Betancourt Synthesis for Three-Position Problem in Mechanism Design

Juan Ignacio Cuadrado Iglesias[1](✉) [ID] and Marco Ceccarelli[2] [ID]

[1] Polytechnic University of Valencia, Camino de Vera s/n, 46022 Valencia, Spain
`jcuadrad@mcm.upv.es`
[2] University of Rome Tor Vergata, Via del Politecnico, 00133 Rome, Italy
`marco.ceccarelli@uniroma2.it`

Abstract. Between the late 18th century and the early 19th century, the Spanish engineer Agustín de Betancourt y Molina (1758–1824) left an indelible mark on the field of kinematic design of mechanisms through two groundbreaking contributions to the formulation and resolution of trajectory synthesis problems, particularly within the realm of steam engine design. These pivotal contributions are encapsulated in two distinct works: the "Mémoire sur une machine à vapeur à double effet," presented at the French Académie Royale des Sciences in 1789, and the collaborative effort "Essai sur la composition des machines," co-authored with José Maria de Lanz y Zaldívar (1764–1839) published in Paris in 1808. The here-in proposed comprehensive examination delves into the intricacies, advancements, limitations, and nuances of both of Betancourt's seminal works. Today, with accessible and thoroughly studied documentation at our disposal, it is evident that these contributions represent the earliest formulations and resolutions of what would later evolve into the synthesis of a four-bar linkage, wherein the coupler point exhibits approximately rectilinear motion with three points of precision.

Keywords: History of Mechanism and Machine Science · Mechanism Design · Synthesis of trajectory generation · History of Kinematics of Mechanisms · Agustin de Betancourt

1 Introduction

The modern history of the kinematics of mechanisms generally refers to publications that have had considerable diffusion and success in design practices, especially starting from the early 1800s with extensive developments during the Industrial Revolution. The processes of kinematic synthesis of the mechanisms are therefore mostly ascribed to researchers and designers, who had the opportunity to document these procedures with results both in publications in editorial fields and in patents and even in mechanical realizations as documented in a large technical-scientific literature but also in the history of science and technology with particular reference to mechanical engineering. However, in these areas, some contributions are not considered or overlooked which, although fundamental to these documented developments, have not been documented

or referenced in practical procedures or in this vast literature. Emblematic of this gap is the work proposed in this paper which refers to a kinematic synthesis procedure for three positions of a coupler point well before its definition in scientific literature and professional practice. The paper aims to present the content of a work by Agustín de Betancourt from 1789 which, having remained at an epistolary level, has not so far been considered in the scientific literature but surprisingly contains a first dimensional synthesis procedure of mechanisms for trajectory generation through three points of precision with formulation and contents that well anticipate what was formulated in the second half of the nineteenth century.

2   Biographical Notes

Agustín de Betancourt y Molina (in short Betancourt), Fig. 1, [1–3], was born on February 1, 1758, in Puerto de la Cruz (Santa Cruz de Tenerife, Spain). In 1778, he moved to the mainland and pursued scientific studies at the Reales Estudios de San Isidro and artistic studies at the Academia de Bellas Artes of San Fernando in Madrid, from 1778 to 1784.

In a pivotal turn of events in 1784, Betancourt secured a scholarship that facilitated his engagement with the École des Ponts et Chaussées in Paris, where he dedicated himself to the pursuit of a degree in Hydraulic Engineering. Simultaneously, he assumed the responsibility of establishing a Cabinet of Machines, a venture aimed at incorporating models and machines of broad utility for both public works and industry.

Fig. 1. A portrait of young Agustín de Betancourt (1758–1824)

For an in-depth exploration of Betancourt's life, contributions to engineering, and his profound influence on industrial development, refer to the comprehensive biographical sources [1–3]. These sources provide rich and detailed insights into Agustín de Betancourt's legacy.

Amidst the establishment of the Cabinet of Machines, Betancourt embarked on a significant journey to England in 1788. There, he encountered the remarkable advancements pioneered by James Watt (1736–1819) in steam engine development. Watt's trailblazing innovations, detailed in sources [4–6], played a transformative role in significantly enhancing the performance of the steam engine.

3 The Problem

On April 28, 1784, James Watt introduced a remarkable invention through a patent that featured diverse solutions for achieving rectilinear guidance of the piston and its connection to the rocker arm. Among these innovations, one particularly noteworthy solution stood out: the incorporation of an articulated quadrilateral, where a point on the connecting rod traced an almost perfectly straight trajectory. Subsequently, Watt elevated the sophistication of his design by introducing a pantograph mechanism to complement the articulated quadrilateral. These two distinctive solutions are commonly known as Watt's singular mechanism (see Fig. 2) and the extended mechanism (see Fig. 3).
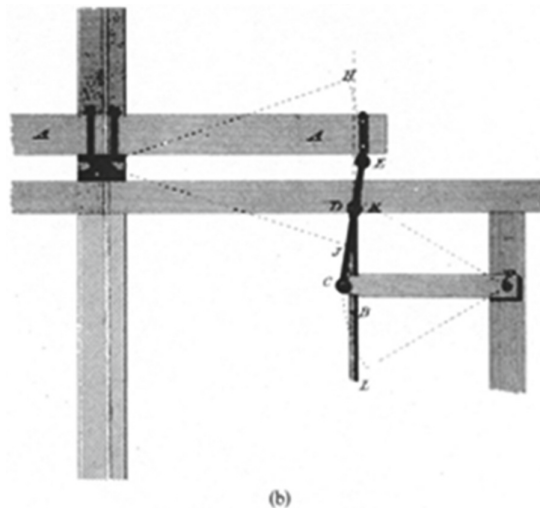

(b)

Fig. 2. Rectilinear guidance of the piston and its connection to the rocker in the Patent submitted by Watt in 1784. (This second solution (b) constitutes the so-called Watt's singular mechanism. In Muirhead, The origin and progress of the Mechanical Inventions of James Watt (1854), vol. III, pl. XXII and XXIII.)

The design in Fig. 3 can be considered as a Watt's singular mechanism with a pantograph-shaped extension. Since EDGB is a parallelogram, point E describes a curve
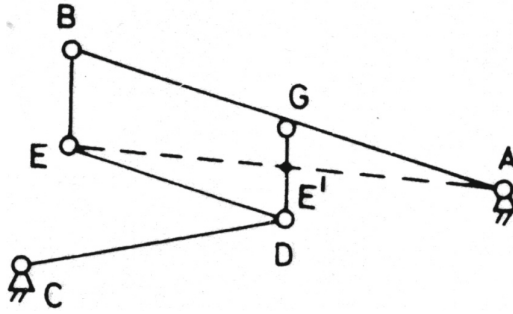
Fig. 3. Extended Watt mechanism used by Watt in his steam engines.

similar to the one described by the intersection of EA and GD, that is, point E'. Therefore, if AGDC form a Watt's singular mechanism, point E' will approximately describe a straight line, and consequently, point E will also do so, as noted by Koetsier in [6]. The two designs in Figs. 2 and 3 together, they mark significant strides in the evolution of steam engine design, showcasing Watt's ingenuity and lasting impact on technological advancement.

This innovation empowered the piston to apply force to the rocker arm not only during its upward stroke but also during the downward motion. It is this dual functionality that led Betancourt to designate the novel steam engine as a "double-acting machine" (double effect).

Despite facing several challenges in gaining access to these innovative machines, Betancourt persevered and managed to observe a segment of the apparatus. Through this limited exposure, he discerned the profound implications that the inventive design held for enhancing overall performance (see Fig. 4).

Betancourt, among other things, points out:

"To begin with, I was surprised to see that the chain connected to the rocker arm, which suspended the piston inside the steam cylinder, had been replaced by a parallelogram. I will provide a more detailed description of this later on (…).

The day after witnessing this machine, I departed for France. Upon returning home, I dedicated myself to faithfully recalling all the parts I had seen and endeavored to understand their purpose. I drew various plans and profiles in an attempt to decipher their function. Eventually, I conceived a double-acting machine. From that very moment, I embarked on constructing a model that surpassed my expectations.

Recognizing the immense usefulness of this machine in mechanical arts, as well as its economical construction and fuel consumption advantages, I believed that the Academy would be pleased to receive the forthcoming description."

Upon returning to Paris, Betancourt presents the "Memoire sur une machine à vapeur à double effet" on December 15, 1789, and signs it as "Le chevalier de Betancourt Capitaine au service d'Espagne" (Knight of Betancourt, Captain in the service of Spain). The session records of December 16, 1789, of the Royal Academy of Sciences confirm that "Mr. Betancourt has presented a Memoire on a double-acting steam engine" and that the commissioners Jean Charles Borda (1733–1799) and Gaspard Monge (1746–1818)
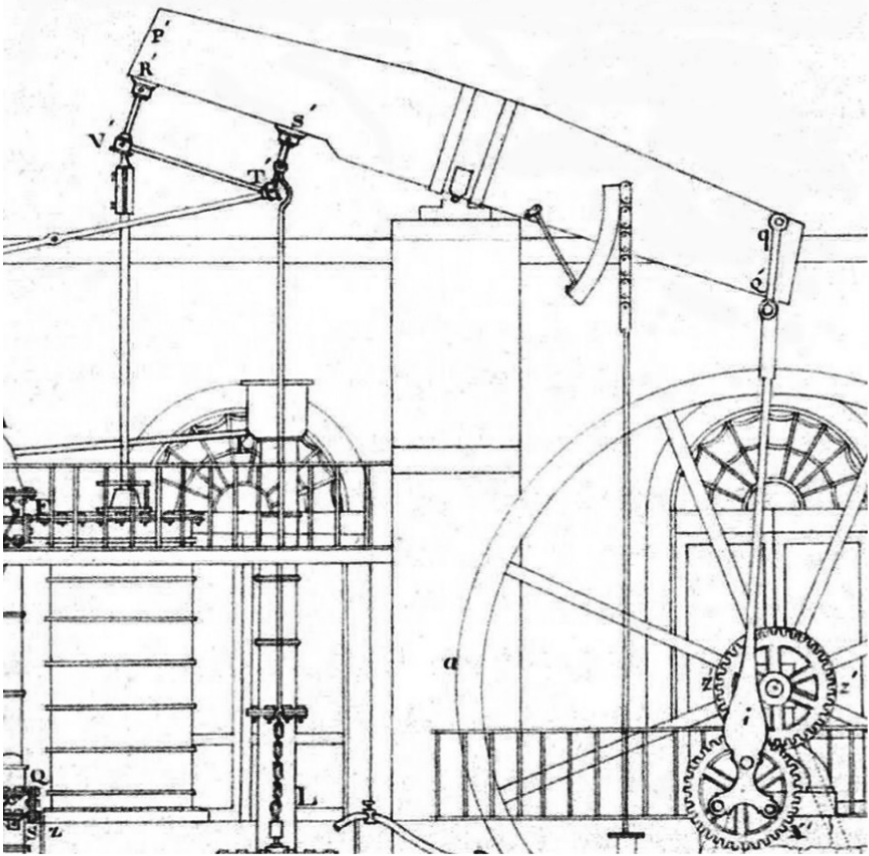
Fig. 4. Watt's double-acting steam engine. (In Betancourt, Mémoire sur une machine à vapeur à double effet (1789), plate III).

have been appointed to report on said Memoire. In the session on February 10, 1790, the commissioners' report concludes as follows:

> "We believe the Academy should applaud Mr. Betancourt's enthusiasm and expertise for introducing to France a discovery whose knowledge would not have naturally reached him until much later. The Memoire he has presented, which deserves our approval, should be published in the collection of works by foreign scholars."

In this study, we will meticulously examine specific drawings and mathematical advancements that Betancourt incorporated into the "Mémoire," focusing particularly on those pertaining to the mechanism of rectilinear guidance.

4   The Betancourt Procedure

The "Mémoire" [7], a 31-page manuscript depicted in Fig. 6, serves as a comprehensive document encompassing various facets. It commences with an introduction to the evolution of the steam engine, detailing Betancourt's expedition to England to glean insights into advancements in steam engine design. During this visit, he established connections with luminaries like Watt and Boulton, and toured facilities showcasing Watt's cutting-edge designs. Despite being limited to partial observations of the machine, Betancourt astutely outlines the machine's components, emphasizing Watt's ingenious mechanism.

The manuscript not only delves into the description of the machine's parts and its operational intricacies but also offers a formulation and resolution for what Betancourt terms the specific case of the parallelogram mechanism. This formulation enables the calculation of the length of one of the beams. Complementing the textual content, seven plates within the manuscript visually reproduce segments of the steam engine, featuring a representation of the parallelogram mechanism in three distinct positions. This representation serves as the foundational basis for the earlier mentioned calculation of the beam's length.

Our focus will now shift to this concluding section, encompassing the formulation articulated in the manuscript, auxiliary mathematical developments elucidating Betancourt's methodology, and an analysis of inherent challenges within this formulation that is based on the drawing in Fig. 5. The crux of the problem posed by Betancourt involves determining the length of the rocker arm XE, given the mechanism's three designated positions, the associated angles of the rocker arm in those positions, and the lengths of all movable bars except for XE. In contemporary terms, the quandary Betancourt presents aligns with a trajectory generation synthesis problem featuring three points of precision.

Betancourt strategically represents the mechanism in three distinct positions that correspond to varied orientations of the rocker arm: the first with an angle above the horizontal, the second in a horizontal alignment, and the third with an angle below the horizontal, symmetrically mirroring the first position.

The represented mechanism corresponds to what we have called the extended Watt mechanism, formed by two rocker arms: in the first position, AC rotates around the fixed-point A and XE rotates around the fixed-point X. Additionally, there is a coupling bar EB, so that the bars AB, BE, and XE, all joined by kinematic pairs of rotation, correspond to a singular Watt mechanism. In this mechanism, the midpoint of the coupling bar traces an approximately straight path under certain conditions. The pantograph is added to this mechanism, formed by the extension of the bar AB to form the bar AC, and the bars CD and DE parallel to BE and BC, respectively. Point D, aligned with the center of the coupling bar BE and with point A, will also trace an approximately straight path.

The choice of the two symmetric extreme positions of the rocker arm allows Betancourt to ensure that the segments DR, EM, and BP are parallel to each other and perpendicular to the horizontal line AO. Moreover, the segments EM and BP have the same length.

Although Betancourt does not mention it in the Memoire, once the length of the crank is calculated and the perpendicular bisector of the segment EL is represented, the rotation center of the rocker arm, denoted by point X in the figure, can be located.
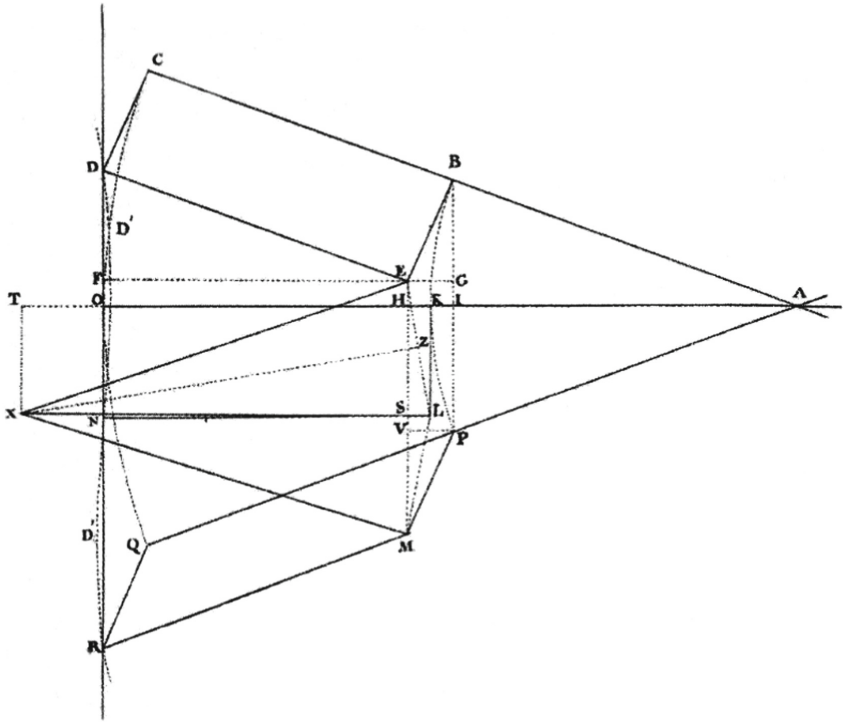
Fig. 5. Geometrical scheme used by Betancourt to describe the operation of the Watt´s mechanism. In Betancourt, Memoire sur une machine à vapeur à double effet (1789), Plate IV.

Betancourt's solution is a geometric resolution primarily based on trigonometric principles. As observed in the figure, the selected positions, along with a set of auxiliary lines, generate a significant number of right triangles that contribute to the resolution.

On Plate IV (see Fig. 5), the mechanism is depicted in three positions, and different letters are assigned to the locations of the different kinematic pairs.

Notably, points B, K, and P, on one hand, and points C, O, and Q, on the other hand, lie on two circles centered at point A, while points E, L, and M are located on a circle centered at point X. Points D, O, and R lie on a line perpendicular to segment AO, passing through point O. The location of the rocker arm in position 2 is represented by AO.

The points A, B, E, and X form an articulated quadrilateral, where points A and X indicate the positions of the Revolute joints with the fixed bar. Additionally, the bars CD and DE are added to this mechanism to form a parallelogram that constitutes a pantograph mechanism. Under specific conditions, point D traces an almost straight segment.

To distinguish the formulas included by Betancourt in the report from the auxiliary formulas added to facilitate Betancourt's approach, an asterisk has been added to the formula of the former.

The starting data are given as

$$AB = a(*) \tag{1}$$

$$BC = b(*) \tag{2}$$

$$CD = c(*) \tag{3}$$

$$CAO$$

$$\text{angle}(CAO) = \alpha(*) \tag{4}$$

The question at hand is to calculate the length of the bar XE based on the lengths of segments EL and LM.

He begins by calculating the length of segment HK. H represents the vertical projection of points E and M onto the horizontal position of segment OA, which aligns with the second position of the rocker arm. K corresponds to the position occupied by point B in the second position of the mechanism:

$$HK = OK - OH \tag{5}$$

$$OH = FE = DE\cos\alpha \tag{6}$$

$$OK = DE = b \tag{7}$$

$$HK = b(1 - \cos\alpha) = bm \tag{8}$$

Calculating the length of segment KI, I is the vertical projection of points B and P onto the horizontal position of segment OA coinciding with the second position of the rocker:

$$KI = KA - IA \tag{9}$$

$$IA = AB\cos\alpha \tag{10}$$

$$KA = AB = a \tag{11}$$

$$KI = a(1 - \cos\alpha) = am(*) \tag{12}$$

The length of segment HI will be the sum of segments HK and KI:

$$HI = HK - KI = EG \tag{13}$$

$$HI = (a + b)m(*) \tag{14}$$

From the right triangle EGB, calculate the length of the segment BG:

$$BG = \pm\sqrt{BE^2 - EG^2} \tag{15}$$

$$BE = c \tag{16}$$

$$BG = \pm\sqrt{\left(c^2 - (a+b)^2 m^2\right)}(*) \tag{17}$$

Being S the vertical projection of points E and M onto the horizontal position of segment NL coinciding with the second position of bar DE, which, when the points OHSN form a parallelogram, will have a horizontal position, the length of segment ES will be:

$$ES = HS + IB - BG \tag{18}$$

$$HS = c \tag{19}$$

$$IB = AB\sin\alpha = a\sin\alpha \tag{20}$$

$$ES = c + a\sin\alpha \mp \sqrt{\left(c^2 - (a+b)^2 m^2\right)}(*) \tag{21}$$

From the right triangle ESL, calculate the length of segment EL:

$$EL = \pm\sqrt{SL^2 + ES^2} \tag{22}$$

$$SL = HK \tag{23}$$

$$EL = \pm\sqrt{b^2 m^2 + \left(c + a\sin\alpha \mp \sqrt{\left(c^2 - (b+a)^2 m^2\right)}\right)^2} = A(*) \tag{24}$$

In the text of the Memoire, the calculation expression for the length of segment ML is included, and it is indicated that the same steps have been followed as to obtain the previous expression. We will now proceed to follow these steps in detail.

The length of segment MS will be:

$$MS = MV - HS + VH \tag{25}$$

$$MV = BG \tag{26}$$

$$HS = c \tag{27}$$

$$VH = PI = -AP\sin\alpha \tag{28}$$

$$AP = a \tag{29}$$

$$MS = \sqrt{\left(c^2 - (b+a)^2 m^2\right)} - c - a\sin\alpha \tag{30}$$

From the right triangle MSL, we calculate the length of segment ML:

$$ML = \pm\sqrt{SL^2 + MS^2} \tag{31}$$

$$SL = HK \tag{32}$$

$$ML = \pm\sqrt{b^2 m^2 + \left(\sqrt{\left(c^2 - (b+a)^2 m^2\right)} - c - a\sin\alpha\right)^2} = B(*) \tag{33}$$

Next, based on triangles MLS and LXZ, he calculates the radius of the rocker arm R, identified by the length of segment XL:

$$R = \frac{B \times A}{2bm}(*) \tag{34}$$

By identifying the segments depicted in the graph, we will be able to determine:

$$\frac{XL}{ZL} = \frac{ML}{SL} \tag{35}$$

This relationship is only true if the triangles MLS and LXZ are similar. However, that similarity does not always occur. It only occurs if:

$$ES = \frac{EM}{2} \tag{36}$$

In this case, the triangles MLS and ESL will be equal. Since the perpendicular bisectors have the same inclination but opposite directions, their intersection must occur at a point X that lies on the same horizontal line as point S. Therefore, the segment XL will be horizontal.

In this case, the triangles ESL and XLZ will be similar, and consequently, MLS and LXZ will also be similar:

$$ES = HS + EH$$

$$= HS + BA\sin\alpha - BG = c + a\sin\alpha - \sqrt{\left(c^2 - (a+b)^2 m^2\right)} \tag{37}$$

$$EM = BP = 2a\sin\alpha \tag{38}$$

That is to say, the formula deduced by Betancourt only holds if:

$$c + a\sin\alpha - \sqrt{\left(c^2 - (a+b)^2 m^2\right)} = a\sin\alpha \tag{39}$$

In other words, if:

$$c = \sqrt{\left(c^2 - (a+b)^2 m^2\right)} \tag{40}$$

Condition that is only fulfilled if:

$$m = (1 - cos\alpha) = 0 \tag{41}$$

Which, in turn, is only satisfied if:

$$\alpha = 0 \tag{42}$$

Which is incompatible with the design conditions; therefore, no case of similarity between said triangles can be established, and the deduced formula would not be applicable for design purposes.

In the subsequent comments to the mathematical formulation and as a consequence of the analysis of the obtained expressions, Betancourt states that the curve traced by point D will fit more closely to the straight-line DR as the sine of the rocker's rotation angle decreases, and therefore, as the rotated angle decreases.

On the other hand, in the second position of the mechanism, he points out that if a line is drawn from point N through the center of motion A, all points on this line, whose motion depends on that of the parallelogram, will deviate as little as possible from the corresponding vertical. Betancourt has established the dimensions of the extended Watt mechanism so that the end of the parallelogram passes through three points contained in a straight line, and he deduces that the midpoint of the coupler KL will also be contained in a straight line for those positions. Therefore, point D will be connected to the end of the rod piston, and the midpoint of the coupler will be connected to the air pump that supplies water to the boiler.

It is surprising that Betancourt exclusively uses the trigonometric procedure, makes a mistake in formulating the similarity condition, and does not use more direct geometric procedures such as obtaining the crank radius and locating its center by applying the condition of locating points E, L, and M on a circle. However, it is undeniable that he has laid the foundations for planning a synthesis of generating rectilinear trajectories with three precision points.

The approximation using a circle will appear in an example from the book "Essai sur la composition des machines," which we will discuss next.

5   A Short Historical Bibliographic Note

The history of dimensional kinematic synthesis for the generation of coupler curves is generally summarized in the development of algorithms for the design of planar mechanisms, especially the four-bar linkage, with the conditions of use of the possible number of precision points assignable as function of the degree of the coupler curve. The problem of this generation of coupler curves passing through assigned precision points is historically developed as a function of the number of precision points that can be considered assigned up to a maximum number that allows admissible solutions both in
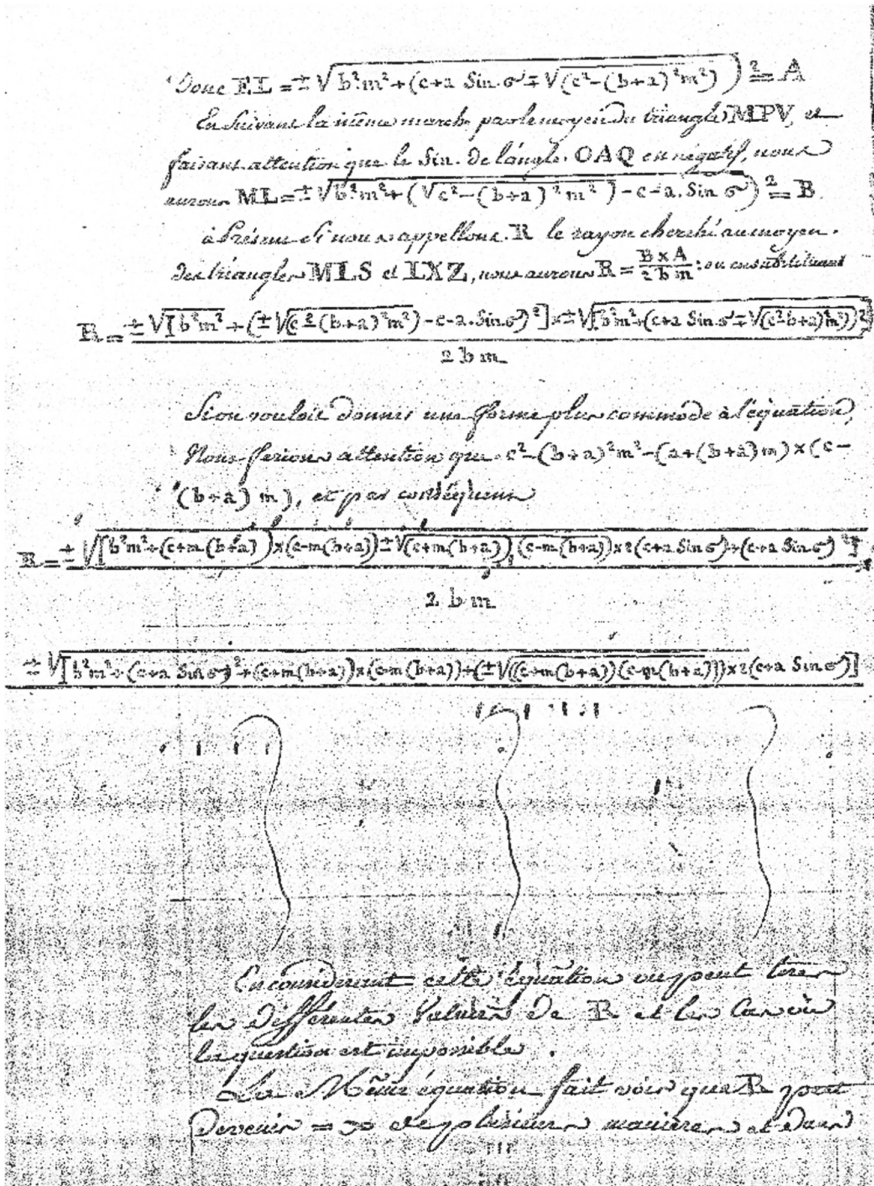
Fig. 6. Handwritten page of the memoir with some of the mathematical expressions referenced in the contribution. (In Betancourt, Mémoire sur une machine à vapeur à double effet (1789), [7]).

algebraic closed-form and in iterative numerical algorithms. These procedures are now well identified and formulated in today's technical-scientific literature, but they are not always indicated with reference to the scientists or designers who have proposed them

over the years. This is the case of the algorithm reported above by Betancourt that is not mentioned in any of these technical-scientific or historical-bibliographical references.

The history of the kinematics of mechanisms has been and is still the subject of historical study for the identification of the major sources in the development of these algorithms. In this activity we can mention the compilation of bibliographic and procedural sources in works such as [8, 9], as well as bibliographies in texts and monographs such as that by Robert Cayley, [10] and later Kurt Hain, [11], in which considerable attention is dedicated to an historical compilation of references with a remarkably rich and vast bibliography. It should be noted that even in the technical-scientific texts of the Spanish or Iberian community the work of Betancourt is not mentioned as, for example, in the book by Justo Nieto, [12] considered fundamental in the kinematics of mechanisms not only in the Spanish world but at international.

6 Conclusions

In this paper, we have meticulously elucidated a pioneering methodology articulated by Agustín de Betancourt in the "Mémoire sur une machine à vapeur à double effet," published in 1789. This innovative procedure specifically addresses the design intricacies of the extended Watt mechanism, with a focus on calculating one of the rocker arms. The approach is rooted in the condition that a point within the mechanism follows a straight path in three distinct positions. Betancourt employs geometric-analytical approximations, and despite encountering formulation errors in one aspect, we can confidently assert its groundbreaking nature at the time of publication. Notably, it surged more than half a century ahead of analogous contributions, constituting the earliest exploration into the synthesis of generating straight-line paths with three points of precision. In the case of the "Mémoire," its diffusion was limited. Despite receiving approval for publication from the French Academy of Sciences, the tumultuous revolutionary events in France likely impeded its release. Consequently, only the manuscript we have engaged with, now preserved at the École des Ponts et Chaussées, has endured the test of time.

References

1. Cuadrado, J.I., Ceccarelli, M.: El nacimiento de la teoría de Máquinas y Betancourt. In: Silva, M. (ed.) Técnica e Ingeniería en España, vol. III. El siglo de las luces, Real Academia de Ingeniería, Zaragoza (2005)
2. Erogova, O., Ceccarelli, M., Cuadrado, J.I., López-Cajún, C.S., Pavlov, V.E.: Agustín betancourt: an early modern scientist and engineer in TMM. In: Proceedings of ASME IDETC/CIE 2006 Mechanisms & Robotics Conference, Paper No. DETC2006-99198, Philadelphia (2006)
3. Iglesias, J.I.C.: Agustín de Betancourt y Molina (1758–1824). In: Ceccarelli, M. (ed.) Distinguished Figures in Mechanism and Machine Science. History of Mechanism and Machine Science, vol. 1, pp. 31–60. Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-6366-4_2
4. Ferguson, E.S.: Kinematics of mechanisms from the time of Watt. In: Contributions from the Museum of History and Technology, Paper 27, Washington, pp. 186–230 (1962)
5. Muirhead, J.P.: The Origin and Progress of the Mechanical and Inventions of James Watt, 3 volumes. Murray, London (1854)

6. Koetsier, T.: A contribution to the history of kinematics – I. Watt straight-line linkages and the early French contributions to the theory of the planar 4-bar coupler curve. Mech. Mach. Theory **18**, 37–42 (1983)

7. Betancourt, A.: Mémoire sur une machine à vapeur à double effet. Bibliothèque de l'École National des ponts et chaussés, Paris (1789)

8. De Groot, J.: Bibliography on Kinematics. Eindhoven University of Technology, Eindhoven (1970)

9. Nolle, H.: Linkage coupler curve synthesis: a historical review — I. Developments up to 1875. In: Mechanism and Machine Theory, vol. 9, pp. 147–168 and pp. 325–348 (1974). https://doi.org/10.1016/0094-114X(74)90034-2

10. Cayley: On three-bar motion. Proc. Lond. Math. Soc. **s1–7**(1), 136–166 (1875). https://doi.org/10.1112/plms/s1-7.1.136

11. Hain, K.: Applied Kinematics. McGraw-Hill, New York (1967)

12. Nieto, N.J.: Sintesis de Mecanismos. Editorial AC, Madrid (1977). (in Spanish)