Port-Hamiltonian Systems' Modelling in Electrical Engineering

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Abstract. The port-Hamiltonian (pH) modelling framework allows for models that preserve essential physical properties such as energy conservation or dissipative inequalities. If all subsystems are modelled as pH systems and the inputs are related to the output in a linear manner, the overall system can be modelled as a pH system, too, which preserves the properties of the underlying subsystems. If the coupling is given by a skew-symmetric matrix, as usual in many applications, the overall system can be easily derived from the subsystems without the need of introducing dummy variables and therefore artificially increasing the complexity of the system. Hence the framework of pH systems is especially suitable for modelling multiphysical systems.

In this paper, we show that pH systems are a natural generalization of Hamiltonian systems, define coupled pH systems as ordinary and differential-algebraic equations. To highlight the suitability for electrical engineering applications, we derive pH models for MNA network equations, electromagnetic devices and coupled systems thereof.

1 Port-Hamiltonian Systems Modelling in a Nutshell

Port-Hamiltonian (pH) systems are a generalization of Hamiltonian systems

$$\dot{x} = J \cdot \nabla H(x), \quad x(0) = x_0 \tag{1}$$

with $x = (p, q)$ consisting of generalized position $q(t) \in \mathbb{R}^n$ and momentum $p(t) \in \mathbb{R}^n$ (where $t \in [0, T]$), the skew-symmetric matrix J given by

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

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and the Hamiltonian $H(x) = H(p,q) = U(p) + V(q)$ given as the sum of potential and kinetic energy, which maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and is twice continuously differentiable. The Hamiltonian flow $\varphi(t; x_0))$, i.e., the solution of (1) at time point t, starting at the initial value $x(0) = x_0$, is characterized by four geometric properties:

1. *Preservation of the Hamiltonian*:
$$\frac{\mathrm{d}}{\mathrm{d}t} H(\varphi(t;x_0)) = (\nabla H(\varphi(t;x_0)))^\top J(\nabla H(\varphi(t;x_0))) = 0.$$
2. *Time-reversibility*:
$$\rho \circ \varphi(t;x_0) \circ \rho \circ \varphi(t;x_0) = x_0,$$
with $\rho(p,q) = (-p,q)$, which is a direct consequence of the ρ-reversibility of the Hamiltonian flow: $\rho \circ J\nabla H(\varphi(t;x_0))) = -J\nabla H(\rho \circ \varphi(t;x_0)))$.
3. *Symplectic structure of the Hamiltonian flow*:
$$\Psi(t)^\top J^{-1} \Psi(t) = J^{-1}, \qquad \Psi(t) := \tfrac{\partial \varphi(t;x_0)}{\partial x_0},$$
which is a direct consequence of the skew-symmetry of J.
4. *Volume-preservation*:
$$(\det \Psi(t))^2 = 1,$$
which follows immediately from the symplectic structure in 3.

First generalization step: arbitrary skew-symmetric matrices J

If we replace in (1) J by an arbitrary skew-symmetric matrix, the Hamiltonian is still preserved. As x will loose its characterization as generalized positions and momenta of classical mechanics, time-reversibility will generally not hold anymore. However, the symplectic structure of the flow still holds in the case of a regular J, and volume preservation is still a consequence of the Hamiltonian flow.

Second generalization step: adding dissipation to the system

Allowing the flow to become dissipative, we may generalize (1) to the dissipative Hamiltonian system

$$\dot{x} = (J - R) \cdot \nabla H(x), \quad x(0) = x_0 \tag{2}$$

with $R \geq 0$ being symmetric and positive semi-definite. In this case, the flow will neither be symplectic nor volume preserving, and the preservation of the Hamiltonian is replaced by the dissipativity condition

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = (\nabla H(x))^\top \dot{x} = -(\nabla H(x))^\top R \nabla H(x) \leq 0 \\ &\Rightarrow H(x(t)) = H(x_0) - \int_0^t (\nabla H(x(\tau)))^\top R \nabla H(x(\tau))\, \mathrm{d}\tau \leq H(x_0). \end{aligned}$$

Third generalization step: coupling to the environment via inputs and outputs

Allowing for inputs and outputs to couple the system to the environment, we end up with linear pH system characterized by

$$\begin{aligned}\dot{x} &= (J-R)\cdot\nabla H(x) + Bu(t), \quad x(0)=x_0,\\ y &= B^\top \nabla H(x)\end{aligned}$$

with inputs $u(t)\in\mathbb{R}^p$, outputs $y(t)\in\mathbb{R}^p$ and port-matrices $B\in\mathbb{R}^{n\times p}$. The dissipativity inequality now reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(x(t)) &= (\nabla H(x))^\top \dot{x} = -(\nabla H(x))^\top R\nabla H(x) + (\nabla H(x))^\top Bu(t))\\ &= -(\nabla H(x))^\top R\nabla H(x) + y(t)^\top u(t) \le y(t)^\top u(t)\\ \Rightarrow H(x(t)) &= H(x_0) - \int_0^t (\nabla H(x(\tau)))^\top R\nabla H(x(\tau))\,\mathrm{d}\tau + \int_0^t y(\tau)^\top u(\tau)\mathrm{d}\tau\\ &\le H(x_0) + \int_0^t y(\tau)^\top u(\tau)\mathrm{d}\tau.\end{aligned}$$

Fourth generalization step: pH-DAE systems

Linear pH systems can be easily generalized to port-Hamiltonian differential-algebraic equations (pH-DAEs) given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ex) = (J-R)\cdot z(x) + Bu(t), \quad x(0)=x_0, \tag{3a}$$

$$y = B^\top z(x) \tag{3b}$$

with a possibly singular matrix $E\in\mathbb{R}^{n\times n}$ and the nonlinear mapping $z:\mathbb{R}^n\to\mathbb{R}^n$ fulfilling the compatibility condition $E^\top z = \nabla H$. Now the dissipativity condition reads

$$\begin{aligned}H(x(t)) &= H(x_0) - \int_0^t z(x(\tau))^\top R\nabla z(x(\tau))\,\mathrm{d}\tau + \int_0^t y(\tau)^\top u(\tau)\,\mathrm{d}\tau\\ &\le H(x_0) + \int_0^t y(\tau)^\top u(\tau)\,\mathrm{d}\tau.\end{aligned}$$

The key point in pH modelling is the following: there is an easy way to couple arbitrary many pH-DAE system such that the overall system is still a pH-DAE system, which preserves a dissipativity inequality.

Let us consider r autonomous pH-DAE systems

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_i x_i) = (J_i - R_i)z_i(x_i) + B_i u_i, \tag{4a}$$

$$y_i = B_i^\top z_i(x_i) \tag{4b}$$

with r Hamiltonians $H_1, H_2, \ldots, H_r$ and compatibility conditions $E_i^\top z_i = \nabla H_i$. If the inputs and outputs fulfill a linear interconnection relation $Mu + Ny = 0$ for the aggregated input $u = (u_1, u_2, \ldots, u_r)$ and output $y = (y_1, y_2, \ldots, y_r)$, it has been shown in [13] that one can write the aggregated system as a joint pH-DAE system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{u} \\ \hat{y} \end{bmatrix}\right) = \begin{bmatrix} J-R & B & 0 & 0 \\ -B^\top & 0 & I_m & -M^\top \\ 0 & -I_m & 0 & -N^\top \\ 0 & M & N & 0 \end{bmatrix} \begin{bmatrix} z(x) \\ \hat{u} \\ \hat{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I_m \\ 0 \end{bmatrix} u,$$
$$y = \hat{y},$$

with $z(x)^\top = (z_1(x_1)^\top, z_2(x_2)^\top, \ldots, z_r(x_r)^\top)$, new dummy variables $\hat{u}, \hat{y}$ and setting $X = \mathrm{diag}(X_1, X_2, \ldots, X_r)$ for $X \in \{E, J, R, B\}$. This coupling property of pH-DAE systems makes the pH modelling framework well suited for multi-physical applications.

Now, we consider external, time dependent inputs. To this end, we split the inputs and outputs into external (bar-notation) and internal ones (hat-notation), i.e., $B_i u_i$ is split into $\bar{B}_i \bar{u}_i + \hat{B}_i \hat{u}_i$. Then, the subsystem (4) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_i x_i) = (J_i - R_i) z_i(x_i) + \hat{B}_i \hat{u}_i + \bar{B}_i \bar{u}_i, \tag{5a}$$
$$\hat{y}_i = \hat{B}_i^\top z_i(x_i), \tag{5b}$$
$$\bar{y}_i = \bar{B}_i^\top z_i(x_i). \tag{5c}$$

For the coupling relation (of the internal quantities) $\hat{u} + C\hat{y} = 0$ with a skew-symmetric matrix $C = -C^\top$ (which often arises in application), these systems can be written as a joint pH-DAE system in condensed form [8]:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ex) = (\tilde{J} - R) z(x) + \bar{B}\bar{u}, \tag{6a}$$
$$\bar{y} = \bar{B}^\top z(x) \tag{6b}$$

with the condensed skew-symmetric matrix $\tilde{J} = J - \hat{B} C \hat{B}^\top$. Note that in this case all internal coupling modelled via the port-matrices $\hat{B}_i$ has now been transferred into the off-block diagonal elements of the skew-symmetric matrix $\tilde{J}$, i.e., $-\hat{B} C \hat{B}^\top$.

A systems theoretic treatment of pH systems goes back to Bernhard Maschke and Arjan van der Schaft (see [12,14] for an overview), where nonlinear systems governed by ordinary differential equations are treated. For simplicity of presentation, we will (a) not follow the differential geometric path via Dirac structures, (b) neglect a feed-through from input to output and (c) only consider finite dimensional systems, i.e., ordinary (ODEs) and differential-algebraic equations (DAEs), but no partial differential equations (PDEs). For simulation, the latter are usually transformed into ODEs and DAEs by spatial semi-discretization. For a differential geometric setting of pH systems see [15] and an introduction into pH-PDEs see [11].

The paper is organized as follows: In the next, section we introduce pH-DAEs, which allow for a general nonlinear dissipative part. A pH-DAE formulation of the modified nodal analysis (MNA) network equations is derived in Sect. 3, and for electromagnetic devices in Sect. 4. Section 5 discusses formulations of pH systems of coupled EM/circuit systems, which allow for monolithic as well as weak coupling simulation approaches. Section 6 finishes with conclusions.

2 pH-DAE Systems

When dealing with applications in electrical engineering, the concept of pH modelling has to be generalized to coupled differential-algebraic equations, which (a) allow for a general nonlinear resistive part $r(z)$ instead of a quasilinear setting Rz as in the approach of [13] and (b) has only to be accretive on a subspace $\mathcal{V} \subset \mathbb{R}$ according to the constraints of the system.

A differential-algebraic equation of the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Jz(x(t)) - r(z(x(t))) + Bu(t), \\ y(t) &= B^\top z(x(t)) \end{aligned} \tag{7}$$

is called a *port-Hamiltonian differential-algebraic equation* (pH-DAE) [8] if the following holds:

- $E \in \mathbb{R}^{n\times n}$, $J \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $z, r : \mathbb{R}^n \to \mathbb{R}^n$.
- There exists a subspace $\mathcal{V} \subset \mathbb{R}^n$ with the following properties:
  - (i) for all intervals $\mathcal{I} \subset \mathbb{R}$ and functions $u : \mathcal{I} \to \mathbb{R}^m$ such that (7) has a solution $x : \mathcal{I} \to \mathbb{R}^n$, it holds $z(x(t)) \in \mathcal{V}$ for all $t \in \mathcal{I}$.
  - (ii) J is skew-symmetric on $\mathcal{V}$. That is, $v^\top Jw = -w^\top Jv$ for all $v, w \in \mathcal{V}$.
  - (iii) r is accretive on $\mathcal{V}$. That is, $v^\top r(v) \geq 0$ for all $v \in \mathcal{V}$.
- There exists some function $H \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $\nabla H(x) = E^\top z(x)$ for all $x \in z^{-1}(\mathcal{V})$.

Remark 1. a) The pH-DAE (7) system provides the usual energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = -z(x(t))^\top r(z(x(t)))) + y(t)^\top u(t) \leq y(t)^\top u(t).$$

b) pH-DAE subsystems now read

$$\frac{\mathrm{d}}{\mathrm{d}t} E_i x_i(t) = J_i z_i(x_i(t)) - r_i\big(z_i(x_i(t))\big) + B_i u_i(t), \tag{8a}$$

$$y_i(t) = B_i^\top z_i\big(x_i(t)\big) \tag{8b}$$

instead of (4), and if they are coupled by a skew-symmetric coupling relation $\hat{u} + C\hat{y} = 0$ with a skew-symmetric matrix $C = -C^\top$ as before, they can be condensed into an overall pH-DAE system

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex = \hat{J}z - r + \bar{B}\bar{u}, \tag{9a}$$

$$\bar{y} = \bar{B}^\top z \tag{9b}$$

with the skew-symmetric matrix $\hat{J}$ again given by $\hat{J} = J - \hat{B}\hat{C}\hat{B}^\top$.

3 Electrical Networks

We consider the classical charge-/flux oriented MNA network equations [8,9]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}0&0&0&A_C&0\\0&0&0&0&I\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}e\\ \jmath_L\\ \jmath_V\\ q_C\\ \phi_L\end{bmatrix}=\begin{bmatrix}0&-A_L&-A_V&0&0\\A_L^\top&0&0&0&0\\A_V^\top&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}e\\ \jmath_L\\ \jmath_V\\ q_C\\ \phi_L\end{bmatrix}$$
$$-\begin{bmatrix}A_R g(A_R^\top e)\\0\\0\\q_C-q(A_C^\top e)\\ \phi_L-\phi(\jmath_L)\end{bmatrix}+\begin{bmatrix}-A_I&0\\0&0\\0&-I\\0&0\\0&0\end{bmatrix}\begin{bmatrix}\imath(t)\\ v(t)\end{bmatrix}$$

with $e, \jmath_L$ and $\jmath_V$ denoting node potentials and currents through flux storing elements and voltages sources, q_C and Φ_l charge and flux-storing elements, $i(t)$ and $v(t)$ independent current and voltage sources, the resistive currents g and the incidence matrices A_C, A_L, A_R, A_V, A_I for charge- and flux storing elements, resistive elements, voltage and current sources, and seek a formulation as a pH-DAE system. For this, we need the following assumptions, which naturally occur in circuit simulation, see [8]:

(a) **Soundness.** The circuit graph has at least one branch and is connected. Furthermore, it contains neither V-loops nor I-cutsets. Equivalently, A_V and $(A_C\, A_R\, A_L\, A_V)^\top$ have full column rank.
(b) **Passivity.** The functions q, ϕ and g fulfill
  (i) $q : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and $\phi : \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$ are bijective, continuously differentiable, and their Jacobians

$$\widetilde{C}(u_C) := \frac{\mathrm{d}q}{\mathrm{d}u_C}(u_C), \qquad \widetilde{L}(\jmath_L) := \frac{\mathrm{d}\phi}{\mathrm{d}\jmath_L}(\jmath_L)$$

  are symmetric and positive definite for all $u_C \in \mathbb{R}^{n_C}$, $\jmath_L \in \mathbb{R}^{n_L}$.
  (ii) $g : \mathbb{R}^{n_R} \to \mathbb{R}^{n_R}$ is continuously differentiable, and its Jacobian has the property that $\frac{\mathrm{d}g}{\mathrm{d}u_R}(u_R) + \frac{\mathrm{d}g}{\mathrm{d}u_R}(u_R)^\top$ is positive definite for all $u_R \in \mathbb{R}^{n_R}$.

If $q : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and $\phi : \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$ fulfill these assumptions, then there exist twice continuously differentiable and non-negative functions $V_C : \mathbb{R}^{n_C} \to \mathbb{R}$, $V_L : \mathbb{R}^{n_L} \to \mathbb{R}$ with the following property: the gradients of V_C and V_L are, respectively, the inverse functions of q and ϕ. That is,

$$\forall q_C \in \mathbb{R}^{n_C} : \nabla V_C(q_C) = q^{-1}(q_C), \quad \forall \phi_L \in \mathbb{R}^{n_L} : \nabla V_L(\phi_L) = \phi^{-1}(\phi_L).$$

With this setting, the pH-DAE MNA network equations can now be derived as follows: we first eliminate the equation $\phi_L - \phi(\jmath_L) = 0$: $\jmath_L = \phi^{-1}(\phi_L)$; secondly,

we replace the equation $q_C - q(A_C^\top e) = 0$ by $A_C^\top e - q^{-1}(q_C) = 0$. We end up with

$$\begin{aligned}\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E :=} \underbrace{\begin{bmatrix} q_C \\ \phi_L \\ e \\ \jmath_V \end{bmatrix}}_{x :=} &= \underbrace{\begin{bmatrix} 0 & -A_L & 0 & -A_V \\ A_L^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 \end{bmatrix}}_{J :=} \underbrace{\begin{bmatrix} e \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ \jmath_V \end{bmatrix}}_{z(x)} \\ &\quad - \underbrace{\begin{bmatrix} A_R g(A_R^\top e) \\ 0 \\ A_C^\top e - q^{-1}(q_C) \\ 0 \end{bmatrix}}_{r(z(x)) :=} + \underbrace{\begin{bmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix}}_{B :=} \underbrace{\begin{bmatrix} \imath(t) \\ v(t) \end{bmatrix}}_{u(t) :=},\end{aligned} \tag{10}$$

which is a pH-DAE of type (7) with subspace $\mathcal{V}$ and Hamiltonian $H(x)$ given by $H(x) = V_C(q_C) + V_L(\phi_L), \mathcal{V} = \left\{ \left(e,\ \jmath_L,\ u_C,\ \jmath_V\right)^\top \in \mathbb{R}^n \,\middle|\, A_C^\top e = u_C \right\}$.

Remark 2. a) The pH-DAE formulation shares the index properties of charge/flux-oriented MNA network equations, if the assumption on soundness and passivity hold: the index is one if, and only if, it neither contains LI-cutsets nor CV-loops except for C-loops; otherwise it is two.

b) If r subcircuits given as pH-DAE MNA network equations are coupled via voltage/current sources, the overall system can be written as a pH-DAE MNA of type (10).

4 Electromagnetic Devices

In [5], the Maxwell grid equations for an electromagnetic device have been developed as a linear pH-DAE system provided that (a) the three-dimensional domain of the device is connected, bounded and surrounded by perfectly conducting material, (b) the permittivity ϵ, the permeability μ are symmetric positive definite, and the conductivity σ is symmetric positive semi-definite, and (c) finite integration technique [6] has been used for the spatial discretization with orthogonal staggered cells:

$$\begin{bmatrix} M_\mu & 0 \\ 0 & M_\epsilon \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \hat{h} \\ \hat{e} \end{bmatrix} = \left(\begin{bmatrix} 0 & -C \\ C^\top & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & M_\sigma \end{bmatrix} \right) \begin{bmatrix} \hat{h} \\ \hat{e} \end{bmatrix} + \begin{bmatrix} 0 \\ X_S \end{bmatrix} \hat{u}_2, \tag{11a}$$

$$\hat{y}_2 = \begin{bmatrix} 0 \\ X_S \end{bmatrix}^\top \begin{bmatrix} \hat{h} \\ \hat{e} \end{bmatrix} = X_S^\top \hat{e}. \tag{11b}$$

Here C denotes the discrete curl operator, the material matrices M_ϵ, M_μ and M_σ represent the discretized permittivity, permeability and conductivity distributions, $\hat{e}$ is vector of the electric mesh voltages e, $\hat{h}$ the vector of the magnetic mesh voltages h, and the (dual grid facet) source current $\hat{u}_2$ as input. This

input is allocated at positions X_S. In fact, X_S maps the interior mesh links onto the exterior mesh nodes. Furthermore, the respective electric mesh voltage $\hat{y}_2$ forms the output. The Hamiltonian of the electromagnetic device is given by $H_1 = \frac{1}{2}(\tilde{e}^\top M_\epsilon \tilde{e} + \tilde{h}^\top M_\mu \tilde{h})$.

5 Coupled EM/circuit System

When coupling an electromagnetic device with an electric circuit, it remains only to define the inputs, outputs and the coupling equation. For the circuit, the electromagnetic device produces the current $\jmath_E$ flowing into the network, which is assembled at the respective nodes of the circuit via an incidence matrix A_E. Hence the circuit part reads (where we split inputs again in external inputs $\imath$, v, and internal ones):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q_C \\ \phi_L \\ e \\ \jmath_V \\ \jmath_E \end{bmatrix} = \begin{bmatrix} 0 & -A_L & 0 & -A_V & -A_E \\ A_L^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \\ A_E^\top & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \jmath_L \\ u_C \\ \jmath_V \\ \jmath_E \end{bmatrix} \tag{12a}$$
$$- \begin{bmatrix} A_R g(A_R^\top e) \\ 0 \\ A_C^\top e - u_C \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \imath(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\hat{u}_1,$$

$$\begin{bmatrix} \bar{y}_{1,1} \\ \bar{y}_{1,2} \\ \hat{y}_1 \end{bmatrix} = \begin{bmatrix} -A_I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top \cdot \begin{bmatrix} e \\ \jmath_L \\ u_C \\ \jmath_V \\ \jmath_E \end{bmatrix} = \begin{bmatrix} -A_I^\top e \\ -\jmath_V \\ \jmath_E \end{bmatrix} \tag{12b}$$

with the Hamiltonian: $H_2 = V_C(q_C) + V_L(\phi_L)$.

The coupling is as follows [5]: the input $\hat{u}_1$ (of the electric circuit) is given by the voltage drop at the electromagnetic device, which reads $\hat{u}_1 = -X_S^\top \tilde{e} = -\hat{y}_2$; on the other hand, the input $\hat{u}_2$ (of the magnetic device) is given by the current $\hat{u}_2 = \jmath_E = \hat{y}_1$. Overall, we get the following skew-symmetric relation between inputs and outputs:

$$0 = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} + \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix}. \tag{13}$$

As we have a system consisting of two pH-DAE systems (11) and (12) with a skew-symmetric linear coupling condition (13), the overall system can be written as a condensed pH-DAE system (9) with Hamiltonian $H = H_1 + H_2$ and enlarged matrices as above.

6 Simulation Strategies

Generally, for simulating the coupled EM/circuit system numerically, two approaches are feasible:

a) *Monolithic approach.* The condensed system (9) can be solved by any integration scheme suitable for index-1 and index-2 systems, depending on the index. To preserve the dissipation inequality also on a discrete level, collocation schemes [13] and discrete gradient schemes tracing back to [7] are the methods-of choice. This strategy is also referred to as strong coupling.

b) *Monolithic multirate approach.* In fact, we are facing models, where the subsystems can have widely separated time scales. This can create so-called multirate potential, where it is beneficial to employ schemes, which use inherent step sizes for each subsystem. In this way, each subsystem can be sampled on its time scale. See e.g. [2,10].

c) *Weak coupling.* Since the coupling equations is merely the one-to-one identification of output and input, we can insert this. Furthermore, omitting outputs due to external sources, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q_C \\ \phi_L \\ e \\ \jmath_V \\ \jmath_E \end{bmatrix} = \begin{bmatrix} 0 & -A_L & 0 & -A_V & -A_E \\ A_L^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \\ A_E^\top & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \jmath_L \\ u_C \\ \jmath_V \\ \jmath_E \end{bmatrix} \tag{14a}$$

$$- \begin{bmatrix} A_R g(A_R^\top e) \\ 0 \\ A_C^\top e - u_C \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \imath(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\hat{y}_2,$$

$$\hat{y}_1 = \jmath_E \tag{14b}$$

and

$$\begin{bmatrix} M_\mu & 0 \\ 0 & M_\epsilon \end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \tilde{h} \\ \tilde{e} \end{bmatrix} = \left(\begin{bmatrix} 0 & -C \\ C^\top & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & M_\sigma \end{bmatrix}\right)\begin{bmatrix} \tilde{h} \\ \tilde{e} \end{bmatrix} + \begin{bmatrix} 0 \\ X_S \end{bmatrix}\hat{y}_1 \tag{15a}$$

$$\hat{y}_2 = X_S\tilde{e}. \tag{15b}$$

Here dynamic iteration schemes [1] are the methods-of choice, as due to the ODE-DAE coupling no stability constraints occur [4]. In addition, each step of a Jacobi or Gauß-Seidel iteration scheme defines a pH-DAE system by its own [8].

Operator splitting approaches are not generally feasible for differential-algebraic equations, which can easily be seen for the linear pH-DAE (3a) with $z(x) = x$ and $B = 0$. A Lie-Trotter splitting approach may read

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ex) = Jx, \quad x(0) = x_0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(Ew) = -Rw. \quad w(0) = x(T),$$

allowing for using a symplectic integrator for the first step, and a dissipative scheme for the second one. However, the matrix pencil $\{E, J\}$ or $\{E, R\}$ may be singular and thus not define a unique solution for the respective subproblem, even if the matrix pencil $\{E, J - R\}$ of the overall system is regular. Even if this does not happen, the first problem, for example, may not allow for a unique solution for arbitrary choices of consistent initial values. For

$$E = \operatorname{diag}(1, 0, 1), \quad J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \operatorname{diag}(0, 1, 1), \quad x_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

all matrix pencils $\{E, J - R\}$, $\{E, J\}$ and $\{E, R\}$ are regular, but the first step yields $x_1 \equiv 0 \neq 1$.
One may overcome the problem by rewriting the DAE in terms of an underlying ODE and subsequent algebraic variables given by explicit evaluations. For network equations a branch oriented loop-cutset approach is an option for defining such a pH-DAE system, see [5]. Another way to avoid the problems above is to follow an operator splitting based approach for dynamic iteration. In the latter case, no stability problems occur and a monotone convergence can be obtained [3].

7 Conclusions

Port-Hamiltionian (pH) systems provide a modelling framework which preserves essential physical properties. It is especially suited for multiphysical applications, as the proper coupling of pH subsystems yields an overall pH system. In electrical engineering, we have shown that electrical networks and electromagnetic devices can be written as pH systems, and coupled EM/circuit system yield coupled pH systems with a skew-symmetric coupling, which can be rewritten as an overall pH system. For simulation, a monolithic approach is suitable for the former, and weak coupling methods for the latter. There are still many unresolved questions, such as how to adequately integrate distributed ports into the pH system's modeling.

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