A Historical Review of Polyhedral Linkages

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Abstract. Polyhedral linkages are linkages that resemble polyhedral shapes at different configurations. This paper summarizes the necessary geometrical fundamentals of polyhedral geometry and presents a historical and critical review of the polyhedral linkage designs available in the literature. Basic definitions of polyhedral geometry and operations are needed to comprehend and design polyhedral linkages. First, early works on polyhedral linkages are presented, where flexible polyhedra with rigid faces and flexible edges are issued. The final part is reserved to conformal polyhedral linkages, which go through shape transformations while plane, dihedral and solid angles are preserved. Conformal polyhedral linkages are examined in four categories: 1) Jitterbug-like linkages with screwing polygonal links connected to each other with dihedral angle preserving links, 2) polyhedral linkages with planar kinematic chains in radial motion planes, 3) polyhedral linkages with planar kinematic chains on faces, that are connected to each other with dihedral angle preserving links, and 4) other conformal polyhedral linkages.

Keywords: Polyhedral Geometry · Flexible Polyhedra · Conformal Polyhedral Linkages

1 Introduction

A polyhedron can be thought as a linear approximation of a solid. First, polyhedra appeared as solids in ancient times. Hundreds of carved stone spheres, believed to date to around 2000 BC, have been found in Scotland. Some are carved with lines corresponding to the edges of regular polyhedra [1]. Platon, Euclid, Archimedes and many other famous ancient thinkers worked on polyhedral shapes. Folding lattices to obtain polyhedra was first presented in detail by Dürer in 1525 [2]. A detailed book on historical development and fundamentals of polyhedra is written by Cromwell [3].

Polyhedral linkages are linkages that resemble polyhedral shapes at different configurations. In the 19th century, polyhedral linkages appeared as a consequence of a discussion of mathematicians who were trying to answer whether all polyhedra are rigid, or else how flexible polyhedra can be obtained. The name "polyhedral linkage" was first coined by Goldberg in 1942 [4]. In 20th century inventors, scientists and engineers came up with many different types of polyhedral linkages.

Polyhedral linkages have been a hot topic among mathematicians and engineers in the last three decades. To comprehend and design the transformations for such linkages

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one needs to understand basics of polyhedral geometry and operations. After summarizing the necessary geometrical fundamentals of polyhedral geometry, this paper presents a historical and critical review of the available polyhedral linkage designs. The fundamental definitions of polyhedral geometry are given in Sect. 2. Section 3 presents the early works on polyhedral linkages. Section 4 presents conformal polyhedral linkages. Section 5 concludes the paper.

2 Fundamental Definitions of Polyhedral Geometry

Just as a thorough understanding in plane kinematics necessitates a good knowledge on polygons and circles, in order to study spatial kinematics, one needs to master the geometry of polyhedra and the sphere. The name *polyhedron* comes from the two Greek words *poly*, meaning many, and *hedron*, meaning face. Polyhedra are three-dimensional compact figures bounded by planes. The planar bounds constitute polygonal *faces*. The planes intersect along line segments called *edges* and edges meet at points called *vertices*. The number of edges that meet at a vertex is called the *valency* of a vertex and a vertex with valency *n* is called an *n-valent* vertex. A *vertex figure* is a planar or spherical polygon which describes how the faces are arranged around a vertex [3]. A 3-valent vertex figure is depicted in Fig. 1. There are three different fundamental angles of polyhedra. A *plane angle* is the inner angle of a polygonal face. The plane angle of a face *A* is shown, but the plane angles for faces *B* and *C* are not designated in Fig. 1. A *dihedral angle* is the angle between two adjacent face of a polyhedron. The dihedral angle between faces *B* and *C* is shown, but the dihedral angles between face pairs *A*-*B* and *A*-*C* are not designated in Fig. 1. A *solid angle* is a quantity assigned to a vertex as the area of the unit sphere portion corresponding to the vertex. Its unit is *steradian*s. The angle by which the sum of the plane angles around a solid angle is less than 2π is called its *deficiency* [3]. Descartes proved that the sum of deficiencies of the solid angles of a convex polyhedron is 2π [5].

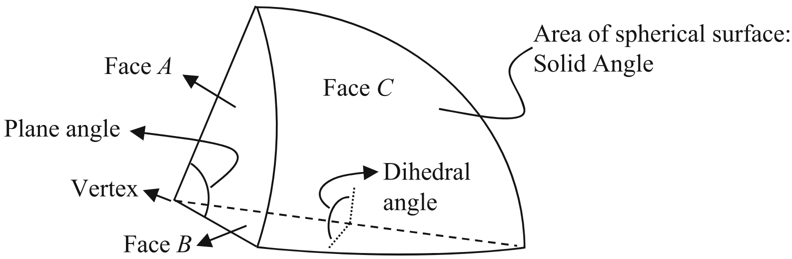

Fig. 1. A plane angle, a dihedral angle and solid angle around a vertex

Polyhedra that possess some kinds of symmetry are of special interest in polyhedral linkage design, just as many other applications of polyhedra. The notion of transitivity makes explicit the intuitive idea that vertices, edges and faces are equivalent or indistinguishable, that they all look the same no matter which is focused on. A polyhedron is said to be *face-transitive* or *isohedral* if, for any pair of faces, there is a symmetry of the polyhedron which carries the first face onto the second. This means that the polyhedron looks the same when viewed face on, no matter which face is presented to the eye. A polyhedron is *vertex-transitive* or *isogonal* if any vertex can be carried to any other by a symmetry operation. A polyhedron is *edge-transitive* or *isotoxal* if any edge can be carried to any other by a symmetry operation. Every edge-transitive polyhedron has to be either face transitive or vertex transitive [3].

If all vertices of a polyhedron lie on a sphere then the sphere is called its *circumscribed sphere* of a *circumsphere.* If all faces of a polyhedron are tangent to a sphere, the sphere is called its *inscribed sphere* or *insphere.* If there is a sphere that is tangent to the midpoints of all edges of a polyhedron, it is called a *midsphere* [6]. Michon calls a polyhedron with a circumsphere as an *equiradial polyhedron*, a polyhedron with a midsphere as a *canonical polyhedron* and a polyhedron with an insphere as an *orthohedral polyhedron* [7] (these definitions are not common in the literature). Another name for an orthohedral polyhedron is a *tangential polyhedron* [8, 9].

Descartes observed that, for an equiradial polyhedron, all faces have circumscribed circles, i.e. they are cyclic polygons. Cyclic faces is a necessary, but not sufficient condition for an equiradial polyhedron. But it is a sufficient condition for *trilinear* (or *simple*) polyhedra, which comprises trivalent vertices only. On the other hand, polyhedra with triangular faces only are called *simplicial* polyhedra [10]. These two types of polyhedra turn out to be important to construct some polyhedral linkages. Recently Wohlhart [11] coined a name for polyhedra with cyclic faces: *cyclic polyhedra.*

There are several relations between polyhedra. A polyhedron can be derived from another polyhedron via certain operations. An important relation between polyhedra is *duality*. In general, duality is a mathematical concept such that if dual of A is B, then dual of B is A. The dual of a polyhedron is obtained by replacing the faces of a polyhedron with vertices and vice versa. Theoretically all polyhedra have duals, but not all are finite polyhedra. If we disregard metric properties, this is combinatorial duality. Metric properties can be considered via projective duality by applying reciprocation with respect to a sphere [12]. Some dual polyhedra are illustrated in Fig. 2.

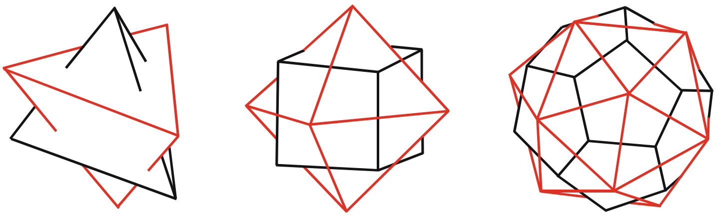

Fig. 2. Tetrahedron (self dual), cube-octahedron duals and dodecahedron-icosahedron duals

Dual polyhedra can be obtained from each other by means of *truncation*s (cuttings of pyramids on each vertex) and *augmentation*s (or *cumulation*s) (assembling pyramids on each face) [3]. For the dual polyhedra P and P', this fact is illustrated in Fig. 3, where $P1$-$P1'$, $P2$-$P2'$ and tP-aP are duals of each other; $P1$-$P1'$, $P2$-$P2'$ are forms in between P, P' and tP, aP [13]. The truncation/augmentation series of cube-octahedron dual pair is illustrated in Fig. 4. There is also *edge truncation*, which replaces edges with new rectangular faces, while n-valent vertices are replaced with new n-gons [14].

Besides truncation and augmentation, there are several other polyhedral operations to obtain new polyhedra: rectification (or complete truncation), bitrancation, omnitruncation, alternation (or partial truncation), elevation, stellation, snubbification, cantellation (or expansion), contraction, etc. (see chapter 21 or [15]). These operations can be used to design a family of polyhedral linkages (as in [16]) and also they frequently appear as configuration transformations of polyhedral shapes in a polyhedral linkage.

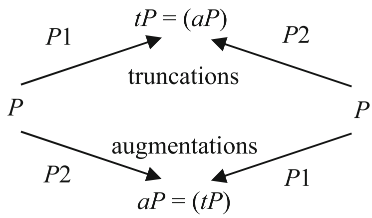

Fig. 3. Truncation/augmentation sequence diagram (adopted from [13])

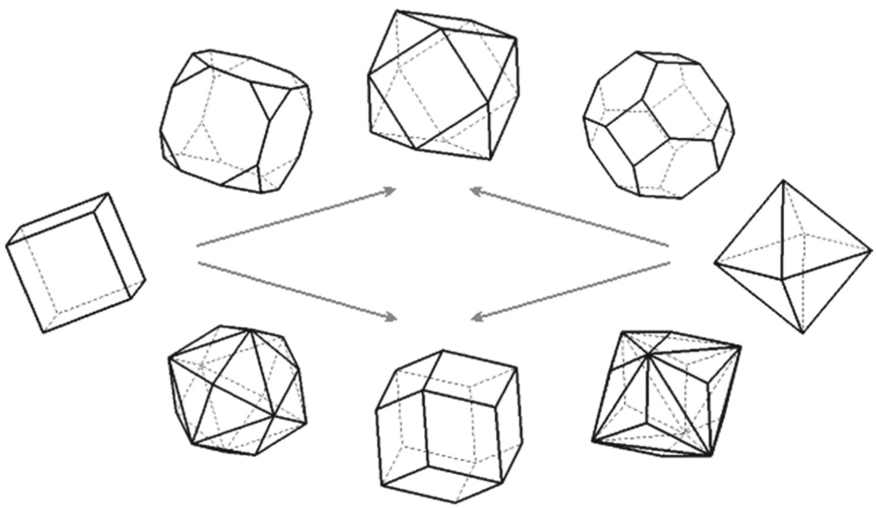

Fig. 4. Truncation/augmentation series between the cube and the octahedron [13]

If the line segment connecting any two points inside a polyhedron completely remains within the polyhedron, it is called a *convex* polyhedron; otherwise it is called *non-convex* or *concave*. A *regular polygon* is an equilateral and equiangular polygon. A vertex transitive polyhedron with regular polygonal faces is called a *uniform polyhedron*. Face- and edge-transitive uniform polyhedra are isohedral and are called *regular* [17]. The 5 regular convex polyhedra are the *Platonic solid*s. Duals of Platonic solids are Platonic as well: tetrahedron is self-dual, cube and octahedron are duals and dodecahedron and icosahedron are duals (Fig. 2). Each Platonic solid has a circumsphere, a midsphere and an insphere. The concave regular polyhedra are the four star polyhedra called *Kepler-Poinsot polyhedra*. There are various other definitions such as semi-regular, quasi-regular, half-regular versi-regular etc. (see for ex. [3] and [18]), but these definitions differ in different resources. Following [19], there are 77 kinds of uniform polyhedra: the 5 Platonic solids (convex regular polyhedra), the 13 *Archimedean solids* (convex semi-regular polyhedra), the 4 Kepler-Poinsot star polyhedra (concave regular polyhedra), the 53 non-regular star polyhedra, and the 2 infinite families of uniform *prism*s and *anti-prism*s. The 13 convex *Archimedean solids* are great rhombicuboctahedron, icosidodecahedron, small rhombicosidodecahedron, small rhombicuboctahedron, snub cube, snub dodecahedron, truncated cube, truncated dodecahedron, truncated icosahedron, truncated octahedron, truncated tetrahedron. The duals of Archimedean solids are the 13 *Catalan solids*. Archimedean solids are isogonal and equiradial, whereas Catalan solids are isohedral and tangential. Two of the Archimedean solids, the cuboctahedron and the icosidodecahedron are also isotoxal. Convex polyhedra with regular faces only are the infinite family of *prism*s, *anti-prism*s and the 92 *Johnson solid*s [20]. A polyhedron with congruent faces (not necessarily regular) is called a *homohedron*. Isohedra are special types of homohedra.

For some polyhedra we encounter, left- or right-handedness. A *chiral polyhedron* is one that has two distinct forms or *enantiomorph*s: *laevo* (left-handed) and *dextro* (right-handed) version [21]. The two Archimedean solids, the snub cube and the snub dodecahedron are well-known examples of chiral polyhedra.

Euler (1707–1773) investigated topological properties of polyhedra and came up with a general simple formula, which can be derived from Descartes's theorem on deficiencies: $V - E + F = 2$, where V, E and F are number of vertices, edges and faces of a polyhedron, respectively [22]. Among several applications of this formula, in mechanism science it is used to evaluate the number of independent loops of a mechanism, where V, E and F correspond to number of joints, links and loops, respectively. Euler's formula is valid for most polyhedra, but not all. A polyhedron that satisfies Euler's formula can be deformed into a sphere, and vice versa. Euler's formula can be modified as $V - E + F = 2 - 2g$, where g is, roughly speaking, the number of tunnels through the polyhedron and is called the *genus*. For example, the genus of a torus-like polyhedron is 1. $\chi = V - E + F$ is called the *Euler characteristics* of the polyhedron [3]. Some fundamentals of polyhedral geometry, that are frequently required in polyhedral linkage design are presented in this section. The reader can refer to [1, 3, 5] and [14] for further details of polyhedral geometry.

3 First Works on Polyhedral Linkages: Flexible Polyhedra

Euler assumed that two polyhedra are identical if they have the same faces [3]. Then people started questioning whether all polyhedra are rigid, or whether there are flexible polyhedra. Here, what we mean by a flexible polyhedron is a polyhedral surface, where the faces are rigid, but the edges are allowed to flex as if there is a hinge along the edge. In 1813, Cauchy proved that a closed convex polyhedron is rigid (rigidity theorem) [23]. In 1987, Bricard presented first examples of flexible polyhedra, which are the three types of concave octahedra with self-intersecting faces: line-symmetric octahedra, plane-symmetric octahedra and doubly collapsible (or skew flexible) octahedra [24]. Each type of octahedron comprises 8 faces, 12 edges and 6 vertices (just as a regular octahedron) and if two of the faces are removed, an over-constrained linkage with 6 revolute (R) joints (corresponding to the remaining 6 edges) is obtained. Later, Bricard added three more over-constrained 6R linkages to his list [25]. Goldberg examined Bricard linkages along with other spatial linkages made of rigid flat plates hinged together [4]. The plates may be considered as the faces of a polyhedron, not necessarily convex, nor even closed. This definition would result in linkages with only revolute joints, where all joints axes of a link are necessarily coplanar (Fig. 5).

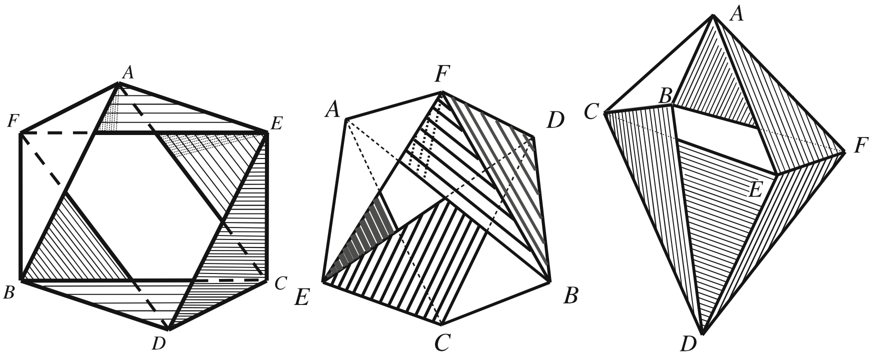

Fig. 5. From left to write: line-symmetric, plane-symmetric and doubly collapsible octahedra of Bricard [24]. For each octahedron the faces are ABC, DEF, BCD, CAE, ABF, AEF, BFD and CDE.

Bennett presented infinitesimally deformable octahedra [26] and Stachel generalized the theory to higher order flexibility of octahedra [27]. Goldberg worked on other infinitesimally deformable polyhedra, and called them *shaky polyhedra* [28]. In 1977, Connelly discovered a mobile polyhedron with nonintersecting rigid faces [29]. Right after Connelly, Steffen found a simpler example with only 9 vertices and 14 faces [3, 30]. Maksimov proved that all polyhedra with nonintersecting triangular faces and fewer than nine vertices are rigid, which implies that Steffen's flexible polyhedron has as few vertices as possible [3, 31]. Connelly et al. proved that the volume of orientable flexible polyhedra are constant during the flexion [32]. Roth worked on rigid and flexible frameworks (rods connected but freely pivoting at vertices – not necessarily representing polyhedra with planar faces) and proved that a framework given by a convex polyhedron is rigid if and only if all faces are triangular, i.e. a convex simplicial polyhedron [33].

4 Conformal Polyhedral Linkages

Bricard discovered his polyhedral linkage while seeking the answer to the following question: "Do there exist polyhedra with invariant faces that are susceptible to an infinite family of transformations that only alter solid angles and dihedrals?" [24]. In such polyhedral linkages, faces remain rigid, whereas dihedral angles change. In the second half of 20$^{\text{th}}$ century, several over-constrained linkages were invented such that the linkages resemble some polyhedral forms, where all angles remain constant, but the size and/or shape of the faces change. Let us call such linkages as *conformal polyhedral linkage*s. These polyhedral linkages can be classified into categories as follows: 1) Jitterbug-like linkages, 2) Polyhedral linkages with radial motion planes, 3) Polyhedral linkages with planar link groups, 4) Other conformal polyhedral linkages.

4.1 Jitterbug-like Linkages

The first conformal polyhedral linkage is discovered in 1948 by Fuller, which he calls the *Jitterbug*, since the motion looks like the famous ballroom dance of the time [34]. Fuller discovered the Jitterbug while he was working on sphere packing. The Jitterbug comprises 24 struts that form 8 triangles and the assembly has a single degree-of-freedom (dof) motion in which the triangles make congruent motions along their axes [35]. Fuller just mentions flexible joints, but does not describe the joint type explicitly. Yet, he describes the motion from a cuboctahedron to first an icosahedron and finally to an octahedron in detail (Fig. 6) [36]. Note that the motion can result in a laevo or dextro version of the linkage. Fuller also mentions that the assembly can be further deformed to a double tetrahedron and even an eight-fold triangle. Hence, he found an assembly transforming between the symmetrical figures of Plato and Archimedes [35].

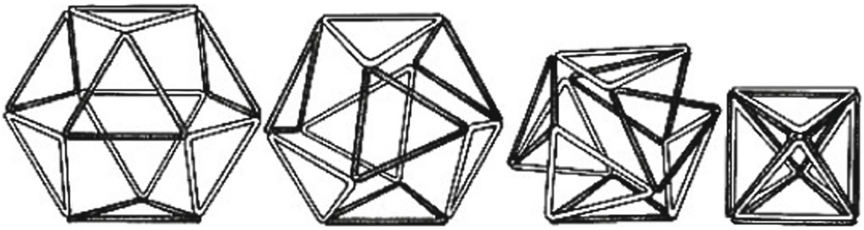

Fig. 6. The Jitterbug motion from maximal to minimal configuration [33]

In 1963, Stuart generalized the Jitterbug transformation as transformations from a regular/semiregular polyhedron to another and demonstrated that the vertices would move along intersection of two cylinders with intersecting axes and equal radius, that is an ellipse [37]. Clinton describes this transformation as follows: By allowing each surface to rotate about its axis, translate along its axis, and maintain connection with one of its paired vertices; the surfaces enclosing the polyhedron will transform into another polyhedral form [38].

In 1979, Verheyen introduced the concept of *dipolygonid*s for Jitterbug-transformers as a finite set of regular polygons of two types connected one to another by a common vertex and as a set having a finite group of rotations as its symmetry group [39] and in 1989 he presented a complete classification of all dipolygonids [40]. If a polygon rotates clockwise, the adjacent one rotates counter-clockwise. So, the polygons around a vertex should be even in number. The polyhedra with odd-valent vertices are self-enantiomorphus (contain both laevo and dextro chiral forms), i.e. the double version of the polyhedron should be used (such as a double-tetrahedron – which is actually an octahedron with 4 pairs of coplanar faces). Recently Roelofs [41] presented a detailed discussion on even valence requirement for Jitterbug-like linkage and double polyhedra, where he concluded that "For the Jitterbug transformation to be possible, we should replace the requirement that all vertices have even valence by the requirement that the polyhedron is two-colourable. This is a stronger requirement.". Kovács et al. used one of Verheyen's dipolygonids to simulate the motion of polyhedral viruses [42].

In 1974, Dreher designed a two-dof joint, so-called *constant dihedral hinge*, for the Jitterbug. The Jitterbug can be constructed with cubic links instead of triangles, such that the three vertices of the cube correspond to a triangular link of the Jitterbug, as demonstrated by Tomura and called the Tom cube. De Clippeleir discovered a rhombic dodecahedral Jitterbug-like linkage [43]. A giant kinetic sculpture with a triangular side length of 8 m was constructed by Schwabe's initiation and demonstrated during the Zurich Expo fair in 1991. The sculpture was called the *Heureka octahedron* and after three months in operation it collapsed into the tetrahedral form [34].

After the exhibition of the Heureka octahedron, many mathematicians and engineers started working on this incredibly over-constrained multi-loop spatial linkage [44–54]. Before this, over-contrained spatial linkages would typically be comprised of a single loop. The Jitterbug not only has multi loops, but also all loops have multi-dof, yet the whole assembly has single-dof. For 8 triangular links connected with 12 two-dof joints, the Chebyshev-Grübler-Kutzbach formula predicts a dof of –6, hence the degree of over-constaint is 7. The first exact mobility calculation of the 1-dof Jitterbug motion is given by Stachel and he also demonstrates the double tetrahedral linkage as a different assembly mode of the Heureka octahedron [44, 45]. Zsombor-Murray presented the *Brussels folding cube*, which is a Jitterbug-like linkage [46]. Wohlhart introduced *turning tower*s and *screwing tower*s as generalizations of the Jitterbug, where the polygonal faces are not necessarily identical or even regular [47–50, 55, 56]. Screw joints replace revolute joints in the screwing towers. In [49], Wohlhart introduced the *octoid* as a generalization of the Heureka octahedron with non-regular faces and in [50] he introduced the *toroidal linkages* which consist of arbitrary number of equal double octoids to form a torus; the *breathing ball* as an assembly of six octoids; the *spheroid* and the *star-cube* as the outer and inner part of the breathing ball and finally the *Fulleroid* which is obtained by simplifying the spheroid. Kiper further generalized the Fulleroid by making use of the cumulation series of the cube and also presented *rombohedral*, *dipyramidal* and *stellated* linkages [16]. The symmetrical 8R linkage examined in [16] was used as a building block for polyhedral linkages by Wei and Dai [57].

Röschel presented a methodology to design new Jitterbug-like linkages by combining an axial Darboux motion or a planar equiform motion with a congruent motion [51–54]. Röschel's designs include torus-like linkages as well [54]. Verheyen and Röschel's approach to synthesize Jitterbug-like polyhedral linkages was to consider the in-plane motion of a polygonal element and transfer this motion spatially by transforming them into neighboring planes. As a different approach, Kiper and Söylemez examined the spatial loops that comprise the linkages, proposed a formal definition for Jitterbug-like linkages and listed several properties of these linkages [58, 59]. The definition is as follows: Let E, F, V, n_i denote the number of edges, faces, vertices and valency of i^{th} vertex ($i = 1, \ldots, V$) for a polyhedral shape P. A *Jitterbug-like linkage* is a mobile linkage F-many polygonal links such that the polygonal links remain parallel to faces, and dap links that remain perpendicular to the edges of P with the joint axes intersection of a dap link on the corresponding edge. No polygonal (or dap) link is directly connected to another polygonal (or dap) link. The joint axes of a polygonal link are all parallel to each other. In any configuration, the planes defined by intersections of joint axis bound a finite volume, called the *supporting polyhedron*, and this shape can be obtained from P by a conformal transformation. The topological shape of the supporting polyhedron is invariant and it is called the *base polyhedron.* If all side lengths of the supporting polyhedron vary proportionally, such linkages are called *homothetic* Jitterbug-like linkages. A *wholly representative Jitterbug-like linkage* is obtained if the number of dap links is E, hence to each vertex i of P there corresponds a spatial loop with $2n_i$ revolute joints. Based on these definition, it is shown that the spherical indicatrix of the linkage is immobile and the link dimensions of the indicatrix of a loop are uniquely given by the associated dihedral and plane angles. If the base polyhedron has genus zero, it contains at least one 4-valent vertex. The polygonal links have Schönflies motion and the dap links are in pure translation. Several results are concluded for homothetic Jitterbug-like linkages: 1) The instantaneous rotation axes of neighboring faces intersect each other; 2) A mobile linkage can be obtained if and only if two homothety centers of neighboring faces are in symmetrical position with respect to the common edge; 3) The homothety centers on the faces around a vertex are equidistant to the vertex and the plane angles around a vertex are interrelated; 4) A polyhedron can be used as a base polyhedron for a homothetic Jitterbug-like linkage if and only if one can locate spheres centered at vertices of the polyhedron such that on all faces, the associated spheres meet at a common point. Also, some results are deduced for special cases such as if the supporting polyhedron is a homohedron, a tangential polyhedron, a trilinear polyhedron or a polyhedron with 4-valent vertices. Kiper and Söylemez used these results to analyze Wren platforms, that can be considered as polyhedral shapes with two n-gons and n diagonal links [60]. They also presented polyhedral linkages obtained by adding links around the vertices or inside a homothetic Jitterbug-like linkage using Cardan motion of the polygonal links [61, 62]. Recently Warisaya et al. [63] presented a design methodology for Jitterbug-like linkages to obtain auxetic behavior for polyhedral approximation of an arbitrary surface – open or closed. The works of Röschel, Kiper and Sözlemez and Warisaya et al. set certain rules for possible base polyhedra of Jitterbug-like linkages, but a certain classification is still an open problem.

4.2 Polyhedral Linkages with Radial Motion Planes

A polyhedral linkage with radial motion plane expands and contracts with links moving in radial planes, typically all passing through the center of a polyhedral shape. Such linkages were first presented by Wohlhart – the star transformers [49, 64] (Fig. 7). The polygonal links of a star-transformer reciprocates between dual polyhedral forms.

The most famous polyhedral linkage with radial motion planes is a toy: Hoberman's sphere, where Hoberman introduced the *angulated element* pairs that constitute the edges of the polyhedral shape [65]. As oppose to the star-transformers, in Hoberman sphere, all links radially expand or contract simultaneously. For both type of linkages, the links connecting the polygonal links realize Cardan motion in their respective radial

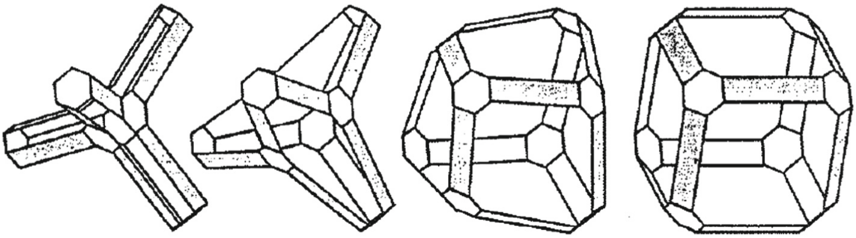

Fig. 7. The star-cube transformer [49]

planes [66]. Wohlhart presented a family of linkages that are similar to the Hoberman sphere, but with regular scissors along edges rather than angulated scissors [67]. The scissors may be along radial planes or planes perpendicular to that, such that an edge-truncation type of transformation occurs. Hoberman also invented the switch-ball toy that can reconfigure between two different color modes [68]. The kinematic structure of the switch-ball mechanism is equivalent to that of the star-cube transformer shown in Fig. 7, except that gear pairs between binary links are added to avoid assembly mode change. Wei and Dai coined the name polyhedral linkages with *radially reciprocating motion* and published a series of papers for analysis and design of these linkages [56, 69, 70]. Li et al. presented reconfigurable polyhedral linkages that have a Hoberman sphere-like mode and a star transformer-like mode [71].

Recently, Wohlhart introduced *cyclic polyhedra* and described how to design a polyhedral linkage by inserting appropriate *streching stars* into the faces of a cyclic polyhedron [11, 70]. Star transformers are a special case of these linkages, but in the general case, the binary links do not necessarily move in planes that all intersect at a center. These linkages also transform in between dual polyhedral shapes. In [72], Wohlhart points out that two neighboring cyclic polygonal face centers form a pair of reflection points via a reflection plane through the common edge. This is the same geometrical condition presented in [58] for homothetic Jitterbug-like linkages.

4.3 Polyhedral Linkages with Planar Link Groups

Several conformal polyhedral linkages are reported such that open or closed planar kinematic chains are used to transform the shapes of the faces. Early examples are presented by Wohlhart [73–77]. The linkages in [73] comprise planar open kinematic chains with n-many peripheral binary links connected around an n-gonal central link for scaling an n-gonal face of a regular polyhedron. Binary links meet each other via dap links (with as many revolute joints as the vertex valency) representing the vertices. In [74], Wohlhart generalized the method in [73] to irregular polyhedra. Wei et al. presented polyhedral linkages for which the planar kinematic chains on the faces are obtained by combining a planar chain of [73] with its mirror image such that the laevo and dextro pairs of central polygonal links and binary links constitute parallelogram loops [78]. Li et al. [79] showed that there is an assembly mode of these linkages where the corresponding pair of links coincide and the linkages becomes equivalent to the linkages in [73]. The planar kinematic chains of [75] are similar to the ones in [73], but instead of binary links there are triangular links meeting mirror image links on adjacent faces via dap links moving along edges of the supporting polyhedron. In [76], the multi-dof planar single-loop kinematic chains on the faces comprise $2n$ many right triangles, which are obtained by dissecting an n-gonal regular face into $2n$ equal pieces. The planar kinematic chains of [76] are also used in [75] for double pyramidal linkages with irregular triangular faces. In [77], Wohlhart made use of Kempe's single-dof over-constrained planar double-chain linkages with parallelogram loops and a second type of planar double-ring linkage comprising Hoberman's angulated elements. Kiper and Söylemez [80] also presented polyhedral linkages with planar loops of angulated elements on faces, but the linkages in different faces are connected to each other via dap links at vertices, whereas the connections in [77] are a pair of dap links on the edges.

Gosselin and Gagnon-Lachance presented polyhedral linkages with single-dof closed kinematic chains on the faces, where a chain on an n-gonal face comprises n many 6-link sub-assembly with two of the links common to all sub-assemblies [81]. Kiper and Söylemez presented a general methodology to design polyhedral linkages with planar kinematic chains on faces, and they demonstrated the method with three different link groups used for a homohedral and two tangential polyhedral linkages [82]. The three examples of kinematic chains are selected to represent three different kinds of deployment of central and peripheral link groups: 1) variable + fixed distance, 2) fixed + variable distance, 3) variable + variable distance from the center to the side of a face. The planar kinematic chains may be open or closed, as well as single-dof or multi-dof, but when they are constrained with dap links at vertices or edges, the resulting polyhedral linkage has single-dof. A recent linkage design by Broeren et al. has skew pantographs on triangulated faces of an arbitrary closed surface [83].

4.4 Other Conformal Polyhedral Linkages

Agrawal et al. used prismatic joints along edges, and along diagonals of faces if necessary, to obtain conformal regular and semi-regular polyhedral linkages [84]. Besides [41], in order to model motions of trilinear polyhedral viruses Kovács et al. designed a polyhedral linkage by connecting regular polygons via a pair of parallel binary links using spherical joints [85]. Kiper and Söylemez assembled equilateral Bennett 4R loops along sides of faces of regular and semi-regular polyhedra with triangular faces (square and pentagonal faces are feasible alongside with triangular faces) [86].

Wang and Kong presented several regular polyhedral linkages with 6R, 8R or 10R spatial loops on faces connected by spherical joints [87, 88] or non-intersecting 3R chains [89]. Similarly, Chen et al. designed a tetrahedral and a cubic linkage comprising threefold symmetric Bricard 6R loops and spherical and revolute joints at other vertices, which move like a dipolygonid [90, 91]. Zhang et al. obtained cubic linkages by placing 8 threefold-symmetric Bricard loops at the vertices of a cube and connecting the 8 loops in 4 different ways [92]. Wang et al. implemented Wren platforms at the faces of regular polyhedra to obtain conformal polyhedral linkages [93]. Gu and Chen constrained the motion of a 9R loop as an assembly of three 3R chains with parallel axes (9R version of a Sarrus 6R) using 3 pairs of spherical 4R loops of each face of a 3-valent vertex to obtain trilinear origami polyhedral linkages [94]. Gu et al. used Sarrus 6R loops along edges to obtain conformal regular polyhedral linkages [95].

5 Conclusions

The number of works on polyhedral linkages has been rapidly increasing especially in the last 30 years and the trend shows that there is much more to do. Most of these recent works are done by engineers, who first need to properly comprehend polyhedral geometry. This paper may serve as an initial read to examine the fundamentals of polyhedral geometry and operations, such as regularity, duality, truncations and augmentations, that are necessary to design polyhedral linkages, and also presents a historical review.

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