Strong Edge Coloring of Subquartic Graphs

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Abstract. A strong k-edge coloring of a graph G is a mapping $c : E(G) \to \{1, 2, 3, ..., k\}$ such that for any two edges e and e' with distance at most two, $c(e) \neq c(e')$. The strong chromatic index of G, written $\chi'_s(G)$, is the smallest integer k such that G has a strong k-edge coloring. In this paper, using color exchange method and discharging method, we prove that for a subquartic graph G, $\chi'_s(G) \leq 11$ if $mad(G) < \frac{8}{3}$, where $mad(G) = \max\{\frac{2|E(G)|}{|V(G)|}, H \subseteq G\}$.

Keywords: subquartic graph · strong edge coloring · maximum average degree

1 Introduction

To solve the Channel Assignment Problem in wireless communication networks, Fouquet and Jolivet [8] first introduced the notion of strong edge coloring in 1983. A strong k-edge coloring of a graph G is a mapping $c : E(G) \to \{1, 2, 3, \cdots, k\}$ such that $c(e) \neq c(e')$ for any two edges e and e' with distance at most two. The smallest integer k such that G has a strong k-edge coloring of G is called the strong chromatic index of G, written $\chi'_s(G)$. By greedy algorithm, it is easy to see that $2\Delta^2 - 2\Delta + 1$ is a trivial upper bound on $\chi'_s(G)$, where Δ is the maximum degree of G. However, it is NP-complete to decide wether $\chi'_s(G) = k$ holds for a general graph G [14]. In 1989, Erdős and Nešetřil [7] proposed the following important conjecture while studying the strong edge coloring of graphs.

Conjecture 1. [7] For any graph G with maximum degree Δ, $\chi'_s(G) \leq \frac{5}{4}\Delta^2$ if Δ is even, $\chi'_s(G) \leq \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$ if Δ is odd.

In [7], Erdős and Nešetřil constructed two classes of graphs satisfying $\chi'_s(G) = \chi'(G) = |E(G)|$ while $|E(G)|$ attains the upper bound in Conjecture 1. This illustrate that the upper bound is sharp if Conjecture 1 is true. Also, they asked a question: For a general graph G, is there any positive number ε such that $\chi'_s(G) \leq (2-\epsilon)\Delta^2$, where Δ is the maximum degree of G. As yet, there are many research results on strong edge coloring. For a graph G with sufficient large Δ, Molloy

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and Reed [15] proved that $\chi'_s(G) \le 1.998\Delta^2$ using probabilistic methods. In the next decides, this result was improved to $1.93\Delta^2$ by Bruhn and Joos [4], $1.835\Delta^2$ by Bonamy, Perrett and Postle [3]. For graphs with small Δ, scholars also made a lot of research works. It is an obvious result that $\chi'_s(G) \le 5 = \frac{5}{4}\Delta^2$ while $\Delta = 2$. For subcubic graphs, the above conjecture was verified by Andersen [1], and independently by Horák, Qing, Trotter [10]. For subquartic graphs, $\chi'_s(G) \le 22$ was proven by Cranston [6] using algorithms. Huang, Santana and Yu [11] reduced 22 to 21. For graphs with $\Delta = 5$, Zang [18] confirmed that $\chi'_s(G) \le 37$.

For graphs with maximum average degree restriction, there are also a mount of results. The maximum average degree of a graph G, written $mad(G)$, is the largest average degree of its subgraph. In other words, $mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$. In 2013, Hocquard [9] studied the strong chromatic index of subcubic graphs with maximum average degree and obtained the following theorem.

Theorem 1. *[9] Let G be a graph with $\Delta(G) = 3$.*

(1) If $mad(G) < \frac{7}{3}$, then $\chi'_s(G) \le 6$;
(2) If $mad(G) < \frac{5}{2}$, then $\chi'_s(G) \le 7$;
(3) If $mad(G) < \frac{8}{3}$, then $\chi'_s(G) \le 8$;
(4) If $mad(G) < \frac{20}{7}$, then $\chi'_s(G) \le 9$.

The given upper bound on $mad(G)$ in Theorem 1(1)(2)(4) is optimal since there exist subcubic graphs with $mad(G) = \frac{7}{3}$ (or $mad(G) = \frac{5}{2}, \frac{20}{7}$) and $\chi'_s(G) > 6$ (or $\chi'_s(G) > 7, 9$), see Fig. 1.

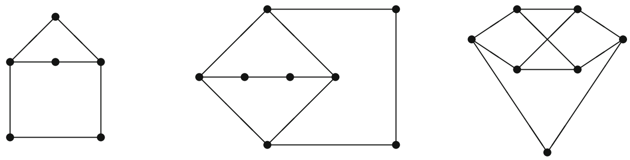

Fig. 1. $mad(G) = \frac{7}{3}$ (or $\frac{5}{2}, \frac{20}{7}$) and $\chi'_s(G) = 7$ (or $\chi'_s(G) = 8, 10$)

For subquartic graphs with bounded maximum average degree, Lv et al. [13] gave out the following theorem, which improved the corresponding upper bound on $mad(G)$ due to Bensmail et al. [2].

Theorem 2. *[13] Let G be a graph with $\Delta(G) = 4$.*

(1) If $mad(G) < \frac{61}{18}$, then $\chi'_s(G) \le 16$;
(2) If $mad(G) < \frac{7}{2}$, then $\chi'_s(G) \le 17$;
(3) If $mad(G) < \frac{18}{5}$, then $\chi'_s(G) \le 18$;
(4) If $mad(G) < \frac{26}{7}$, then $\chi'_s(G) \le 19$;
(5) If $mad(G) < \frac{51}{13}$, then $\chi'_s(G) \le 20$.

Ruksasakchai and Wang [17] studied the strong edge coloring of graphs with $\Delta(G) \le 4$ and $mad(G) < 3$ and obtained the following theorem.

Theorem 3. *[17] If G is a graphs G with maximum degree $\Delta \leq 4$ and $mad(G) < 3$, then $\chi'_s(G) \leq 3\Delta + 1$.*

For graphs with maximum degree 5 and bounded maximum average degree, Qin et al. [16] obtained the following theorem.

Theorem 4. *[16] Let G be a graph with $\Delta(G) = 5$.*

(1) If $mad(G) < \frac{8}{3}$, then $\chi'_s(G) \leq 13$;
(2) If $mad(G) < \frac{14}{5}$, then $\chi'_s(G) \leq 14$.

Additionally, Choi et al. [5] studied the strong edge coloring of graphs with maximum degree $\Delta \geq 7$ and bounded maximum average degree. They obtained a theorem as follows.

Theorem 5. *[5] Let G be a graph with maximum degree Δ.*

(1) If $\Delta \geq 9$ and $mad(G) < \frac{8}{3}$, then $\chi'_s(G) \leq 3\Delta - 3$;
(2) If $\Delta \geq 7$ and $mad(G) < 3$, then $\chi'_s(G) \leq 3\Delta$.

Recently, Li et al. [12] studied the strong edge coloring of graphs with maximum degree $\Delta \geq 6$ and bounded maximum average degree. The following theorem is given in [12].

Theorem 6. *[12] Let G be a graph with maximum degree Δ.*

(1) If $\Delta \geq 6$ and $mad(G) < \frac{23}{8}$, then $\chi'_s(G) \leq 3\Delta - 1$;
(2) If $\Delta \geq 7$ and $mad(G) < \frac{26}{9}$, then $\chi'_s(G) \leq 3\Delta - 1$.

In this paper, we further consider the strong edge coloring of subquartic graphs by using color exchange method and discharging method. We obtained the following theorem.

Theorem 7. *If G is a graph with $\Delta(G) = 4$ and $mad(G) < \frac{8}{3}$, then $\chi'_s(G) \leq 11$.*

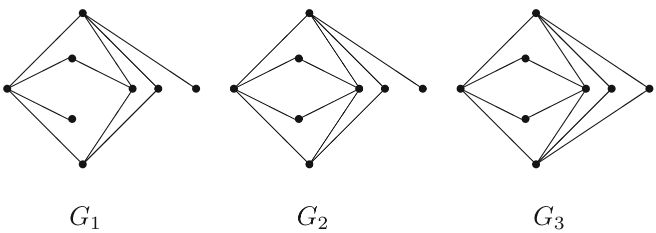

Fig. 2. Subquartic graphs.

G_1, G_2, G_3 in Fig. 2 are subquartic graphs, where $mad(G_1) = \frac{8}{3}$, $\chi'_s(G_1) = 10$; $mad(G_2) = \frac{20}{7}$, $\chi'_s(G_2) = 11$ and $mad(G_3) = 3$, $\chi'_s(G_3) = 12$ (we can take the graph obtained from G_1 by deleting two 1-vertices, G_2 by deleting the 1-vertex and G_3 as subgraphs, respectively). We do not know whether the upper bound $mad(G) < \frac{8}{3}$ in Theorem 7 is optimal. However, due to the graph G_3 in Fig. 2, we know that there exists a graph G with $\Delta(G) = 4$, $mad(G) = 3$ and $\chi'_s(G) = 12$.

For the strong edge coloring of subquartic graphs, Theorem 2 gives out some sufficient conditions for $\chi'_s(G) \leq 16$ (respectively 17,18,19,20). Theorem 3 indicates that any graph G with $\Delta(G) = 4$ and $mad(G) < 3$ satisfies $\chi'_s(G) \leq 13$. Therefore, Theorem 7 enriches the results of strong edge coloring for subquartic graphs.

2 Notations

All graphs considered here are finite undirected simple graphs. For a graph G, $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ denote its vertex set, edge set, maximum degree and minimum degree respectively. For $v \in V(G)$, $d_G(v)$ (abbreviated by $d(v)$) denotes the degree of v in G. v is a i (or i^+, i^-)-vertex if $d(v) = i$ (or $d(v) \geq i$, $d(v) \leq i$). For a vertex v, a i-neighbor of v is a i-vertex in $N(v)$. A i_j-vertex is a i-vertex adjacent to exactly j 2-vertices. A 2-vertex is bad if it is adjacent to a 2-vertex, semi-bad if it is adjacent to a 3_2-vertex. A 2-vertex is good if it is neither bad nor semi-bad. For an edge e, $F(e)$ denotes the set of forbidden colors for it.

3 Proof of Theorem 7

Suppose G is a counterexample with minimum 2^+-vertices and then with minimum edges. Let H be the graph obtained from G by deleting all 1-vertices. Obviously, $H \subseteq G$ and then $mad(H) \leq mad(G) < \frac{8}{3}$. In the following, we first illustrate some properties of H.

Lemma 1. *H does not have vertices of degree 1.*

Proof. Suppose v is a 1-vertex in H and $uv \in E(H)$. Since H is the graph obtained from obtained from G by deleting all 1-vertices, $d_G(v) > 1$ and v has at least one 1-neighbor v_1 in G. Compared with G, $G - v_1$ has the same 2^+-vertices but fewer edges. By the minimality of G, $\chi'_s(G - v_1) \leq 11$. Note that in G, $|F(vv_1)| \leq 6$. Thus, vv_1 can be colored, which leads to a contradiction.

Lemma 2. *If $d_H(v) = 2$, then $d_G(v) = 2$.*

Proof. Suppose $d_G(v) > 2$. Then, v has at least one 1-neighbor v_1 in G. Compared with G, $G - v_1$ has fewer edges while the same 2^+-vertices. By the minimality of G, $\chi'_s(G - v_1) \leq 11$. Note that in G, $|F(vv_1)| \leq 9$. Thus, vv_1 can be colored, which leads to a contradiction.

Lemma 3. *If v is a 3_i-vertex in H, where $i \geq 1$, then $d_G(v) = 3$.*

Proof. Suppose $d_G(v) > 3$. Then, v has at least one 1-neighbor v' in G. Let v_1 be a 2-neighbor of v in H, By Lemma 2, $d_G(v_1) = 2$. Let $G' = G - v'$. Compared with G, G' has the same 2^+-vertices but fewer edges. By the minimality of G, $\chi'_s(G - v_1) \leq 11$. Note that in G, $|F(vv')| \leq 10$. Thus, vv' can be colored, which leads to a contradiction.

Lemma 4. *Every bad vertex in H is adjacent to a 4-vertex.*

Proof. Suppose v is a bad vertex in H and it is adjacent to a 2-vertex u and a 3^--vertex w. By Lemma 2, $d_G(u) = d_G(v) = 2$. Denote $N_G(u) = \{u_1, v\}$. Note that $2 \leq d_H(w) \leq 3$. If $d_H(w) = 2$, then $d_G(w) = 2$ by Lemma 2. If $d_H(w) = 3$, then by Lemma 3, $d_G(w) = 3$ since $d_H(v) = 2$. Let $G' = G - uv + ww_1$, where ww_1 is a pendent edge incident with w. Note that $3 \leq d_{G'}(w) \leq 4$ and G' has fewer 2^+-vertices than G. By the definition of maximum average degree, $mad(G') < 2$ if $mad(G) < 2$ and $mad(G') \leq mad(G) < \frac{8}{3}$ if $2 \leq mad(G) < \frac{8}{3}$. By the minimality of G, $\chi'_s(G') \leq 11$. Let c be a strong 11-edge coloring of G'. Note that in G, $|F(uv)| \leq 8$. If $c(uu_1) \neq c(vw)$, then uv can be colored, which is a contradiction. If $c(uu_1) = c(vw)$, then we first exchange the colors on pendant edges wv and ww_1 in G'. After that, uv can be colored, which leads to a contradiction.

Lemma 5. *H does not have 3_3-vertices.*

Proof. Suppose v is a 3_3-vertex in H and $N_H(v) = \{v_1, v_2, v_3\}$. By Lemma 2, $d_G(v_i) = 2$, $i = 1, 2, 3$. By Lemma 3, $d_G(v) = 3$. Let $G' = G - v$. Note that G' has fewer 2^+-vertices than G. By the minimality of G, $\chi'_s(G') \leq 11$. Note that in G, $|F(vv_i)| \leq 6$, $i = 1, 2, 3$, vv_i can be colored, which is a contradiction.

Lemma 6. *Every semi-bad vertex in H is adjacent to a 4-vertex.*

Proof. Suppose v is a semi-bad vertex in H and it is adjacent to a 3_2-vertex u and a 3^--vertex w. Let $N_H(u) = \{u_1, u_2, v\}$, where $d_H(u_1) = 2$ (see Fig. 3). By Lemma 2, $d_G(u_1) = d_G(v) = 2$. By Lemma 3, $d_G(u) = 3$. Note that $2 \leq d_H(w) \leq 3$ and $d_H(v) = 2$, we have $d_G(w) = d_H(w)$ by Lemma 2 and Lemma 3. Let $G' = G - uv + ww_1$, where ww_1 is a pendant edge incident with w. Note that G' has fewer 2^+-vertices than G, by the definition of maximum average degree, $mad(G') < 2$ if $mad(G) < 2$ and $mad(G') \leq mad(G) < \frac{8}{3}$ if $2 \leq mad(G) < \frac{8}{3}$. By

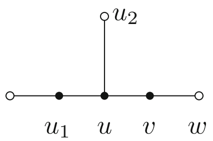

Fig. 3. .

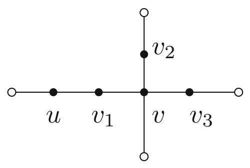

Fig. 4. .

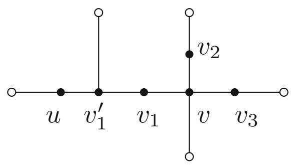

Fig. 5. .

the minimality of G, $\chi'_s(G') \leq 11$. Let c be a strong 11-edge coloring of G'. Erase on color on uu_1. Note that in G, $|F(uu_1)| \leq 9$, $|F(uv)| \leq 9$. If $c(uu_2) \neq c(vw)$, then uu_1, uv can be colored, which is a contradiction. If $c(uu_2) = c(vw)$, then we first exchange the colors on pendant edges wv and ww_1 in G'. After that, uu_1 and uv can be colored, which leads to a contradiction.

Lemma 7. *Let v be a 4_i-vertex in H, where $i \geq 3$. Then its 2-neighbors are all good vertices.*

Proof. Suppose that v_1, v_2, v_3 are 2-neighbors of v and at least one of them is not good. Without loss of generality, we assume that v_1 is not a good vertex. This implies that v_1 is adjacent to a 2-vertex or a 3_2-vertex.

If v_1 is adjacent to a 2-vertex u (see Fig. 4), then by Lemma 2, $d_G(v_i) = d_G(u) = 2$, $i = 1, 2, 3$. Let $G' = G - v_1$. Note that G' has fewer 2^+-vertices than G. By the minimality of G, $\chi'_s(G') \leq 11$. Note that in G, $|F(uv_1)| \leq 7$, $|F(vv_1)| \leq 9$, uv_1, vv_1 can be colored, which is a contradiction.

If v_1 is adjacent to a 3_2-vertex v'_1 and $u \neq v_1$ is the other 2-neighbor of v'_1 (see Fig. 5). By Lemma 2, $d_G(v_i) = d_G(u) = 2$, $i = 1, 2, 3$. By Lemma 3, $d_G(v'_1) = 3$. Let $G' = G - v_1$. Note that $G - v_1$ has fewer 2^+-vertices than G. By the minimality of G, $\chi'_s(G') \leq 11$. Note that in G, $|F(v_1v'_1)| \leq 9$, $|F(vv_1)| \leq 10$. Thus, $vv_1, v_1v'_1$ can be colored in order, which is a contradiction.

Proof of Theorem 7: We define weight function $w(v) = d(v)$ for each $v \in V(H)$ and we define five discharging rules R1-R5 as follows. Let $w'(v)$ be the final weight function while discharging finished. As we know, the sum weigh is fixed. However, we shall prove that $w'(v) \geq \frac{8}{3}$ for each $v \in V(H)$. This will lead to a contradiction as follow.

$$\frac{8}{3}|V(H)| \leq \sum_{v \in V(H)} w'(v) = \sum_{v \in V(H)} w(v) \leq mad(H)|V(H)| < \frac{8}{3}|V(H)|.$$

Discharging Rules:

R1 Each 4-vertex gives $\frac{2}{3}$ to each adjacent bad vertex.
R2 Each 4-vertex gives $\frac{1}{2}$ to each adjacent semi-bad vertex.
R3 Each 4-vertex gives $\frac{1}{3}$ to each adjacent good vertex.
R4 Each 3_2-vertex gives $\frac{1}{6}$ to each adjacent semi-bad vertex.
R5 Each 3_1-vertex gives $\frac{1}{3}$ to each adjacent good vertex.

In the following, we shall verify that $w'(v) \geq \frac{8}{3}$ for each $v \in V(H)$.
By Lemma 1, $\delta(H) \geq 2$.

- $d(v) = 2$

If v is bad, then by Lemma 4, v is adjacent to a 4-vertex. By R1, $w'(v) = 2 + \frac{2}{3} = \frac{8}{3}$.

If v is semi-bad, then by Lemma 6, v is adjacent to a 4-vertex. By R2 and R4, $w'(v) = 2 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}$.

If v is good, then by the definition of good vertex and Lemma 5, each neighbor of v is either 3_1-vertex or 4-vertex. By R3 and R5, $w'(v) = 2 + \frac{1}{3} \times 2 = \frac{8}{3}$.

- $d(v) = 3$

  By Lemma 5, v is 3_i-vertex, where $0 \leq i \leq 2$.
  If v is a 3_2-vertex, then by R4, $w'(v) \geq 3 - \frac{1}{6} \times 2 = \frac{8}{3}$.
  If v is a 3_1-vertex, then by R5, $w'(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$.
  If v is a 3_0-vertex, then $w'(v) = w(v) = 3$.

- $d(v) = 4$

If v is a 4_i-vertex, where $i \geq 3$, then by Lemma 7, the 2-neighbors of v are good. Thus, $w'(v) \geq 4 - \frac{1}{3} \times 4 = \frac{8}{3}$ by R3.

If v is a 4_i-vertex, where $0 \leq i \leq 2$, then by R1-R3, $w'(v) \geq 4 - \frac{2}{3} \times 2 = \frac{8}{3}$.

Therefore, for each $v \in V(H)$, $w'(v) \geq \frac{8}{3}$ and the proof of Theorem 7 is finished. □

4 Further Considered Problems

Theorem 7 illustrates that $\chi'_s(G) \leq 11$ holds for any graph G with $\Delta(G) = 4$ and $mad(G) < \frac{8}{3}$. For the graph G_3 in Fig. 2, it satisfies that $\Delta(G_3) = 4$, $mad(G_3) = 3$ and $\chi'_s(G_3) = 12$. Then a question follows out naturally. What is the supremum M such that any graph G with $\Delta(G) = 4$ and $mad(G) < M$ satisfying $\chi'_s(G) \leq 11$?

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Declaration of Competing Interest. We declare that we have no conflicts of interest to this work. We also declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

References

1. Andersen, L.D.: The strong chromatic index of a cubic graph is at most 10. Discrete Math. **108**(1–3), 231–252 (1992)
2. Bensmail, J., Bonamy, M., Hocquard, H.: Strong edge coloring sparse graphs. Electron. Note Discrete Math. **49**, 773–778 (2015)
3. Bonamy, M., Perrett, T., Postle, L.: Colouring graphs with sparse neighbourhoods: bounds and applications. J. Combin. Theory Ser. B **155**, 278–317 (2022)
4. Bruhn, H., Joos, F.: A strong bound for the strong chromatic index. Combin. Probab. Comput. **27**(1), 21–43 (2018)
5. Choi, I., Kim, J., Kostochka, A.V., Raspaud, A.: Strong edge-colorings of sparse graphs with large maximum degree. European J. Combin. **67**, 21–39 (2018)
6. Cranston, D.W.: A strong bound edge-colouring of graphs with maximum degree 4 using 22 colours. Discrete Math. **306**, 2772–2778 (2006)
7. Erdős, P., Nešetřil, J., Halász, G.: Irregularities of Partitions, pp. 161–349. Springer, Berlin (1989). https://doi.org/10.1007/978-3-642-61324-1

8. Fouquet, J.L., Jolivet, J.L.: Strong edge-colorings of graphs and applications to multi-k-gons. Ars Combin. **16**, 141–150 (1983)
9. Hocquard, H., Montassier, M., Raspaud, A., Valicov, P.: On strong edge-colouring of subcubic graphs. Discrete Appl. Math. **161**(16–17), 2467–2479 (2013)
10. Horák, P., Qing, H., Trotter, W.T.: Induced matching in cubic graphs. J. Graph Theory **17**(2), 151–160 (1993)
11. Huang, M.F., Santana, M., Yu, G.X.: Strong chromatic index of graphs with maximum degree four. Electron. J. Combin. **25**(3), 3–31 (2018)
12. Li, X.W., Li, Y.F., Lv, J.B., Wang, T.: Strong edge-colorings of sparse graphs with $3\varDelta - 1$ colors. Inform. Process. Lett. **179**, 106313 (2023)
13. Lv, J.B., Li, X.W., Yu, G.X.: On strong edge-coloring of graphs with maximum degree 4. Discrete Appl. Math. **235**, 142–153 (2018)
14. Mahdian, M.: On the computational complexity of strong edge-coloring. Discrete Appl. Math. **118**(3), 239–248 (2002)
15. Molloy, M., Reed, B.: A bound on the strong chromatic index of a graph. J. Combin. Theory Ser. B **69**(2), 103–109 (1997)
16. Qin, L.Z., Lv, J.B., Li, J.X.: Strong edge-coloring of some sparse graphs. Adv. Math. (China) **51**(1), 41–52 (2022)
17. Ruksasakchai, W., Wang, T.: List strong edge coloring of some classes of graphs. Australas. J. Combin. **68**, 106–117 (2017)
18. Zang C.Y.: The strong chromatic index of graphs with maximum degree $\varDelta$. arXiV1510.00785vl (2015)