11 implies that there is a subset S of the vertices such that the number of odd components of $\Gamma \setminus S$ is larger than $|S|$. An odd component of $\Gamma \setminus S$ is either an isolated vertex in $N(S \cap X)$ or one other large component with vertex set C. Note that Hall's condition implies that every vertex in A is connected to some vertex in X, so if there is a vertex in A not in C, then it is in $N(S \cap X)$.

Let J be the subset of $N(S \cap X)$ which are not in C. Then $N(J) \subseteq S \cap X$. Let $T = N(J)$.

Any vertex in $(S \cap X) \setminus T$ is joined to a vertex in C and so is in C. Its removal may change the status of C as an odd component, but it does not change the status of any other odd components. Therefore, it is possible that $\mathrm{oc}(\Gamma \setminus T)$ is one less than $\mathrm{oc}(\Gamma \setminus S)$ but then $|T|$ is one less than $|S|$, so we have that T also has the property that the number of odd components of $\Gamma \setminus T$ is more than $|T|$.

If $|C|$ is even then

$$|J| = \mathrm{oc}(\Gamma \setminus T) > |T| = |N(J)|.$$

If $|C|$ is odd then $|T| \neq |J|$ (since $C = (A \cup X) \setminus (J \cup T)$ and $|A| = |X|$), so

$$|J| = \mathrm{oc}(\Gamma \setminus T) - 1 \geqslant |T| = |N(J)|,$$

implies $|J| > |N(J)|$.

5.11

i. Counting vertices modulo 2.
ii. By Tutte's theorem, there exists a subset S of the vertices such that

$$|\mathrm{oc}(\Gamma \setminus S)| - |S| > 0.$$

By counting edge-vertex pairs, it follows that a cubic graph has an even number of vertices. Now apply i.
iii. Suppose that Γ does not have a perfect matching.
By ii., there exists an S such that

$$|\mathrm{oc}(\Gamma \setminus S)| - |S| \geqslant 2.$$

Each odd component of $\Gamma \setminus S$ has an odd number of vertices and so there are an odd number of edges from $\Gamma \setminus S$ to S. If this number is 1 then the edge is a bridge. If there are at most two bridges we reach a contradiction.

5.12
Hint: Consider what happens to the number of components with an odd number of edges if we remove an edge of a graph.
5.13

i. In every unbalanced even component there are at least two vertices joined by the matching to a vertex in S, giving one direction. The converse follows by Tutte's theorem.
ii. If Γ is bipartite then odd connected components must be unbalanced. Therefore,

$$\mathrm{uc}(\Gamma) = \mathrm{oc}(\Gamma) + \mathrm{euc}(\Gamma) \leqslant \mathrm{oc}(\Gamma) + 2\mathrm{euc}(\Gamma)$$

and the condition is necessary by part (i). Sufficiency again holds by Tutte's theorem.
iii. The graph $\Gamma = K_{1,2,3}$ has a perfect matching. Let S be one of the vertices in the stable set with two vertices. Then $\Gamma \setminus S$ has one odd component and one severely unbalanced component.

6.2
Hint: If Γ/e is not 2-connected then the vertices of e have a neighbour in all components of $\Gamma \setminus e$, since Γ is 2-connected.
Hint: If $\Gamma \setminus e$ is not 2-connected then it has a cut-vertex v. Since Γ is 2-connected, the edge e must join vertices in distinct components of $\Gamma \setminus v$.

6.3

Hint: Suppose z is a cut-vertex for $\Gamma \setminus \{x, y\}$. Prove that $\{z, v_{xy}\}$ is a cut-set of Γ/xy, where v_{xy} is the vertex obtained by contracting the edge xy.
Hint: Suppose $\{z, w\}$ is a cut-set of Γ/xy. Argue that $w = v_{xy}$.

6.4

Hint: By induction on k and apply Menger's theorem.

6.6

Hint: Let C_i be a connected component of $G \setminus S$ such that $C_i \cap S' = \emptyset$. Every vertex of S has a neighbour in C_i, since otherwise S is not a minimal separating set. Thus, $S \setminus S'$ is in the same connected component as C_i in $G \setminus S'$.

6.9

Hint: We need to prove that a separator set has size at least n, where $|A| = n$ and A, B is the bipartite partition of the bipartite graph Γ. Suppose there is a separator set $S_A \cup S_B$ of size at most $n - 1$, where $S_A \subseteq A$ and $S_B \subseteq B$. Prove that Hall's condition fails for $J = A \setminus S_A$.

6.10

By induction on k. Note that Q_k splits into two copies of Q_{k-1}, those whose first coordinate is zero and those whose first coordinate is 1. Argue that to separate Q_k we must separate each of these copies of Q_{k-1} and use that fact that $(1, x)$ is joined to $(0, x)$.

6.12

iii. $K_3 + E_5$.

7.1

The average degree is smaller than 6. If G is triangle free, the average degree is smaller than 4.

7.2

Every face is bounded by a cycle. If a cycle is not a face, consider the subgraph which has this cycle as outer face.

7.4

All vertices must be in the cycle of the outer face. Some vertex has degree two.

7.5

Add one vertex in the outer face connected to all vertices of Γ and apply Kuratowski's theorem.

7.6

i. Use Euler's formula and double counting faces and edges.
ii. If the end vertices of an edge share more than two neighbours then Γ has a separating triangle. Use induction on $|V(\Gamma)|$.
iii. Suppose that $S = \{x, y\}$ separates Γ and let X and Y be two connected components of $\Gamma - S$. Contract edges until we eventually reach a maximal planar graph with four vertices and get a contradiction.

7.7

Let $H \subset \Gamma$ be a subdivision of K_5. If there is a vertex v not in H, there are three independent paths from v to three branching vertices of H and a subdivision of $K_{3,3}$ appears. Otherwise, choose a vertex $v \in V(H)$ in a path P joining two branching vertices x and y of H. There is a path from v to a third branching vertex different from x and y not using edges from P (by 3-connectedness) and again a subdivision of $K_{3,3}$ can be formed.

7.8

Let $S = \{x, y\}$ be a set of two vertices and a, b arbitrary vertices from the skeleton different from x and y. By an appropriate projective transformation we may assume that the supporting planes through a and b are parallel. Let P be a plane parallel to them containing x and let z be a point on P interior to the polyhedra. The projection of the polyhedra to a plane orthogonal to the line xz sends a, b and y to the boundary polygon of the projection and x to its interior. Since a polygon is 2-connected, there is a path joining the images of a and b avoiding the images of x and y. Lifting this path back to the polyhedra we obtain a path from a to b avoiding S.

7.9

First show that a maximal planar graph with an extra edge contains a subdivision of K_5. If xy is an edge not in G, consider the neighborhood of x in G. It induces a cycle (with x a wheel) with $k \geqslant 3$ vertices. Select three vertices in the boundary and three independent paths joining them to y (by 3-connectivity of G). This already gives K_5^- (the complete graph minus one edge) as a subdivision in G. The added edge xy provides the subdivision of K_5. (It was proved by Mader that any graph with $3n-5$ edges contains K_5 as a minor, this is more difficult without the planarity condition of the graph minus one edge).

7.10

By induction on the number n of vertices, starting with $n = 4$. Choose an edge $e = xy$ such that Γ/e is maximal planar (a former exercise shows that such an edge exists). In a decomposition of three trees of Γ/e consider the cases where the edges av_{xy} and $v_{xy}b$, where a and b are the only common neighbours of x and y in Γ and v_{xy} is the vertex arising from the contraction, belong to the same tree or to two distinct trees. Show that the decomposition can be extended to G in both cases.

8.1

Order the vertices with a lex order from a given r-colouring of Γ. For the second part use an ordering where vertices joined by the edges of the matching are consecutive.

8.3

Colour $\chi(x, y) = \chi_\Gamma(x) + \chi_{\Gamma'}(y) \pmod{\max(\chi(\Gamma), \chi(\Gamma')\})}$.

8.4

There is a graph homomorphism from $\Gamma \times \Gamma'$ to K_r with $r = \min\{\chi(\Gamma), \chi(\Gamma')\}$.

8.5

An $(r-1)$ colouring could be extended from the connected components of an $(r-2)$ edge-cut.

8.8

Given an r-colouring orient every edge from smaller to larger colour of its end vertices. For the converse, given the orientation, show that colouring v with the length of the longest oriented path ending at v provides a proper colouring.

8.9

Hint: Use the first part of the exercise and induction.

8.10

[iii.] $f_{K_n}(x) = \prod_{i=0}^{n-1}(x-i), \quad f_T(x) = x(x-1)^{n-1}$.

8.11

If the color classes are $C_1, \ldots, C_r$ then $m \leqslant \prod_{i<j} |C_i| \cdot |C_j|$. This function is maximized under $\sum_i |C_i| = n$ by taking all C_i with almost the same size.

8.12

It is 2-degenerated.

8.13

Every colour class induces a matching and misses at least one vertex.

8.14

Each connected component of $\Gamma - e$ with e a bridge has odd order.

8.15

For n odd label the vertices $\{0, 1, \ldots n-1\}$ and colour the edges $\chi(xy) = x + y \pmod{n}$. For n even colour K_{n-1} and add one vertex adjacent to all of them. For each vertex of K_n one colour is available for the edge joining it to the new vertex.

8.17

For $k \geqslant 2$ let $n = \binom{2k-1}{k}$ and consider the list assignment of $K_{n,n}$ where vertices in each stable set have pairwise distinct k-subsets of $\{1, \ldots, n\}$. There is no list colouring with such an assignment of lists.

8.20

i. Run the greedy coloring algorithm on the following order: x_n is a vertex with degree at most k; given $x_n, \ldots, x_{n-i+1}$ choose a vertex of degree of degree at most k in $G[V \setminus \{x_n, \ldots, x_{n-i+1}\}]$ (which exists by degeneracy). At each step of the algorithm at most k colors are forbidden.
ii. Prove by induction. Note that we can add an edge from x to $G - x$ to obtain a k-degenerated graph G' (every subgraph containing x has $\delta(G') \leqslant d_{G'}(x) = k$ while a subgraph not containing x is still k degenerated).
iii. An outer planar graph is 2-degenerate.

9.4

Hint: Consider subsets of size n_0 of the vertices of Γ. Prove that at least $\frac{1}{2}\epsilon\binom{n}{n_0}$ of them have at least $(\rho + \frac{1}{2}\epsilon)\binom{n_0}{2}$ edges.

9.5

Each vertex has $m+1$ neighbours unless x is an absolute point, i.e. $b(u, u) = 0$, in which case $x = \langle u \rangle$ has m neighbours. There are $m+1$ absolute points.

9.6

Consider two vertices of the graph u and w. The common neighbours of u and w are points which are incident with the planes $\sigma(u)$ and $\sigma(w)$. These two planes intersect in a line. A line intersects the ellipsoidal cone either in at most two points or lies on the cone. Note that the planes $\sigma(u)$ and $\sigma(w)$ are not incident with the origin, so the line is not lying on the cone. Therefore, u and w have at most two common neighbours and so G contains no $K_{2,3}$ subgraph.

9.7

Let Γ be a bipartite graph on n vertices which contains no $K_{t,s}$. Let V_0 be the least large of the two stable sets, so $n_0 = |V_0| \leqslant \frac{1}{2}n$. Let N be the number of copies of $K_{1,t}$ in which t of the vertices are in V_0.

Then, since Γ contains no $K_{t,s}$,

$$N \leqslant \binom{n_0}{t}(s-1).$$

Furthermore,

$$N = \sum_{u \in V_0} \binom{d(u)}{t} \geqslant n_0 \lfloor (m/n_0) \rfloor t \geqslant n_0((m/n_0) - t)^t / t!.$$

Combine these two inequalities and take n_0 large enough,

9.8

Hint: consider AG$(3, q)$ as a linear space.

9.10

Construct a graph whose vertices are the points of S and where two vertices are joined by an edge if and only if the distance between them in in D.

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