Harmonic Balance with Small Signal Perturbation

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Abstract. Investigating perturbations of a periodic steady state of an electric circuit is of interest e.g. for small signal responses, noise analysis or the generation of X-parameter models. We present a method based on Harmonic Balance, to compute the Fourier coefficients of the circuit response for a small signal perturbation of the input. The relation to two-tone Harmonic Balance is investigated and it is shown that under suitable conditions the perturbation method can approximate the full two-tone solution at extremely lower costs. The method is tested on a Gilbert mixer circuit.

1 Introduction

Many problems require the computation of the distortion of the periodic steady state (PSS) of a circuit if a small signal perturbation is applied. For instance, if the small signal response itself is of interest [1]. Furthermore, noise analysis [2] is based on the injection of small random signals. Another application is the generation of X-parameters [3,4], which provide a behavioral model for a neighborhood of the PSS.

Here, we present a frequency domain method, which is based on Harmonic Balance (HB) [5]. If HB is suitable for the simulated circuit then the perturbed solution is obtained easily, with only little extra cost, compared to the HB for the unperturbed PSS. In Sect. 2 we reformulate the problem as an infinite system of equations for the Fourier coefficients. Simplifications for real-valued signals are given in Sect. 3. In Sect. 4 we introduce a discretization scheme, which results in a linear system of equations, for a truncated sequence of Fourier coefficients. The relation to multi-tone HB is described in Sect. 5. In Sect. 6 we describe how our method can be utilized for the computation of X-parameters. The method is tested on a Gilbert cell mixer circuit in Sect. 7.

2 Small Signal Distortion of a Periodic Steady State

Consider the circuit equations from modified nodal analysis

$$\frac{d}{dt}q\big(x(t)\big)+f\big(x(t)\big)+s(t)=0, \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the vector of unknowns, $f(x)\in\mathbb{R}^n$ the vector sums of static currents entering each node, $q(x)\in\mathbb{R}^n$ the vector sums of charges and magnetic fluxes, and $s(t)\in\mathbb{R}^n$ is the vector of time-dependent sources.

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Let $x(t)$ be a PSS, i.e., $x(t)$ solves (1) and for all t

$$s(t) = s(t+T),\ x(t) = x(t+T).$$

A perturbation $\tilde{s}(t) = s(t) + \Delta s(t)$ will lead to the perturbed solution $\tilde{x}(t) = x(t) + \Delta x(t)$. For small $\Delta s(t)$ and $\Delta x(t)$ one obtains by linearization of q and f an approximated version of the circuit equations, namely

$$\tfrac{d}{dt} q\big(x(t)\big) + \tfrac{d}{dt}\big(C(t)\,\Delta x(t)\big) + f\big(x(t)\big) + G(t)\,\Delta x(t) + s(t) + \Delta s(t) = 0, \tag{2}$$

where the Jacobians $C(t) := \frac{dq}{dx}\big(x(t)\big)$ and $G(t) := \frac{df}{dx}\big(x(t)\big)$ are T-periodic matrix-valued functions. Taking the difference of (2) and (1) one obtains the linear, time-variant differential algebraic equation

$$\tfrac{d}{dt}\big(C(t)\,\Delta x(t)\big) + G(t)\,\Delta x(t) + \Delta s(t) = 0 \tag{3}$$

for the perturbation Δx of the PSS $x(t)$.

Now we consider a harmonic perturbation $\Delta s(t) = \hat{s}\, e^{i(m\omega + \Delta\omega)t}$, where $\omega = \frac{2\pi}{T}$ is the angular frequency, the amplitude $\hat{s} \in \mathbb{C}^N$ is small, and $\Delta\omega \in \mathbb{R}$ is a frequency offset which can be chosen such that $|\Delta\omega| \leq \frac{\omega}{2}$ for suitable $m \in \mathbb{Z}$. With the Fourier expansions

$$C(t) = \sum_{k\in\mathbb{Z}} C_k\, e^{ik\omega t}, \quad G(t) = \sum_{k\in\mathbb{Z}} G_k\, e^{ik\omega t}, \quad \Delta x(t) = \sum_{k\in\mathbb{Z}} X_k e^{i(k\omega+\Delta\omega)t}$$

one obtains from (3)

$$\frac{d}{dt}\Big(\sum_{\ell\in\mathbb{Z}} e^{i(\ell\omega+\Delta\omega)t} \sum_{k\in\mathbb{Z}} C_{\ell-k} X_k\Big) + \sum_{\ell\in\mathbb{Z}} e^{i(\ell\omega+\Delta\omega)t} \sum_{k\in\mathbb{Z}} G_{\ell-k} X_k + \hat{s}\, e^{i(m\omega+\Delta\omega)t} = 0, \tag{4}$$

For the expansion of Δx we have to assume that the homogeneous equations (i.e. for $\Delta s(t) = 0$) have only the trivial solution. This means the circuit does not contain an oscillator. Equating coefficients in (4) now yields

$$i(\omega\ell + \Delta\omega) \sum_{k\in\mathbb{Z}} C_{\ell-k} X_k + \sum_{k\in\mathbb{Z}} G_{\ell-k} X_k + \hat{s}\,\delta_{\ell,m} = 0, \qquad \ell \in \mathbb{Z}, \tag{5}$$

where $\delta_{\ell,m}$ is the Kronecker delta.

3 Real-Valued Signals

In practice, we can assume that $q(x)$ and $f(x)$ are real-valued functions. This implies that $C(t)$ and $G(t)$ are real-valued, too, with Fourier coefficients satisfying $C_{-k} = \overline{C_k}$ and $G_{-k} = \overline{G_k}$. Since the stimulus $s(t)$ is in practice also a real-valued (typically sinusoidal) signal, it is of interest how the solution of (3) can be used for the solution of real-valued problems. It turns out that this requires only little extra effort.

In a preliminary step, solving (3) for the conjugate complex $\overline{\Delta s(t)}$ leads in (5) to the equations

$$i(\omega\ell - \Delta\omega) \sum_{k\in\mathbb{Z}} C_{\ell-k} X_k^* + \sum_{k\in\mathbb{Z}} G_{\ell-k} X_k^* + \bar{\hat{s}}\,\delta_{\ell,-m} = 0,$$

with the solution coefficient X_k^*. Substituting $\ell \to -\ell$ and $k \to -k$ we obtain due to $C_{-k} = \overline{C_k}$ and $G_{-k} = \overline{G_k}$ that

$$\overline{i(\omega\ell + \Delta\omega) \sum_{k\in\mathbb{Z}} C_{\ell-k} \overline{X_{-k}^*} + \sum_{k\in\mathbb{Z}} G_{\ell-k} \overline{X_{-k}^*} + \hat{s}\,\delta_{\ell,m}} = 0,$$

i.e. $X_{-k}^* = \overline{X_k}$ and thus the corresponding solution of (3) for $\overline{\Delta s(t)}$ is

$$\Delta x^*(t) = \sum_{k\in\mathbb{Z}} \overline{X_k}\, e^{-i(k\omega+\Delta\omega)t} = \overline{\Delta x(t)}.$$

Therefore, $\mathrm{Re}(\Delta x(t))$ and $\mathrm{Im}(\Delta x(t))$ are the solutions for the real-valued perturbations $\mathrm{Re}(\Delta s(t))$ and $\mathrm{Im}(\Delta s(t))$, respectively.

4 Discretization

Since (5) is an infinite system one needs to approximate it by a finite system of equations. Because one can expect the Fourier coefficients to decay for $|k| \to \infty$, we set $X_k = 0$, $|m_1 - k| > K$ for some $m_1 \in \mathbb{Z}$. To determine the remaining coefficients X_k, $k = m_1 - K, \ldots, m_1 + K$ one chooses $2K+1$ equations from (5), namely

$$i(\omega\ell + \Delta\omega) \sum_{k=m_1-K}^{m_1+K} C_{\ell-k} X_k + \sum_{k=m_1-K}^{m_1+K} G_{\ell-k} X_k + \hat{s}\,\delta_{\ell,m} = 0, \quad \ell = m_2 - K, \ldots, m_2 + K, \quad (6)$$

for some $m_2 \in \mathbb{Z}$. A possible choice would be $m_1 = m_2 = m$, to adapt to the center frequency. However, if the computation has to be repeated for several values of m, fixed values of m_1 and m_2 may be used to speed up computations, since only one LU-factorization has to be performed.

The regularity of the system matrix is equivalent to the fact that there is only the trivial solution to the homogeneous system. Since we have assumed this property for the original system (5), it should typically hold for a sufficiently good approximation (4).

The matrices C_k and G_k can be computed numerical integration, namely the trapezoidal rule, i.e.,

$$G_k = \frac{1}{T} \int_0^T G(t)\, e^{-ik\omega t}\, dt \approx \frac{1}{TN} \sum_{\ell=0}^{N-1} G\left(\tfrac{\ell T}{N}\right) e^{-2\pi i k\ell/N},$$

where $N > 2K+1$ to avoid aliasing. Therefore, an efficient computation of G_k, C_k, and $x_\ell := x(\frac{\ell T}{N})$ can be done by employing the Fast Fourier Transform, if a suitable N is chosen, e.g. a power of two.

In contrast to the method in [1], where the linear, time-variant system (3) is solved in the time domain by classical time stepping methods, the presented method works in the frequency domain. It is well suited if the matrix sequences (C_k) and (G_k) are decaying fast, which implies a fast decay of (X_k), too. This is a case if the PSS $x(t)$ is nearly sinusoidal, i.e., if the PSS can be computed by an HB efficiently, than the described perturbation method will perform very well, too.

5 Relation to Two-Tone Harmonic Balance (HB)

Two-tone signals are of the form $x(t) = \hat{x}(t,t)$, where

$$\hat{x}(t_1,t_2) = \hat{x}(t_1 + T_1, t_2) = \hat{x}(t_1, t_2 + T_2), \quad t_1, t_2 \in \mathbb{R}.$$

They have a bi-variate Fourier expansion of the form

$$x(t) = \sum_{k,\ell \in \mathbb{Z}} \hat{X}_{k,\ell}\, e^{2\pi i (k/T_1 + \ell/T_2)t}.$$

With the substitution $k \to k - n\ell$ and $X_{k,\ell} = \hat{X}_{k-n\ell,\ell}$ this becomes

$$x(t) = \sum_{k,\ell \in \mathbb{Z}} X_{k,\ell}\, e^{i(k\omega + \ell\Delta\omega)t}, \tag{7}$$

where $\omega = \frac{2\pi}{T_1}$ and $\Delta\omega = \frac{2\pi}{T_2} - n\omega$. Typically, $n \in \mathbb{Z}$ is chosen to obtain a small frequency offset $\Delta\omega$, i.e., $|\Delta\omega| \le \frac{\omega}{2}$, which corresponds to the setting in Sect. 2. By a multi-rate HB [6,7] one can compute the coefficients $X_{k,\ell}$ for the PSS of a circuits driven by a two-tone signal.

The two-tone solution of (1) with the real-valued source term

$$\tilde{s}(t) = s(t) + \hat{s}\, e^{i(n\omega + \Delta\omega)t} + \overline{\hat{s}}\, e^{-i(n\omega + \Delta\omega)t}$$

with a T_1-periodic single-tone signal $s(t)$, can be approximated using the solution of (5) for sufficiently small $\hat{s}$. Due to linearity one solves for $s(t)$ by a single-tone HB, then solves (5) for $\Delta s(t) = \hat{s}\, e^{i(n\omega+\Delta\omega)t}$. The solution for $\overline{\Delta s(t)} = \overline{\hat{s}}\, e^{-i(n\omega+\Delta\omega)t}$ follows immediately from Sect. 3. By linear combination one obtains the solution

$$x(t) = \sum_{k \in \mathbb{Z}} \sum_{\ell=-1}^{1} \tilde{X}_{k,\ell}\, e^{i(k\omega + \ell\Delta\omega)t}, \tag{8}$$

where $\tilde{X}_{k,0}$ is the result of a single-tone HB with source $s(t)$, while $\tilde{X}_{k,1} = X_k$, $\tilde{X}_{k,-1} = \overline{X_{-k}}$ (cf. Sect. 3) are obtained from the subsequent perturbation approach (5). Obviously, for small $\hat{s}$ the expansion (8) will be a good approximation of the two-tone signal (7).

A general multi-tone analysis can be performed by computing the perturbed solution for several harmonic perturbations $\hat{s}_\ell\, e^{i(k_\ell \omega + \ell\Delta\omega_\ell)t}$ separately, and using the linearity for superpositions of the harmonics.

Note, that the computational cost of the perturbation approach is much smaller than for the full two-tone HB since it requires only the computation of a single-tone PSS with an essentially smaller system of equations to be solved. The cost for the final step of solving the linear system (5) equals essentially the cost of one Newton step in the preciding single-tone HB.

6 Extraction of X-Parameter Models

X-parameters [3,4] are a generalization of S-Parameters to describe the relation of power waves in electronic circuits or devices. While S-Parameter are used as behavioral model for linear systems, X-parameters describe the behavior of a non-linear network in the neighborhood of a PSS. For an N-port system with a large signal incident wave of fixed amplitude at port 1 the X-parameter model reads as

$$
\begin{aligned}
B_{p,k} &= X^{(FB)}_{p,k}(|A_{1,1}|, f_0)\,P^k \\
&+ \sum_{\substack{q=1,\ell=1 \\ (q,\ell)\neq(1,1)}}^{q=N,\ell=K} X^{(S)}_{p,k,q,\ell}(|A_{1,1}|, f_0)\,A_{q,\ell}\,P^{k-\ell} + X^{(T)}_{p,k,q,\ell}(|A_{1,1}|, f_0)\,\overline{A}_{q,\ell}\,P^{k+\ell},
\end{aligned}
$$

where $A_{p,k}$ and $B_{p,k}$ denote the k-th Fourier coefficient of the incident and scattered wave at the p-th port, respectively. The first term describes the contribution of the large signal input $DC + A_{1,1}\,e^{2\pi i f_0} + \overline{A_{1,1}}\,e^{-2\pi i f_0}$, where $P = e^{i\,\arg(A_{1,1})}$ and the amplitude $|A_{1,1}|$ is fixed. The remaining terms contain variable small signal contributions.

Since the incident and scattered waves are related to voltages and currents at the ports by

$$
a_p = \frac{U_p + I_p Z_p}{2\sqrt{\mathrm{Re}(Z_p)}} \qquad b_p = \frac{U_p - I_p \overline{Z_p}}{2\sqrt{\mathrm{Re}(Z_p)}}, \tag{9}
$$

with the port impedance Z_p (often $50\,\Omega$), we can obtain the X-parameters of a given circuit with N ports as follows. The ports are connected to voltage sources with internal impedance Z_p. To obtain the large signal contribution $X^{(FB)}_{p,k}(|A_{1,1}|, f_0)$, a HB with input signal

$$
V(t) = U_0 \cos(2\pi f_0 t)
$$

at port 1, while all other sources are set to zero. The amplitude U_0 is chosen to get the proper value for $|A_{1,1}|$ as described e.g. in [4, Eq. (14)]. Using (9) we obtain

$$
X^{(FB)}_{p,k}(|A_{1,1}|, f_0) = \frac{U_{p,k} - I_{p,k}\overline{Z_p}}{2\sqrt{\mathrm{Re}(Z_p)}},
$$

where $U_{p,k}$ and $I_{p,k}$ are the k-th Fourier coefficient of voltage and current at the p-th port, respectively, which can be extracted immediately from the HB solution.

To compute the remaining X-parameters one can now use the perturbation method described in Sect. 2. We reformulate the problem as an infinite system of equations for the Fourier coefficients. Simplifications for real-valued signals are given in Sect. 3 and 4. First the matrices G_k and C_k are computed. Now, to determine $X^{(S)}_{p,k,q,\ell}$ and $X^{(T)}_{p,k,q,\ell}$ we solve (6) with $\Delta\omega = 0$ and ℓ chosen as the corresponding X-parameter index. The source term $\hat{s}$ is chosen as if all voltage sources are set to zero except for q-th port (as it is done for the computation of S-Parameters). Due to linearity, the voltage at q-th port can be any non-zero value, e.g. 1V is usually a good choice. Using again the relation (9) we obtain the X-parameters as

$$
X^{(S)}_{p,k,q,\ell} = \frac{U_{p,k} - I_{p,k}\overline{Z_p}}{U_{q,\ell} + I_{q,\ell} Z_q} \qquad X^{(T)}_{p,k,q,\ell} = \frac{\overline{U_{p,-k}} - \overline{I_{p,-k}}\,Z_p}{U_{q,\ell} + I_{q,\ell} Z_q}.
$$

One should take into account, that the system (6) has to be solved for the same ℓ for different ports q, i.e., the same matrix for several right hand sides. To save computation time, one should therefore first set up the system matrix for (6) (for each $\ell = 1,\dots,K$), do an LU-factorization, and solve for each right hand side (port) by forward and backward substitution.

7 Numerical Test

We have tested the method on a Gilbert cell mixer [8, Fig. 2.8] with a local oscillator input of 100 MHz and an radio frequency (RF) input of 99.9 MHz yielding a frequency offset $\Delta\omega = -0.1$ MHz. The perturbation method as well as the 2-tone HB were performed with a cutoff of the Fourier series after $K = 31$ and $N = 64$ sampling points for the trapezoidal rule. The RF signal is treated as perturbed input and the amplitude is swept from 0.1mV to 0.4V. In Fig. 1 one can see the absolute value of Fourier coefficients of the output signal for local oscillator amplitude 1 V plotted against the RF amplitude. As one can see the coefficients $\tilde{X}_{0,1}$ and $X_{0,1}$ agree very well for RF input almost up to 0.1 V. The coefficient $X_{0,2}$ (only obtained by two-tone HB) is included as a measure of non-linearity.

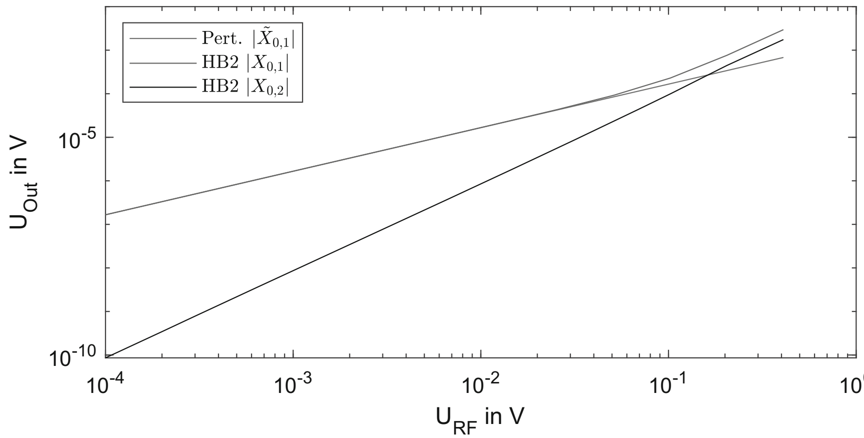

Fig. 1. Fourier coefficient $X_{0,1}$ for perturbation technique and two-tone HB, as well as $X_{0,2}$ for two-tone HB as reference

The above result is confirmed by the relative ℓ^2-error (i.e. in the Euclidean norm) over all coefficients (Fig. 2), obtained by comparison to a two-tone HB of high precision.

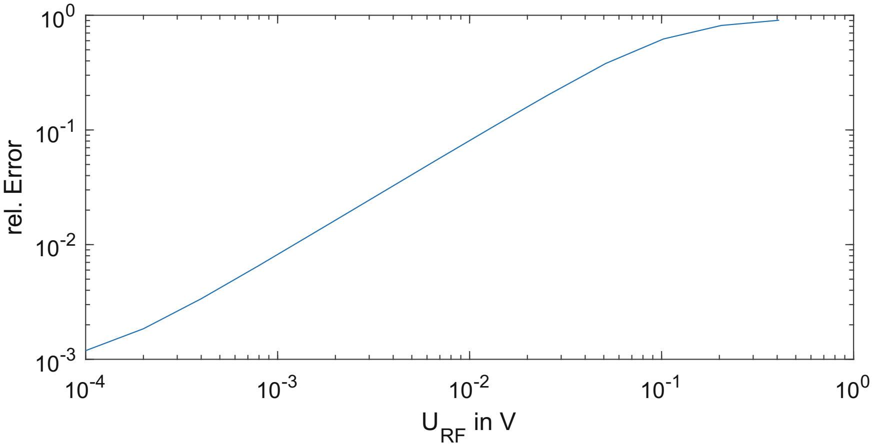

Fig. 2. ℓ_2-error of perturbation technique.

8 Conclusion

The presented HB perturbation method permits the efficient analysis of the distortion of a PSS by a small signal. The results allow a first assessment of the qualitative characteristics of RF circuits with moderate nonlinear behavior.

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