Intensional Harmony as Isomorphism

Paolo Pistone and Luca Tranchini

Abstract In the present paper we discuss a recent suggestion of Schroeder-Heister concerning the possibility of defining an intensional notion of harmony using isomorphism in second-order propositional logic. The latter is not an absolute notion, but its definition is relative to the choice of criteria for identity of proofs. In the paper, it is argued that in order to attain a satisfactory account of harmony, one has to consider a notion of identity stronger than the usual one (based on β- and η-conversions) that the authors have investigated in recent work.

Key words: identity of proofs, equivalence of derivations, second-order logic, parametricity, System F, quantification

1 Introduction

The inferentialist thesis that the meaning of a logical constant is determined by its inferences rules has been famously challenged by Prior (1960) who put forward the following pair of introduction and elimination rules for the binary connective tonk:

$$\frac{A}{A \;\mathtt{tonk}\; B}\,\mathtt{tonkI} \qquad\qquad \frac{A \;\mathtt{tonk}\; B}{B}\,\mathtt{tonkE}$$

The strong intuition that tonk is semantically deficient has been taken to require a qualification of the inferentialist thesis: Not any arbitrary collection of inference rules

---

Paolo Pistone
Department of Computer Science and Engineering, University of Bologna, Italy,
e-mail: paolo.pistone@uniroma3.it

Luca Tranchini
Department of Computer Science, University of Tübingen, Germany,
e-mail: luca.tranchini@gmail.com

T. Piecha and K. F. Wehmeier (eds.), *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic 29,
https://doi.org/10.1007/978-3-031-50981-0_10

can determine the meaning of a logical constant but only those satisfying a certain requirement, that (following Dummett, 1991) is commonly referred to as *harmony*.

The exact significance of harmony is open to interpretation. In particular, there is no agreement as to whether harmony should be regarded as a descriptive or normative criterion; nor as to whether harmony should be regarded as a criterion of "meaningfulness" or merely of "logicality" (i.e., whether expressions governed by rules which are not in harmony should be regarded as meaningless; or as meaningful, but not belonging to the logical vocabulary), or possibly something else. Moreover, different characterizations of the notion of harmony have been proposed in the literature, some of them being fully formal, some of them being less precise.

Harmony via second-order translations. In the context of natural deduction (Gentzen, 1935; Prawitz, 1965a), harmony is usually described by reference to the "perfect balance" between the introduction and elimination rules of the connectives of intuitionistic propositional logic NI (see Table 1).

Table 1 The natural deduction system NI.

$$\frac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I} \qquad\qquad \frac{A \wedge B}{A}\,\wedge\mathrm{E}_1 \quad \frac{A \wedge B}{B}\,\wedge\mathrm{E}_2$$

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}\,\supset\mathrm{I} \qquad\qquad \frac{A \supset B \qquad A}{B}\,\supset\mathrm{E}$$

$$\frac{A}{A \vee B}\,\vee\mathrm{I}_1 \quad \frac{B}{A \vee B}\,\vee\mathrm{I}_2 \qquad\qquad \frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\,\vee\mathrm{E}$$

$$\frac{}{\top}\,\top\mathrm{I} \qquad\qquad \frac{\bot}{C}\,\bot\mathrm{E}$$

But what does this "perfect balance" consist in, exactly? In spite of some attempts to answer the question in a precise way (see, e.g., Belnap, 1962; Tennant, 1978; Read, 2010) a first fully formal definition of harmony has been proposed only recently by Peter Schroeder-Heister (2014a; 2014b).

In a nutshell, Schroeder-Heister's proposal is that of characterizing collections of introduction rules and collections of elimination rules with a formula of quantified propositional intuitionistic logic NI^2, the extension of NI with universal quantification over propositions, governed by the following rules:

$$\frac{A}{\forall X.A}\,\forall\mathrm{I} \qquad\qquad \frac{\forall X.A}{A[C/X]}\,\forall\mathrm{E}$$

Schroeder-Heister's proposal is that two collections of introduction and elimination rules for a connective are in harmony if and only if their characteristic formulas are *interderivable* in NI^2.

Towards intensional harmony. Although this proposal represents a long-awaited step forward in the understanding of harmony, there are reasons of dissatisfaction with it. These hinge upon the fact that collections of rules that are interderivable are characterized by interderivable formulas. For instance, given $\wedge E_1$, the rule $\wedge E_2$ is obviously interderivable with the following rule:

$$\frac{A \wedge B \qquad A}{B}\ \wedge E_2'$$

and the formulas $X \wedge Y$ and $X \wedge (X \supset Y)$ — characterizing the collections of elimination rules for $\wedge$ consisting in the pairs of $\wedge E_1$, $\wedge E_2$ and of $\wedge E_1$, $\wedge E_2'$ respectively — are obviously interderivable. Thus both pairs of rules qualify as in harmony with $\wedge I$ on Schroeder-Heister's criterion. However, one has a clear intuition that the pair $\wedge E_1$, $\wedge E_2'$ is "less" in harmony with $\wedge I$ than the pair $\wedge E_1$, $\wedge E_2$.

This has prompted the second author (see Tranchini, 2016a) to regard Schroeder-Heister's criterion as a characterization of a "weak" notion of harmony, and to call for a strengthening of it capable of capturing a notion of harmony on which only the pair $\wedge E_1$, $\wedge E_2$ (and not the pair $\wedge E_1$, $\wedge E_2'$) qualify as in harmony with $\wedge I$. What distinguishes such a would-be stronger notion of harmony — not just from Schroeder-Heister's weak harmony, but also from other proposals, such as those of Belnap (1962), and of Tennant (1978) — is its (hyper-)*intensional* nature, that is, its being capable of discriminating among collection of rules which are indistinguishable in terms of derivability.

Intensional inferentialism. The idea that a semantic framework should be able to draw (hyper-)intensional distinctions has a long tradition, going back at least to Carnap (1956), who notably regarded logical equivalence as too coarse a criterion for synonymy, and proposed instead to characterize synonymy using the notion of intensional isomorphism.

In the context of inferential accounts of meaning, especially in the proof-theoretic semantics tradition of Dummett and Prawitz, (hyper-)intensional aspects have been largely ignored. One of the reasons for this is that these theories of meaning have been shaped in analogy with traditional ones, by replacing the notion of truth conditions with the one of assertibility conditions. Consider for instance a language containing two distinct binary connectives $\sharp$ and $\flat$ whose inferential behaviour is described as follows: a proof of $A \sharp B$ is a triple consisting of a proof of A, a method to transform proofs of A into proofs of B and a method to transform proofs of B into proofs of A; a proof of $A \flat B$ differs from a proof $A \sharp B$ in that its first member is a proof of B. Clearly, $A \sharp B$ and $A \flat B$ are assertible under the same conditions: Whenever one has a proof of $A \sharp B$, one knows that a proof of $A \flat B$ could be constructed, and vice versa. However, to be in possession of a proof of $A \sharp B$ is clearly a different epistemic state from being in possession of a proof of $A \flat B$. If we take meaning to consist not only of what Frege called *Bedeutung* (the portion of reality referred to by an expression) but also of *Sinn* (the epistemic content of an expression), then an inferentialist account of meaning should be able to distinguish between the meanings of $A \sharp B$ and $A \flat B$.

As a matter of fact, proof theory offers a wide range of formal tools to analyze such issues, and in the present paper we will discuss the prospects of using some of these tools to deliver an intensional account of the notion of harmony.

Harmony via isomorphism. An obvious way of attaining a notion of harmony stronger than Schroeder-Heister's would be to require the characteristic formula of the collection of introduction rules to be *the same as* that of the collection of eliminations. But this would be too much. It is true that on such a strengthening the pair $\wedge\mathrm{E}_1$, $\wedge\mathrm{E}_2'$ would not count as in harmony with $\wedge$I. However, neither would $\vee$E count as in harmony with the pair $\vee\mathrm{I}_1$, $\vee\mathrm{I}_2$: The characteristic formula (see below for precise definition) of the former is $\forall X((Y \supset X) \wedge (Z \supset X)) \supset X$ while that of the latter is $Y \vee Z$, i.e., two distinct, though interderivable NI^2-formulas. In other words, by adopting this notion of harmony (we dub it *strict harmony*) one would be led to deny that the rules of disjunction are harmonious.

The notion of formula isomorphism coming from categorial proof-theory and the study of typed lambda-calculi provides a middle ground between interderivability and identity. Inspired by the work of Došen (see, e.g., Došen, 2003), the second author (see again Tranchini, 2016a) used the notion of isomorphism to clarify the exact sense in which merely weakly harmonious rules are harmful (see also below Section 3) therefore pointing to the relevance of isomorphism for a characterization of harmony.

Different options as to defining harmony using isomorphism have been tentatively put forward by Schroeder-Heister (2016). Among the different options proposed, there is that of defining strong harmony by replacing interderivability (resp. identity) in the definition of weak (resp. strict) harmony with that of isomorphism. However, this proposal is discarded as inappropriate, due to the fact that — at least *prima facie* — $\forall Z((X \supset Z) \wedge (Y \supset Z)) \supset Z$ and $X \vee Y$ are not isomorphic in NI^2.

Main contribution. The isomorphism of two formulas in a given system is not an absolute notion, but it is relative to the choice of a notion of identity of proofs (that is, of an equational theory on the derivations of the system). Building on well-established results in the categorial semantics of second-order logic, in recent work the authors have introduced an equational theory stronger than the usual one using a class of equations referred to as ε-equations (see Tranchini, Pistone, and Petrolo, 2019; Pistone, Tranchini, and Petrolo, 2021; Pistone and Tranchini, 2021). The class of isomorphisms relative to ε-equivalence is rich enough to overcome the problem mentioned above (in particular $\forall X((Y \supset X) \wedge (Z \supset X)) \supset X$ and $Y \vee Z$ are ε-isomorphic formulas in NI^2). Moreover, although the equational theory induced by ε-equations (together with the standard conversions for NI^2) is not maximal on the whole of NI^2, it is the maximum equational theory of certain weak fragments of NI^2. Among these fragments there is the one whose formulas correspond to the "encodings" of collections of introduction and elimination rules for propositional connectives. In this paper, we present in an informal way the notion of identity of proofs captured by the ε-equations and show how the results obtained about them provide a firm footing for an intensional account of harmony between weak and strict harmony.

2 From reductions and expansions to isomorphism

The "perfect balance" between introduction and elimination rules that the notion of harmony aims at capturing has been described as obtaining when

> what can be inferred from a logically complex sentence by means of the *elimination rules* for its main connective is *no more* and *no less* than what has to be established in order to infer that very logically complex sentence using the *introduction rules* for its main connective.

When the rules for a connective are in harmony, two kinds of deductive patterns can be exhibited.

Patterns of the first kind are those giving rise to *maximal formulas occurrences*, that is formula occurrences which are the major premise of an application of an elimination rule (i.e., the premise whose main connective is the one to be eliminated) and that are the conclusion of an application of one of the introduction rules. Prawitz (1965b) defined certain operations on derivations called *reductions*. Reductions allow rewriting a derivation into another one thereby getting rid of a single maximal formula occurrence (though new ones may be generated in the process): In the case of conjunction, we have the following two reductions:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\quad\begin{matrix}\mathscr{D}_2\\ B\end{matrix}}{A\wedge B}\,\wedge\mathrm{I}}{A}\,\wedge\mathrm{E}_1 \quad \text{reduces to} \quad \begin{matrix}\mathscr{D}_1\\ A\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A\end{matrix}\quad\begin{matrix}\mathscr{D}_2\\ B\end{matrix}}{A\wedge B}\,\wedge\mathrm{I}}{B}\,\wedge\mathrm{E}_2 \quad \text{reduces to} \quad \begin{matrix}\mathscr{D}_2\\ B\end{matrix}$$

Prawitz showed how — by successively applying reductions in a certain order — any given NI-derivation can be to transformed into one in *normal* form, that is one with no maximal formula occurrences.

The other kinds of patterns are those in which the premises of applications of introduction rules have been obtained by applying the corresponding elimination rules. Prawitz (1971) defined operations that are, in a sense, the dual of reductions, called immediate expansions. In the case of conjunction, the expansion looks as follows:

$$\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix} \quad \text{expands to} \quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix}}{A}\,\wedge\mathrm{E}_1 \qquad \dfrac{\begin{matrix}\mathscr{D}\\ A\wedge B\end{matrix}}{B}\,\wedge\mathrm{E}_2}{A\wedge B}\,\wedge\mathrm{I}$$

Prawitz showed that by successively applying expansions it is possible to transform any given normal derivation in NI into one in *long normal* form, i.e., into a derivation in which all minimal formula occurrences (those that are the conclusion of an elimination and the premise of an introduction rule) are atomic.

The reduction and expansion associated to the rules of implication are the following:

$$\dfrac{(u)\ \dfrac{\begin{matrix}\overset{u}{[A]}\\ \mathscr{D}\\ B\end{matrix}}{A \supset B}\supset\mathrm{I} \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B} \quad \text{reduces to} \quad \begin{matrix}\mathscr{D}'\\ [A]\\ \mathscr{D}\\ B\end{matrix}$$

$$\begin{matrix}\mathscr{D}\\ A \supset B\end{matrix} \quad \text{expands to} \quad (u)\ \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ A \supset B\end{matrix} \qquad \overset{u}{A}}{B}\supset\mathrm{E}}{A \supset B}\supset\mathrm{I}$$

Via the Curry-Howard correspondence between derivations in the implicational fragment of NI and terms of the simply typed λ-calculus, these rewriting operations on derivations correspond (respectively) to those of β-reduction and η-expansion on λ-terms:

$$(\lambda x.t)s \overset{\beta}{\rightsquigarrow} t[s/x] \qquad t \overset{\eta}{\rightsquigarrow} (\lambda x.t)x$$

Like in λ-calculus, reductions and expansions can be used to define an equivalence relation on natural deduction derivations. Two derivations $\mathscr{D}$ and $\mathscr{D}'$ are equivalent if and only if one can be obtained from the other by applying a finite number of times (β-)reduction, (η-)expansion and their inverse operations to $\mathscr{D}$ and $\mathscr{D}'$ or their sub-derivations (we indicate that $\mathscr{D}$ and $\mathscr{D}'$ are $\beta\eta$-equivalent as $\mathscr{D} \overset{\beta\eta}{\equiv} \mathscr{D}'$, and more in general given an equivalence relation E, we indicate E-equivalence as $\mathscr{D} \overset{E}{\equiv} \mathscr{D}'$).[1] As equivalent λ-terms can be seen as different ways of representing the same function, Prawitz (1971) observed — following a suggestion by Martin-Löf — that equivalent derivations can be seen as different linguistic representations of the same proof (where proofs are understood as abstract entities informally characterized by the so-called BHK-clauses; see Tranchini 2012; 2016b; 2019).

Given an equivalence relation on derivations, it is possible to use it to define an equivalence relation on formulas that is, in general, stricter than interderivability and that it is commonly referred to as *isomorphism*. Let E be an equivalence relation on derivations of a natural deduction system S. Two formulas are E-isomorphic (notation $A \overset{E}{\simeq} B$) iff

1. there exist two S-derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of A from B and of B from A respectively (i.e., A and B are interderivable in S);
2. such that their compositions are E-equivalent to the derivations consisting only of the assumptions of A and of B respectively:

$$B \overset{E}{\equiv} \begin{matrix}[B]\\ \mathscr{D}_1\\ [A]\\ \mathscr{D}_2\\ B\end{matrix} \qquad\qquad \begin{matrix}[A]\\ \mathscr{D}_2\\ [B]\\ \mathscr{D}_1\\ A\end{matrix} \overset{E}{\equiv} A$$

[1] We will always implicitly identify derivations up to renaming of discharge indexes, which corresponds to α-equivalence on λ-terms.

The derivation consisting of the assumption of a formula A can be viewed as representing the identity function on the set of proofs of A. Hence, the second condition of the definition of isomorphism can be expressed by saying that the two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ represent two functions from proofs of A to proofs of B and vice versa which are the inverse of each other. This in turn means that the set of proofs of A and of B are in bijection.

Typical examples of $\beta\eta$-isomorphic formulas in NI are pairs of formulas of the form $(A \wedge B) \wedge C$ and $A \wedge (B \wedge C)$, or $(A \wedge B) \supset C$ and $A \supset (B \supset C)$, whereas typical examples of interderivable but non-$\beta\eta$-isomorphic formulas are pairs of formulas of the form A and $A \wedge A$, or $A \wedge B$ and $A \wedge (A \supset B)$.

The notion of isomorphism has been proposed (notably by Došen, 2003) as a formal counterpart of the informal notion of synonymy, i.e., identity of meaning. Intuitively, interderivability is only a necessary, but not sufficient condition for synonymy. From an inferential perspective, isomorphic formulas can be regarded as synonymous in the sense that:

> "they behave exactly in the same manner in proofs: by composing, we can always extend proofs involving one of them, either as assumption or as conclusion, to proofs involving the other, so that nothing is lost, nor gained. There is always a way back. By composing further with the inverses, we return to the original proofs." (Došen, 2003, p. 498)

Clearly, a necessary condition for some notion of E-isomorphism not to collapse on that of interderivability is that the equivalence relation E used in the definition is non-trivial (i.e., there must be at least one formula A and two derivations of A belonging to distinct equivalence classes). In particular, if any two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of any formula A from itself were E-equivalent, the second condition of the definition of E-isomorphism would be vacuously satisfied.

The notion of $\beta\eta$-equivalence (and consequently that of $\beta\eta$-isomorphism) plays a distinguished role in NI, since $\beta\eta$-equivalence is the maximum non-trivial equivalence relation definable on NI-derivations. As Došen (2003) and Widebäck (2001) argued, the maximality of an equivalence relation E on the derivation of a system S can be taken as supporting the claim that it is the *correct* way of analyzing the notion of identity of proofs underlying S.

For the $\{\supset, \wedge, \top\}$-fragment of NI, $\beta\eta$-equivalence and $\beta\eta$-isomorphism are well-understood: the decidability of $\beta\eta$-equivalence is an immediate consequence of normalization and confluence of $\beta\eta$-reduction in the $\{\supset, \wedge\}$-fragment, and its maximality was established by Statman (1983), Došen and Petrić (2001), and Widebäck (2001). Moreover, $\beta\eta$-isomorphism in this fragment is decidable and it has been fully axiomatized by Solov'ev (1983).

The extension of these results to richer language fragments has proven a difficult task. In presence of disjunction the decidability and maximality of $\beta\eta$-equivalence have been established only recently by Scherer (2017); the decidability of $\beta\eta$-isomorphism was established by Ilik (2014). In this case the difficulty were due to the form of the η-expansion for disjunction:

$$\begin{array}{c}\mathscr{D}'\\ [A\vee B]\\ \mathscr{D}''\\ C\end{array}\quad \text{expands to}\quad \dfrac{\begin{array}{c}\mathscr{D}'\\ A\vee B\end{array}\quad \begin{array}{c}\dfrac{\stackrel{n}{A}}{[A\vee B]}\,{\vee}\mathrm{I}\\ \mathscr{D}''\\ C\end{array}\quad \begin{array}{c}\dfrac{\stackrel{m}{B}}{[A\vee B]}\,{\vee}\mathrm{I}\\ \mathscr{D}''\\ C\end{array}}{C}\,(n,m)$$

(with n and m fresh for $\mathscr{D}'$)

which can be seen as the composition of the simpler form of expansion proposed by Prawitz

$$\begin{array}{c}\mathscr{D}\\ A\vee B\end{array}\quad \text{expands to}\quad \dfrac{\begin{array}{c}\mathscr{D}\\ A\vee B\end{array}\quad \dfrac{\stackrel{n}{A}}{A\vee B}\,{\vee}\mathrm{I}_1\quad \dfrac{\stackrel{m}{B}}{A\vee B}\,{\vee}\mathrm{I}_2}{A\vee B}\,{\vee}\mathrm{E}\ (n,m)$$

(with n and m *fresh*[2] for $\mathscr{D}$)

and a generalization of the permutative conversions used in establishing the subformula property of normal derivations in NI:

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathscr{D}\\ A\vee B\end{array}\quad \begin{array}{c}\stackrel{n}{[A]}\\ \mathscr{D}_1\\ C\end{array}\quad \begin{array}{c}\stackrel{m}{[B]}\\ \mathscr{D}_2\\ C\end{array}}{[C]}\,{\vee}\mathrm{E}\ (n,m)\\ \mathscr{D}_3\\ D\end{array}\quad \rightsquigarrow_{\gamma^+}\quad \dfrac{\begin{array}{c}\mathscr{D}\\ A\vee B\end{array}\quad \begin{array}{c}\stackrel{n}{[A]}\\ \mathscr{D}_1\\ [C]\\ \mathscr{D}_3\\ D\end{array}\quad \begin{array}{c}\stackrel{m}{[B]}\\ \mathscr{D}_2\\ [C]\\ \mathscr{D}_3\\ D\end{array}}{D}\,{\vee}\mathrm{E}\ (n,m)$$

(see, for a discussion, Tranchini 2016a; 2018).[3]

Whereas the maximality and decidability of $\beta\eta$-equality also hold in presence of $\bot$ (hence for the full language of NI), the decidability of $\beta\eta$-isomorphism in presence of $\bot$ is still an open problem.

3 Weak harmony and its limits

In order to define harmony formally, a useful preliminary move is that of identifying rules — that are usually taken to be meta-linguistic schemata — with expressions belonging to an object language of the appropriate kind. Slightly reformulating

[3] In this work, the first author proposed the existence of reduction and expansions of this more general form as an informal notion of harmony. Although informal, the requirement is enough to rule out "quantum disjunction" (i.e., the connective whose rules are almost the same as those of disjunction in NI, the only difference being that the elimination rule can be applied only when its minor premises depend on no assumption other than those discharged by the rule application; see Dummett, 1991) as disharmonious, since the restriction on the elimination rule blocks the possibility of defining an expansion of the more general form. See also concluding remarks in Section 5.

insights of Schroeder-Heister (2014a; 2014b), we propose to identify the *rules* of an arbitrary natural deduction system with a particular class of *formulas* of the second-order language extending that of the system NI with

1. universal quantification over propositions
2. for each n, denumerably many variables for n-ary connectives (to be indicated with $\dagger^n$, possibly with subscripts, and other *ad hoc* symbols), so that if $A_1, \ldots, A_n$ are formulas and $\dagger^n$ is an n-ary connective variable, then $\dagger^n(A_1, \ldots, A_n)$ is also a formula (the superscript n indicating the arity of $\dagger$ will be mostly omitted).

More precisely, given denumerably many propositional variables (to be indicated with X, Y, Z possibly with subscripts) and denumerably many connective variables as above, we define the set of formulas $\mathcal{L}$, to be indicated with A, B, C, possibly with subscripts, by the following grammar (we use A^n for a sequence of n comma-separated formulas):[4]

$$A ::= X \mid \top \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \dagger^n(A^n) \mid \forall X.A$$

(we will indicate with $\mathcal{L}^2$ the fragment of $\mathcal{L}$ lacking variables for connectives, and with $\mathcal{L}^{2\supset}$ fragment of $\mathcal{L}^2$ lacking all connectives apart from $\supset$).

The idea of identifying a rule with a formula may appear odd at first, but it is actually very natural. For instance, the rules $\vee \mathrm{I}_1$, $\vee \mathrm{I}_2$ and $\vee$E of NI can be identified with the following $\mathcal{L}$-formulas:

$$(\vee \mathrm{i}_1) \qquad \forall XY.X \supset \dagger(X, Y)$$

$$(\vee \mathrm{i}_2) \qquad \forall XY.Y \supset \dagger(X, Y)$$

$$(\vee \mathrm{e}) \qquad \forall XYZ.(\dagger(Y, Z) \wedge (Y \supset X) \wedge (Z \supset X)) \supset X$$

Observe that the standard propositional connectives and the universal quantifier are used to “encode” the different “structural features” implicit in natural deduction rules (i.e., conjunction “encodes” multiplicity of premises, implication “encodes” the passage from premises to conclusions, and universal quantification “encodes” the schematic nature of rules). As we are using $\mathcal{L}$ as a meta-language to investigate the notion of rule, disregarding the fact that rules can be associated to a specific piece of vocabulary, we replaced accordingly the disjunction of the rules of NI with the (binary) connective variable $\dagger$. In this way, the conjunction of the three formulas ($\vee \mathrm{i}_1$), ($\vee \mathrm{i}_2$) and ($\vee$e) above is an $\mathcal{L}$-formula with a free connective variable that we can take to express the predicate “being a disjunction”. Similarly, we can identify the rule $\supset$E with the formula $\forall XY.(\dagger(X, Y) \wedge X) \supset Y$ and $\wedge$I with the formula $\forall XY.(X \wedge Y) \supset \dagger(X, Y)$.

More generally, we call *structural formulas* (to be indicated with S, $S_1, \ldots$) those formulas constructed using only propositional variables and connective variables.

[4] In other words, we are actually working in the fragment of the third-order language of the system F_1 of Girard (1986), in which variables for connectives occur only free. Thus the natural deduction system over $\mathscr{L}$ consists of the rules of the second-order natural deduction system NI^2, the extension of Girard’s System F with primitive rules for $\top$, $\bot$, $\wedge$ and $\vee$.

More precisely, the set of structural formulas is the subset $\mathscr{L}^s$ of $\mathscr{L}$ defined as follows (we use S^n for a list of n comma-separated structural formulas):

$$S ::= X \mid \dagger^n(S^n)$$

and we call *rule formulas* (or simply rules, to be indicated with $R, R_1, \ldots$) the $\mathscr{L}$-formulas constructed according to the following grammar (we indicate the set of rule formulas as $\mathscr{L}^r$):

$$R ::= S \mid \forall\overrightarrow{X}(\textstyle\bigwedge_{i=1}^n R_i \supset S)$$

where $\forall\overrightarrow{X} = \forall X_1 \ldots \forall X_m$ if $m > 0$ or it is empty otherwise, and $\bigwedge_{i=1}^n A_i = A_1 \wedge (A_2 \wedge (\ldots \wedge A_n) \ldots)$ if $n > 0$ or $\bigwedge_1^n A_i = \top$ otherwise. The *level of a rule* R, indicated with $\ell(R)$ is the maximum number of nested implications in R, so that $\ell(S) = 0$ and $\ell(\forall\overrightarrow{X}(\bigwedge_{i=1}^n R_i \supset S)) = \mathtt{max}(\ell(R_i)) + 1$.

By an *introduction rule* for an n-ary connective $\dagger$ we understand a rule of the form

(INTRO) $$\forall\overrightarrow{X}\left(\textstyle\bigwedge_{i=1}^n R_i \supset \dagger(\overrightarrow{X})\right)$$

satisfying the following two conditions:

1. in each R_i neither the universal quantifier nor connective variables occurs at all;
2. no propositional variable occurs free (i.e., the only propositional variables occurring in each R_i are among those in $\overrightarrow{X}$).

If R is an introduction rule for $\dagger$ of the above form, we define the *content of* R (notation $\mathscr{C}(R)$) to be the $\mathscr{L}^2$-formula $\bigwedge_{i=1}^n R_i$. If $\mathscr{I}\dagger = \langle R_{I1}, \ldots, R_{Im_I}\rangle$ is a list of introduction rules for $\dagger$, we define the *content of* $\mathscr{I}\dagger$ (notation $\mathscr{C}(\mathscr{I}\dagger)$) to be the $\mathscr{L}^2$-formula $\bigvee_{j=1}^{m_I} \mathscr{C}(R_{Ii})$, where $\bigvee_{i=1}^n A_i = A_1 \vee (A_2 \vee (\ldots \vee A_n) \ldots)$ if $n > 0$ or $\bigvee_{i=1}^n A_i = \bot$ otherwise.

By an *elimination rule* for an n-ary connective $\dagger$ we understand a rule of the form

(ELIM) $$\forall X \forall\overrightarrow{Y}\forall\overrightarrow{X}\left(\left(\dagger(\overrightarrow{X}) \wedge \textstyle\bigwedge_1^n R_i\right) \supset X\right)$$

satisfying the following three conditions:

1. in each R_i neither the universal quantifier nor connective variables occurs at all;
2. no propositional variable occurs free (i.e., $\overrightarrow{Y}$ contains all propositional variables occurring in any of the R_i other than X and those in $\overrightarrow{X}$);
3. if X occurs in any of the R_i it occurs in rightmost position.[5]

If R is an elimination rule for $\dagger$ of the above form, its *content* $\mathscr{C}(R)$ is the $\mathscr{L}^2$-formula $\forall X \forall\overrightarrow{Y}\left(\bigwedge_{i=1}^n R_i \supset X\right)$. If $\mathscr{E}\dagger = \langle R_{E1}, \ldots, R_{Em_E}\rangle$ is a list of elimination rules for $\dagger$, we define the *content of* $\mathscr{E}\dagger$ (notation $\mathscr{C}(\mathscr{E}\dagger)$) to be the $\mathscr{L}^2$-formula $\bigwedge_{k=1}^{m_E} \mathscr{C}(R_{Ek})$.

[5] With the exception of this last conditions, the definitions of introduction and elimination rules follow those given in Schroeder-Heister (2014a). The reason for considering this final restriction will be discussed at the end of Section 4.

Suppose now we are given a list $\mathscr{I}\dagger$ of introduction rules and a list $\mathscr{E}\dagger$ of elimination rules for an n-ary connective $\dagger$. We say that the two collections of rules are in *weak harmony* if and only if the following holds in NI^2:[6]

$$\mathscr{C}(\mathscr{I}\dagger) \dashv\vdash \mathscr{C}(\mathscr{E}\dagger)$$

For example, the rule $\wedge\mathrm{I}$, $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2'$ discussed in the introduction are the formulas (we use the connective variable $\dagger$ for conjunction) $\forall XY.(X \wedge Y) \supset \dagger(X, Y)$, $\forall XY.\dagger(X, Y) \supset X$ and $\forall XY.(\dagger(X, Y) \wedge X) \supset Y$ respectively. Thus, the content of the collection of introduction rules for conjunction consisting of $\wedge\mathrm{I}$ and of the collection of elimination rules for conjunction consisting of $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}_2'$ are $X \wedge Y$ and $X \wedge (X \supset Y)$ respectively. As these two formulas are intederivable in NI^2, the two collections of rules are in weak harmony. As the reader can check, the collections of introduction and elimination rules consisting of the rules of NI are in weak harmony as well.

Prawitz (1979) and Schroeder-Heister (1981; 1984; 2014b) proposed a simple method to construct a weakly harmonious collection of elimination rules by "inverting" a collection of introduction rules for a given connective $\dagger$. In particular, let $\mathscr{I}\dagger$ be a sequence of m distinct introduction rules of the above form, i.e., $\mathscr{I}\dagger = \langle R_1, \ldots, R_m \rangle$, with $R_j = \forall\overrightarrow{X}\left(\bigwedge_{i=1}^{n_j} R_{ji} \supset \dagger(\overrightarrow{X})\right)$ for all $1 \leq j \leq m$. The Prawitz-Schroeder-Heister collection of canonical elimination rule associated to $\mathscr{I}\dagger$, to be indicated as $\mathtt{PSH}(\mathscr{I}\dagger)$ is the list containing only one element, namely the elimination rule $\forall X\forall\overrightarrow{X}.\left(\dagger(\overrightarrow{X}) \wedge \bigwedge_{j=1}^{m}\left(\bigwedge_{i=1}^{n_j} R_{ji} \supset X\right)\right) \supset X$ that we indicate with $\dagger\mathrm{E}_{\mathtt{PSH}(\mathscr{I}\dagger)}$. The collections of introduction rules $\mathscr{I}\dagger$ and that of elimination rules $\mathtt{PSH}(\mathscr{I}\dagger)$ are in weak harmony since in NI^2 the content of $\mathscr{I}\dagger$ is interderivable with that of $\mathtt{PSH}(\mathscr{I}\dagger)$ (i.e., with the content of $\dagger\mathrm{E}_{\mathtt{PSH}(\mathscr{I}\dagger)}$):

$$\bigvee_{j=1}^{m}\bigwedge_{i=1}^{n_j} R_{ji} \dashv\vdash \forall X.\left(\bigwedge_{j=1}^{m}\left(\bigwedge_{i=1}^{n_j} R_{ji} \supset X\right)\right) \supset X$$

As Schroeder-Heister (2014a) shows, the left-to-right direction of harmony (called "conservativity" criterion) warrants that the addition of the rules for $\dagger$ (understood as meta-linguistic schemata) to a given natural deduction system N yields a conservative extension of N; and that the right-to-left direction warrants the uniqueness of $\dagger$ (where conservativity and uniqueness are understood in the sense of Belnap, 1962).

As shown in Schroeder-Heister (1981) one can define reduction procedures to get rid of consecutive applications of an introduction rule for a connective followed immediately by the Prawitz-Schroeder-Heister elimination rule, and the same is true for expansions (see Tranchini, 2016a). Actually, reduction and expansions are available not only when the elimination rules follow the pattern of Prawitz and Schroeder-Heister: If two collections of introduction and elimination rules are in weak harmony, it is possible to equip them with expansions and reductions as well. A formalization of this claim hinges on a formal characterization of what sort of

[6] Observe that the two formulas contain no occurrence of connective variables, that is they belong to the proper second-order system NI^2. Cf. note 4 above.

operations can qualify as reductions and expansions. Here we limit to an informal sketch of how reduction and expansions for harmonious rules can be obtained from those associated to the "canonical pair" consisting of a collection of introduction rule and the Prawitz-Schroeder-Heister collection of elimination rules, and to discuss some examples.

Observe first that, if a collection of introduction rules $\mathscr{I}\dagger$ and a collection of elimination rules $\mathscr{E}\dagger = \langle R_1, \ldots, R_m \rangle$ are in weak harmony, then the content $\mathscr{E}$ is interderivable with the content of the Prawitz-Schroeder-Heister elimination rule and hence the content of each elimination rule in $\mathscr{E}\dagger$ is derivable from that of $\dagger\mathrm{E}_{\mathrm{PSH}(\mathscr{I}\dagger)}$. From each possible way of deriving the content of any of the rules R_j in $\mathscr{E}\dagger$ from that of $\dagger\mathrm{E}_{\mathrm{PSH}(\mathscr{I}\dagger)}$ one can "extract" a reduction procedure to get rid of consecutive applications of any introduction rule in $\mathscr{I}\dagger$ followed immediately by R_j. Moreover, from the derivation of the content of $\dagger\mathrm{E}_{\mathrm{PSH}(\mathscr{I}\dagger)}$ from the content of $\mathscr{E}\dagger$ we can "extract" an expansion for the collections of rules $\mathscr{I}\dagger$ and $\mathscr{E}\dagger$.

For example, let us reconsider the collection of introduction rules simply consisting of $\wedge$I and the "deviant" collection of elimination rules consisting of $\wedge\mathrm{E}_1$ and $\wedge\mathrm{E}'_2$ that we discussed in the introduction. We can obviously define the following reductions and expansion:

$$\dfrac{\dfrac{\stackrel{\mathscr{D}_1}{A} \qquad \stackrel{\mathscr{D}_2}{B}}{A \wedge B}\,{\scriptstyle\wedge\mathrm{I}}}{A}\,{\scriptstyle\wedge\mathrm{E}_1} \quad \overset{\beta-\wedge_1}{\rhd} \quad \stackrel{\mathscr{D}_1}{A} \qquad\qquad \dfrac{\dfrac{\stackrel{\mathscr{D}_1}{A} \qquad \stackrel{\mathscr{D}_2}{B}}{A \wedge B}\,{\scriptstyle\wedge\mathrm{I}} \qquad \stackrel{\mathscr{D}'}{A}}{B}\,{\scriptstyle\wedge\mathrm{E}'_2} \quad \overset{\beta-\wedge'_2}{\rhd} \quad \stackrel{\mathscr{D}_2}{B}$$

$$\stackrel{\mathscr{D}}{A \wedge B} \quad \text{expands to} \quad \dfrac{\dfrac{\stackrel{\mathscr{D}}{A \wedge B}}{A}\,{\scriptstyle\wedge\mathrm{E}_1} \qquad \dfrac{\stackrel{\mathscr{D}}{A \wedge B} \qquad \dfrac{\stackrel{\mathscr{D}}{A \wedge B}}{A}\,{\scriptstyle\wedge\mathrm{E}_1}}{B}\,{\scriptstyle\wedge\mathrm{E}'_2}}{A \wedge B}\,{\scriptstyle\wedge\mathrm{I}}$$

Using these transformation one can prove a normalization theorem and the atomization of minimal formulas in normal derivation for the natural deduction system NI' obtained by replacing $\wedge\mathrm{E}_2$ with the deviant $\wedge\mathrm{E}'_2$ by opportunely modifying the standard proofs of Prawitz (1965b; 1971) (from which conservativity and uniqueness results for the conjunction governed by these rules follow).

As we remarked in the introduction, however, there are reasons for regarding this as a characterization of a weak notion of harmony, and for looking for a stricter criterion capturing a notion of strong harmony.

The problem with weak harmony is that although one can equip weakly harmonious rules with reductions and expansions, the resulting notion of equivalence on derivation might collapse the notion of isomorphism on that of interderivability. Tranchini (2016a) considers the following collection of weakly harmonious rules:

$$\frac{A \natural B \quad A}{B}\,\natural\mathrm{E}_1$$

$$\frac{\begin{array}{cc}[A] & [B]\\ B & A\end{array} \quad A}{A \natural B}\,\natural\mathrm{I} \qquad \frac{A \natural B \quad B}{A}\,\natural\mathrm{E}_2$$

$$\frac{A \natural B}{B}\,\natural\mathrm{E}_3$$

In spite of the mismatch between the third premise of the introduction and the conclusion of the third elimination, it is not difficult to devise three reductions and an expansion for these rules as well. However, Tranchini shows that any two derivations of A from itself can be shown to belong to the same equivalence class induced by these operations. From this, it immediately follows that any two closed derivations of the same formula are equivalent. Hence, although the rules are weakly harmonious and thus conservative in Belnap's sense (i.e., with respect to derivability), they are non-conservative with respect to identity of proofs. Let S' be the result of extending a system S (including the rules for $\supset$) with the rules for $\natural$, and let E' be the smallest equivalence relation on derivations of S' extending an equivalence relation E on derivations on S (that is closed under the reduction for $\supset$) and closed under the reductions and expansions for $\natural$.[7] For any pair of closed derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of S we have that $\mathscr{D}_1 \overset{E'}{\equiv} \mathscr{D}_2$. Thus if E is non-trivial E' is a non-conservative extension of E, in fact a trivial non-conservative extension in which any two derivations (of the same formula) are equated. Similar consideration apply to isomorphism. The notion of isomorphism relative to the equational theory E' is trivial in the sense that any two interderivable formula qualify as E'-isomorphic. Even if the notion of E-isomorphism is non-trivial (i.e even if there are interderivable formulas which are not E-isomorphic), E'-isomorphism is trivial. Thus, we can say that weakly harmonious rules are unacceptable since their addition to a system has the consequence of blurring meaning distinctions.

4 Strong harmony via isomorphism

As recalled in the introduction, a tighter connection between introduction and elimination rules could be achieved by replacing interderivability with syntactic identity in Schroeder-Heister's definition of weak harmony. The resulting notion could be dubbed *strict harmony* since not only the standard introduction rule for conjunction and the deviant collection of elimination rules are not in strict harmony, but not even the two standard introductions rules for disjunction and its standard elimination rule are in strict harmony. One may expect that a middle ground between weak and strict harmony–a notion of *strong harmony* — could be obtained by using isomorphism instead of derivability in the definition of harmony. At first, it might

[7] The existence of E' is warranted by the fact that the class of equivalence relations with these properties are closed under infinite intersection.

seem that this move does not bring us very far. If strong harmony is defined using $\beta\eta$-isomorphism, not only the rules for $\natural$ would fail to qualify as harmonious, but also those for disjunction, since $Y \vee Z \overset{\beta\eta}{\not\simeq} \forall X.((Y \supset X) \wedge (Z \supset X)) \supset X$.

However, this counter-example does not rule out a definition of strong harmony based on isomorphism *per se*, but only one based on $\beta\eta$-isomorphism. Actually, there are independent reasons for adopting a different notion of equivalence and of isomorphism when working in NI^2. Both in NI and NI^2 the maximum non-trivial notion of equivalence can be defined as *contextual equivalence* (notation $\overset{\mathcal{CE}}{\equiv}$). In the setting of natural deduction this notion can be defined as follows: two derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of A are contextually equivalent iff for every derivation $\mathscr{D}$ of $\top \vee \top$ such that

1. the assumption A occurs exactly once in $\mathscr{D}$;
2. the result of replacing the assumption A in $\mathscr{D}$ with $\mathscr{D}_1$ and $\mathscr{D}_2$ (respectively) are closed derivations;

the following holds:[8]

$$\begin{array}{c} \mathscr{D}_1 \\ [A] \\ \mathscr{D} \\ \top \vee \top \end{array} \quad \overset{\beta}{\equiv} \quad \begin{array}{c} \mathscr{D}_2 \\ [A] \\ \mathscr{D} \\ \top \vee \top \end{array}$$

In contrast to what happens in NI, where $\beta\eta$-equivalence and contextual equivalence coincide (Scherer, 2017), in NI^2 $\beta\eta$-equivalence is much weaker than contextual equivalence (so much that the former is decidable, while contextual equivalence is undecidable; this is well-known, for a detailed proof of the latter claim see, e.g., Pistone and Tranchini, 2021). Similarly, the notions of isomorphism arising by contextual equivalence is strictly richer than $\beta\eta$-isomorphisms. In particular $Y \vee Z$ does qualify as $\mathcal{CE}$-isomorphic to $\forall X.((Y \supset X) \wedge (Z \supset X)) \supset X$.

From these considerations, a natural proposal would be that of defining strong harmony by replacing derivability with $\mathcal{CE}$-isomorphism in Schroeder-Heister definition of weak harmony. The resulting notion would allow to overcome the problems of weak harmony. However, also this notion is not entirely satisfactory. In general, it is very hard to decide whether two derivations of a formula A are contextually equivalent, since one has to consider *all* derivations of $\top \vee \top$ from A. Moreover, due to the undecidability of contextual equivalence there are few hopes for the decidability of the notion of $\mathcal{CE}$-isomorphism in NI^2 and hence for that of a notion of strong harmony defined in its terms.

For this reason Tranchini, Pistone, and Petrolo (2019) investigated notions of equivalence lying between $\beta\eta$-equivalence and contextual equivalence, in the hope of finding a notion more manageable than $\mathcal{CE}$-equivalence but still suitable for the definition of the notion of strong harmony. In particular, the authors focused on the

[8] The derivation $\mathscr{D}$ can be seen as a context, and thus $\mathscr{D}_1 \overset{\mathcal{CE}}{\equiv} \mathscr{D}_2$ means that $\mathscr{D}_1$ and $\mathscr{D}_2$ are (β-)equivalent in any context of type $\top \vee \top$. Note that $\top \vee \top$ is a proposition with exactly two distinct proofs, since $\top$ is the proposition with a unique (trivial) proof, and a proof of a disjunction is a proof of either of the disjuncts (together with a bit of information telling which of the two disjuncts is proven). Thus contextual *inequivalence* means that two proofs can be distinguished in some context in which the possible results of evaluating them are only two.

one arising from the functorial interpretation of second-order formulas (Bainbridge, Freyd, Scedrov, and Scott, 1990) and established some results suggesting that this notion fits the needs of the study of generalized intuitionistic connectives in the second-order setting (Pistone and Tranchini, 2021).

The key idea underlying this extension of $\beta\eta$-equivalence can be informally introduced starting from the statement of the proof-conditions of a formula of the form $\forall X.A$ in the style of the BHK-clauses:

> A proof of $\forall X.A$ (henceforth a *universal proof*) is a function that applied to a proposition B yields a proof $A[B/X]$.

As in the case of a proof of $A \supset B$ (taken to be a function from proofs of A to proofs of B) a universal proof cannot be merely understood as an infinite list of ordered pairs (each consisting of a proposition B and of a proof of $A[B/X]$) on pain of making the notion of proof epistemically unsurveyable, and thereby contradicting the assumptions that intuitionistic proofs are the result of an activity of mental construction performed by a knowing subject. Rather, these functions have to be understood as given in such a way as to make it possible for a knowing subject to grasp them. A way of meeting this demand is that of assuming that a universal proof of $\forall X.A$ is a function f that associates a proof of $A[B/X]$ to each proposition B "in a uniform way".[9]

To ask that the values of a universal proof f are assigned to its arguments in a uniform way amounts to requiring that the values of f themselves must be defined in a uniform manner. We take this to mean that in defining each value of f (i.e., the proofs of $A[B/X]$) no knowledge about the argument of f (i.e., the propositions B) should be assumed (for a more thorough discussion, see Pistone, 2018).

Consider for example the proposition $\forall X.X \supset X$. A proof of this proposition is a function f mapping each proposition B onto a proof fB of $B \supset B$, which in turn is a function from the set of proofs of B onto itself.[10] It is true that there may be different ways of mapping the set of proofs of a certain proposition onto itself. However, there does not seem to be many options for defining such a map without assuming any knowledge about the proposition and hence, about its sets of proofs. In fact it seems that the only function one can come up with is the identity function. In other words, if we assume the proofs of $\forall X.X \supset X$ to be uniform functions, there seems to be only one such proof, namely the one associating to each proposition B the identity function $\mathtt{id}_B$ on the set of proofs of B. Although this informal argument is non-conclusive, being based on considerations of a heuristic nature, it turns out that the strengthening of $\beta\eta$-equivalence considered by the authors captures exactly these intuitions.

That the assumption of uniformity has consequences for identity of proofs is not as surprising as it may appear at first. Consider proofs of propositions of the form

[9] The notion of uniformity has been widely investigated in theoretical computer science under the name "parametricity" (Strachey, 1967; Reynolds, 1983; Hermida, Reddy, and Robinson, 2014) and it is in direct line of descent from the "schematic" (as opposed to the "numerical") interpretation of second-order quantification (see, e.g., Carnap, 1931).

[10] Following the common notation in λ-calculus, we indicate the application of a function f to its argument a with (fa), where outermost parentheses will be dropped and application is assumed to be left associative, so that fgh is short for $((fg)h)$.

$(\forall X.A) \supset B$, i.e., functions from proofs of $\forall X.A$ to proofs of B. To assume that universal proofs are uniform functions means that one is restricting the domain of the proofs of $(\forall X.A) \supset B$. If two such proofs assign the same value when taking an arbitrary, but uniform, universal proof as argument, then they denote the same proof under the assumption that all universal proofs are uniform. Yet, it might still be possible that these two proofs of $\forall X.A \supset B$ assign a different value to some (non-uniform) proof of $\forall X.A$, so that they would no more denote the same function without the assumption.[11]

To see how the assumption of uniformity can be used to justify new equations between proofs, consider for example the following two derivations:

$$\dfrac{B \supset C \qquad \dfrac{\dfrac{\forall X.X \supset X}{B \supset B} \qquad B}{B}}{C} \qquad\qquad \dfrac{\dfrac{\forall X.X \supset X}{C \supset C} \qquad \dfrac{B \supset C \qquad B}{C}}{C}$$

On the assumption that the proofs of $\forall X.X \supset X$ are uniform, the two derivations should denote the same proof. This is best appreciated when the derivations are decorated with proofs terms:

$$\dfrac{h : B \supset C \qquad \dfrac{\dfrac{f : \forall X.X \supset X}{fB : B \supset B} \qquad b : B}{fBb : B}}{h(fBb) : C} \qquad\qquad \dfrac{\dfrac{f : \forall X.X \supset X}{fC : C \supset C} \quad \dfrac{h : B \supset C \quad b : B}{hb : C}}{fC(hb) : C}$$

Uniformity warrants that fB and fC are the identity functions $\mathtt{id}_B$ and $\mathtt{id}_C$ on the sets of proofs of B and C respectively, and thus the two derivation encode (for any f, h, b, B and C) the same proof of C:

$$h(fBb) = h(\mathtt{id}_B b) = hb = \mathtt{id}_C(hb) = fC(hb)$$

It easy to see that $\beta\eta$-equivalence fails to capture the consequences of uniformity. The two derivations above are $\beta\eta$-normal and thus (as a consequence of the Church-Rosser theorem for $\beta\eta$-reduction in NI^2) they belong to two distinct $\beta\eta$-equivalence classes. Hence in order to capture the uniformity of the proofs of $\forall X.X \supset X$ we need to strengthen the equivalence relation on derivations by requiring it to be closed under the following scheme, the instances of which will be referred to as ε-equations (observe that the left-to-right orientation of these equations can be seen as an operation that permutes-up the derivation $\mathscr{D}'$ across the application of $\forall$E):

[11] It may also be worth observing that the restriction to uniform proofs does not require to modify in any way the rules for the second order quantifier in NI^2: in fact, the variable condition on the rule $\forall$I (that ensures that when inferring $\forall X.A$ from A no assumption is made on X) can be seen as a syntactic counterpart of the uniformity requirement informally described above. In other words, all NI^2-derivations of formulas of the form $\forall X.A$ actually denote uniform universal proofs. Moreover, it is well-known that extensions of the syntax of NI^2 with non-uniform constructors, although possible, might lead to inconsistencies (see, e.g., Harper and Mitchell, 1999).

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \forall X.X\supset X\end{matrix}}{B\supset B}\qquad B}{\begin{matrix}B\\ \mathscr{D}'\\ C\end{matrix}} \quad \overset{\varepsilon}{\equiv} \quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \forall X.X\supset X\end{matrix}}{C\supset C}\qquad \begin{matrix}B\\ \mathscr{D}'\\ C\end{matrix}}{C}$$

Hence, the ε-equations are justified by the assumption that the only proof of $\forall X.X\supset X$ is the function associating to each proposition the identity function on its sets of proofs. Conversely, in every categorial model of NI^2 in which the ε-equations are satisfied, $\forall X.X\supset X$ has exactly one proof.

Analogous informal considerations show that the set of uniform proofs of $\forall X.(A\supset X)\supset X$ must be in bijection with the set of proofs of A itself (provided X does not occur free in A). In particular, a proof of $\forall X.(A\supset X)\supset X$ associates to each proposition B a function from proofs of $(A\supset B)$ (which in turn are functions from proofs of A into proofs of B) to proofs of B. But the only way to define such a proof in a uniform manner consists in taking a proof a of A (if any is available) and associate to each proposition B the function that maps each proof f of $A\supset B$ onto fa.

Syntactically, uniformity can be again expressed as the possibility of permuting-up a derivation across an application of $\forall$E with premise $\forall X.(A\supset X)\supset X$ using the ε-equations obtained from the scheme below. In this case observe that the derivation $\mathscr{D}'$ cannot be permuted as it stands, on pain of changing the open assumptions of the derivation, and due to the mismatch between the conclusion of $\mathscr{D}'$ and the minor premise required to apply $\supset$E. The mismatch can however be resolved by "surrounding" $\mathscr{D}'$ (whose conclusion is D and whose undischarged assumptions are C and possibly further assumptions Δ) with some applications of elimination and introduction rules yielding a derivation, that we indicate with $(A\supset X)\{\mathscr{D}'\}$, of $(A\supset X)[D/X]$ from $(A\supset X)[C/X],\Delta$:

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \forall X.(A\supset X)\supset X\end{matrix}}{((A\supset X)\supset X)[C/X]}\quad (A\supset X)[C/X]}{\begin{matrix}C\\ \mathscr{D}'\\ D\end{matrix}} \quad \overset{\varepsilon}{\equiv} \quad \dfrac{\dfrac{\begin{matrix}\mathscr{D}\\ \forall X.(A\supset X)\supset X\end{matrix}}{((A\supset X)\supset X)[D/X]}\quad \begin{matrix}(A\supset X)[C/X]\\ \boxed{(A\supset X)\{\mathscr{D}'\}}\\ (A\supset X)[D/X]\end{matrix}}{D}$$

where

$$(A\supset X)\{\mathscr{D}'\} = \dfrac{\begin{matrix}\dfrac{(A\supset X)[C/X]\qquad \overset{n}{A}}{C}\\ \mathscr{D}\\ D\end{matrix}}{(A\supset X)[D/X]}\,(n)$$

Since the uniform proofs of $\forall X.(A\supset X)\supset X$ are in bijection with those of A, it should be possible, syntactically, to show that $\forall X.(A\supset X)\supset X$ and A are E-isomorphic on any notion of equivalence E that is strong enough to encode the uniformity of universal proofs. This is actually the case: taken the two derivations

$$\mathscr{D}_1 \quad = \quad (n)\, \cfrac{\cfrac{\overset{n}{A \supset X} \qquad A}{X}}{\cfrac{(A \supset X) \supset X}{\forall X.(A \supset X) \supset X}} \qquad\qquad \cfrac{\cfrac{\forall X.(A \supset X) \supset X}{(A \supset A) \supset A} \qquad (n)\,\cfrac{\overset{n}{A}}{A \supset A}}{A} \quad = \quad \mathscr{D}_2$$

it is easy to show that the composition $\begin{array}{c}\mathscr{D}_1\\ \forall X.(A \supset X) \supset X\\ \mathscr{D}_2\end{array}$ β-reduces to the derivation consisting only of the assumption of A and that the composition $\begin{array}{c}\mathscr{D}_2\\ A\\ \mathscr{D}_1\end{array}$, after the application of an ε-permutation, η-reduces to the derivation consisting only of the assumption of $\forall X.(A \supset X) \supset X$.

In a similar way, we can define ε-permutations for the formula $\forall X.(A \supset X) \supset ((B \supset X) \supset X)$ (provided X does not occur free in A and B) encoding the uniformity of the proofs of this proposition. On the one hand, this formula is $\beta\eta$-isomorphic to $\forall X.((A \supset X) \wedge (B \supset X)) \supset X$, and on the other hand, using the ε-permutation we can show that $\forall X.(A \supset X) \supset (B \supset X) \supset X \overset{\varepsilon}{\simeq} A \vee B$. Hence, we have that $A \vee B \overset{\varepsilon}{\simeq} \forall X.((A \supset X) \wedge (B \supset X)) \supset X$, that is, that by defining strong harmony using ε-isomorphism, the standard rules for $\vee$ qualify as strongly harmonious.

More in general, we can establish that:

$$\bigvee_{j=1}^{m} \bigwedge_{i=1}^{n_j} R_{ji} \overset{\varepsilon}{\simeq} \forall X. \left(\bigwedge_{j=1}^{m} \left(\bigwedge_{i=1}^{n_j} R_{ji} \supset X\right)\right) \supset X$$

and hence that any collection of introduction rules and its Prawitz–Schroeder-Heister collection of elimination rules are in strong harmony, by defining ε-permutations for all quantified formulas $\forall X.A$ in which A has a distinctively simple form that we call *nested sp-X* (see, for details, Tranchini, Pistone, and Petrolo, 2019). An $\mathcal{L}^{2\supset}$-formula A is *strictly positive in* X iff X does not occur to the left of $\supset$ in A, and a nested sp-X formula is a formula of the form $A_1 \supset (\ldots (A_n \supset X) \ldots)$ where each A_i is sp-X for all $1 \leq i \leq n$. The above isomorphism is established by showing that the right-hand side formula is $\beta\eta$-isomorphic to the nested sp-X formula

$$\forall X.\Big(R_{11} \supset (\ldots (R_{1n_1} \supset X) \ldots)\Big) \supset \Big(\ldots \Big(\Big(R_{m1} \supset (\ldots (R_{mn_m} \supset X) \ldots)\Big) \supset X\Big) \ldots\Big)$$

which in turn is ε-isomorphic to the left-hand side formula.

Let $\mathcal{L}^{2\supset}_{sp}$ be the fragment of $\mathcal{L}^{2\supset}$ obtained by allowing to prefix a formula A with $\forall X$ only if A is nested sp-X. The content of any introduction and elimination rule of the form we considered above are ε-isomorphic to formulas in $\mathcal{L}^{2\supset}_{sp}$, and so are the content of collections of introduction and the elimination rules.

Let $\mathsf{NI}^{2\supset}_{sp}$ be the restriction of NI^2 to the language $\mathcal{L}^{2\supset}_{sp}$. As shown by the authors (Pistone and Tranchini, 2021), the ε-equational theory has characteristically strong properties in this fragment, namely it is decidable and the maximum equivalence extending β-equivalence.[12] Thus, taking the stance of Došen and Widebäck, ε-equivalence (resp. ε-isomorphism) can be considered as the canonical notion of equivalence (resp. isomorphism) in $\mathsf{NI}^{2\supset}_{sp}$.

These decidability and maximality results are based on the fact that the derivations of $\mathsf{NI}^{2\supset}_{sp}$ modulo ε-equivalence form a category equivalent to that of the derivations of NI modulo $\beta\eta$-equivalence. This in turns implies that the question of the decidability of isomorphism in $\mathsf{NI}^{2\supset}_{sp}$ is equivalent to that of the decidability of isomorphism in NI, which is (as remarked in the previous section) still open.

The foregoing results speak in favor of defining strong harmony using ε-isomorphism rather than $\beta\eta$- or $\mathcal{CE}$-isomorphism, at least when the form of introduction and elimination rules follows the schemata given above.

Whenever confronted with two collections of introduction and elimination rules for $\dagger$, we are not in general capable of telling whether they are in harmony (since $\beta\eta$-isomorphism in NI, and hence ε-isomorphism in $\mathsf{NI}^{2\supset}_{sp}$ is — of today — not known to be decidable), but we can decide whether certain derivations do or do not testify their isomorphism.

5 Concluding remarks

The account of strong harmony using ε-isomorphism delivers a satisfactory sharpening of the notion of weak harmony developed by Schroeder-Heister for introduction and elimination rules of the form discussed above.

It is worth stressing, however, that Schroeder-Heister (2014b) considers introduction and elimination rules of a more general form, namely the following:

$$\text{(INTRO}^*\text{)} \qquad \forall\overrightarrow{X}\forall\overrightarrow{Y}\left(\bigwedge_{i=1}^{n} R_i \supset \dagger(\overrightarrow{X})\right)$$

$$\text{(ELIM}^*\text{)} \qquad \forall X\forall\overrightarrow{Y}\forall\overrightarrow{X}\left(\left(\dagger(\overrightarrow{X}) \wedge \bigwedge_{i=1}^{n} R_i\right) \supset X\right)$$

satisfying the following two conditions:

1. in each R_i no connective variable occurs at all;
2. no propositional variable occurs free (i.e., $\overrightarrow{Y}$ contains all propositional variables occurring free in any of the R_i other than X and those in $\overrightarrow{X}$);

By dropping the third condition on elimination rules (see footnote 5 above) and allowing nested quantification inside introduction rules and elimination rules, these less restricted forms of introduction and elimination rules significantly enrich the class of connectives amenable of a characterization in terms of "pure" introduction and elimination rules (i.e., introduction rules in which no connective occurs apart from the one being "defined"). For instance, Schroeder-Heister (2014b) observes that

[12] Moreover, it allows to show that any derivation is equivalent to one in which applications of $\forall$E are have an atomic witness (Pistone, Tranchini, and Petrolo, 2021).

it is possible to formulate an introduction rule for negation which does not mention $\bot$, namely $\forall X.(\forall Y.X \supset Y) \supset \dagger(X)$.

This more general form of introduction rules is however much more expressive than that, as it allows for instance to formulate an introduction rule for a zero-place connective:

$$(\forall Y.(Y \supset Y) \wedge Y \supset Y) \supset \dagger$$

which is essentially the impredicative encoding of the natural number predicate in NI^2.

In contrast to what we observed in the case of introduction and elimination of the more restricted form we considered throughout the paper, the content of introduction and elimination rules of this more general form are formulas which cannot be shown to be ε-isomorphic to formulas in the fragment $\mathcal{L}^{2\supset}_{sp}$.

It is true that the ε-equations can be formulated for any formula of the form $\forall X.A$. However, in contrast to what happens in the restricted fragment so far considered, as soon as one allows for encodings of inductive types the ε-equational theory is not decidable (Pistone and Tranchini, 2021) and might not even be maximal (this is suggested by the fact that the equivalence between the ε-equational theory for the fragment of $\mathsf{NI}^{2\supset}$ containing the encoding of the natural number predicate is related to a (non-maximal) equational theory for Gödel's System T investigated by Okada and Scott, 1999).[13]

Thus, on this more general understanding of introduction and elimination rules, it may be more appropriate to define the notion of strong harmony using an equational theory stronger than ε. Moreover, since the ε-equational theory is undecidable outside the $\mathcal{L}^{2\supset}_{sp}$ language fragment, there are few hopes for a decidable notion of strong harmony when introduction and elimination rules of this more general form are taken into consideration.

We conclude by observing that the notions of weak and strong harmony as defined in the work of Schroeder-Heister and in the present paper are not directly applicable to some prominent examples discussed in the literature, such as the rules for quantum-disjunction[14] and rules whose formulation requires first-order structure (as, e.g., those

---

[13] Equational theories stronger than ε for the whole of $\mathsf{NI}^{2\supset}$ have been studied, among others, by (Longo, Milsted, and Soloviev, 1993).

[14] As to "quantum disjunction" (see footnote 3 above), due to the restriction on its elimination rule, it does not seem that its elimination content is expressible using an NI^2 formula, and hence its rules fall outside the scope of weak (and hence strong) harmony. One could however consider extensions of NI^2 capable of expressing rules with restrictions of the kind displayed by the quantum disjunction elimination rule. The most natural possibility would be that of extending NI^2 with an implication of the kind described by Dummett (1991) (see also Tranchini, 2018), i.e., in which the introduction rule is restricted so that it can be applied only if its premise depends on no other assumption than those to be discharged by the rule. Using $\overline{\supset}$ for this connective, the content of the collection of elimination rule for quantum disjunction would be expressible as $\forall X.((Y \mathbin{\overline{\supset}} X) \wedge (Z \mathbin{\overline{\supset}} X)) \supset X$. Perhaps unsurprisingly, this formula is interderivable with $A \vee B$ in the envisaged extension of NI^2 and thus the rules of quantum disjunction would qualify as weakly harmonious. The question of strong harmony is harder to address, since it would require the definition of an appropriate equivalence relation on derivations for the system considered (and this might not be obvious, since no expansion

of the identity predicate). The extension of the notions of weak and strong harmony to a first-order setting is an interesting topic for further research.

References

Bainbridge, E. S., P. J. Freyd, A. Scedrov, and P. J. Scott (1990). Functorial polymorphism. *Theoretical Computer Science* 70, 35–64.

Belnap, N. D. (1962). Tonk, plonk and plink. *Analysis* 22, 130–134.

Carnap, R. (1931). Die logizistische grundlegung der mathematik. *Erkenntnis* 2, 91–105.

– (1956). *Meaning and Necessity*. Chicago Univerisity Press.

Došen, K. (2003). Identity of proofs based on normalization and generality. *Bulletin of Symbolic Logic* 9, 477–503.

Došen, K. and Z. Petrić (2001). The typed Böhm theorem. *Electronic Notes in Theoretical Computer Science* 50. BOTH 2001, Böhm's theorem: applications to Computer Science Theory (Satellite Workshop of ICALP 2001), 117–129.

Dummett, M. (1991). *The Logical Basis of Metaphysics*. London: Duckworth.

Gentzen, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift* 39. eng. transl. 'Investigations into logical deduction', in M. E. Szabo (ed.), *The collected papers of Gerhard Gentzen*, North-Holland, 1969, pp. 68–131.

Girard, J.-Y. (1986). The System F of variable types, fifteen years later. *Theoretical Computer Science* 45, 159–192.

Harper, R. and J. C. Mitchell (1999). Parametricity and variants of Girard's J operator. *Information Processing Letters* 70 (1), 1–5.

Hermida, C., U. S. Reddy, and E. P. Robinson (2014). Logical relations and parametricity – A Reynolds programme for category theory and programming languages. In: *Proceedings of the Workshop on Algebra, Coalgebra and Topology (WACT 2013)*. Vol. 303. Electronic Notes in Theoretical Computer Science. Elsevier, 149–180.

Ilik, D. (2014). Axioms and decidability for type isomorphism in the presence of sums. In: *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. CSL-LICS '14. New York, NY, USA: Association for Computing Machinery. DOI: 10.1145/2603088.2603115. URL: https://doi.org/10.1145/2603088.2603115.

Longo, G., K. Milsted, and S. Soloviev (1993). The genericity theorem and the notion of parametricity in the polymorphic λ-calculus. *Theoretical Computer Science* 121, 323–349.

Okada, M. and P. J. Scott (1999). A note on rewriting theory for uniqueness of iteration. *Theory and Applications of Categories* 6, 47–64.

Pistone, P. (2018). Polymorphism and the obstinate circularity of second order logic, a victims' tale. *The Bulletin of Symbolic Logic* 24, 1–52.

seems to be available for the implication with restricted introduction rule; see Tranchini, 2018). This is however not the only option, and the issue requires further investigation.

Pistone, P. and L. Tranchini (2021). The Yoneda Reduction of Polymorphic Types. In: *29th EACSL Annual Conference on Computer Science Logic (CSL 2021)*. Ed. by C. Baier and J. Goubault-Larrecq. Vol. 183. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 35:1–35:22.

Pistone, P., L. Tranchini, and M. Petrolo (2021). *The naturality of natural deduction (II). Some remarks on atomic polymorphism*. To appear *Studia Logica*. Pre-print available at https://arxiv.org/abs/1908.11353.

Prawitz, D. (1965a). *Natural deduction, a proof-theoretical study*. Almqvist & Wiskell.

– (1971). Ideas and results in proof theory. In: *Proceedings of the Second Scandinavian Logic Symposium*. Ed. by J. E. Fenstad. Vol. 63. Studies in Logic and the Foundations of Mathematics. Elsevier, 235–307.

– (1979). Proofs and the meaning and completeness of the logical constants. In: *Essays on Mathematical and Philosophical Logic: Proceedings of the Fourth Scandinavian Logic Symposium and the First Soviet-Finnish Logic Conference, Jyväskylä, Finland, June 29–July 6, 1976*. Ed. by J. Hintikka, I. Niiniluoto, and E. Saarinen. Dordrecht: Kluwer, 25–40.

Prior, A. N. (1960). The runabout inference-ticket. *Analysis* 21, 38–39.

Reynolds, J. C. (1983). Types, abstraction and parametric polymorphism. In: *Information Processing '83*. Ed. by R. E. A. Mason. North-Holland, 513–523.

Scherer, G. (2017). Deciding equivalence with sums and the empty type. In: *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages, POPL 2017, Paris, France, January 18-20, 2017*. Ed. by G. Castagna and A. D. Gordon. ACM, 374–386. URL: http://dl.acm.org/citation.cfm?id=3009901.

Schroeder-Heister, P. (1981). Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen. PhD thesis. Bonn University.

– (Dec. 1984). A natural extension of natural deduction. *Journal of Symbolic Logic* 49, 1284–1300.

– (2014a). Harmony in proof-theoretic semantics: A reductive analysis. In: *Dag Prawitz on Proof and Meaning*. Ed. by H. Wansing. Vol. 7. Outstanding Contributions to Logic. Springer.

– (2014b). The calculus of higher-level rules, propositional quantification, and the foundational approach to proof-theoretic harmony. English. *Studia Logica* 102, 1185–1216. DOI: 10.1007/s11225-014-9562-3.

– (2016). Open problems in proof-theoretic semantics. In: *Advances in Proof-Theoretic Semantics*. Ed. by T. Piecha and P. Schroeder-Heister. Springer International Publishing, 253–283. DOI: 10.1007/978-3-319-22686-6_16. URL: https://doi.org/10.1007/978-3-319-22686-6_16.

Solov'ev, S. V. (1983). The category of finite sets and cartesian closed categories. *Journal of Soviet Mathematics* 22, 1387–1400.

Statman, R. (1983). λ-definable functionals and $\beta\eta$-conversion. *Archiv für mathematische Logik und Grundlagenforschung* 23, 21–26.

Strachey, C. (1967). Fundamental concepts in programming languages. *Higher Order and Symbolic Computation* 13, 11–49.

Tennant, N. (1978). *Natural Logic*. Edimburgh: Edimburgh University Press.

Tranchini, L. (2012). Truth from a proof-theoretic perspective. *Topoi* 31, 47–57.
– (2016a). Proof-theoretic harmony: towards an intensional account. *Synthese*. DOI: 10.1007/s11229-016-1200-3.
– (2016b). Proof-theoretic semantics, paradoxes, and the distinction between sense and denotation. *Journal of Logic and Computation* 26, 495–512.
– (2018). Stabilizing quantum disjunction. *Journal of Philosophical Logic* 47, 1029–1047.
– (2019). Proof, meaning and paradox: some remarks. *Topoi* 38, 591–603.
Tranchini, L., P. Pistone, and M. Petrolo (2019). The naturality of natural deduction. *Studia Logica* 107, 195–231.
Widebäck, F. (2001). *Identity of proofs*. Stockholm: Almqvist & Wiksell.