Complex Analysis and Operator Theory (2022) 16:117
https://doi.org/10.1007/s11785-022-01292-x

Complex Analysis and Operator Theory

Functions and Operators in Real, Quaternionic, and Cliffordian Contexts

Florian-Horia Vasilescu[1]

Received: 6 May 2022 / Accepted: 14 October 2022 / Published online: 5 November 2022

Abstract

This is an expository paper mainly based on certain works due to the author himself. After some introductory sections, we discuss the transformation of vector valued stem functions, defined on sets in the complex plane into quaternionic and Cliffordian valued function, using functional calculi, algebraically or derived via a Cauchy type kernel. Then we consider large families of quaternionic and Cliffordian linear operators, regarded as special classes of real linear operators, extended via a complexification procedure, and thus having the spectrum in the complex plane, which permits the construction of functional calculi with adequate analytic functions, in a classical manner, recaptured by restriction.

Keywords Real, Hamilton and Clifford algebras · Spectral and Cauchy transformations · Clifford and quaternionic operators · Analytic functional calculus

Mathematics Subject Classification Primary 47A10; Secondary 47A20 · 47A60 · 30G35 · 15A66

1 Introduction

The aim of the present work is to exhibit an overview of some of the main results from the author's works [22, 24, 25], adding certain relevant comments and new remarks.

Dedicated to the memory of Jörg Eschmeier.

Communicated by Mihai Putinar.

This article is part of the topical collection "Multivariable Operator Theory. The Jörg Eschmeier Memorial" edited by Raul Curto, Michael Hartz, Mihai Putinar and Ernst Albrecht.

✉ Florian-Horia Vasilescu
florian.vasilescu@univ-lille.fr

1 Department of Mathematics, University of Lille, 59655 Villeneuve d'Ascq, France

From historical point of view, the algebra of quaternions, denoted in the following by $\mathbb{H}$ and often called Hamilton algebra, was introduced in mathematics by W. R. Hamilton as early as 1843. It is a unital non commutative division algebra, with numerous applications not only in mathematics but also in physics, and in other domains as well. The celebrated Frobenius theorem, proved in 1877, placed the algebra of quaternions among the only three finite dimensional division algebras over the real numbers, which is a remarkable feature shared with the real and complex fields. In physics, an early suggestion towards a quaternionic quantum mechanics can be found in a paper by G. Birkhoff and J. von Neumann [2], and later, a quaternionic quantum mechanics was firstly developed in [9].

A Clifford algebra is a unital associative algebra, whose elements generalize the real numbers, complex numbers, quaternions and other hypercomplex systems of numbers. Its theory is connected with that of quadratic forms and orthogonal transformations, with applications in many domains, as analysis, geometry and theoretical physics. This concept is named after the English mathematician William Kingdon Clifford, who had several contributions in this area (see the paper [3] as a first reference). In the present work, we use a special type of a Clifford algebra, which is unital and has n generators, denoted by $\mathfrak{C}_n$ (see Subsect. 3.1).

Going back to quaternions, an important investigation in their context has been to find a convenient manner to express the ”analyticity“ of functions depending on them. Among the pioneer contributions in this direction one should mention the works [18] and [10].

More recently, a concept of *slice regularity (or hyperholomorphy)* for functions of one quaternionic variable has been introduced in [12] (see also [11]), leading to a vast development sythesized in [6], which contains a large list of references. Moreover, functions defined on other domains, in particular on sets in a Clifford agebra $\mathfrak{C}_n$, were considered from this point of view (see [7, 13], etc.). As a matter of fact, we shall mainly deal with functions in the Cliffordian context, deriving the corresponding results for the quaternionic context as (almost) particular cases.

Unlike in [12] and in the articles following, in an early preprint cited and partially presented in the work [23], the regularity of a quaternionic-valued function was investigated in the context of matrix quaternions. In fact, each matrix quaternion was regarded as a normal operator, having a spectrum, which was used to define various compatible functional calculi, including the analytic one. This discussion was continued and refined to the context of the abstract Hamilton algebra in [22], by embedding the real C^*-algebra of quaternions into its complexification, organized as a complex C^*-algebra. The new arguments were not only much simpler but the framework was intrisic, that is, it did not depend of any representation of Hamilton's algebra as a matrix algebra. Our main tools in [22] were the complexification and conjugation, also intensively used in the subsequent works. Similar ideas also appear in some previous works (see for instance [13]) but the systematic use of the concept of spectrum of a quaternion, seemingly not used so far, is essential for our development, leading to a new approach to regularity, equivalent but different from that based on slice regularity, in both quaternionic and Cliffordian contexts. In fact, the regularity of quaternionic-valued functions, or functions with values in a Clifford algebra, becomes a consequence of their representation via a Cauchy type integral (see for instance Proposition 1 from [22]).

The scientific necessities of the quantum mechanics, as well as those of other domains like partial differential equations, imposed the study of vector spaces, which are modules over the Hamilton and Clifford algebras, and the associated operators, which are linear with respect to these algebras. Unlike in works by other authors, many of them quoted in the monographs [6, 7], in order to construct an analytic functional calculus for quaternionic linear operators, the class of the quaternionic slice regular functions, defined on subsets in the quaternionic algebra, is replaced by a class of vector-valued holomorphic functions, called *stem functions*, defined on subsets in the complex plane. In fact, these two classes are isomorphic via a Cauchy type transform (see Theorem 6 from [22]), and we use the latter to construct an analytic functional calculus for what are called quaternionic linear operators. Similar properties are valid in the Cliffordian context, which are exhibited in the work [24], and some of them will be recalled in what follows.

The spectral theory for quaternionic or Cliffordian linear operators has been already discussed in numerous work, in particular in the monographs [6] and [7], where the construction of an analytic functional calculus (called *S-analytic functional calculus*) amounts to associate to a fixed quaternionic or Cliffordian linear operator and to a function from the class of *slice hyperholomorphic* (or *slice regular functions*), an "extension" using a specific noncommutative kernel.

Unlike in these works, our idea is to first consider the case of real operators on real Banach spaces, whose complex spectrum is in the complex plane, and to perform the construction of an analytic functional calculus for them, using some classical ideas. Then, regarding the quaternionic or Clifford operators as particular cases of real ones, this framework is extended to them, showing that the approach from the real case can be adapted to that more intricate situation. Unlike in [6] or [7], the analytic functional calculus is obtained via a Riesz-Dunford-Gelfand formula, defined in a partially commutatative context, rather than the non-commutative Cauchy type formula used by previous authors. This is possible because the S-spectrum, introduced by F. Colombo and I. Sabadini (see [5]), can be replaced by a spectrum in the complex plane. Moreover, we show the analytic functional calculus obtained with some vector-valued stem functions, defined in the complex plane, is equivalent to the analytic functional calculus obtained with slice holomorphic functions in [6] or [7], in the sense that the images of these functional calculi coincide (see Remark 8 from [25] and Remark 17 from [24]).

Let us briefly describe the contents of this work. The next section is dedicated to some (more or less known) results valid in abstract real algrebras. We present with its proof Theorem 1, which reveals the importance for our approach of the functions preserving the conjugacy, and recall the statement of the general analytic functional calculus in real algebras (Theorem 2), which can be used, with no major changes, also in quaternionic and Cliffordian contexts. A Cauchy transformation, a general concept of complex spectrum, and an abstract slice regularity are also mentioned in this section.

The third section presents Hamilton's and Clifford's algebras, as real algebras, insisting on the concept of the complex spectrum in these frameworks.

Inspired by some elementary properties of certain particular spectral operators (see [8], Part III), the fourth section starts with a discrete functional calculus, called *spectral transformation*, applied to some vector valued functions, defined in the complex plane, leading to Clifford or quaternionic valued functions (see Theorem 3), whose properties are stated by Theorem 3.

The slice regularity of such functions is also recalled, the vector valued commutative Cauchy kernel of type Riesz-Dunford is considered, and its the regularity is proved. Example 3 shows that what the non-commutative kernel used in most of the preceding works of other authors is given by the Cauchy transform the Riesz-Dunford kernel.

The fifth section introduces quaternionic and Clifford operators, discussing their complex spectrum, as compared with the S-spectrum.

The final section is dedicated to the analytic functional calculus for Clifford and Hamilton operators, presenting a general statement, and proving the equivalence between the present functional calculus with functions defined in the complex plane and that developed by the previous authors, with functions defined on domains in the Clifford or Hamilton algeberas.

Thanks are due to the referee for a careful reading of the manuscript and for several suggestions.

2 Some Preliminary Results in Real Algebras

In this section we mainly present some elements of spectral theory in real algebras, that is, algebras over the real field $\mathbb{R}$ (see [1, 15, 17, 19, 21] etc.). We also present a general form of what is called in the literature the *slice regularity* (see [12] for the original concept in the quaternionic context), introduced in real alternative algebras in [14]. Some concrete examples will be later discussed.

2.1 Spectrum and Analytic Functional Calculus

Let $\mathcal{A}$ be a unital real Banach algebra, and let $\mathcal{A}_{\mathbb{C}}$ be its complexification, that is, $\mathcal{A}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$, identified with the direct sum $\mathcal{A} + i\mathcal{A}$, where i is the imaginary unit. Endowed with the norm $\|c\| = \|a\| + \|b\|$, where $c = a + ib,\ b, c \in \mathcal{A}$, the algebra $\mathcal{A}_{\mathbb{C}}$ becomes a unital complex Banach algebra. Moreover, the complex field $\mathbb{C}$ can be identified with a subalgebra of $\mathcal{A}_{\mathbb{C}}$, consisting of multiples of the unit.

Using an idea going back to Kaplansky (see [17] or [15]), the (complex) spectrum of an element $a \in \mathcal{A}$ is given by

$$\sigma_{\mathbb{C}}(a) = \{u + iv; u, v \in \mathbb{R}, (u - a)^2 + v^2 \text{ not invertible in } \mathcal{A}\}. \tag{1}$$

Note that the set $\sigma_{\mathbb{C}}(a)$ is *conjugate symmetric*, that is, $u + iv \in \sigma_{\mathbb{C}}(a)$ if and only if $u - iv \in \sigma_{\mathbb{C}}(a)$.

The map $\mathcal{A}_{\mathbb{C}} \ni a + ib \mapsto a - ib \in \mathcal{A}_{\mathbb{C}}$ is a *conjugation* of $\mathcal{A}_{\mathbb{C}}$, meanig that it is a real unital automorphism of $\mathcal{A}_{\mathbb{C}}$, whose square is the identity. If $\mathbf{s} = a + ib$ we usually put $\bar{\mathbf{s}} = a - ib$ for all $\mathbf{s} \in \mathcal{A}_{\mathbb{C}}$. In particular, an element $\mathbf{s} \in \mathcal{A}_{\mathbb{C}}$ is invertible if and only if the element $\bar{\mathbf{s}} \in \mathcal{A}_{\mathbb{C}}$ is invertible. Of course, the conjugation of $\mathcal{A}_{\mathbb{C}}$ extends the usual conjugation of the complex plane $\mathbb{C}$.

For an element $\mathbf{a} \in \mathcal{A}_{\mathbb{C}}$, we denote by $\sigma(\mathbf{a})$ its usual spectrum. Note that $\lambda \in \sigma(\mathbf{a})$ if and only if $\bar{\lambda} \in \sigma(\bar{\mathbf{a}})$.

When $a \in \mathcal{A}$ is regarded as an element of $\mathcal{A}_{\mathbb{C}}$, formula (1) can be rewritten as

$$\sigma_{\mathbb{C}}(a) = \{\lambda \in \mathbb{C}, \lambda - a \text{ not invertible in } \mathcal{A}_{\mathbb{C}}\}, \tag{2}$$

because $|\lambda|^2 - 2a\Re(\lambda) + a^2 = (\lambda - a)(\bar{\lambda} - a)$ is not invertible in $\mathcal{A}_{\mathbb{C}}$ is equivalent to the fact that neither $\lambda - a$ nor $\bar{\lambda} - a$ is invertible in $\mathcal{A}_{\mathbb{C}}$. Consequently, the complex spectrum $\sigma_{\mathbb{C}}(a)$ coincides with the spectrum $\sigma(a)$, computed in $\mathcal{A}_{\mathbb{C}}$.

Having a concept of spectrum, we may also discuss a concept of analytic functional calculus. We start with some necessary notation.

If $U \subset \mathbb{C}$ is an open set, we denote by $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ the algebra of all analytic $\mathcal{A}_{\mathbb{C}}$-valued functions. If U is conjugate symmetric, and $\mathcal{A}_{\mathbb{C}} \ni \mathbf{a} \mapsto \bar{\mathbf{a}} \in \mathcal{A}_{\mathbb{C}}$ is its natural conjugation, we denote by $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ the real subalgebra of $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ consisting of those functions f with the property $f(\bar{\zeta}) = \overline{f(\zeta)}$ for all $\zeta \in U$. Following [14], such functions will be called ($\mathcal{A}_{\mathbb{C}}$*-valued*) *stem functions*. In fact, this definition has an old origin, seemingly going back to [10].

When $\mathcal{A} = \mathbb{R}$, so $\mathcal{A}_{\mathbb{C}} = \mathbb{C}$, the space $\mathcal{O}_s(U, \mathbb{C})$ will be denoted by $\mathcal{O}_s(U)$, which is a real algebra. Note that $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ is also a two-sided $\mathcal{O}_s(U)$-module.

Let $U \subset \mathbb{C}$, and let $\mathfrak{S}(U)$ be the set $\{\mathbf{a} \in \mathcal{A}_{\mathbb{C}}; \sigma(\mathbf{a}) \subset U\}$. If U is open, then $\mathfrak{S}(U)$ is also open, via the upper semicontinuity of the spectrum (see [8]). We put $\mathcal{A}_{\mathbb{C}}(U) = \{\mathbf{s} \in \mathcal{A}_{\mathbb{C}}; \sigma(\mathbf{s}) \subset U\}$, and consider the usual analytic functional calculus for an element $\mathbf{a} \in \mathcal{A}_{\mathbb{C}}(U)$:

$$\mathcal{O}(U, \mathcal{A}_{\mathbb{C}}) \ni f \mapsto f(\mathbf{a}) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - \mathbf{a})^{-1} d\lambda \in \mathcal{A}_{\mathbb{C}},$$

where Γ is the boundary of a Cauchy domain (that is, Γ is a finite union of Jordan piecewise smooth closed curves) containing $\sigma(\mathbf{a})$ in its interior.

A natural question is to find a condition insuring the inclusion $C[f](a) \in \mathcal{A}$ for f in a sufficiently large subspace of $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$, whenever $a \in \mathcal{A}$ such that $\sigma_{\mathbb{C}}(a) \subset U$.

The next result is a more general version of Theorem 1 from [25] or of Theorem 6 from [24].

Theorem 1 *Let $U \subset \mathbb{C}$ be open and conjugate symmetric. If $f \in \mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$, we have $\overline{f(\mathbf{a})} = f(\bar{\mathbf{a}})$ for all $\mathbf{a}$ with $\sigma(\mathbf{a}) \subset U$.*

Proof The proof follows the lines of the proof of Theorem 1 from [25]. We put $\Gamma_{\pm} := \Gamma \cap \mathbb{C}_{\pm}$, where $\mathbb{C}_{+}$ (resp. $\mathbb{C}_{-}$) equals to $\{\lambda \in \mathbb{C}; \Im\lambda \geq 0\}$ (resp. $\{\lambda \in \mathbb{C}; \Im\lambda \leq 0\}$). We write $\Gamma_{+} = \cup_{j=1}^{m} \Gamma_{j+}$, where Γ_{j+} are the connected components of Γ_{+}. Similarly, we write $\Gamma_{-} = \cup_{j=1}^{m} \Gamma_{j-}$, where Γ_{j-} are the connected components of Γ_{-}, and Γ_{j-} is the reflexion of Γ_{j+} with respect of the real axis. □

We fix a function $f \in \mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$). As Γ is the boundary of a Cauchy domain, for each index j we have a parametrization $\phi_j : [0, 1] \mapsto \mathbb{C}$, positively oriented, such that

$\phi_j([0,1]) = \Gamma_{j+}$. Taking into account that the function $t \mapsto \overline{\phi_j(t)}$ is a parametrization of Γ_{j-} negatively oriented, and setting $\Gamma_j = \Gamma_{j+} \cup \Gamma_{j-}$, we define

$$f_j(\mathbf{a}) := \frac{1}{2\pi i} \int\limits_{\Gamma_j} f(\zeta)(\zeta - \mathbf{a})^{-1} d\zeta$$

$$= \frac{1}{2\pi i} \int\limits_0^1 f(\phi_j(t))(\phi_j(t) - \mathbf{a})^{-1} \phi_j'(t) dt$$

$$- \frac{1}{2\pi i} \int\limits_0^1 f(\overline{\phi_j(t)})(\overline{\phi_j(t)} - \mathbf{a})^{-1} \overline{\phi_j'(t)} dt.$$

Therefore,

$$\overline{f_j(\mathbf{a})} = -\frac{1}{2\pi i} \int\limits_0^1 \overline{f(\phi_j(t))}(\overline{\phi_j(t)} - \bar{\mathbf{a}})^{-1} \overline{\phi_j'(t)} dt$$

$$+ \frac{1}{2\pi i} \int\limits_0^1 f(\phi_j(t))(\phi_j(t) - \bar{\mathbf{a}})^{-1} \phi_j'(t) dt.$$

via our assumption on the function f. Consequently, $f_j(\mathbf{a}) = \overline{f_j(\bar{\mathbf{a}})}$ for all j, and so

$$\overline{f(\mathbf{a})} = \sum_{j=1}^m \overline{f_j(\mathbf{a})} = \sum_{j=1}^m f_j(\bar{\mathbf{a}}) = f(\bar{\mathbf{a}}).$$

Remark 1 According to Theorem 1, given a conjugate symmetric open set $U \in \mathbb{C}$, and setting $\mathcal{A}(U) = \{a \in \mathcal{A}; \sigma_{\mathbb{C}}(a) \subset U\}$, we have a map

$$\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}}) \times \mathcal{A}(U) \ni (f, a) \mapsto f(a) \in \mathcal{A}$$

given by

$$\mathcal{O}(U, \mathcal{A}_{\mathbb{C}}) \ni f \mapsto f(a) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - a)^{-1} d\lambda$$

where Γ is the boundary of a Cauchy domain containing $\sigma_{\mathbb{C}}(a)$ in its interior.

The properties of the map $f \mapsto f(a)$, which can be called the *(left) analytic functional calculus of a*, are given by the following.

Theorem 2 *Let $\mathcal{A}$ be a unital real Banach algebra and let $\mathcal{A}_{\mathbb{C}}$ be its complexification. Let also $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $a \in \mathcal{A}$, with $\sigma_{\mathbb{C}}(a) \subset U$. Then the assignment*

$$\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}}) \ni f \mapsto f(a) \in \mathcal{A}$$

is an $\mathbb{R}$-linear map, and its restriction

$$\mathcal{O}_s(U) \ni f \mapsto f(a) \in \mathcal{A}$$

is a unital real algebra morphism.

Moreover, the following properties are true:

(1) For all $f \in \mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$, $h \in \mathcal{O}_s(U)$, we have $(fh)(a) = f(a)h(a)$.
(2) For every polynomial $P(\zeta) = \sum_{n=0}^{m} c_n \zeta^n$, $\zeta \in \mathbb{C}$, with $c_n \in \mathcal{A}$ for all $n = 0, 1, \ldots, m$, we have $P(a) = \sum_{n=0}^{m} c_n a^n$.

Proof The arguments are more or less standard (see [8]). The $\mathbb{R}$-linearity of the maps

$$\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}}) \ni f \mapsto f(a) \in \mathcal{A},\ \mathcal{O}_s(U) \ni f \mapsto f(a) \in \mathcal{A},$$

is clear. The second one is actually multiplicative, which follows from the multiplicativity of the usual analytic functional calculus of a.

The more general property, specifically $(fh)(a) = f(a)h(a)$ for all $f \in \mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$, $h \in \mathcal{O}_s(U)$ follows from the equalities,

$$\frac{1}{2\pi i}\int_{\Gamma_0} f(\zeta)h(\zeta)(\zeta - a)^{-1}d\zeta$$
$$= \left(\frac{1}{2\pi i}\int_{\Gamma_0} f(\zeta)(\zeta - a)^{-1}d\zeta\right)\left(\frac{1}{2\pi i}\int_{\Gamma} f(\eta)(\eta - a)^{-1}d\eta\right),$$

obtained as in the classical case (see [8], Section VII.3), which holds because h is $\mathbb{C}$-valued and commutes with the elements of $\mathcal{A}$. Here Γ, Γ_0 are the boundaries of two Cauchy domains Δ, Δ_0 respectively, such that Δ contains the closure of Δ_0, and Δ_0 contains $\sigma(a)$.

In particular, for every polynomial $P(\zeta) = \sum_{n=0}^{m} c_n \zeta^n$ with $c_n \in \mathcal{A}$ for all $n = 0, 1, \ldots, m$, we have $P(a) = \sum_{n=0}^{m} c_n a^n \in \mathcal{B}(\mathcal{V})$ for all $a \in \mathcal{A}$. □

Definition 1 Let $\mathcal{A}$ be a unital real Banach algebra, and let $U \subset \mathbb{C}$ be a conjugate symmetric open set. Let also $\mathcal{A}_{\mathbb{C}}$ be the complexification of $\mathcal{A}$, let $\mathcal{A}(U) = \{a \in \mathcal{A}; \sigma_{\mathbb{C}}(a) \subset U\}$, and let $\mathcal{R}(\mathcal{A}(U), \mathcal{A}) = \{\mathcal{C}[f] : \mathcal{A}(U) \mapsto \mathcal{A}; \mathcal{C}[f](a) = f(a), f \in \mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}}), a \in \mathcal{A}(U)\}$. The $\mathbb{R}$-linear map

$$\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}}) \ni f \mapsto \mathcal{C}[f] \in \mathcal{R}(\mathcal{A}(U), \mathcal{A})$$

is called the $\mathcal{A}$-*Cauchy transformation* on $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$

2.2 Algebra of Real Linear Operators

An important unital real algebra is that consisiting of bounded $\mathbb{R}$-linear operators, which will be discussed in the following.

Let $\mathcal{V}$ be a real vector space. We denote by $\mathcal{V}_{\mathbb{C}}$ the complexification of $\mathcal{V}$, identified with the direct sum $\mathcal{V} + i\mathcal{V}$, where i is the imaginary unit of the complex plane. As in the case of algebras, the map $\mathcal{V}_{\mathbb{C}} \ni x + iy \mapsto x - iy \in \mathcal{V}_{\mathbb{C}}$ is said to be the (natural) *conjugation* of $\mathcal{V}_{\mathbb{C}}$, which is $\mathbb{R}$-linear, and whose square is the identity. It will be usually denoted by C. If $\mathcal{V}$ is a real Banach space, then $\mathcal{V}_{\mathbb{C}}$ is a complex Banach space, for which we fix the norm $\|x + iy\| = \|x\| + \|y\|$ for all $x + iy \in \mathcal{V}_{\mathbb{C}}$ with $x, y \in \mathcal{V}$, where $\| * \|$ is the norm of $\mathcal{V}$. In this way, the conjugation C is an isometry.

Let $\mathcal{B}(\mathcal{V})$ (resp. $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$) be the Banach algebra of all $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operators acting on $\mathcal{V}$ (resp. of $\mathcal{V}_{\mathbb{C}}$).

We have an unital injective algebra morphism $\mathcal{B}(\mathcal{V}) \ni T \mapsto T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ given by $T_{\mathbb{C}}(x + iy) = Tx + iTy$ for all $T \in \mathcal{B}(\mathcal{V})$, which $\mathbb{R}$-linear.

The operator $T_{\mathbb{C}}$ will be called the *complex extension* of T.

The natural conjugation of $\mathcal{V}_{\mathbb{C}}$ induces a conjugation on $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$ via the equality $S^{\flat} = CSC$ for all $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$. We set $\mathcal{B}_c(\mathcal{V}_{\mathbb{C}}) = \{S = S^{\flat}; S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})\}$, which is a unital real algebra. Then we have the direct sum $\mathcal{B}(\mathcal{V}_{\mathbb{C}}) = \mathcal{B}_c(\mathcal{V}_{\mathbb{C}}) + i\mathcal{B}_c(\mathcal{V}_{\mathbb{C}})$, via the remark that $S + S^{\flat}, i(S - S^{\flat}) \in \mathcal{B}_c(\mathcal{V}_{\mathbb{C}})$. Clearly, the real algebras $\mathcal{B}(\mathcal{V})$ and $\mathcal{B}_c(\mathcal{V}_{\mathbb{C}})$ are isomorphic, also as $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$ and $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$.

As a particular case of formula (2), the complex spectrum of an operator $T \in \mathcal{B}(\mathcal{V})$, regarded as an element of $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$, is given by

$$\sigma_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}, \lambda - T \text{ not invertible in } \mathcal{B}(\mathcal{V})_{\mathbb{C}}\}.$$

2.3 Abstract Slice Regularity

Let $\mathbb{S}_{\mathcal{A}}$ be the set $\{e \in \mathcal{A}; e^2 = -1, \|e\| = 1\}$, which will be called the *imaginary sphere* of $\mathcal{A}$, and which is supposed to be nonempty. To simplify the notation, we put $\mathbb{S} = \mathbb{S}_{\mathcal{A}}$. Each element $e \in \mathbb{S}$ is said to be an *imaginary unit*. For such an element $e \in \mathbb{S}$, we define the set $\mathbb{C}_e := \{x + ye; x, y \in \mathbb{R}\}$, which is isomorphic to the complex plane.

Let $\Omega \subset \mathcal{A}$, and let $F : \Omega \mapsto \mathcal{A}_{\mathbb{C}}$. For a fixed imaginary unit $e \in \mathbb{S}$, we assume that the set $\mathbb{C}_e \cap \Omega$ is nonempty and open, and that the restriction of F to $\mathbb{C}_e \cap \Omega$ is differentiable as a functions of x, y, with $x + ye \in \mathbb{C}_e \cap \Omega$. For such a function F, we define the operator

$$\bar{\partial}_e = \frac{1}{2}\left(\frac{\partial}{\partial x} + R_e \frac{\partial}{\partial y}\right) F(x + ye), \;\; x + ye \in \mathbb{C}_e \cap \Omega,$$

where R_e is the right multiplication of the elements of $A_{\mathbb{C}}$ by e. Such a definition, introducing the classical $\bar{\partial}$-operator in a noncommuting framework, firstly appears in [12], in the quaternionic context (see also [4, 6, 7], etc.). For an abstract similar concept, valid in real alternative algebras, see the paper [14]. In this spirit, we say that

F is *slice regular along the imaginary unit* e if $\bar{\partial}_e F(x+ye) = 0$ on the set $\mathbb{C}_e \cap \Omega$. When F is slight regular along all $e \in \mathbb{S}$, we say that F is *slice regular*.

Unlike in [6] and in other works, we have some reasons, related to the action of certain operators, to use the right slice regularity rather than the left one. Nevertheless, a left slice regularity can also be defined via the left multiplication of the elements of $\mathcal{A}_{\mathbb{C}}$ by elements from $\mathbb{S}$. In what follows, the right slice regularity will be simply called *slice regularity*.

Example 1 Fixing $\mathbf{a} \in \mathcal{A}_{\mathbb{C}}$, we denote by Ω the the set of those $\mathbf{s} \in \mathcal{A}$ such that $\mathbf{a} - \mathbf{s}$ is invertble in $\mathcal{A}_{\mathbb{C}}$. Choosing a imaginary unit $e \in \mathbb{S}$ such that $\mathbf{a}e = e\mathbf{a}$, and assuming that $\mathbb{C}_e \cap \Omega$ is nonempty, we can write the equalities

$$\frac{\partial}{\partial x}(\mathbf{a} - x - ye)^{-1} = (\mathbf{a} - x - ye)^{-2},$$
$$R_{\mathfrak{s}}\frac{\partial}{\partial y}(\mathbf{a} - x - ye)^{-1} = -(\mathbf{a} - x - ye)^{-2},$$

because $e^2 = -1$, and $\mathbf{a}$, e and $(\mathbf{a} - x - ye)^{-1}$ commute in $\mathcal{A}_{\mathbb{C}}$. Therefore, $\bar{\partial}_e((\mathbf{a} - x - ye)^{-1}) = 0$, implying that the function $\Omega \ni \mathbf{s} \mapsto (\mathbf{a} - \mathbf{s})^{-1} \in \mathcal{A}_{\mathbb{C}}$ is slice regular.

3 Hamilton and Clifford Real Algebras

The results from the previous section will be applied especially to Hamilton and Clifford algebras, regarded as real algebras. In this section we mainly discuss the specific framework of these algebras, dealing especially with elements of spectral theory.

The next subsection is inspired by the Subsect.2.1 from [24].

3.1 Preliminaries for Clifford Algebras

In the following, a *Clifford algebra*, denoted by $\mathfrak{C}_n$ for a fixed integer $n \geq 0$, is a unital associative real algebra having $n+1$ generators $e_0 = 1, e_1, \ldots, e_n$, satisfying the relations $e_j^2 = -1$, $e_j e_k = -e_k e_j$ for all $j, k = 1, \ldots, n$, $j \neq k$ (see also [6, 16, 20, 24] etc.). In particular, $\mathfrak{C}_0 = \mathbb{R}$, $\mathfrak{C}_1 = \mathbb{C}$, and $\mathfrak{C}_2 = \mathbb{H}$, that is, the the real, complex and quaternionic algebras are special cases of Clifford algebras.

If $\mathbb{N}_n = \{1, 2, \ldots, n\}$, for every subset $J = \{j_1, j_2, \ldots, j_p\} \subset \mathbb{N}_n$, with $j_1 < j_2 < \cdots < j_p$ and $1 \leq p \leq n$, we put $e_J = e_{j_1} e_{j_2} \cdots e_{j_p}$. We use the symbol $J \prec \mathbb{N}_n$ to indicate that J is an oredered set as above. Assuming also that $\emptyset \prec \mathbb{N}_n$, $e_\emptyset = 1$, $e_{\{j\}} = e_j$, $j = 1, \ldots, n$, and that the family $\{e_J\}_{J \prec \mathbb{N}_n}$ is a basis of the vector space $\mathfrak{C}_n$, an arbitrary element $\mathbf{a} \in \mathfrak{C}_n$ can be written as

$$\mathbf{a} = \sum_{J \prec \mathbb{N}_n} a_J e_J, \tag{3}$$

where $a_J \in \mathbb{R}$ are uniquely determined for all $J \prec \mathbb{N}_n$. To simplify the notation, we also put $e_0 = e_\emptyset = 1$, and $a_0 = a_\emptyset$. The elements of the Clifford algebra $\mathfrak{C}_n$ will be called *Clifford vectors*, or briefly, *Cl-vectors*.

The real linear subspace of $\mathfrak{C}_n$ spanned by $\{e_k\}_{k=0}^n$ will be denoted by $\mathfrak{P}_n$, playing an important role in the following. The Cl-vectors from the subspace $\mathfrak{P}_n$, which have the form $\mathbf{a} = a_0 + \sum_{k=0}^n a_k e_k$ with $a_k \in \mathbb{R}$ for all $k = 0, \ldots, n$, will be called *paravectors* (as in [6]). The linear subspace $\mathfrak{P}_n$ will be often identified with the Euclidean space $\mathbb{R}^{n+1}$, via the linear isomorphism

$$\mathfrak{P}_n \ni \sum_{k=0}^n a_k e_k \mapsto (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}.$$

For every $\mathbf{a} = \sum_{J \prec \mathbb{N}_n} a_J e_J \in \mathfrak{C}_n$ we have a decomposition $\mathbf{a} = \Re(\mathbf{a}) + \Im(\mathbf{a})$, where $\Re(\mathbf{a}) = a_0$ and $\Im(\mathbf{a}) = \sum_{\emptyset \neq J \prec \mathbb{N}_n} a_J e_J$, that is, the *real part* and the *imaginary part* of the Cl-vector $\mathbf{a} \in \mathfrak{C}_n$, respectively.

The algebra $\mathfrak{C}_n$ has a norm defined by;

$$|\mathbf{a}|^2 = \sum_{J \prec \mathbb{N}_n} a_J^2, \tag{4}$$

where $\mathbf{a}$ is given by (3).

The algebra $\mathfrak{C}_n$ also has an *involution* $\mathfrak{C}_n \ni \mathbf{a} \mapsto \mathbf{a}^* \in \mathfrak{C}_n$, which is defined via the conditions $e_j^* = -e_j$ $(j = 1, \ldots, n)$, $r^* = r \in \mathbb{R}$, $(\mathbf{ab})^* = \mathbf{b}^*\mathbf{a}^*$ for all $\mathbf{a}, \mathbf{b} \in \mathfrak{C}_n$ (see [6], Definition 2.1.11). According to Proposition 2.1.12 from [6], we therefore have $(\mathbf{a}^*)^* = \mathbf{a}$, $(\mathbf{a} + \mathbf{b})^* = \mathbf{a}^* + \mathbf{b}^*$. Particularly, if $\mathbf{a} = a_0 + \sum_{j=1}^n a_j e_j$, then $\mathbf{a}^* = a_0 - \sum_{j=1}^n a_j e_j$, for all $\mathbf{a} \in \mathfrak{P}_n$.

Considering the complexification $\mathfrak{K}_n = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{C}_n$ of $\mathfrak{C}_n$, identified with the direct sum $\mathfrak{C}_n + i\mathfrak{C}_n$, we have a unital algebra with the involution

$$\mathfrak{K}_n \ni \mathbf{c} = \mathbf{a} + i\mathbf{b} \mapsto \mathbf{c}^* = \mathbf{a}^* - i\mathbf{b}^* \in \mathfrak{K}_n,\ \ \mathbf{a}, \mathbf{b} \in \mathfrak{C}_n,$$

which extends the involution of $\mathfrak{C}_n$.

As in the abstract case (see Subsect. 2.1), we have a *conjugation* on the complexification $\mathfrak{K}_n$. Specifically, the $\mathbb{R}$-linear map

$$\mathfrak{K}_n \ni \mathbf{c} = \mathbf{a} + i\mathbf{b} \mapsto \bar{\mathbf{c}} := \mathbf{a} - i\mathbf{b} \in \mathfrak{K}_n$$

is a *conjugation* on $\mathfrak{K}_n$, so it is a real automorphism of unital algebras, whose square is the identity. Note that the elements of the real subalgebra $\mathfrak{C}_n$ commute with the complex numbers in the algebra $\mathfrak{K}_n$. As we have, $\bar{\mathbf{a}} = \mathbf{a}$ if and only if $\mathbf{a} \in \mathfrak{C}_n$, we get a useful criterion to identify the elements of $\mathfrak{C}_n$ among those from $\mathfrak{K}_n$.

The next subsection is partially inspired by the corresponding part from Sect. 3 in [24].

3.2 Spectrum of a Paravector

Because $\mathfrak{C}_n$ is a real algebra, we have natural concept of (complex) spectrum, via the coreresponding concept in the complex algebra $\mathfrak{K}_n$, as noticed in the previous section. For an element $\kappa \in \mathfrak{C}_n$, we denote by $\sigma_{\mathbb{C}}(\kappa)$ its spectrum, when κ is identified with an element of $\mathfrak{K}_n$. Let also $\rho_{\mathbb{C}}(\kappa) := \mathbb{C} \setminus \sigma_{\mathbb{C}}(\kappa)$.We are particularly interested in the spectrum of a paravector, which can be completely described.

Lemma 1 *Let $\kappa \in \mathfrak{P}_n$ and let $\lambda \in \mathbb{C}$. The following conditions are equivalent:*

(i) $\lambda \in \rho_{\mathbb{C}}(\kappa)$;
(ii) $\lambda^2 - \lambda(\kappa + \kappa^*) + |\kappa|^2 \neq 0$;
(iii) $|\lambda|^2 - 2\kappa\Re(\lambda) + \kappa^2$ *invertible in* $\mathfrak{C}_n$.

Proof Because κ commutes with κ^* and with λ, the equivalence $(i) \Leftrightarrow (ii)$ follows from the equality

$$(\lambda - \kappa^*)(\lambda - \kappa) = (\lambda - \kappa)(\lambda - \kappa^*) = \lambda^2 - \lambda(\kappa + \kappa^*) + |\kappa|^2 \in \mathbb{C}. \tag{5}$$

The equivalence $(i) \Leftrightarrow (iii)$ follows from the equality

$$(\lambda - \kappa)(\bar{\lambda} - \kappa) = |\lambda|^2 - 2\kappa\Re(\lambda) + \kappa^2, \tag{6}$$

using the fact that $\lambda - \kappa$ is invertible if and only if $\bar{\lambda} - \kappa$ is invertible.

Remark 2 We follow the lines of Remark 1 from [22].

(1) It follows from Lemma 1 that $\lambda - \kappa$ is invertible if and only if the complex number $\lambda^2 - 2\lambda\Re(\kappa) + |\kappa|^2$ is nonnull. Therefore

$$(\lambda - \kappa)^{-1} = \frac{1}{\lambda^2 - 2\lambda\Re(\kappa) + |\kappa|^2}(\lambda - \kappa^*).$$

Hence, the element $\lambda - \kappa \in \mathfrak{K}_n$ is not invertible if and only if $\lambda = \Re(\kappa) \pm i|\Im(\kappa)|$. In this way, the *spectrum* of a paravector κ is given by the equality $\sigma_{\mathbb{C}}(\kappa) = \{s_\pm(\kappa)\}$, where $s_\pm(\kappa) = \Re(\kappa) \pm i|\Im(\kappa)|$ are the *eigenvalues* of κ.

(2) As usually, the function

$$\rho_{\mathbb{C}}(\kappa) \ni \lambda \mapsto (\lambda - \kappa)^{-1} \in \mathfrak{K}_n$$

is called the *resolvent* (function) of κ, and it is a $\mathfrak{K}_n$-valued analytic function on $\rho_{\mathbb{C}}(\kappa)$.

(3) Two paravectors $\kappa, \tau \in \mathfrak{P}_n$ have the same spectrum if and only if $\Re(\kappa) = \Re(\tau)$ and $|\Im(\kappa)| = |\Im(\tau)|$. In fact, the equality $\sigma_{\mathbb{C}}(\kappa) = \sigma_{\mathbb{C}}(\tau)$ is an equivalence relation in $\mathfrak{P}_n$, and the equivalence class of an element $\kappa_0 = x_0 + y_0\mathfrak{s}_0 \in \mathfrak{P}_n$ is given by $\{x_0 + y_0\mathfrak{s};\ \mathfrak{s} \in \mathbb{S}_n\}$, where $\mathbb{S}_n$ the unit sphere of imaginary paravectors.

(4) Every paravector $\kappa \in \mathfrak{P}_n \setminus \mathbb{R}$ can be written as $\kappa = x + y\mathfrak{s}$, where x, y are real numbers, with $x = \Re(\kappa)$, $y \in \{\pm|\Im(\kappa)|\}$, and $\mathfrak{s} \in \{\pm\Im(\kappa)/|\Im(\kappa)|\} \subset \mathbb{S}_n$. Clearly, we have $\sigma_{\mathbb{C}}(\kappa) = \{x \pm iy\}$, because $\Im(\kappa) = y\mathfrak{s}$, and the spectrum of κ does not depend on $\mathfrak{s}$. Thus, for every $\lambda = u + iv \in \mathbb{C}$ with $u, v \in \mathbb{R}$, we have $\sigma(u + v\mathfrak{s}) = \{\lambda, \bar{\lambda}\}$ for all $\mathfrak{s} \in \mathbb{S}_n$.

(5) Fixing an element $\mathfrak{s} \in \mathbb{S}_n$, we have an isometric $\mathbb{R}$-linear map from the complex plane $\mathbb{C}$ into the space $\mathfrak{P}_n$, say $\theta_{\mathfrak{s}}$, defined by $\theta_{\mathfrak{s}}(u + iv) = u + v\mathfrak{s}$, $u, v \in \mathbb{R}$. For every subset $A \subset \mathbb{C}$, we put

$$A_{\mathfrak{s}} = \{x + y\mathfrak{s}; x, y \in \mathbb{R}, x + iy \in A\} = \theta_{\mathfrak{s}}(A). \tag{7}$$

Note that, if A is open in $\mathbb{C}$, then $A_{\mathfrak{s}}$ is open in the $\mathbb{R}$-vector space $\mathbb{C}_{\mathfrak{s}} \subset \mathfrak{P}_n$.

The next subsection reproduces parts from Subsect. 2.1 and Section 3 from [22].

3.3 Hamilton Algebra

The algebra of quaternions $\mathbb{H}$, or Hamilton algebra, is the Clifford algebra $\mathfrak{C}_2$, generated as an algebra by the family $\{e_0, e_1, e_2\}$. We put $e_1 = \mathbf{j}$, $e_2 = \mathbf{k}$, and setting $\mathbf{l} = \mathbf{jk}$, we obtain the equalities

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{l},\ \mathbf{kl} = -\mathbf{lk} = \mathbf{j},\ \mathbf{lj} = -\mathbf{jl} = \mathbf{k},\ \mathbf{jj} = \mathbf{kk} = \mathbf{ll} = -1.$$

The norm of $\mathbb{H}$ defined by (4) and given by

$$|\mathbf{x}| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2},\ \ \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l},\ \ x_0, x_1, x_2, x_3 \in \mathbb{R},$$

is multiplicative, and its involution is

$$\mathbb{H} \ni \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} \mapsto \mathbf{x}^* = x_0 - x_1\mathbf{j} - x_2\mathbf{k} - x_3\mathbf{l} \in \mathbb{H},$$

satisfying $\mathbf{x}^*\mathbf{x} = \mathbf{x}\mathbf{x}^* = |\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{H}$. In particular, every element $\mathbf{x} \in \mathbb{H}\setminus\{0\}$ is invertible, and $\mathbf{x}^{-1} = |\mathbf{x}|^{-2}\mathbf{x}^*$.

As in the case of Cl-vectors, for an arbitrary quaternion $\mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l}$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$, we set $\Re\mathbf{x} = x_0 = (\mathbf{x} + \mathbf{x}^*)/2$, and $\Im\mathbf{x} = x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} = (\mathbf{x} - \mathbf{x}^*)/2$, that is, the *real* and the *imaginary part* of $\mathbf{x}$, respectively.

Let us note that the space of paravectors $\mathfrak{P}_2$ of the Clifford algebra $\mathbb{H} = \mathfrak{C}_2$ is the vector space spanned by $\{1, \mathbf{j}, \mathbf{k}\} \neq \mathbb{H}$, but when working with quaternions, we replace $\mathfrak{P}_2$ by $\mathbb{H}$. Nevertheless, most of the properties of the paravectors hold true for quaternions.

The complexification $\mathbb{H} + i\mathbb{H}$ of $\mathbb{H}$ will be denoted by $\mathbb{M}$. In the complex algebra $\mathbb{M}$ we use the natural concept of spectrum, which can be described in the case of quaternions, as in the case of paravectors.

Remark 3 (1) Because of the identities

$$(\lambda - \mathbf{x}^*)(\lambda - \mathbf{x}) = (\lambda - \mathbf{x})(\lambda - \mathbf{x}^*) = \lambda^2 - \lambda(\mathbf{x} + \mathbf{x}^*) + |\mathbf{x}|^2 \in \mathbb{C},$$

for all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{H}$, implying that the element $\lambda - \mathbf{x} \in \mathbb{M}$ is invertible if and only if the complex number $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ is nonnull, we obtain that

$$(\lambda - \mathbf{x})^{-1} = \frac{1}{\lambda^2 - 2\lambda\Re\mathbf{x} + |\mathbf{x}|^2}(\lambda - \mathbf{x}^*).$$

Therefore, the *spectrum* of a quaternion $\mathbf{x} \in \mathbb{H}$ is given by the equality $\sigma(\mathbf{x}) = \{s_\pm(\mathbf{x})\}$, where $s_\pm(\mathbf{x}) = \Re\mathbf{x} \pm \mathbf{i}|\Im\mathbf{x}|$ are the *eigenvalues* of $\mathbf{x}$.

(2) As usually, the *resolvent set* $\rho(\mathbf{x})$ of a quaternion $\mathbf{x} \in \mathbb{H}$ is the set $\mathbb{C} \setminus \sigma(\mathbf{x})$, while the function

$$\rho(\mathbf{x}) \ni \lambda \mapsto (\lambda - \mathbf{x})^{-1} \in \mathbb{M}$$

is the *resolvent* (function) of $\mathbf{x}$, which is an $\mathbb{M}$-valued analytic function on $\rho(\mathbf{x})$.

(3) Note that two quaternions $\mathbf{x}, \mathbf{y} \in \mathbb{H}$ have the same spectrum if and only if $\Re\mathbf{x} = \Re\mathbf{y}$ and $|\Im\mathbf{x}| = |\Im\mathbf{y}|$. Moreover, the equality $\sigma(\mathbf{x}) = \sigma(\mathbf{y})$ is an equivalence relation.

(4) Let $\mathbb{S}$ be the unit sphere of imaginary quaternions. Every quaternion $\mathbf{q} \in \mathbb{H} \setminus \mathbb{R}$ can be written as $\mathbf{q} = x + y\mathfrak{s}$, where x, y are real numbers, with $x = \Re\mathbf{q}$, $y = \pm|\Im\mathbf{q}|$, and $\mathfrak{s} = \pm\Im\mathbf{q}/|\Im\mathbf{q}| \in \mathbb{S}$, and we have $\sigma(\mathbf{q}) = \{x \pm iy\}$. In fact, for every $\lambda = u + iv \in \mathbb{C}$ with $u, v \in \mathbb{R}$, we have $\sigma(u + v\mathfrak{s}) = \{\lambda, \bar{\lambda}\}$ for all $\mathfrak{s} \in \mathbb{S}$.

Remark 4 The properties of interest of the quaternionic algebra $\mathbb{H}$ are not always direct consequences of those of the Clifford algebra $\mathfrak{C}_2$. Indeed, in this case of $\mathbb{H}$, we are mainly interested in $\mathbb{H}$-valued functions, defined on subsets of $\mathbb{H}$. The algebra $\mathfrak{C}_2$ is generated by $\{1, e_1, e_2\}$, and the vector space $\mathfrak{P}_2$ generated by this set is strictly included in $\mathbb{H}$. As the algebra $\mathbb{H}$ is also generated by the set $\{1, e_1, e_2, e_3\}$, where $e_3 = e_1e_2$, it is isomorphic to the quotient of the algebra $\mathfrak{C}_3$ by the two-sided ideal generated by $e_3 - e_1e_2$. Consequently, a separate approach concerning the quaternion algebra $\mathbb{H}$ (as in [6]), rather than that in the framework of Clifford algebras, is often more appropriate. Note also that for $\mathbb{C} = \mathfrak{C}_1$ we have $\mathfrak{P}_1 = \mathbb{C}$.

4 Spectral Transformation of Some Vector-Valued Stem Functions

The space of stem $\mathfrak{K}_n$-valued functions, defined on subsets of the complex plane, may be associated with spaces of functions, defined on subsets of $\mathfrak{P}_n$, taking values in the Clifford algebra $\mathfrak{C}_n$, using spectral methods and functional calculi. This operation may be regarded either as an ”extension“ or as a general functional calculus, with arbitrary functions. Similar results are also valid for some $\mathbb{M}$-valued functions. Nevertheless, we shall mainly deal with $\mathfrak{K}_n$-valued functions, as developed in [24], Sect. 4, providing references also for the case of $\mathbb{M}$-valued functions whenever necessary, which can be found in [22].

4.1 Stem Functions and Spectral Extensions

The main idea of the following approach comes from the theory of spectral operators (see [8], Part III), and it can be applied to define an appropriate functional calculus, also useful in the Cliffordian (or quaternionic) context. Specifically, regarding the algebra $\mathfrak{K}_n$ as a (complex) Banach space, and considering the Banach space $\mathcal{B}(\mathfrak{K}_n)$ of all linear operators acting on $\mathfrak{K}_n$, the operator L_κ, $\kappa \in \mathfrak{P}_n$, which is the left multiplication operator on $\mathfrak{K}_n$ by the paravector κ, is a particular case of a *scalar type* operator, as defined in [8], Part III, XV.4.1. Its resolution of the identity consists of four projections $\{0, P_\pm(\kappa), \mathrm{I}\}$, including the null operator 0 and the identity I, where $P_\pm(\kappa)$ are the spectral projections of L_κ, whose spectrum coincides with that of κ, and its integral representation is given by

$$L_\kappa = s_+(\kappa)P_+(\kappa) + s_-(\kappa)P_-(\kappa) \in \mathcal{B}(\mathfrak{K}_n),$$

provided by Corollary 1.

For every function $f : \sigma(\kappa) \mapsto \mathbb{C}$ we may define the operator

$$f(L_\kappa) = f(s_+(\kappa))P_+(\kappa) + f(s_-(\kappa))P_-(\kappa) \in \mathcal{B}(\mathfrak{K}_n), \tag{8}$$

which provides a functional calculus with arbitrary functions on the spectrum. One can even replace the scalar function f by a function $F : \sigma(\kappa) \mapsto \mathcal{B}(\mathfrak{K}_n)$, getting what may be called a ”left functional calculus“, not multiplicative, in general. It is this idea which leads us to try to define some $\mathfrak{C}_n$-valued functions on subsets of $\mathfrak{P}_n$ via certain $\mathfrak{K}_n$-valued functions, defined on subsets of $\mathbb{C}$. The next concept is is quoted from [24], Definition 2(2).

Definition 2 A subset $A \subset \mathfrak{P}_n$ is said to be *spectrally saturated* (as in [22], Definition 2) if whenever $\sigma(\theta) = \sigma(\kappa)$ for some $\theta \in \mathfrak{P}_n$ and $\kappa \in A$, we also have $\theta \in A$.

For an arbitrary $A \subset \mathfrak{P}_n$, we put $\mathfrak{S}(A) = \cup_{\kappa \in A}\sigma(\kappa) \subset \mathbb{C}$.

Conversely, for an arbitrary subset $S \subset \mathbb{C}$, we put $S_\sigma = \{\kappa \in \mathfrak{P}_n; \sigma(\kappa) \subset S\}$.

From [22], Remark 4, we quote the following properties.

Remark 5 (1) If $A \subset \mathfrak{P}_n$ is spectrally saturated, then $S = \mathfrak{S}(A)$ is conjugate symmetric, and conversely, if $S \subset \mathbb{C}$ is conjugate symmetric, then S_σ is spectrally saturated, which can be easily seen. Moreover, the assignment $S \mapsto S_\sigma$ is injective. Similarly, the assignment $A \mapsto \mathfrak{S}(A)$ is injective and $A = S_\sigma$ if and only if $S = \mathfrak{S}(A)$.

(2) If $\Omega \subset \mathfrak{P}_n$ is an open spectraly saturated set, then $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open, using a direct argument. Conversely, if $U \subset \mathbb{C}$ is open and conjugate symmetric, the set U_σ is also open via the upper semi-continuity of the spectrum (see [8], Part I, Lemma VII.6.3.). An important particular case is when $U = \mathbb{D}_r := \{\zeta \in \mathbb{C}; |\zeta| < r\}$, for some $r > 0$. Then $U_\sigma = \{\kappa \in \mathfrak{P}_n; |\kappa| < r\}$. Indeed, if $|\kappa| < r$ and θ has the property $\sigma(\kappa) = \sigma(\theta)$, from the equality $\{\Re(\kappa) \pm i|\Im(\kappa)|\} = \{\Re(\theta) \pm i|\Im(\theta)|\}$ it follows that $|\theta| < r$.

(3) A subset $\Omega \subset \mathfrak{C}_n$ is said to be *axially symmetric* if for every $\kappa_0 = u_0 + v_0\mathfrak{s}_0 \in \Omega$ with $u_0, v_0 \in \mathbb{R}$ and $\mathfrak{s}_0 \in \mathbb{S}_n$, we also have $\kappa = u_0 + v_0\mathfrak{s} \in \Omega$ for all $\mathfrak{s} \in \mathbb{S}_n$. This concept is introduced in [6], Definition 2.2.17. In fact, we have the following.

Lemma 2 *A subset $\Omega \subset \mathfrak{C}_n$ is axially symmetric if and only if it is spectrally saturated.*

The assertion follows easily from the fact that the equality $\sigma(\kappa) = \sigma(\tau)$ is an equivalence relation in $\mathfrak{P}_n$ (see Remark 2(3)).

We prefer to use the expression "spectrally saturated set" to designate an "axially symmetric set", because the former name is more compatible with our spectral approach.

As in the general case (see Sect. 2), the algebra $\mathfrak{K}_n$ is endowed with a conjugation given by $\bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, when $\mathbf{a} = \mathbf{b} + i\mathbf{c}$, with $\mathbf{b}, \mathbf{c} \in \mathfrak{C}_n$. Note also that, because $\mathbb{C}$ is a subalgebra of $\mathfrak{K}_n$, the conjugation of $\mathfrak{K}_n$ restricted to $\mathbb{C}$ is precisely the usual complex conjugation.

Let $U \subset \mathbb{C}$ be conjugate symmetric, and let $F : U \mapsto \mathfrak{K}_n$ be a $\mathfrak{K}_n$-valued stem function (that is, $F(\bar{\lambda}) = \overline{F(\lambda)}$ for all $\lambda \in U$).

For an arbitrary conjugate symmetric subset $U \subset \mathbb{C}$, we put

$$\mathcal{S}(U, \mathfrak{K}_n) = \{F : U \mapsto \mathfrak{K}_n;\ F(\bar{\zeta}) = \overline{F(\zeta)},\ \zeta \in U\}, \tag{9}$$

that is, the $\mathbb{R}$-vector space of all $\mathfrak{K}_n$-valued stem functions on U. Replacing $\mathfrak{K}_n$ by $\mathbb{C}$, we denote by $\mathcal{S}(U)$ the real algebra of all $\mathbb{C}$-valued stem functions, which is an $\mathbb{R}$-subalgebra in $\mathcal{S}(U, \mathfrak{K}_n)$. In addition, the space $\mathcal{S}(U, \mathfrak{K}_n)$ is a two-sided $\mathcal{S}(U)$-module.

Remark 6 As in [25], Remark 4, every paravector $\mathfrak{s} \in \mathbb{S}_n$ may be associated with two elements $\iota_\pm(\mathfrak{s}) = (1 \mp i\mathfrak{s})/2$ in $\mathfrak{K}_n$, which are commuting idempotents such that $\iota_+(\mathfrak{s}) + \iota_-(\mathfrak{s}) = 1$ and $\iota_+(\mathfrak{s})\iota_-(\mathfrak{s}) = 0$. For this reason, setting $\mathfrak{K}_\pm^{\mathfrak{s}} = \iota_\pm(\mathfrak{s})\mathfrak{C}_n$, we have a direct sum decomposition $\mathfrak{K}_n = \mathfrak{K}_+^{\mathfrak{s}} + \mathfrak{K}_-^{\mathfrak{s}}$. Explicitly, if $\mathbf{a} = \mathbf{u} + i\mathbf{v}$, with $\mathbf{u}, \mathbf{v} \in \mathfrak{C}_\mathbf{n}$, the equation $\iota_+(\mathfrak{s})\mathbf{x} + \iota_-(\mathfrak{s})\mathbf{y} = \mathbf{a}$ has the unique solution $\mathbf{x} = \mathbf{u} + \mathfrak{s}\mathbf{v},\ \mathbf{y} = \mathbf{u} - \mathfrak{s}\mathbf{v} \in \mathfrak{C}_n$, because $\mathfrak{s}^{-1} = -\mathfrak{s}$.

In particular, if $\kappa \in \mathfrak{P}_n$ and $\Im(\kappa) \neq 0$, setting $\mathfrak{s}_{\tilde{\kappa}} = \tilde{\kappa}|\tilde{\kappa}|^{-1}$, where $\tilde{\kappa} = \Im(\kappa)$, the elements $\iota_\pm(\mathfrak{s}_{\tilde{\kappa}})$ are idempotents, as above.

The next result provides explicit formulas of the spectral projections (see [8], Part I, Section VII.1) associated to the element $\kappa \in \mathfrak{P}_n$, regarded as a left multiplication operator on $\mathfrak{K}_n$. They are not trivial only if $\kappa \in \mathfrak{P}_n \setminus \mathbb{R}$ because if $\kappa \in \mathbb{R}$, its spectrum is this real singleton, and the only spectral projection is the identity.

The statement of the result corresponding to Lemma 1 from [25] looks like that:

Lemma 3 *Let $\kappa \in \mathfrak{P}_n \setminus \mathbb{R}$ be fixed. The spectral projections associated to the eigenvalues $s_\pm(\kappa)$ are given by*

$$P_\pm(\kappa)\mathbf{a} = \iota_\pm(\mathfrak{s}_{\tilde{\kappa}})\mathbf{a},\ \ \mathbf{a} \in \mathfrak{K}_n. \tag{10}$$

Moreover, $P_+(\kappa)P_-(\kappa) = P_-(\kappa)P_+(\kappa) = 0$, and $P_+(\kappa) + P_-(\kappa)$ is the identity on $\mathfrak{K}_n$.

When $\kappa \in \mathbb{R}$, the corresponding spectral projection is the identity on $\mathfrak{K}_n$.

Corollary 1 *For every $\kappa \in \mathfrak{P}_n$ and $\mathbf{a} \in \mathfrak{K}_n$ we have*

$$L_\kappa \mathbf{a} = s_+(\kappa)P_+(\kappa)\mathbf{a} + s_-(\kappa)P_-(\kappa)\mathbf{a}.$$

We now define the concept of *spectral transformation* for some $\mathfrak{K}$-valued functions (see Definition 4 from [24]).

Definition 3 Let $U \subset \mathbb{C}$ be conjugate symmetric. For every $F : U \mapsto \mathfrak{K}_n$ and all $\kappa \in U_\sigma$ we define a function $F_\sigma : U_\sigma \mapsto \mathfrak{K}_n$, via the assignment

$$U_\sigma \setminus \mathbb{R} \ni \kappa \mapsto F_\sigma(\kappa) = F(s_+(\kappa))\iota_+(\mathfrak{s}_{\tilde{\kappa}}) + F(s_-(\kappa))\iota_-(\mathfrak{s}_{\tilde{\kappa}}) \in \mathfrak{K}_n, \qquad (11)$$

where $\tilde{\kappa} = \Im(\kappa)$, $\mathfrak{s}_{\tilde{\kappa}} = |\tilde{\kappa}|^{-1}\tilde{\kappa}$, and $\iota_\pm(\mathfrak{s}_{\tilde{\kappa}}) = 2^{-1}(1 \mp i\mathfrak{s}_{\tilde{\kappa}})$, and $F_\sigma(r) = F(r)$, if $r \in U_\sigma \cap \mathbb{R}$.

Formula (11) is strongly related to formula (8), via Remark 6.

Theorem 3 *Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $F : U \mapsto \mathfrak{K}_n$. The element $F_\sigma(\kappa)$ belongs to $\mathfrak{C}_n$ for all $\kappa \in U_\sigma$ if and only if $F \in \mathcal{S}(U, \mathfrak{K}_n)$.*

For the proof we refer to Theorem 1 from [24].

Corollary 2 *Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $f : U \mapsto \mathbb{C}$. The following conditions are equivalent;*

(1) $f \in \mathcal{S}(U)$;
(2) $f_\sigma(\kappa)$ belongs to $\mathbb{C}_\mathfrak{s}$, and $f_\sigma(\kappa^*) = f_\sigma(\kappa)^*$ for all $\kappa \in U_\sigma \cap \mathbb{C}_\mathfrak{s}$, where $\mathbb{C}_\mathfrak{s} = \{u + v\mathfrak{s}; u, v \in \mathbb{R}\}$, and $\mathfrak{s} \in \mathbb{S}_n$.

This result is Corollary 2 from [24].

Remark 7 This is a description of the zeros of the functions obtained via Theorem 3, corresponding to Remark 7 from [24]. We shall sketch the proof of one inclusion.

Let $U \subset \mathbb{C}$ be a conjugate symmetric set and let $F \in \mathcal{S}(U, \mathfrak{K}_n)$ be arbitrary. We describe the zeros of F_σ in the following way. If $F_\sigma(\kappa) = F(s_+(\kappa))\iota_+(\tilde{\kappa}) + F(s_-(\kappa))\iota_-(\tilde{\kappa}) = 0$, we deduce that $F(s_+(\kappa)) = 0$ and $F(s_-(\kappa)) = 0$, via a manipulation with the idempotents $\iota_\pm(\tilde{\kappa})$. Consequently, setting $\mathcal{Z}(F) := \{\lambda \in U; F(\lambda) = 0\}$, and $\mathcal{Z}(F_\sigma) := \{\kappa \in U_\sigma; F_\sigma(\kappa) = 0\}$, we must have

$$\mathcal{Z}(F_\sigma) = \{\kappa \in U_\sigma; \sigma(\kappa) \subset \mathcal{Z}(F)\}.$$

For every subset $\Omega \subset \mathfrak{P}_n$, we denote by $\mathcal{F}(\Omega, \mathfrak{C}_n)$ the set of all $\mathfrak{C}_n$-valued functions on Ω. Let also

$$\mathcal{IF}(\Omega, \mathfrak{C}_n) = \{g : \mathcal{F}(\Omega, \mathfrak{C}_n); g(\kappa^*) = g(\kappa)^* \in \mathbb{C}_\mathfrak{s}, \kappa \in \Omega \cap \mathbb{C}_\mathfrak{s},\ \mathfrak{s} \in \mathbb{S}_n\}, \quad (12)$$

which is a unital commutative subalgebra of the algebra $\mathcal{F}(\Omega, \mathfrak{C}_n)$. The functions from the space $\mathcal{IF}(\Omega, \mathfrak{P}_n)$ are similar to those called *intrinsic functions*, appering in [6], Definition 3.5.1, or in [7], Definition 2.1.2.

The next result provides a $\mathfrak{C}_n$-*valued general functional calculus* for arbitrary stem functions, stated and proved in [24] as Theorem 2 (for the quaternionic case see [22], Theorem 3).

Theorem 4 *Let $\Omega \subset \mathfrak{P}_n$ be a spectrally saturated set, and let $U = \mathfrak{S}(\Omega)$. The map*

$$\mathcal{S}(U, \mathfrak{K}_n) \ni F \mapsto F_\sigma \in \mathcal{F}(\Omega, \mathfrak{C}_n)$$

is $\mathbb{R}$-linear, injective, and has the property $(Ff)_\sigma = F_\sigma f_\sigma$ for all $F \in \mathcal{S}(U, \mathfrak{K}_n)$ and $f \in \mathcal{S}(U)$. Moreover, the restricted map

$$\mathcal{S}(U) \ni f \mapsto f_\sigma \in \mathcal{IF}(\Omega, \mathfrak{C}_n)$$

is unital and multiplicative.

Proof We give the proof of this result because of its simplicity.

The map $F \mapsto F_\sigma$ is clearly $\mathbb{R}$-linear. The injectivity of this map follows from Remark 7. Note also that

$$\begin{aligned} F_\sigma(\kappa) f_\sigma(\kappa) &= (F(s_+(\kappa))\iota_+(\mathfrak{s}_{\tilde{\kappa}}) + F(s_-(\kappa))\iota_-(\mathfrak{s}_{\tilde{\kappa}})) \\ &\quad \times (f(s_+(\kappa))\iota_+(\mathfrak{s}_{\tilde{\kappa}}) + f(s_-(\kappa))\iota_-(\mathfrak{s}_{\tilde{\kappa}}) \\ &= (Ff)(s_+(\kappa))\iota_+(\mathfrak{s}_{\tilde{\kappa}}) + (Ff)(s_-(\kappa))\iota_-(\mathfrak{s}_{\tilde{\kappa}}) = (Ff)_\sigma(\kappa), \end{aligned}$$

because f is complex valued, and using the properties of the idempotents $\iota_\pm(\mathfrak{s}_{\tilde{\kappa}})$ In particular, this computation shows that if $f, g \in \mathcal{S}(U)$, and so $f_\sigma, g_\sigma \in \mathcal{IF}(\Omega, \mathfrak{C}_n)$ by Corollary 2, we have $(fg)_\sigma = f_\sigma g_\sigma = g_\sigma f_\sigma$, thus the map $f \mapsto f_\sigma$ is multiplicative. It is also clearly unital. □

4.2 Slice Regular $\mathfrak{K}_n$- and $\mathbb{M}$- Valued Functions

We have dealt so far with arbitrary stem functions. We continue our discussion with stem functions having regularity properties. We first discuss the adaptation of the abstract concept of slice regularity, as introduced in Subsect. 2.3, to the case of $\mathfrak{K}_n$- and $\mathbb{M}$-valued functions.

Fixing a Clifford algebra $\mathfrak{C}_n$, the subspace $\mathfrak{P}_n \subset \mathfrak{C}_n$ of paravectors plays an important role in this context. We are particularly interested in $\mathfrak{C}_n$-valued functions defined on open subsets of $\mathfrak{P}_n$ (which is identified with $\mathbb{R}^{n+1}$).

For $\mathfrak{K}_n$-valued functions defined on subsets of $\mathfrak{P}_n$, the concept of *slice regularity* is defined as follows (see also [6, 24]).

As before, let $\mathbb{S}_n$ be the unit sphere of imaginary elements of $\mathfrak{P}_n$. It is clear that $\mathfrak{s}^* = -\mathfrak{s}$, $\mathfrak{s}^2 = -1$, $\mathfrak{s}^{-1} = -\mathfrak{s}$, and $|\mathfrak{s}| = 1$ for all $\mathfrak{s} \in \mathbb{S}_n$. Moreover, every nonnull paravector $\mathbf{a}$ can be written as $\mathbf{a} = \Re(\mathbf{a}) + |\mathbf{a}|\mathfrak{s}_\mathbf{a}$, with $\mathfrak{s}_\mathbf{a} = |\mathbf{a}|^{-1}\Im(\mathbf{a}) \in \mathbb{S}_n$.

Now, let $\Omega \subset \mathfrak{P}_n$ be an open set, and let $F : \Omega \mapsto \mathfrak{K}_n$ be a differentiable function. In the spirit of [6], we say that F is *right slice regular* on Ω if for all $\mathfrak{s} \in \mathbb{S}_n$,

$$\bar{\partial}_\mathfrak{s} F(x + y\mathfrak{s}) := \frac{1}{2}\left(\frac{\partial}{\partial x} + R_\mathfrak{s}\frac{\partial}{\partial y}\right) F(x + y\mathfrak{s}) = 0,$$

on the set $\Omega \cap (\mathbb{R} + \mathbb{R}\mathfrak{s})$, where $R_\mathfrak{s}$ is the right multiplication of the elements of $\mathfrak{C}_n$ by $\mathfrak{s}$.

Although we prefer the right slice regularity, a left slice regularity can also be defined via the left multiplication of the elements of $\mathfrak{K}_n$ by elements from $\mathbb{S}_n$. In what follows, the right slice regularity will be simply called *slice regularity*.

We are particularly interested in the slice regularity of $\mathfrak{C}_n$-valued functions, but the concept is valid for $\mathfrak{K}_n$-valued functions and plays an important role in our discussion. For $\mathbb{M}$ valued functions, similar results are valid (see Subsect. 2.3 from [22]).

Example 2 (1) The convergent series of the form $\sum_{m\ge0} \mathbf{a}_m \kappa^m$, on balls $\{\kappa \in \mathfrak{P}_n; |\kappa| < r\}$, with $r > 0$ and $\mathbf{a}_m \in \mathfrak{K}_n$ for all $m \ge 0$, are $\mathfrak{K}_n$-valued slice regular on their domain of definition. Of course, when $\mathbf{a}_m \in \mathfrak{C}_n$, such functions are $\mathfrak{C}_n$-valued slice regular on their domain of definition.

(2) We give an example slightly different from Example 1. As in Example 2 from [24], or Example 2 from [22], we use the following (see Definition 1 from [24] or Definition 1 from [22]).

Definition 4 The $\mathfrak{K}_n$-*valued Cauchy kernel* on the open set $\Omega \subset \mathfrak{P}_n$ is given by

$$\rho(\kappa) \times \Omega \ni (\zeta, \kappa) \mapsto (\zeta - \kappa)^{-1} \in \mathfrak{K}_n. \tag{13}$$

The $\mathfrak{K}_n$-*valued Cauchy kernel* on the open set $\Omega \subset \mathfrak{P}_n$ is slice regular. Specifically, choosing an arbitrary relatively open set $V \subset \Omega \cap (\mathbb{R} + \mathbb{R}\mathfrak{s})$, and fixing $\zeta \in \cap_{\kappa \in V} \rho(\kappa)$, we can write for $\kappa \in V$ the equalities

$$\frac{\partial}{\partial x}(\zeta - x - y\mathfrak{s})^{-1} = (\zeta - x - y\mathfrak{s})^{-2},$$
$$R_\mathfrak{s} \frac{\partial}{\partial y}(\zeta - x - y\mathfrak{s})^{-1} = -(\zeta - x - y\mathfrak{s})^{-2},$$

because $\mathfrak{s}^2 = -1$, and $\zeta, \mathfrak{s}$ and $(\zeta - x - y\mathfrak{s})^{-1}$ commute in $\mathfrak{K}_n$, implying the assertion.

4.3 A Cauchy Transformation in the Clifford or Hamilton Algebra Context

Having the $\mathfrak{K}_n$-valued Cauchy kernel (see Definition 4), we may introduce a concept of a Cauchy transform (as in [24], Definition 5; see also [22] in the quaternionic context), whose some useful properties will be recalled in this subsection.

For a given open set $U \subset \mathbb{C}$, we recall that $\mathcal{O}(U, \mathfrak{K}_n)$ is the complex algebra of all $\mathfrak{K}_n$-valued analytic functions on U, and if U is also conjugate symmetric, $\mathcal{O}_s(U, \mathfrak{K}_n)$ is the real subalgebra of $\mathcal{O}(U, \mathfrak{K}_n)$ consisting of all stem functions from $\mathcal{O}(U, \mathfrak{K}_n)$.

Because $\mathbb{C} \subset \mathfrak{K}_n$, we have $\mathcal{O}(U) \subset \mathcal{O}(U, \mathfrak{K}_n)$, where $\mathcal{O}(U)$ is the complex algebra of all complex-valued analytic functions on the open set U. Similarly, when $U \subset \mathbb{C}$ is also conjugate symmetric, $\mathcal{O}_s(U) \subset \mathcal{O}_s(U, \mathfrak{K}_n)$, where $\mathcal{O}_s(U)$ is the real subalgebra consisting of all functions f from $\mathcal{O}(U)$ which are stem functions.

Definition 5 Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $F \in \mathcal{O}(U, \mathfrak{K}_n)$. For every $\kappa \in U_\sigma$ we set

$$C[F](\kappa) = \frac{1}{2\pi i} \int_\Gamma F(\zeta)(\zeta - \kappa)^{-1} d\zeta, \tag{14}$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(\kappa)$. The function $C[F] : U_\sigma \mapsto \mathfrak{K}_n$ is called the ($\mathfrak{K}_n$*-valued*) *Cauchy transform* of the function $F \in \mathcal{O}(U, \mathfrak{K}_n)$. Clearly, the function $C[F]$ does not depend on the choice of Γ.

We shall put

$$\mathcal{R}(U_\sigma, \mathfrak{K}_n) = \{C[F]; F \in \mathcal{O}(U, \mathfrak{K}_n)\}. \tag{15}$$

Proposition 1 *Let $U \subset \mathbb{C}$ be open and conjugate symmetric, and let $F \in \mathcal{O}(U, \mathfrak{K}_n)$. Then function $C[F] \in \mathcal{R}(U_\sigma, \mathfrak{K}_n)$ is slice regular on U_σ.*

For the proof see Proposition 1 from [24].

Let $\Omega \subset \mathfrak{P}_n$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) \subset \mathbb{C}$, which is conjugate symmetric and also open. We introduce the notation

$$\mathcal{R}_{s,n}(\Omega) = \{C[f]; f \in \mathcal{O}_s(U)\},$$
$$\mathcal{R}_s(\Omega, \mathfrak{C}_n) = \{C[F]; F \in \mathcal{O}_s(U, \mathfrak{K}_n)\},$$

which are $\mathbb{R}$-vector spaces.

In fact, these $\mathbb{R}$-vector spaces have some important properties, as already noticed in a quaternionic version of the next theorem (see Theorem 5 in [22]).

Theorem 5 *Let $\Omega \subset \mathfrak{P}_n$ be a spectrally saturated open set, and let $U \subset \mathbb{C}$ be given by $U_\sigma = \Omega$. The space $\mathcal{R}_{s,n}(\Omega)$ is a unital commutative $\mathbb{R}$-algebra, the space $\mathcal{R}_s(\Omega, \mathfrak{C}_n)$ is a right $\mathcal{R}_{s,n}(\Omega)$-module, the linear map*

$$\mathcal{O}_s(U, \mathfrak{K}_n) \ni F \mapsto C[F] \in \mathcal{R}_s(\Omega, \mathfrak{C}_n)$$

is a right module isomorphism, and its restriction

$$\mathcal{O}_s(U) \ni f \mapsto C[f] \in \mathcal{R}_{s,n}(\Omega)$$

is an $\mathbb{R}$-algebra isomorphism.

Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^m a_n \zeta^n$, $\zeta \in \mathbb{C}$, with $a_n \in \mathfrak{C}_n$ for all $n = 0, 1, \ldots, m$, we have $P_\sigma(\kappa) = \sum_{n=0}^m a_n \kappa^n \in \mathfrak{C}_n$ for all $\kappa \in \mathfrak{P}_n$.

Remark 8 If $F \in \mathcal{O}_s(U, \mathfrak{K}_n)$, then $C[F]$ takes values actually in $\mathfrak{C}_n$. As a version of Definition 1, the map $F \mapsto C[F]$ is a *Cauchy transformation*, defined on $\mathcal{O}_s(U, \mathfrak{K}_n)$. Let us also remark that the element $C[F](\kappa)$ is equal to $F_\sigma(\kappa)$ for all $\kappa \in \Omega$, where tha latter is defined by formula (11). This follows from the proof of Theorem 3 from

[24], as noticed in Remark 9 from the same paper, showing the compatibiliry of those constructions.

It is also useful to mention that the space $\mathcal{R}_s(\Omega, \mathfrak{C}_n)$ coincides with the space of all slice regular functions on Ω, as given by Theorem 5 from [24].

Similar assertions are also valid in the quaternionic context. See for instance Theorem 6 from [22].

Similar results, valid in a quaternionic context, can be found in Sect. 5, from [22].

Example 3 This result shows that the non-commutative Cauchy kernel from [6] is given by the Cauchy transform of the complex Cauchy kernel associated to a paravector.

Let $\mathbf{s}, \mathbf{a} \in \mathfrak{P}_n$ with $\sigma(\mathbf{s}) \cap \sigma(\mathbf{a}) = \emptyset$, and so $\mathbf{s} \neq \mathbf{a}$. In particular, the paravector $\mathbf{s}^2 - 2\Re(\mathbf{a})\mathbf{s} + |\mathbf{a}|^2$ is invertible by Lemma 1, because if $\zeta = \Re\mathbf{a} + i|\Im\mathbf{a}| \in \sigma(\mathbf{a})$ then $\zeta \notin \sigma(\mathbf{s})$.

Let us consider the equality

$$S_R^{-1}(\mathbf{a}, \mathbf{s}) = -(\mathbf{s} - \mathbf{a}^*)\big(\mathbf{s}^2 - 2\Re(\mathbf{a})\mathbf{s} + |\mathbf{a}|^2\big)^{-1},$$

which is the *right noncommutative Cauchy kernel* (see [6], Definition 2.7.5 for the left version of this kernel).

Note also that the function $\rho(\mathbf{a}) \ni \zeta \mapsto (\zeta - \mathbf{a})^{-1} \in \mathfrak{K}_n$ is in the space $\mathcal{O}_s(\rho(\mathbf{a}), \mathfrak{K}_n)$, because $\overline{(\zeta - \mathbf{a})^{-1}} = (\bar{\zeta} - \mathbf{a})^{-1}$.

We can show the equality

$$-S_R^{-1}(\mathbf{a}, \mathbf{s}) = \frac{1}{2\pi i}\int_{\Gamma_\mathbf{s}} (\zeta - \mathbf{a})^{-1}(\zeta - \mathbf{s})^{-1} d\zeta,$$

where $\Gamma_\mathbf{s}$ surrounds a Cauchy domain containing $\sigma(\mathbf{s})$, whose closure is disjoint of $\sigma(\mathbf{a})$. Indeed,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_\mathbf{s}} (\zeta - \mathbf{a})^{-1}(\zeta - \mathbf{s})^{-1} d\zeta \\
&= \frac{1}{2\pi i}\int_{\Gamma_\mathbf{s}} [(\zeta - \mathbf{a})(\zeta - \mathbf{a}^*)]^{-1}(\zeta - \mathbf{a}^*)(\zeta - \mathbf{s})^{-1} d\zeta \\
&= \frac{1}{2\pi i}\int_{\Gamma_\mathbf{s}} (\zeta^2 - 2\zeta\Re(\mathbf{a}) + |\mathbf{a}|^2)^{-1}(\zeta - \mathbf{a}^*)(\zeta - \mathbf{s})^{-1} d\zeta \\
&= (\mathbf{s} - \mathbf{a}^*)(\mathbf{s}^2 - 2\Re(\mathbf{a})\mathbf{s} + |\mathbf{a}|^2)^{-1},
\end{aligned}$$

showing that the kernel $S_R^{-1}(\mathbf{s}, \mathbf{a})$ is the Cauchy transform of the function $\rho(\mathbf{a}) \ni \zeta \mapsto -(\zeta - \mathbf{a})^{-1} \in \mathfrak{K}_n$.

5 Clifford and Quaternionic Operators, and Their Spectrum

5.1 Clifford and Quaternionic Spaces, and Their Operators

As in [24] (see also [6]), by a *Clifford space* (or a *Cl-space*) we mean a two-sided module over a given Clifford algebra $\mathfrak{C}_n$. A Clifford space is, in particular, a real vector space, more precisely described in the following.

Let $\mathfrak{C}_n$ be a fixed Clifford algebra, and let $\mathcal{V}$ be a real vector space. The space $\mathcal{V}$ is said to be a *right Cl-space* if it is a right $\mathfrak{C}_n$- module, that is, there exists in $\mathcal{V}$ a right multiplication with the elements of $\mathfrak{C}_n$, such that $x1 = x, (x + y)\mathbf{a} = x\mathbf{a} + y\mathbf{a},\ x(\mathbf{a} + \mathbf{b}) = x\mathbf{a} + x\mathbf{b},\ x(\mathbf{ab}) = (x\mathbf{a})\mathbf{b}$ for all $x, y \in \mathcal{V}$ and $\mathbf{a}, \mathbf{b} \in \mathfrak{C}_n$.

Similarly, we may define a concept of *left Cl-space*, replacing the right multiplication of the vectors of $\mathcal{V}$ with the elements of $\mathfrak{C}_n$ by the left multiplication.

A vector space which is simultaneously left and right Cl-space will be simply called a *Cl-space*.

As before, if $\mathcal{V}$ is a real or complex Banach space, we denote by $\mathcal{B}(\mathcal{V})$ the algebra of all real or complex bounded linear operators, respectively. Let us recall that if $\mathcal{V}$ be a real Banach space, then $\mathcal{V}_{\mathbb{C}} = \mathcal{V} + i\mathcal{V}$ is its complexification, endowed with the norm $\|x + iy\| = \|x\| + \|y\|$, for all $x, y \in \mathcal{V}$, where $\| * \|$ is the norm of $\mathcal{V}$. We denote by C the *conjugation* on $\mathcal{V}_{\mathbb{C}}$, that is, the map $C(x + iy) = x - iy$ for all $x, y \in \mathcal{V}$, which is an $\mathbb{R}$-linear map whose square is the identity.

If $\mathcal{V}$ is a right Cl-space which is also a Banach space with the norm $\| * \|$, such that $\|x\mathbf{a}\| \leq K\|x\||\mathbf{a}|$ for all $x \in \mathcal{V}$ and $\mathbf{a} \in \mathfrak{C}_n$, where K is a positive constant, then $\mathcal{V}$ is said to be a *right Banch Cl-space*.

In a similar way, one defines the concept of a *left Banach Cl-space*.

A real Banach space $\mathcal{V}$ will be said to be a *Banach Cl-space* if it is simultaneously a right and a left Banach Cl-space.

Let $\mathcal{V}$ be a fixed Banach Cl-space. An operator $T \in \mathcal{B}{\Leftarrow}\mathcal{V}{\Rightarrow}$ is said to be *right Cl-linear* if $T(x\mathbf{a}) = T(x)\mathbf{a}$ for all $x \in \mathcal{V}$ and $\mathbf{a} \in \mathfrak{C}_n$. The set of right Cl-linear operators will be denoted by $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, which is, in particular, a unital real Banach algebra.

We shall denote by $R_{\mathbf{a}}$ (resp. $L_{\mathbf{a}}$) the right (resp. left) multiplication operator of the elements of $\mathcal{V}$ with the Cl-vector $\mathbf{a} \in \mathfrak{C}_n$. It is clear that $R_{\mathbf{a}}, L_{\mathbf{a}} \in \mathcal{B}(\mathcal{V})$ for all $\mathbf{a} \in \mathfrak{C}_n$. Note also that

$$\mathcal{B}^{\mathrm{r}}(V) = \{T \in \mathcal{B}(\mathcal{V});\ TR_{\mathbf{a}} = R_{\mathbf{a}}T,\ \mathbf{a} \in \mathfrak{C}_n\}.$$

The elements of the algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ will be sometimes called *right Clifford (or Cl-) operators*. As we work especially with such operators, the word ”right“ will be usually omitted. Note that all operators $L_{\mathbf{a}},\ \mathbf{a} \in \mathfrak{C}_n$, are Cl-operators.

Consider again the complexification $\mathcal{V}_{\mathbb{C}}$ of $\mathcal{V}$. Because $\mathcal{V}$ is a $\mathfrak{C}_n$-bimodule, the space $\mathcal{V}_{\mathbb{C}}$ is actually a two-sided $\mathfrak{K}_n$-module, via the multiplications

$$\begin{aligned}(\mathbf{a} + i\mathbf{b})(x + iy) &= \mathbf{a}x - \mathbf{b}y + i(\mathbf{a}y + \mathbf{b}x),\ (x + iy)(\mathbf{a} + i\mathbf{b})\\ &= x\mathbf{a} - y\mathbf{b} + i(y\mathbf{a} + x\mathbf{b}),\end{aligned}$$

for all $\mathbf{a}, \mathbf{b} \in \mathfrak{C}_n,\ x, y \in \mathcal{V}$.

As in Subsect. 2.2, for every $T \in \mathcal{B}(\mathcal{V})$, we consider its complex extension to $\mathcal{V}_{\mathbb{C}}$ given by $T_{\mathbb{C}}(x+iy) = Tx + iTy$, for all $x, y \in \mathcal{V}$, which is actually $\mathbb{C}$-linear, so $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$. Moreover, the map $\mathcal{B}(\mathcal{V}) \ni T \mapsto T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is a unital injective morphism of real algebras. Moreover, if $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the operator $T_{\mathbb{C}}$ is right $\mathfrak{K}_n$-linear, that is $T_{\mathbb{C}}((x+iy)(\mathbf{a}+i\mathbf{b})) = T_{\mathbb{C}}(x+iy)(\mathbf{a}+i\mathbf{b})$ for all $\mathbf{a}+i\mathbf{b} \in \mathfrak{K}_n,\ x+iy \in \mathcal{V}_{\mathbb{C}}$, via a direct computation.

The left and right multiplications with $\mathbf{a} \in \mathfrak{C}_n$ on $\mathcal{V}_{\mathbb{C}}$ will be still denoted by $L_{\mathbf{a}}, R_{\mathbf{a}}$, respectively, as elements of $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$. We set

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}) = \{S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}});\ SR_{\mathbf{a}} = R_{\mathbf{a}}S,\ \mathbf{a} \in \mathfrak{C}_n\},$$

which is a unital complex algebra, consisting of all right $\mathfrak{K}_n$-linear operators on $\mathcal{V}_{\mathbb{C}}$, containing all operators $L_{\mathbf{a}}, \mathbf{a} \in \mathfrak{C}_n$. It is easily seen that if $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, then $T_{\mathbb{C}} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$.

Assuming that $\mathcal{V}$ is a Banach Cl-space implies that $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is a unital real Banach Cl-algebra (that is, a Banach algebra which also a Banach Cl-space), via the algebraic operations $(\mathbf{a}T)(x) = \mathbf{a}T(x)$, and $(T\mathbf{a})(x) = T(\mathbf{a}x)$ for all $\mathbf{a} \in \mathfrak{C}_n$ and $x \in \mathcal{V}$. The complexification $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is, in particular, a unital complex Banach algebra, with the product $(T_1+iT_2)(S_1+iS_2) = T_1S_1 - T_2S_2 + i(T_1S_2 + T_2S_1),\ T_1, T_2, S_1, S_2 \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, and a fixed norm, say $\|(T_1+iT_2)\| = \|T_1\| + \|T_2\|, T_1, T_2 \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

Also note that the complex numbers, regarded as elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$, commute with the elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

Remark 9 As in Subsect. 2.2, for every $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ we put $S^{\flat} = CSC \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, and $S \mapsto S^{\flat}$ is a conjugate linear automorphism of the algebra $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$, whose square is the identity operator. In fact, the map $S \mapsto S^{\flat}$ is a conjugation of $\mathcal{B}(\mathcal{V})$, induced by C. Moreover, $S^{\flat} = S$ if and only if $S(\mathcal{V}) \subset \mathcal{V}$. In particular, we have $S = S_1 + iS_2$ with $S_j(\mathcal{V}) \subset \mathcal{V},\ j = 1, 2$, uniquely determined. Its action on the space $\mathcal{V}_{\mathbb{C}}$ is given by $S(x+iy) = S_1x - S_2y + i(S_1y + S_2x)$ for all $x, y \in \mathcal{V}$.

Because $CR_{\mathbf{a}} = R_{\mathbf{a}}C$ for all $\mathbf{a} \in \mathfrak{P}_n$, it follows that if $S \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, then $S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$. Moreover, we have $(S + S^{\flat})(\mathcal{V}) \subset \mathcal{V}, i(S - S^{\flat})(\mathcal{V}) \subset \mathcal{V}$, and $(T_{\mathbb{C}})^{\flat} = T_{\mathbb{C}}$ for all $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$. Note also that the map

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}) \ni T \mapsto T_{\mathbb{C}} \in \{S = S^{\flat};\ S \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})\}$$

is actually a real unital algebra isomorphism, since its surjectivity follows from the equality $(S|\mathcal{V})_{\mathbb{C}} = S$ whenever $S = S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$. This implies that the algebras $\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$ and $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ are isomorphic. This isomorphism is given by the assignment

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} \ni T_1 + iT_2 \mapsto T_{1\mathbb{C}} + iT_{2\mathbb{C}} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$$

which is is actually an algebra isomorphism, via a direct calculation.

The continuity of this assignment is also clear, and therefore it is a Banach algebra isomorphism. For this reason, may identify the algebras $\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$ and $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$. As already noticed above, the real algebras $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ and $\{S \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}});\ S = S^{\flat}\}$ may and will be also identified.

The operators from the algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ will be sometimes called *Clifford operators*, or simply *Cl-operators*.

The (complex) spectrum of an operator $T \in B^{\mathrm{r}}(\mathcal{V})$ is defined by $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.

Looking at Definition 3.1.4 from [6] (see also Definition 6 from [24], we give the folowing.

Definition 6 For a given operator $T \in B^{\mathrm{r}}(\mathcal{V})$, we have

$$\sigma_{Cl}(T) := \{\kappa \in \mathfrak{P}_n;\, T^2 - 2\Re(\kappa)T + |\kappa|^2) \text{ not invertible}\}$$

The set $\sigma_{Cl}(T)$ is call it the *Clifford* (or *Cl-*)*spectrum* of T. The complement $\rho_{Cl}(T) = \mathfrak{P}_n \setminus \sigma_{Cl}(T)$ is called the *Clifford* (or *Cl-*)*resolvent of* T.

Note that, if $\mathbf{a} \in \sigma_{Cl}(T)$, then $\{\mathbf{b} \in \mathfrak{P}_n;\, \sigma(\mathbf{b}) = \sigma(\mathbf{a})\} \subset \sigma_{Cl}(T)$. In other words; the subset $\sigma_{Cl}(T)$ spectrally saturated (see Definition 2(2)).

This concept is related to that of S-spectrum, defined in [6]. Definition 6 is given only for historical reasons. In fact, we mainly use the classical concept of complex spectrum. Indeed, since every operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is, in particular, $\mathbb{R}$-linear, we also have a *complex resolvent*, defined by

$$\begin{aligned}\rho_{\mathbb{C}}(T) &= \{\lambda \in \mathbb{C};\, (T^2 - 2\Re(\lambda)T + |\lambda|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})\\ &= \{\lambda \in \mathbb{C};\, (\lambda - T_{\mathbb{C}})^{-1} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})\} = \rho(T_{\mathbb{C}}),\end{aligned}$$

and the associated *complex spectrum* $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$ as well, which will be mainly used in the following.

Note that both sets $\sigma_{\mathbb{C}}(T)$ and $\rho_{\mathbb{C}}(T)$ are conjugate symmetric.

There exists a strong connexion between $\sigma_{Cl}(T)$ and $\sigma_{\mathbb{C}}(T)$. Specifically, we can prove the following (see Lamma 4 from [24]).

Lemma 4 *For every* $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ *we have the equalities*

$$\sigma_{Cl}(T) = \{\mathbf{a} \in \mathfrak{P}_n;\, \sigma_{\mathbb{C}}(T) \cap \sigma(\mathbf{a}) \neq \emptyset\}. \tag{16}$$

and

$$\sigma_{\mathbb{C}}(T) = \{\lambda \in \sigma(\mathbf{a});\, \mathbf{a} \in \sigma_{Cl}(T)\}. \tag{17}$$

Remark 10 (1) As expected, the set $\sigma_{Cl}(T)$ is nonempty and bounded, which follows from Lemma 4. In fact, we have the equality

$$\sigma_{Cl}(T) = \{\Re(\lambda) + |\Im(\lambda)|\mathfrak{s};\, \lambda \in \sigma_{\mathbb{C}}(T), \mathfrak{s} \in \mathbb{S}_n\}.$$

It is also closed, as a consequence of Definition 6, because the set of invertible elements in $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is open.

(2) For quaternionic operators we have similar definitions and assertions. A complete description of this case can be found in [25].

6 Analytic Functional Calculus for Clifford and Quaternionic Operators

Remark 11 If $\mathcal{V}$ is a Banach Cl-space, in particular a real Banach space, each operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$, and so one can use the classical Riesz-Dunford functional calculus, actually replacing the scalar-valued analytic functions by operator-valued analytic ones, which is a well known idea.

Specifically, if $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, then $T_{\mathbb{C}} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, and for later use, if $U \supset \sigma(T_{\mathbb{C}})$ is an open set in $\mathbb{C}$ and $F : U \mapsto B(\mathcal{V}_{\mathbb{C}})$ is analytic, the (left) Riesz-Dunford analytic functional calculus is given by the formula

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain Δ containing $\sigma(T_{\mathbb{C}})$ in U. Moreover, since $\sigma(T_{\mathbb{C}})$ is conjugate symmetric, we may and shall assume that both U and Γ are conjugate symmetric.

A natural question is to find an appropriate condition to have $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$, which implies the invariance of $\mathcal{V}$ under $F(T_{\mathbb{C}})$. This is given by the following (see also Theorem 6 from [24]).

Theorem 6 *Let $U \subset \mathbb{C}$ be open and conjugate symmetric. If $F \in \mathcal{O}_s(U, \mathcal{B}(\mathcal{V}_{\mathbb{C}}))$, we have $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{V})$ with $\sigma_{\mathbb{C}}(T) \subset U$.*

Moreover, if $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}))$, and $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, then $F(T_{\mathbb{C}}) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$.

This assertion is also a particular case of Theorem 1 from Subsect. 2.1, obtained for $\mathcal{A} = \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ (see also Theorem 6 from [25]).

The following result expresses the *(left) analytic functional calculus* of a given operator from $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$-valued stem functions, obtained as a particular case of Theorem 2 from Subsect. 2.1, when $\mathcal{A} = \mathcal{B}^{\mathrm{r}}(\mathcal{V})$. It can be found as Theorem 7 from [24], and it is a version of Theorem 4 from [25], proved in a quaternionic context.

When $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}))$, and $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, we set $F(T) := F(T_{\mathbb{C}})|\mathcal{V}$, where $\mathcal{V}$ is regarded as a real subspace of $\mathcal{V}_{\mathbb{C}}$, which is invariant under $F(T_{\mathbb{C}})$, as stated in Theorem 5.

Theorem 7 *Let $\mathcal{V}$ be a Banach Cl-space, let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, with $\sigma_{\mathbb{C}}(T) \subset U$. Then the assignment*

$$\mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is an $\mathbb{R}$-linear map, and the map

$$\mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is a unital real algebra morphism.

Moreover, the following properties hold true:

(1) for all $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathbf{r}}(\mathcal{V})_{\mathbb{C}})$*,* $f \in \mathcal{O}_s(U)$*, we have* $(Ff)(T) = F(T)f(T)$*.*
(2) for every polynomial $P(\zeta) = \sum_{n=0}^{m} A_n\zeta^n$*,* $\zeta \in \mathbb{C}$*, with* $A_n \in \mathcal{B}^{\mathbf{r}}(\mathcal{V})$ *for all* $n = 0, 1, \ldots, m$*, we have* $P(T) = \sum_{n=0}^{m} A_n T^n \in \mathcal{B}^{\mathbf{r}}(\mathcal{V})$*.*

Corollary 3 *Let* $\mathcal{V}$ *be a Banach* Cl*-space, let* $U \subset \mathbb{C}$ *be a conjugate symmetric open set, and let* $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$*, with* $\sigma_{\mathbb{C}}(T) \subset U$*. There exists an assignment*

$$\mathcal{O}_s(U, \mathfrak{K}_n) \ni F \mapsto F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}),$$

which is an $\mathbb{R}$*-linear map, such that*

(1) for all $F \in \mathcal{O}_s(U, \mathfrak{K}_n)$*,* $f \in \mathcal{O}_s(U)$*, we have* $(Ff)(T) = F(T)f(T)$*.*
(2) for every polynomial $P(\zeta) = \sum_{n=0}^{m} \mathbf{a}_n\zeta^n$*,* $\zeta \in \mathbb{C}$*, with* $\mathbf{a}_n \in \mathfrak{C}_n$ *for all* $n = 0, 1, \ldots, m$*, we have* $P(T) = \sum_{n=0}^{m} \mathbf{a}_n T^n \in \mathcal{B}^{\mathbf{r}}(\mathcal{V})$*.*

Because the algebra $\mathcal{O}_s(U, \mathfrak{K}_n)$ can be regarded as a subalgebra of the algebra $\mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}})$, whose elements are identified with left multiplication operators, this corollary is a direct consequence of Theorem 7. See also Corollary 3 from [24], and Theorem 5 from [25], stated and proved in the quaternionic context.

Remark 12 The space $\mathcal{R}_s(\Omega, \mathfrak{C}_n)$, introduced in Subsect. 4.3, can be independently defined, and it consists of the set of all $\mathfrak{C}_n$-valued functions, which are *slice monogenic* in the sense of [6], Definition 2.2.2 (or slice regular, as called in this work). They are used in [6] to define a functional calculus for tuples of not necessarily commuting real linear operators. Specifically, with a slightly modified notation, given an arbitrary family $(T_0, T_1, \ldots, T_n)$, acting on the real space $\mathcal{V}$, it is associated with the operator $\mathbf{T} = \sum_{j=0}^{n} T_j \otimes e_j$, acting on the two-sided $\mathfrak{C}_n$-module $\mathcal{V}_n = \mathcal{V} \otimes_{\mathbb{R}} \mathfrak{C}_n$. In fact, the symbol ”⊗” may (and will) be omitted. Moreover, as alluded in [6], page 83, we may work on a Banach Cl- space $\mathcal{V}$, and using operators from $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

Roughly speaking, after fixing a Clifford operator, each regular $\mathfrak{C}_n$-valued function defined in a neighborhood Ω of its Cl-spectrum is associated with another Clifford operator, replacing formally the paravector variable with that operator. This constraction is explained in Chapter 3 of [6].

For an operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the *right S-resolvent* is defined via the formula

$$S_R^{-1}(\mathbf{s}, T) = -(T - \mathbf{s}^*)(T^2 - 2\Re(\mathbf{s})T + |\mathbf{s}|)^{-1}, \ \mathbf{s} \in \rho_{Cl}(T) \tag{18}$$

(which is the right version of formula (3.5) from [6]; see also formula (4.47) from [6]). Fixing an element $\kappa \in \mathbb{S}_n$, and a spectrally saturated open set $\Omega \subset \mathfrak{P}_n$, for $\Phi \in \mathcal{R}_s(\Omega, \mathfrak{C}_n)$ one sets

$$\Phi(T) = \frac{1}{2\pi} \int_{\Sigma_\kappa} \Phi(\mathbf{s}) d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T), \tag{19}$$

where Σ_κ consists of a finite family of closed curves, piecewise smooth, positively oriented, being the boundary of the set $\Theta_\kappa = \{\mathbf{s} = u + v\kappa \in \Theta; u, v \in \mathbb{R}\}$, where $\Theta \subset \Omega$ is a spectrally saturated open set containing $\sigma_{Cl}(T)$, and $d\mathbf{s}_\kappa = -\kappa du \wedge dv$.

Formula (19) is a slight extension of the (right) functional calculus, as defined in [6], Theorem 3.3.2 (see also formula (4.54) from [6]).

Our Corollary 3 constructs, in particular, an analytic functional calculus with functions from $\mathcal{O}_s(U, \mathfrak{K}_n)$, where U is a neighborhood of the complex spectrum of a given Cliffordian operator, leading to another Clifford operator, replacing formally the complex variable with that operator. We can show that those functional calculi are equivalent. This is a consequence of the isomorphism of the spaces $\mathcal{O}_s(U, \mathfrak{K}_n)$ and $\mathcal{R}_s(U_\sigma, \mathfrak{C}_n)$, given by Theorem 5 (see also Remark 8).

Let us give a direct argument concerning the equivalence of those analytic functional calculi. Because the space $\mathcal{V}_{\mathbb{C}}$ is also a Cl-space, we may apply these formulas to the extended operator $T_{\mathbb{C}} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, replacing T by $T_{\mathbb{C}}$ in formulas (18) and (19). In fact, using the properties of the morphism $T \mapsto T_{\mathbb{C}}$ (see beginning of Section 7), we deduce that $S_R^{-1}(\mathbf{s}, T)_{\mathbb{C}} = S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})$.

For the function $\Phi \in \mathcal{R}_s(\Omega, \mathfrak{C}_n)$ there exists a function $F \in \mathcal{O}_s(\Omega, \mathfrak{K}_n)$ such that $F_\sigma = \Phi$, by Theorem 5. Denoting by Γ_κ the boundary of a Cauchy domain in $\mathbb{C}$ containing the compact set $\cup\{\sigma(\mathbf{s}); \mathbf{s} \in \overline{\Theta_\kappa}\}$, we can write

$$\begin{aligned}\Phi(T_{\mathbb{C}}) &= \frac{1}{2\pi}\int_{\Sigma_\kappa}\left(\frac{1}{2\pi i}\int_{\Gamma_\kappa} F(\zeta)(\zeta - \mathbf{s})^{-1} d\zeta\right) d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})\\ &= \frac{1}{2\pi i}\int_{\Gamma_\kappa} F(\zeta)\left(\frac{1}{2\pi}\int_{\Sigma_\kappa}(\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})\right) d\zeta.\end{aligned}$$

It follows from the complex linearity of $S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})$, and via an argument similar to that for getting formula (4.49) in [6], that

$$(\zeta - \mathbf{s}) S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})(\zeta - T_{\mathbb{C}}) - 1,$$

whence

$$(\zeta - \mathbf{s})^{-1} S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})(\zeta - T_{\mathbb{C}})^{-1} + (\zeta - \mathbf{s})^{-1}(\zeta - T_{\mathbb{C}})^{-1},$$

and therefore,

$$\begin{aligned}&\frac{1}{2\pi}\int_{\Sigma_\kappa}(\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = \frac{1}{2\pi}\int_{\Sigma_\kappa} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})(\zeta - T_{\mathbb{C}})^{-1}\\ &+\frac{1}{2\pi}\int_{\Sigma_\kappa}(\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa (\zeta - T_{\mathbb{C}})^{-1} = (\zeta - T_{\mathbb{C}})^{-1},\end{aligned}$$

because

$$\frac{1}{2\pi}\int_{\Sigma_\kappa} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = 1 \text{ and } \frac{1}{2\pi}\int_{\Sigma_\kappa} (\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa = 0,$$

as in Theorem 4.8.11 from [6], since the $\mathfrak{K}_n$-valued function $\mathbf{s} \mapsto (\zeta - \mathbf{s})^{-1}$ is analytic in a neighborhood of the set $\overline{\Theta_\kappa} \subset \mathbb{C}_\kappa$ for each $\zeta \in \Gamma_\kappa$, respectively. Therefore $\Phi(T_{\mathbb{C}}) = \Phi(T)_{\mathbb{C}} = F(T_{\mathbb{C}}) = F(T)_{\mathbb{C}}$, implying $\Phi(T) = F(T)$.

Conversely, choosing a function $F \in \mathcal{O}_s(\Omega, \mathfrak{K}_n)$, and denoting by $\Phi \in \mathcal{R}_s(\Omega, \mathfrak{C}_n)$ its Cauchy transform, the previous computation in reverse order shows that $\Phi(T) = F(T)$. Consequently, for a fixed $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the maps $\Theta : \mathcal{R}_s(\Omega, \mathfrak{C}_n) \mapsto \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, with $\Theta(\Phi) = \Phi(T)$, and $\Psi : \mathcal{O}_s(\Omega, \mathfrak{K}_n) \mapsto \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, with $\Psi(F) = F(T)$, we must have the equality $\Psi = \Theta \circ C[*]$, where $C[*]$ is the Cauchy transform.

Remark 13 Unlike in [6, 7], our approach permits to obtain a version of the *spectral mapping theorem* in a classical stile, via direct arguments. Recalling that $\mathcal{R}_{s,n}(\Omega)$ is the subalgebra of $\mathcal{R}_s(\Omega, \mathfrak{C}_n)$ whose elements are also in $\mathcal{IF}(\Omega, \mathfrak{C}_n)$ (see Theorem 5), for every operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ and every function $\Phi \in \mathcal{R}_{s,n}(\Omega)$ one has $\sigma_{Cl}(\Phi(T)) = \Phi(\sigma_{Cl}(T))$, via Theorem 3.5.9 from [6]. Using our approach, for every function $f \in \mathcal{O}_s(U)$, one has $f(\sigma_{\mathbb{C}}(T)) = \sigma_{\mathbb{C}}(f(T))$, directlly from the corresponding (classical) spectral mapping theorem in [8]. This result is parallel to that from [6] mentioned above, also giving an explanation for the former, via the isomorphism of the spaces $\mathcal{O}_s(U)$ and $\mathcal{R}_{s,n}(\Omega)$

Data Availibility All data generated and analysed during this study are included in this published article.

References

1. Baskakov, A.G., Zagorskii, A.S.: Spectral theory of linear relations on real Banach spaces. Math. Notes (Russian: Matematicheskie Zametki) **81**(1), 15–27 (2007)
2. Birkhoff, G., von Neumann, J.: The Logic of quantum machanics. Ann. Math. **37**, 823 (1936)
3. Clifford, W.K.: Preliminary sketch of bi-quaternions. Proc. Lond. Math. Soc. **1**(381–395), s1-4 (1871)
4. Colombo, F., Gentili, G., Sabadini, I.D.: Struppa, extension results for slice regular functions of a quaternionic variable. Adv. Math. **222**, 1793–1808 (2009)
5. Colombo, F., Sabadini, I., Struppa, D.C.: Slice monogenic functions. Israel J. Math. **171**, 385–403 (2009). https://doi.org/10.1007/s11856-009-0055-4
6. Colombo, F., Sabadini, I., Struppa, D.C.: Noncommutative functional calculus, theory and applications of slice hyperholomorphic functions, progress in mathematics, vol. 28. Birkhäuser/Springer Basel AG, Basel (2011)
7. Colombo, F., Gantner, J., Kimsey, D. P.: Spectral Theory on the S-spectrum for Quaternionic Operators, Birkhäuser, (2018)
8. Dunford, N., N., Schwartz, J. T.: Linear Operators, Part I: General Theory, Interscience Publishers, New York, London,: Part III: Spectral Operators, p. 1971. Wiley Interscience, New York, London, Sydney, Toronto (1958)
9. Finkelstein, D., Jauch, J.M., Schiminovich, S., Speiser, D.: Foundations of quaternion quantum mechanics. J. Math. Phys. **3**(207), 207–220 (1962)
10. Fueter, R.: Über eine Hartogs'schen Satz, Comment. Math. Helv., 12: 75-80, (1939/40)
11. Gentili, G., Stoppato, C., Struppa, D.C.: Regular functions of a quaternionic variable. Springer Monographs in Mathematics, Springer/Heidelberg (2013)

12. Gentili, G., Struppa, D.C.: A new theory of regular functions of a quaternionic variable. Adv. Math. **216**, 279–301 (2007)
13. Ghiloni, R., Moretti, V., Perotti, A.: Continuous slice functional calculus in quaternionic Hilbert spaces. Rev. Math. Phys. **25**, 4 (2013)
14. Ghiloni, R., Perotti, A.: Slice regular functions on real alternative algebras. Adv. Math. **226**, 1662–1691 (2011)
15. Ingelstam, L.: Real Banach algebras. Ark. Mat. **5**, 239–270 (1964)
16. Jefferies, B.: Spectral properties of noncommuting operators. Lecture Notes in Mathematics, vol. 1843. Springer-Verlag, Berlin (2004)
17. Kaplansky, I.: Normed algebras. Duke. Math. J. **16**, 399–418 (1949)
18. Moisil, G., Theodorescu, N.: Fonctions holomorphes dans l'espace. Mathematica (Cluj) **5**, 142–159 (1931)
19. Palmer, T.W.: Real C^*-algebras. Pacific J. Math. **35**(1), 195–204 (1970)
20. Porteous, I.R.: Topological geometry. Van Nostrand Reinhold Company, New York-Toronto-Melbourne (1969)
21. Rosenberg, J.: Structure and applications of real C^*-algebras. Contemp. Math. **671**, 235 (2016)
22. Vasilescu, F.-H.: Quaternionic regularity via analytic functional calculus. Integral Equ. Oper. Theory **92**, 18 (2020). https://doi.org/10.1007/s00020-020-2574-7
23. Vasilescu, F.-H.: Analytic functional calculus in quaternionic framework. Math. Rep. **23**(33), 1–2 (2021)
24. Vasilescu, F.-H.: Spectrum and analytic functional calculus for clifford operators via stem functions. Concr. Oper. **8**, 90–113 (2021)
25. Vasilescu, F.-H.: Spectrum and analytic functional calculus in real and quaternionic frameworks. Pure Appl. Funct. Anal. **7**(1), 389–407 (2022)