On the Theory of Fluctuations of Strongly Nonlinear (vibroimpact) Systems

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Abstract. During the operation of a large group of technical devices and mechanisms with impact interactions, such as clock mechanisms, systems associated with vibro-driven piles and wells drilling for oil and gas production, pneumatic impact mechanisms, vibrating mills, cyclic automation mechanisms, impact interaction mechanisms due to the presence of gaps and others, there exist parameters, such as the restitution coefficient, adjusting gap, that do not remain constant but change randomly from impact to impact. However, most of the research results of strongly nonlinear systems contain a deterministic formulation. In this paper, we propose a numerical-analytical method for studying the dynamics of strongly nonlinear systems, taking into account fluctuations in system parameters. In this case, the mathematical apparatus of the point mappings method of Poincare surfaces is widely used. Random deviations of parameters can be both delta- and non-delta-correlated. The concepts of stochastic stability are introduced, which made it possible to determine not only the optimal values of system parameters, but also to find, to some extent, dangerous/safe areas of the stability regions boundaries of the systems motion modes.

Keywords: Strongly nonlinear systems · delta- and non-delta-correlated random deviations of parameters · Method of point mappings of Poincare surfaces · Stochastic stability

1 Introduction

A large number of experimental and theoretical works are devoted to the study of the dynamics of highly nonlinear (vibration-shock) systems. In this regard, we note that the interest of researchers in such essentially nonlinear systems still does not wane. This is due to the fact that applications of such systems are directly used in many industries and national economies. Along with obvious

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practically important applications such as the construction of various civil facilities, medicine, military construction, etc., systems with impact interactions have found applications in capsule medicine, nanotechnology and other modern areas of science and technology. In the vast majority of problems, Newton's concept of the proportional relationship between the relative pre-impact and post-impact velocities of translationally moving bodies is used to describe direct impact interaction. It is accepted that the proportionality coefficient R, called the coefficient of restoration, depends on the properties of the impacting bodies and does not depend on the impact speed and is a constant value varying within $0 < R < 1$. It is known that in systems with collisions, movements with any finite number of impacts per period are fundamentally possible. Therefore, researchers consider only the movements that are practically the most important. Studies of more complex motion modes indicate a tendency toward a significant narrowing of the areas of existence of stable movements as the motion mode becomes more complex. In a number of cases, the range of feasibility of even two-shock oscillations turns out to be narrower than the stripes of natural dispersion of parameter values. In this paper, we study the dynamics of the vibration-impact mechanism, in the mathematical model of which it is assumed that the coefficient of recovery R varies from blow to blow randomly with a zero average value and a dispersion different from zero. A numerical-analytical technology has been developed based on the point mapping method for finding statistical characteristics of post-impact velocity and impact time. The concept of stochastic stability was introduced and a numerical experiment was carried out to calculate the standard deviations of the speed and times of impacts.

2 Equations of Motion

Consider a vibroimpact mechanism consisting of a mass M, which is affected by an external periodic force $F \sin \omega t$ and a linear friction force $-h\dot{x}$. The mass itself oscillates along the stand attached to the base of the processed material. The physical scheme of the mechanism is presented in [7]. The dynamics of such a mechanism is of independent interest in terms of compaction of various materials (soil, sand, etc.). The research into dynamics of such systems is mainly carried out at constant parameters. Significantly less number of results on studying the dynamics of vibroimpact systems with randomly changing parameters can be observed. It is obvious that during the operation of vibro-impact compaction mechanisms for various materials, the restitution coefficient, when interacting with the medium, changes its initial value from impact to impact affecting the compaction process. In this regard, studying the dynamics of strongly nonlinear (vibro-impact) dynamic systems is a rather topical issue. In what follows, we will assume: 1) energy dissipation occurs through impacts and when friction forces are taken into account; 2) the impact is instantaneous with the restitution coefficient R, varying within $0 \leq R \leq 1$; 3) limiter offset is not taken into account. Under the assumptions made, the differential equation of the system in the interval between impacts $(x > 0)$ can be written as:

$$M\ddot{x} + h\dot{x} = P + F\sin\omega t, \tag{1}$$

where x — is the coordinate of the mass M, P is a constant force acting on the impact mass M, and, generally speaking, $P \neq mg$, h — is the friction coefficient, F, ω —are the amplitude and frequency of the driving force. At $x = 0$, $\dot{x} < 0$ an inelastic impact occurs with the restitution coefficient R, which is described using Newton's hypothesis $\dot{x}^+ = -R\dot{x}^-$, where $\dot{x}^-$, $\dot{x}^+$ are pre-impact and post-impact velocities of a mechanism. Introducing the dimensionless time $\tau = \omega t$ and the coordinate $\xi = \frac{M\omega^2 x}{F}$, Eq. (1) is rewritten as

$$\ddot{\xi} + \frac{h}{M\omega}\dot{\xi} = \frac{P}{F} + \sin\tau, \xi > 0, \tag{2}$$

$$\dot{\xi}^+ = -R\dot{\xi}^-, \xi = 0, \dot{\xi}^- < 0. \tag{3}$$

3 Point Mapping

Let us assume that during the operation of the mechanism, the restitution coefficient changes randomly from impact to impact, and the changes are so small that they do not take the system out of the vicinity of a stable periodic regime. Let in a deterministic system (the restitution coefficient is constant and equal to $R = R_0$) for some values of the parameter $R = R_0$ there is a stable periodic motion. The restitution coefficient for the n-th impact takes on the value: $R_n = R_0 + \mathit{\Delta} R_n$, where $\mathit{\Delta} R_n$ is a small random deviation from R_0 with zero mean value and non-zero variance. The phase space of the considered system (2), (3) in the coordinates $\xi, \dot{\xi}, \tau$ is three-dimensional. The plane $\xi = 0$ divides it into two subspaces: $\mathit{\Phi}_1(\xi \geq 0, \dot{\xi}, \tau)$ and $\mathit{\Phi}_2(\xi < 0, \dot{\xi}, \tau)$. The phase trajectory defined by (2), (3) is located in $\mathit{\Phi}_1$. It follows from the problem formulation that each time the image point falls on the $\xi = 0$ plane at the points $(\dot{\xi}_1, \tau_1), (\dot{\xi}_2, \tau_2), \ldots, (\dot{\xi}_n, \tau_n), \ldots$ with the restitution coefficient $R_1, R_2, \ldots, R_n, \ldots$. Therefore, it can be clearly concluded that the study of the dynamics of system (2), (3) can be carried out using a point mapping of the plane $\xi = 0$ into itself. The point mapping of the plane $\xi = 0$ is written as follows:

$$\begin{aligned}
&\dot{\xi}_n = \mu\Big(e^{-\mu(\tau_{n+1}-\tau_n)} - 1\Big)^{-1} \cdot \Big(e^{-\mu(\tau_{n+1}-\tau_n)} \cdot \Big(\frac{q}{\mu} - \frac{\nu}{\mu}\cos\tau_n + \nu\sin\tau_n\Big) - \frac{q}{\mu} - \\
&- q\tau_n + \frac{1}{\mu}\cos\tau_n + q\tau_{n+1} - \nu\sin\tau_{n+1} - \nu\mu\cos\tau_{n+1}\big), \\
&\dot{\xi}_{n+1} = -R_{n+1}\Big(e^{-\mu(\tau_{n+1}-\tau_n)} \cdot \Big(\dot{\xi}_n - q + \nu\cos\tau_n - \nu\mu\sin\tau_n\Big) + q - \\
&- \nu\cos\tau_{n+1} + \nu\mu\sin\tau_{n+1}\big).
\end{aligned} \tag{4}$$

Here $q = \frac{PM\omega}{Fh}$, $\mu^{-1} = \frac{M\omega}{h}$, $\nu = \frac{M^2\omega^2}{M^2\omega^2+h^2}$ It is directly seen from the point transformation that random sequences of points $\dot{\xi}_1, \tau_1; \dot{\xi}_2, \tau_2; \ldots; \dot{\xi}_n, \tau_n; \ldots$, determined from relations (4) with $R = R_1, R_2, \ldots, R_n, \ldots$, respectively, represent a Markov process to which the theory of Markov processes can be applied. Since, by assumption, the random deviations of the restitution coefficient ΔR_n for each n-th impact from R_0 are small, the set of points $\left\{\dot{\xi}_k, \tau_k\right\}$ will be located in the ϵ neighborhood of the stable periodic regime corresponding to the value of R_0. Therefore, it is possible to linearize point mapping (4) in the vicinity of this stable periodic motion, after which a system of equations in finite differences with respect to post-impact velocity deviations, impact time, and restitution coefficient is obtained in the form:

$$
\begin{aligned}
&\Delta\dot{\xi}_n = \Delta\tau_{n+1}\cdot\left(\frac{\mu(q+\mu\nu)e^{-\mu T}}{e^{-\mu T}-1} - \frac{\mu e^{-\mu T}\cos\tau_0(1-\nu\mu^2-\nu e^{-\mu T})}{(e^{-\mu T}-1)^2} - q\mu - \nu\mu^2\sin\tau_0 + \right.\\
&\left. +\nu\mu\cos\tau_0\right) + \Delta\tau_n\cdot\left(\frac{\mu e^{-\mu T}\cos\tau_0(\nu\mu^2-\nu+1)}{(e^{-\mu T}-1)^2} + \frac{\nu e^{-\mu T}(\sin\tau_0+\mu\cos\tau_0)}{e^{-\mu T}-1} - \frac{+\sin\tau_0}{e^{-\mu T}-1}\right),\\
&\Delta\dot{\xi}_{n+1} = \Delta R_{n+1}\cdot\left((e^{-\mu T}-1)(q-\nu\cos\tau_0+\nu\mu\sin\tau_0) - e^{-\mu T}\dot{\xi}_0\right) +\\
&+\Delta\tau_{n+1}\cdot\left(\mu R_0 e^{-\mu T}(\dot{\xi}_0 - q + \nu\cos\tau_0 - \nu\mu\sin\tau_0) + \nu\sin\tau_0 + \nu\mu\cos\tau_0\right) +\\
&+\Delta\tau_n e^{-\mu T}\cdot\left(-R_0\mu(\dot{\xi}_0 - q + \nu\cos\tau_0 - \nu\mu\sin\tau_0) - \nu\sin\tau_0 - \nu\mu\cos\tau_0\right) +\\
&+e^{-\mu T}\Delta\dot{\xi}_n,
\end{aligned}
\tag{5}
$$

where T denotes the period $2\pi n$. The system of two difference equations of the form (5) is reduced to a linear difference equation with the right side of the form:

$$\Delta\tau_{n+2} + \alpha\Delta\tau_{n+1} + \beta\Delta\tau_n = \gamma\Delta R_{n+1}. \tag{6}$$

Random changes in the restitution coefficient ΔR_{n+1} in Eq. (6) can be selected from the literature sources among the known dependencies.

The solution of the difference equation is sought as the sum of general solution $\Delta\tau_n^0 = C_1 z_1^n + C_2 z_2^n$ (z_1, z_2 — roots of the characteristic equation $z^2+\alpha z+\beta = 0$) of a homogeneous equation and a particular solution Δz_n^* of an inhomogeneous equation, which is found using the method of variation of arbitrary constants.

Since, by assumption, the periodic motion is stable, the roots of the characteristic equation of system (4) satisfy the condition $|z_1| < 1, |z_2| < 1$. With this in mind, the solution of the difference Eq. (6) will be written in the form:

$$\Delta\tau_{n+1} = T\sum_{\nu=-\infty}^{n-1}\alpha_{n,\nu}\Delta R_{\nu+1}, \tag{7}$$

where $\alpha_{n,\nu} = \frac{z_1^{n-\nu-1} - z_2^{n-\nu-1}}{z_1 - z_2}$.

For $\Delta\dot{\xi}_n$ we obtain:

$$\mathit{\Delta}\dot{\xi}_n = \frac{Tp}{1+R}\sum_{\nu=-\infty}^{n-1}\alpha_{n,\nu}\left\{\left(1+\sqrt{\left(\frac{1+R}{p}\right)^2-(1-R)^2(\pi np)^2}\right)\mathit{\Delta}R_{\nu+1}-\right.$$
$$\left.-\mathit{\Delta}R_{\nu+2}\right\}, \tag{8}$$

where p is the ratio of the weight of vibrostriker to the amplitude of driving force.

Assuming that random changes of R are delta-correlated, from (7) and (8), we obtain expressions for standard deviations in the form:

$$\overline{\mathit{\Delta}\tau_n^2} = \frac{T^2(1+z_1z_2)}{(1+z_1z_2)(1-Z_1^2)(1-z_2^2)}\overline{\mathit{\Delta}R^2}, \tag{9}$$

$$\overline{\mathit{\Delta}\dot{\xi}_n^2} = T^2p^2\left[(1+m^2)(1+z_1z_2)\right]\overline{\mathit{\Delta}R^2}, \tag{10}$$

where $m = 1 - \frac{1+R}{p}\cos\tau^*$.

Equalities (7), (8), (9), (9) show that neither $\overline{\mathit{\Delta}\tau_n^2}$, nor $\overline{\mathit{\Delta}\dot{\xi}_n^2}$ do not vanish for any values of the system parameters. This follows from the fact that $|z_i < 1|$, and $1+z1z_2 = 1+R^2$. However, it is possible to specify the values of the system parameters at which the variance $\overline{\mathit{\Delta}\dot{\xi}_n^2}$ equals zero, namely:

$$(1+R^2)\left\{2+\left(\frac{1+R}{p}\cos\tau^*\right)^2-2\frac{1+R}{p}\cos\tau^*\right\}-2\left\{1-\frac{1+R}{p}\cos\tau^*\right\}\cdot$$
$$\cdot\left\{1+R^2-\frac{(1+R)^2}{p}\cos\tau^*\right\}=0. \tag{11}$$

Hence we get the equation:

$$\cos\tau^* = \frac{\left\{2-(1-R)^2\right\}\left\{1-\left(\frac{1-R}{1+R}\pi np\right)^2\right\}}{2p}. \tag{12}$$

For small $R << 1$, which corresponds to an almost absolutely inelastic impact, we obtain

$$\frac{1-(\pi np)^2(1-2R)}{\sqrt{1-(\pi np)^2}} = \frac{2R+1-(\pi np)^2(1-2R)}{2p}. \tag{13}$$

For $R = 0$ formula (13) gives $p = \left(4+(\pi n)^2\right)^{-1/2}$. From the last equality we obtain that for $n = 1$ — $p = 0,276453$, for $n = 2$ — $p = 0,151653$, for $n = 3$ — $p = 0,10379$. For $R = 1$ the value of p is 1. Hence it follows that the curve (12) lies near the upper stability limit.

4 Stochastic Stability

A dynamic system, the input of which is a random process $\xi(\lambda, t)$ and the output is a random process $y(\lambda, t)$, according to [2], will be called stochastically stable if for any $\epsilon > 0$ there exist $\delta > 0$ such that if $r(\xi) \leq \delta$ then $\rho(y) \leq \epsilon(\delta)$ and $\epsilon \to 0$ for $\delta \to 0$, where $r(\xi)$ and $\rho(y)$ are norms. The definition of stochastic stability depends on the way norms $r(\xi)$ and $\rho(y)$ are introduced.

It was shown above that the root mean square deviations of $\overline{\Delta\tau_n^2}$ and $\overline{\Delta\dot{\xi}_n^2}$ can be written as in the form of a product of some constant, which is the function of only the system parameters and the dispersion of the random deviation $\overline{\Delta R^2}$. Assume that $\overline{\Delta R} = 0$ and $\overline{\Delta R^2} \neq 0$. Then, using [2], we can assume that the system will be stochastically stable if there exists a constant C, that depends only on the parameters of the system, such that, for sufficiently large n, the inequalities hold

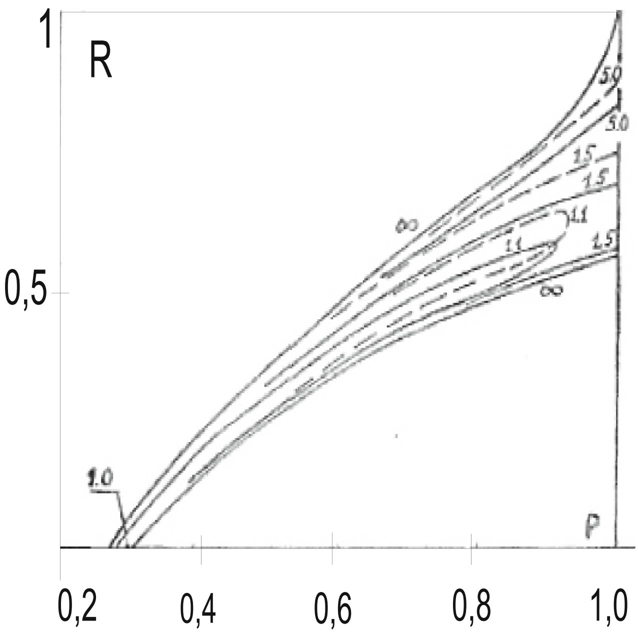

Fig. 1. Numerical experiment

$$\overline{\Delta\tau_n^2} + \overline{\Delta\dot{\xi}_n^2} \leq C\overline{\Delta R^2}. \tag{14}$$

Note that, since the changes of R are small and parameter deviation occurs in the vicinity of such a value at which there is a stable periodic regime in the deterministic case, the value $\overline{\Delta R^2}$ is small, and therefore the expression of the right side of inequality (14) is also a small value ((C is a constant). Hence it follows that in inequality (14), due to the smallness of $C\overline{\Delta R^2}$ the root-mean-square values $\overline{\Delta\tau_n^2}$ and $\overline{\Delta\dot{\xi}_n^2}$ must also be small. In addition, it follows from (14) that if the deviations of the restitution coefficient tend to zero, then the root-mean-square ones also tend to zero if the system is stochastically stable.

5 Numerical Experiment

In this section, we present the results of the numerical experiment in order to identify quantitative characteristics of the effect of correlation functions with a change of R on the value of the root-mean-square deviations.

In Fig. 1 dotted lines are level lines for delta-correlated changes of R, and solid lines are level lines corresponding to non-delta -correlated changes of R.

Inside the stability region there is a boundary curve

$$x = \frac{2(1+R)^2}{\sqrt{4(1+R)^2 + (1-R^2)^2\,T^2}}, \tag{15}$$

meaning as follows. Correlation with a random change of R when accounted for leads to a shift of the level lines to domain of smaller p for $p \le x$ and larger one p for $p > x$.

6 Conclusions

1. A numerical-analytical technique for studying the dynamics of strongly nonlinear (vibro-impact) systems with randomly changing parameters has been developed. The main attention is paid to the study of the influence of the most significant parameters that change in time and depend on the behavior of the systems themselves in time. Thus, for example, the structure of the behavior of a vibro-impact system is studied when the velocity recovery coefficient changes from impact to impact.
2. A refined form of the stochastic stability of vibro-impact systems is given, with the help of which it is possible to determine, in particular, an analogue of dangerous and safe boundaries of stochastic stability.
3. The calculated data on the behavior of maximum velocities upon impact during material processing are given.
4. Exact ratios of solutions of difference equations depending on the choice of the form of random changes in the velocity recovery coefficient upon impact are given.

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