When Advertising Meets Assortment Planning: Joint Advertising and Assortment Optimization Under Multinomial Logit Model

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Abstract. While the topic of assortment optimization has received a significant amount of attention, the relationship between advertising and its impact on this issue has not been well-explored. This paper aims to fill the gap in research by addressing the joint advertising and assortment optimization problem. We propose that advertising can influence product selection by increasing preference for certain products, and the extent of this effect is determined by the product-specific effectiveness of advertising and the resources allocated to advertising for that product. Our goal is to find an optimal solution, which comprises of a combination of advertising strategy and product assortment, that maximizes revenue, taking into account budget constraints on advertising. In this paper, we examine the characteristics of this problem and present efficient methods to solve it under various scenarios. Both the unconstraint and cardinality constraint settings are studied and the joint assortment, pricing, and advertising problem is also examined. We further extend our findings to account for consumer decision-making patterns.

Keywords: budget allocation · advertising effect · assortment optimization

1 Introduction

One of the major challenges faced by both online and offline retailers is the problem of assortment optimization, in which they choose a specific group of products to offer to customers such that their expected revenue can be maximized. The revenue generated by an assortment of products is usually determined by two factors: the revenue generated by selling each individual product, and the purchasing behavior of consumers. The latter is often captured by discrete choice

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models such as a multinomial logit (MNL) model [26] and a nested logit (NL) model [7]. Unlike previous studies that assume *fixed* choice models, we take into account the fact that customer purchasing behavior may be influenced by sophisticated selling practices such as advertising. Specifically, advertising is an important and effective strategy for establishing brand recognition and communicating the value of a product effectively to the public. Given the importance of advertising, determining how to allocate the promotional budget over products and time is a critical aspect of retailers' decision making [13,19,22], hence, it is important for a retailer to consider the impact of advertising on their product choices to increase revenue. To maximize this effect, the retailer should align their advertising and product recommendations.

In this paper, we propose and investigate a joint advertising and assortment optimization problem (JAAOP). We employ the MNL model to understand consumer purchasing behavior, in which every product, including the choice not to purchase, is assigned a random utility. Once presented with a variety of products, the consumer chooses the one with the highest utility. Our study differs from previous research on traditional assortment optimization by taking into account the influence of advertising. That is, rather than just selecting a group of products, we investigate the potential of combining advertising with traditional product selection to enhance the overall optimization. Specifically, we assume that the platform can increase the attractiveness (a.k.a. utility) of a product by advertising it, the effectiveness of which is represented by a product-specific response function and the amount of advertising efforts allocated to that product. With constraints on the advertising budget, our goal is to jointly determine which products to present to consumers and how to allocate the advertising budget among them in order to maximize expected revenue. In one extension of our work, we also examine the sequential choice behavior of consumers [15], a common feature on online shopping platforms such as Amazon and Taobao, where a large number of products are displayed to the consumer in stages. If the consumer does not select any products in a stage, they will move on to the next set of products. This requires the platform to not only select which products to display, but also their positions. We formulate this problem as a joint multi-stage advertising and assortment optimization problem.

1.1 Summary of Contributions

This section summarizes the major contributions of our work.

- We introduce the JAAOP in which the platform must concurrently select (1) an advertising strategy and (2) a set of products to present to consumers. By using the MNL model and assuming no constraint on the maximum number of products that can be displayed to a user, we can obtain an optimal revenue-ordered assortment and an efficient advertising strategy.
- When a constraint on the maximum number of products that can be displayed to a user is present, we analyze the problem under different response functions. If the response function is a log function, the optimal advertising

strategy is to allocate all the advertising budget to a single product. If the response function is a general concave function, we formulate our problem as a nonlinear continuous optimization problem and use McCormick inequalities to convert it into a convex optimization problem. We then develop an efficient algorithm to find the optimal strategy.

– In Appendix A and B, we study several extensions. In one extension of this study, we include the price of each product as a decision variable and consider the joint product assortment, pricing, and advertising optimization problem. We also extend our model to incorporate the multi-stage purchase behavior and investigate the structural properties of the problem. We develop a heuristic method that comes with a performance guarantee.
– We also conduct a series of experiments to evaluate the performance of our solutions and further confirm the value of advertising in Appendix C. Our proposed heuristic method is robust and outperforms other methods in different settings. Specifically, the results suggest that allocating the advertising budget uniformly or greedily leads to substantial revenue loss.

2 Literature Review

Our work is closely related to the assortment optimization problem in revenue management, which aims to select a subset of products to maximize the expected revenue. Various discrete choice models have been proposed to model consumer decision-making behavior, including the MNL model [26], the Nested Logit (NL) model, the *d*-level NL model and so on. Recently, several works have considered sequential choice behavior. For example, Flores et al. [12] investigated a two-stage MNL model, where the consumer sequentially browses two stages that are disjoint in terms of potential products and [17] extended to the multi-stage setting. Moreover, [15] developed a sequential MNL model, where the utility of the no-purchase option is fixed at the beginning instead of being resampled each time, and studied the assortment and pricing problem with impatient customers.

Another related problem is the advertising budget allocation problem. [2] proposed a model to allocate resources among multiple brands in a single period. [9] further considered advertising budget allocation across products and media channels. [11] considered the lagged effect of advertising and studied the dynamic marketing budget allocation problem for a multi-product, multi-country setting. [1] proposed a single-product spatiotemporal model that includes the spatial differences and sale dynamics.

Finally, our work belongs to the growing literature that aims to improve revenue through sophisticated selling practices beyond product selection. The approaches in this area include offering certain items only through lotteries [18] and making certain products unattractive to consumers [4]. [4] studied the refined assortment optimization problem for several regular choice models, including the MNL, latent-class MNL (LC-MNL), and random consideration set (RCS) models. While the authors in [4] focus on *reducing* the utilities of some products to improve revenue, our approach aims to increase revenue by *increasing* the utilities of some products. The main differences are: 1. [4] focus on strategically reducing the utilities of products, whereas our study centers on increasing the utilities of products. 2. [4] assume that changing the utility of a product has no cost, while our model takes into account the cost of increasing the utility of a product through advertising and considers budget constraints in the optimization problem. We also show that the platform has no incentive to decrease the utilities of any products in the MNL model under a cardinality constraint. A similar result was discovered independently by [4] for an unconstrained MNL model.

3 Preliminaries and Problem Formulation

3.1 MNL Model

We list the main notations in Table 1. Generally, the input of our problem is a set of n products $\mathcal{N} = \{1, 2, \cdots, n\}$. In the MNL model, each product $i \in \mathcal{N}$ has a utility $q_i + \epsilon_i$, where q_i is a constant that captures the initial utility of product i, and ϵ_i is a random variable that captures the error term. We assume that ϵ_i follows a Gumbel distribution with a location-scale parameter $(0, 1)$. Let $\mathbf{v}$ denote the preference vector of $\mathcal{N}$, where $v_i := e^{q_i}$ for each $i \in \mathcal{N}$. Given an assortment $S \subseteq \mathcal{N}$ and a preference vector $\mathbf{v}$, for each product $i \in \mathcal{N}$, a consumer will purchase product i with a probability of

$$\phi_i(S, \mathbf{v}) = \begin{cases} \frac{v_i}{1+\sum_{j \in S} v_j} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The no-purchase probability is $\phi_0(S, \mathbf{v}) = \frac{1}{1+\sum_{j \in S} v_j}$. Let $\mathbf{r}$ denote the revenue vector of $\mathcal{N}$, where for each product $i \in \mathcal{N}$, $r_i > 0$ represents the revenue from product i. Based on the above notations, the expected revenue $R(S, \mathbf{v})$ of the assortment S is given by

$$R(S, \mathbf{v}) = \sum_{i \in S} r_i \cdot \phi_i(S, \mathbf{v}) = \frac{\sum_{i \in S} r_i v_i}{1 + \sum_{i \in S} v_i}. \quad (2)$$

3.2 Joint Advertising and Assortment Optimization

We use a vector $\mathbf{x}$ to represent an advertising strategy where for each $i \in \mathcal{N}$, x_i represents the amount of advertising efforts allocated to i. Let $\mathbf{c}$ denote the *advertising effectiveness* of $\mathcal{N}$. We assume that the utility of each product $i \in \mathcal{N}$, increases by $f(c_i x_i)$ if it receives x_i advertising efforts from the platform, where

$f(\cdot)$ is called *response function* and c_i is the advertising effectiveness of product i. Intuitively, $\mathbf{c}$ and $f(\cdot)$ together determine the degree to which a product's preference weight is influenced by the advertising it receives from the platform. For a given preference vector $\mathbf{v}$, the expected revenue $R(S, \mathbf{v}, \mathbf{x})$ of an assortment S under an advertising strategy $\mathbf{x}$ is calculated as

$$R(S, \mathbf{v}, \mathbf{x}) = \frac{\sum_{i \in S} r_i e^{q_i + f(c_i x_i)}}{1 + \sum_{i \in S} e^{q_i + f(c_i x_i)}} = \frac{\sum_{i \in S} r_i v_i g(c_i x_i)}{1 + \sum_{i \in S} v_i g(c_i x_i)}, \tag{3}$$

where $g(\cdot) = e^{f(\cdot)}$. Hence, $R(S, \mathbf{v}) = R(S, \mathbf{v}, 0)$.

We next formally introduce the JAAOP.

Definition 1. *Let $\mathcal{X} = \{\mathbf{x} | \sum_{i=1}^{n} x_i \leq B\}$ denote the set of all feasible advertising strategies subject to the advertising budget B. JAAOP aims to jointly find an assortment S of size at most K and a feasible advertising strategy $\mathbf{x} \in \mathcal{X}$ to maximize the expected revenue, that is,*

$$\max_{\mathbf{x} \in \mathcal{X}} \max_{S : |S| \leq K} R(S, \mathbf{v}, \mathbf{x}). \tag{4}$$

Let S^* and $\mathbf{x}^*$ denote the optimal assortment and advertising strategy, respectively, subject to the advertising budget B and cardinality constraint K. In case of multiple optimal assortments, we select the one with the smallest number of items. For ease of presentation, let $S_{\mathbf{v}}$ denote the optimal assortment when $B = 0$, that is, $S_{\mathbf{v}} = \arg\max_{S : |S| \leq K} R(S, \mathbf{v})$. In this paper, we make two assumptions about $g(\cdot)$.

Assumption 1. *$g(\cdot)$ is differentiable, concave, and monotonically increasing.*

We will now provide the reasoning behind this assumption. Several researchers have investigated the impact of advertising on customer utility, including [10, 25], and [28]. These studies all assumed logarithmic response functions, which imply that market share is a concave function of advertising efforts, meaning that the benefit of incremental advertising decreases as advertising efforts increase. This property, also known as the law of diminishing returns, has been widely used in other works [3, 9, 21]. The assumption we made in our study, known as Assumption 1, captures this property effectively. For the market share of product i in assortment S, that is, $\frac{v_i g(c_i x_i)}{1 + \sum_{i \in S} v_i g(c_i x_i)}$, we can verify the concavity of the market share function = by observing the negativity of the second derivative. The second assumption states that the advertising effect is zero if a product receives zero amount of advertising efforts from the platform.

Assumption 2. *$g(0) = 1$.*

We present a useful lemma that states that there exists an optimal advertising strategy that always uses the entire advertising budget.

Lemma 1. *There exists an optimal advertising strategy $\mathbf{x}^*$ for problem (4) such that $\sum_{i \in S^*} x_i = B$.*

4 Unconstrained JAAOP

We start by examining a special case of the JAAOP, where $K = n$, meaning there is no limit on the assortment size.

In the absence of any size constraints and advertising budget, our problem becomes the standard unconstrained assortment optimization problem. As proven by [26], the optimal assortment in this scenario is a revenue-ordered assortment, i.e. all products generating revenue greater than a certain threshold are included. This threshold, as demonstrated in [24], is the expected optimal revenue.

Lemma 2 *[24, Theorem 3.2].* *If $K = n$ and $B = 0$, there exists an optimal assortment $S_{\mathbf{v}}$ such that $S_{\mathbf{v}} = \{i \in \mathcal{N} | r_i > R(S_{\mathbf{v}}, \mathbf{v})\}$.*

This characteristic has been noted in other contexts as well, such as the joint pricing and assortment optimization problem [27] and the robust assortment optimization problem [24]. The optimal assortment, given a fixed advertising strategy, remains revenue-ordered. Thus, to find the best solution, we find the optimal advertising strategy for each possible revenue-ordered assortment, and choose the one with the highest expected revenue as the final result. The number of possible revenue-ordered assortments is at most n. The efficiency of this algorithm can be improved by taking into consideration the following observations.

Lemma 3. *There exists an optimal assortment S^* such that $S^* \subseteq S_{\mathbf{v}}$.*

Lemma 3 implies that to find the optimal advertising strategy, we must evaluate all the revenue-ordered assortments within $S_{\mathbf{v}}$ and determine the optimal advertising plan. Then, for any revenue-ordered assortment S, we find the optimal advertising strategy to obtain the complete solution, i.e.,

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & \frac{\sum_{i \in S} r_i v_i g(c_i x_i)}{1 + \sum_{i \in S} v_i g(c_i x_i)} \\ \text{s.t.} \quad & \sum_{i \in S} x_i = B. \end{aligned} \tag{5}$$

With $u_i = v_i g(c_i x_i)$, (5) can be rewritten as:

$$\begin{aligned} \max_{\mathbf{u}} \quad & \frac{A(\mathbf{u})}{B(\mathbf{u})} = \frac{\sum_{i \in S} r_i u_i}{1 + \sum_{i \in S} u_i} \\ \text{s.t.} \quad & \mathbf{u} \in \mathcal{U}, \end{aligned} \tag{6}$$

where $\mathcal{U} = \{\mathbf{u} | \sum_{i=1}^{n} m_i(u_i) \leq B, u_i \geq v_i, i = 1, \ldots, n\}$ and $m_i(\cdot) = g^{-1}(\frac{\cdot}{v_i})/c_i$. Here (6) is a single-ratio fractional programming (FP) problem. Before presenting our solution to (6), we show that $\mathcal{U}$ is a convex set.

Lemma 4. *The constraint set $\mathcal{U}$ in (6) is a convex set.*

This part describes our solution to (6) in detail. Lemma 4 indicates that (6) is a concave-convex FP problem. We apply the classical Dinkelbach transform [8] by iteratively solving the following parameterized problem:

$$\begin{aligned} \max_{\mathbf{u}} \quad & h(y) = A(\mathbf{u}) - yB(\mathbf{u}) \\ \text{s.t.} \quad & \mathbf{u} \in \mathcal{U}. \end{aligned} \tag{7}$$

In particular, our algorithm starts with iteration $t = 0$ and $y_0 = \frac{A(\mathbf{v})}{B(\mathbf{v})}$, and in each subsequent iteration $t + 1$, we find $\mathbf{u}_{t+1}$ to maximize $h(y_t)$ by solving (7) and update $y_{t+1} = \frac{A(\mathbf{u}_{t+1})}{B(\mathbf{u}_{t+1})}$. This process iterates until the optimal solution of (7) is 0 and we output the corresponding maximizer $\mathbf{u}^F$. Equation (7) can be solved efficiently because $A(\mathbf{u})$ is a concave function and $B(\mathbf{u})$ is a convex function. This algorithm is guaranteed to converge to the optimal solution [8]. After solving (6) and obtaining $\mathbf{u}^F$, we transform (6) to an optimal advertising strategy such that for each $i \in S$, we set $x_i = m_i(u_i^F)$; that is, we allocate $m_i(u_i^F)$ efforts to i. A detailed description of our solution is presented in Algorithm 1.

Algorithm 1. Optimal Solution for Unconstrained JAAOP

Input: preference weight $\mathbf{v}$, revenue $\mathbf{r}$, advertising effectiveness $\mathbf{c}$, budget B
Output: optimal assortment S^*, advertising strategy $\mathbf{x}^*$
1: Solve the classic unconstrained assortment optimization problem, and obtain the optimal assortment $S_{\mathbf{v}}$ when $B = 0$
2: Solve (6) for each revenue-ordered assortment in $S_{\mathbf{v}}$, and return the best one as the final solution

5 Cardinality-Constrained JAAOP

We next study our problem under a cardinality constraint of $K > 0$. First, we examine a scenario where $g(\cdot)$ is a linear function, and then we delve into the general case where $g(\cdot)$ is a concave function.

5.1 $g(x)$ as Linear Function

We first study the scenario where $g(\cdot)$ is a linear function, expressed as $1 + ax$ for some $a \geq 0$. The next lemma demonstrates the existence of an optimal advertising strategy that allocates the entire budget to a single product. For each $i \in \mathcal{N}$, we define $\mathbf{x}_i$ as a vector in which the i-th element is B and all other elements are zero.

Lemma 5. *For any assortment S, the optimal solution for the following problem is achieved at $\mathbf{x}_i$ for some $i \in S$:*

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & L(S, \mathbf{x}) = \frac{\sum_{i \in S} r_i v_i (1 + a c_i x_i)}{1 + \sum_{i \in S} v_i (1 + a c_i x_i)} \\ \text{s.t.} \quad & \sum_{i \in S} x_i = B. \end{aligned} \tag{8}$$

This lemma implies that to find the optimal advertising strategy, we need to consider at most n candidate advertising strategies: $\{\mathbf{x}_i | i \in \mathcal{N}\}$. Specifically, considering $\mathbf{x}_i$, we replace the original preference weight v_i of i using $v_i g(c_i B)$ and then solve the standard capacity-constrained assortment optimization problem to obtain an optimal assortment. Among the n returned solutions, we return the best one as the final solution. [23] showed that the standard cardinality-constrained assortment optimization problem for each $\mathbf{x}_i$ can be solved in $O(n^2)$ time. Thus, the overall time complexity of our solution is $n \times O(n^2) = O(n^3)$. Assume all products are indexed in non-increasing order of r_i. The next lemma shows that we can further narrow the search space and reduce the time complexity to $O(n^2 T)$, where $T = \max\{i | i \in S_{\mathbf{v}}\}$ represents the index of the product that has the smallest revenue in $S_{\mathbf{v}}$.

Lemma 6. *Assume all products are indexed in non-increasing order of r_i. Let $T = \max\{i | i \in S_{\mathbf{v}}\}$, there exists an optimal assortment S^* such that $S^* \subseteq \{1, 2, ..., T\}$.*

We present the detailed implementation of our algorithm in Algorithm 2.

Algorithm 2. Optimal Cardinality Constrained Solution for Log Response Function

Input: preference weight $\mathbf{v}$, revenue $\mathbf{r}$, cardinality constraint K, advertising effectiveness $\mathbf{c}$, budget B
Output: optimal assortment S^* and advertising strategy $\mathbf{x}^*$
1: Let $T = \max\{i | i \in S_{\mathbf{v}}\}$
2: **for** $i = 1, \ldots, T$ **do**
3: Compute an assortment S_i that maximizes $R(S, \mathbf{v}, \mathbf{x}_i)$
4: **end for**
5: Return the best $(S_j, \mathbf{x}_j)$.

5.2 $g(x)$ as a General Concave Function

We next discuss the general case. Before presenting our solution, we first construct an example to demonstrate that allocating the entire budget to a single product is not necessarily optimal.

Example 1. Consider three products with revenue $\mathbf{r} = (8, 7.5, 2.8)$, preference weight $\mathbf{v} = (1.2, 1, 1.7)$ and the effectiveness $\mathbf{c} = (0.9, 0.8, 1)$. Assume the cardinality constraint is $K = 2$ and the total advertising budget is $B = 10$. We consider a concave function $g(x) = \sqrt{x} + 1$. If we are restricted to allocating the entire budget to a single product, then the optimal advertising strategy is $(10, 0, 0)$, the optimal assortment is composed of the first two products, and the expected revenue of this solution is 6.75. However, the actual optimal advertising strategy is (approximately) $(8.285, 1.715, 0)$, the actual optimal assortment contains the first two products, and it achieves expected revenue of 6.812. The above example shows that the single-product advertising strategy is no longer optimal for a general concave response function.

We next present our solution. Let $u_i = v_i g(c_i x_i)$ and define $m_i(\cdot) = g^{-1}(\frac{\cdot}{v_i})/c_i$ for each $i \in \mathcal{N}$, we first transform (4) to an equivalent nonlinear mixed integer program (9) by replacing $\sum_{i=1}^n x_i = B$ using $\sum_{i=1}^n m_i(u_i) \leq B$,

$$\begin{aligned} \max_{\mathbf{u}\in\mathcal{U}} \quad & \max_{\mathbf{t}\in\{0,1\}^n} \frac{\sum_{j=1}^n u_j r_j t_j}{1+\sum_{j=1}^n u_j t_j} && (9)\\ \text{s.t.} \quad & \sum_{i=1}^n t_i \leq K, \end{aligned}$$

where $\mathcal{U} = \{\mathbf{u} | \sum_{i=1}^n m_i(u_i) \leq B, u_i \geq v_i, i = 1, \ldots, n\}$. We next present a useful lemma from [6].

Lemma 7 *(Theorem 1 [6]). The inner problem of* (9) *is equivalent to the following linear program*

$$\begin{aligned} \max_{\mathbf{w},w_0} \quad & \sum_{j=1}^n r_i w_i && (10)\\ \text{s.t.} \quad & \sum_{i=1}^n w_i + w_0 = 1, && (11)\\ & \sum_{i=1}^n \frac{w_i}{u_i} \leq K w_0, && (12)\\ & 0 \leq \frac{w_i}{u_i} \leq w_0 \qquad \forall i \in \mathcal{N}. && (13) \end{aligned}$$

Notice that (12) and (13) involve some nonlinear constraints if $\mathbf{u}$ is not fixed. Thus we introduce new variables $\ell_i = \frac{w_i}{u_i}$, $i \in \mathcal{N}$ and rewrite (9) as follows:

$$\begin{aligned} \text{(NO)} \quad \max_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{w},\ell,w_0} \quad & \sum_{j=1}^n r_i w_i && (14)\\ \text{s.t.} \quad & \sum_{i=1}^n w_i + w_0 = 1, && (15)\\ & \sum_{i=1}^n \ell_i \leq K w_0, && (16)\\ & 0 \leq \ell_i \leq w_0 \qquad \forall i \in \mathcal{N}, && (17)\\ & w_i = \ell_i u_i \qquad \forall i \in \mathcal{N}. && (18) \end{aligned}$$

We further use the classic McCormick inequalities ([20]) to relax the nonconvex constraints (18):

$$
\text{(MC)} \quad \begin{aligned}
& w_i \geq \ell_i v_i \qquad \forall i \in \mathcal{N}, \\
& w_i \geq u_i + \ell_i v_i g(Bc_i) - v_i g(Bc_i) \qquad \forall i \in \mathcal{N}, \\
& w_i \leq \ell_i v_i g(Bc_i) \qquad \forall i \in \mathcal{N}, \\
& w_i \leq u_i + \ell_i v_i - v_i \qquad \forall i \in \mathcal{N}.
\end{aligned}
$$

Through the above relaxation, we transform (NO) into a convex optimization problem that can be solved efficiently. After solving this relaxed problem and obtaining a solution $\mathbf{w}$, we can compute the final assortment S as follows: we first find the product for which the w_i is strictly larger than 0, that is $S^w = \{i | i \in \mathcal{N}, w_i \neq 0\}$. Then we sort the products in S^w by the value of w_i and choose the first K products. Notice that the advertising strategy that is obtained from solving the previous relaxed problem may not be optimal. One can solve (6) to find the optimal advertising strategy for S. Lastly, if the size of the input is large, we can reduce the problem size by selecting a smaller group of candidate products based on Lemma 6. A detailed description of our solution is listed in Algorithm 3.

Algorithm 3. Cardinality Constrained Solution for General Response Function

Input: preference weight $\mathbf{v}$, revenue $\mathbf{r}$, capacity constraint K, advertising effectiveness $\mathbf{c}$, budget B

Output: assortment S and advertising strategy $\mathbf{x}$

1: Let $T = \max\{i | i \in S_{\mathbf{v}}\}$
2: Solve the optimization problem (NO + MC) for the first T products to find an assortment S
3: Solve problem (6) for S to find the advertising strategy $\mathbf{x}$
4: Return $(S, \mathbf{x})$.

6 Conclusion

This paper considers the JAAOP problem under the MNL model, where the seller decides their advertising strategy for all products to improve the current revenue. We consider both the log and general concave response functions. If there are no capacity constraints, we show that the optimal assortment is still revenue-ordered. However, this result does not hold in the presence of a cardinality constraint. When the response function is a log function, we prove that the optimal advertising strategy is a single-product advertising strategy, thus the optimal solution could be found in polynomial time. For the general concave response function, we develop an efficient algorithm to find a near-optimal solution. We further consider the seller could adjust the price simultaneously, and show that such a problem can be efficiently solvable under unconstrained setting or be transformed as a mixed-integer nonlinear programming for the cardinality constraint setting. Additionally, as an extension, we study the multi-stage MNL choice model, in which the customer browses the assortments sequentially. Our results demonstrate that the seller has no incentive to decrease the utility of any product, even under the capacity constraint. Finally, we conduct extensive experiments to illustrate that the advertising strategy is more effective with small cardinality constraint and large no-purchase utility.

Appendix

A Joint Assortment, Pricing, and Advertising Optimization

In this section, we study the case when the price of each product is also a decision variable. Formally, we assume that the preference weight of each product $i \in \mathcal{N}$ can be represented as $e^{q_i+f(c_i x_i)-p_i}$, whose value is jointly decided by i's initial utility q_i, i's price p_i, and the advertising efforts x_i received from the platform. Hence, the revenue r_i of each product $i \in \mathcal{N}$ is $r_i = p_i - d_i$, where d_i is the production cost of i. Based on the above notations, we can represent the expected revenue $R(S, \mathbf{p}, \mathbf{x})$ of an assortment S as

$$R(S, \mathbf{p}, \mathbf{x}) = \frac{\sum_{i \in S}(p_i - d_i)e^{q_i+f(c_i x_i)-p_i}}{1+\sum_{i \in S} e^{q_i+f(c_i x_i)-p_i}} = \frac{\sum_{i \in S}(p_i - d_i)e^{q_i-p_i}g(c_i x_i)}{1+\sum_{i \in S} e^{q_i-p_i}g(c_i x_i)}. \quad (A.1)$$

A.1 Unconstrained Case

If there is no cardinality constraint, our goal is to solve the following joint advertising, pricing, and assortment optimization problem:

$$\max_{\mathbf{p},\mathbf{x},S} \quad R(S, \mathbf{p}, \mathbf{x}) \quad (A.2)$$
$$\text{s.t.} \quad \sum_{j \in S} x_j \le B.$$

Before describing our solution, we first present a useful lemma from [16].

Lemma A.1 *[16]. Given any assortment S, the optimal price for each product $i \in S$ is $p_i = W(\sum_{i \in S} e^{q_i-d_i-1})g(c_i x_i) + d_i + 1$, where $W(\cdot)$ is the Lambert W function; that is, $W(\cdot)$ is the value of x that satisfies $xe^x = z$. Moreover, the revenue of the optimal solution is $W(\sum_{i \in S} e^{q_i-d_i-1}g(c_i x_i))$.*

For any given advertising strategy $\mathbf{x}$ and assortment S, the optimal price and corresponding expected revenue are explicitly given by Lemma A.1. Because

$W(\cdot)$ is an increasing function, Lemma A.1 implies that the optimal assortment must include all products. Hence, we can transform (A.2) into

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{n} e^{q_i - d_i - 1} g(c_i x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq B, \\ & x_i \geq 0 \quad \forall i \in \mathcal{N}. \end{aligned} \tag{A.3}$$

Denote $\alpha_i = e^{q_i - d_i - 1}$, and rewrite the above problem as

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{n} \alpha_i g(c_i x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq B, \\ & x_i \geq 0 \quad \forall i \in \mathcal{N}. \end{aligned} \tag{A.4}$$

Because $g(\cdot)$ is a concave function (Assumption 1) and $\sum_{i=1}^{n} x_i \leq B$ is a linear constraint, (A.4) is a concave maximization problem with convex constraints. Hence, (A.4) is a convex minimization problem over a convex set, and the problem has efficient solutions [5].

A special case where $g(\cdot)$ is a linear function: If $g(\cdot)$ is a linear function, that is, $g(x) = ax + 1$ for some $a \geq 0$, then the optimal advertising strategy is to allocate the entire advertising budget to a single product.

Lemma A.2. *When $g(\cdot)$ is a linear function, the optimal solution to (A.4) is to allocate the entire advertising budget to the product with the largest $\alpha_i c_i$.*

A.2 Cardinality-Constrained Case

We next consider a case where the size of the assortment is at most $K \geq 0$. Lemma A.1 indicates that the optimal assortment contains the top K products that have the largest $e^{q_i + f(c_i x_i) - d_i - 1}$. We next show that if $e^{f(\cdot)}$ is a linear function, then we only need to consider two possible advertising strategies. Hence, this problem can be solved efficiently.

Lemma A.3. *Let $\alpha_i = e^{q_i - d_i - 1}$ and $g(x) = ax + 1$ for some $a \geq 0$. Assume all products are indexed in non-increasing order of α_i. Let $t_1 = \text{argmax}_{i \in \{1, \cdots, K\}} \{\alpha_i c_i\}$ and $t_2 = \text{argmax}_{j \in \{K+1, \ldots, n\}} \{\alpha_j (a c_j B + 1)\}$, then the optimal advertising strategy is $\mathbf{x}_{t_1}$ or $\mathbf{x}_{t_2}$.*

We next discuss a case with a general response function. For each product $i \in \mathcal{N}$, let $t_j = 1$ if a product j is offered in the assortment and let $t_j = 0$ otherwise. Our problem can be formulated as the following mixed-integer programming problem:

$$
\begin{aligned}
\max_{\mathbf{x},\mathbf{t}} \quad & \sum_{i=1}^{n} \alpha_i t_i g(c_i x_i) \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq B, \\
& \sum_{i=1}^{n} t_i \leq K, \\
& x_i \geq 0 \quad \forall i \in \mathcal{N}, \\
& t_i \in \{0,1\} \quad \forall i \in \mathcal{N}.
\end{aligned} \tag{A.5}
$$

If all products have the same advertising effectiveness, that is, $c_i = c$, for all $i \in \mathcal{N}$, the optimal assortment is to select the top K products that have the largest α_i.

Lemma A.4. *Assume all products are indexed in non-increasing order of α_i and $c_i = c$ for all $i \in \mathcal{N}$. The optimal assortment is $S^* = \{1, \ldots, K\}$, and the optimal advertising strategy $\mathbf{x}^*$ satisfies $x_i^* \geq x_j^*$ $\forall i \leq j$.*

To find the optimal advertising strategy under $S^* = \{1, \ldots, K\}$, we need to solve an optimization problem that is similar to (A.4). Because this problem is a concave maximization problem with convex constraints, it can be solved efficiently.

For a general case, where advertising effectiveness is heterogeneous, the objective function of (A.5) contains the bilinear terms $t_i g(c_i x_i)$. We linearize each of these terms by relaxation. Specifically, for each $t_i e^{f(c_i x_i)}$, we introduce a new continuous variable $w_i = t_i g(c_i x_i)$ and add the inequalities: $g(c_i x_i) - w_i \leq g(c_i B)(1 - t_i)$, $0 \leq w_i \leq g(c_i x_i)$, and $w_i \leq g(c_i B) t_i$. This leads to the following mixed-integer nonlinear programming problem:

$$
\begin{aligned}
\max_{\mathbf{x},\mathbf{t},\mathbf{w}} \quad & \sum_{i=1}^{n} \alpha_i w_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq B, \\
& \sum_{i=1}^{n} t_i \leq K, \\
& g(c_i x_i) - w_i \leq g(c_i B)(1 - t_i) \quad \forall i \in \mathcal{N}, \\
& 0 \leq w_i \leq g(c_i x_i) \quad \forall i \in \mathcal{N}, \\
& w_i \leq g(c_i B) t_i \quad \forall i \in \mathcal{N}, \\
& x_i \geq 0 \quad \forall i \in \mathcal{N}, \\
& t_i \in \{0,1\} \quad \forall i \in \mathcal{N}.
\end{aligned} \tag{A.6}
$$

B Sequential Joint Advertising and Assortment Optimization

In this section, we extend our study to consider a sequential joint advertising and assortment problem. The model put forward by [15] examines the behavior of consumers who may visit multiple product assortments before making a purchase or leaving the store. The consumer is assumed to progress through a sequence of m stages, each featuring a different assortment ($\mathcal{S} = (S_1, \ldots, S_m)$). If the consumer chooses to buy a product in stage i, they will leave the store, but if they do not make a purchase, they will proceed to the next stage. If no product is selected after visiting all m assortments, the consumer exits the store without making a purchase. This choice model is referred to as the sequential multinomial logit (SMNL) choice model. For the purpose of simplicity, we assume that the consumer will continue visiting subsequent assortments if they do not make a purchase in the current stage. However, it should be noted that this assumption can be relaxed to include the factor of consumer patience.

We will now provide a detailed explanation of the SMNL model. Given a sequence of assortments $\mathcal{S}$, the consumer will purchase product i in stage k with a probability of

$$\phi_i^k(\mathcal{S}) = \frac{v_i}{(1+\sum_{\ell=1}^{k-1} V(S_\ell))(1+\sum_{\ell=1}^{k} V(S_\ell))}.$$

Let $V(S) = \sum_{i\in S} v_i$ and $W(S) = \sum_{i\in S} r_i v_i$. The expected revenue is represented as

$$R(\mathcal{S}) = \sum_{k=1}^{m} \frac{W(S_k)}{(1+\sum_{\ell=1}^{k-1} V(S_\ell))(1+\sum_{\ell=1}^{k} V(S_\ell))}.$$

Under the advertising strategy $\mathbf{x}$, the expected revenue increases to

$$R(\mathcal{S},\mathbf{x}) = \sum_{k=1}^{m} \frac{W(S_k,\mathbf{x})}{(1+\sum_{\ell=1}^{k-1} V(S_\ell,\mathbf{x}))(1+\sum_{\ell=1}^{k} V(S_\ell,\mathbf{x}))}, \tag{B.1}$$

where $V(S,\mathbf{x}) = \sum_{i\in S} v_i g(c_i x_i)$ and $W(S,\mathbf{x}) = \sum_{i\in S} r_i v_i g(c_i x_i)$.

Based on the transformation in (5), the optimization problem can be written as

$$\begin{aligned} \max_{\mathbf{u},\mathcal{S}} \quad & \sum_{k=1}^{m} \frac{\sum_{i\in S_k} r_i u_i}{(1+\sum_{l=1}^{k-1}\sum_{i\in S_l} u_i)(1+\sum_{l=1}^{k}\sum_{i\in S_l} u_i)} \\ \text{s.t.} \quad & \mathbf{u} \in \mathcal{U}. \end{aligned} \tag{B.2}$$

We focus on the unconstrained setting. Given an arbitrary advertising strategy, [15] demonstrated that the optimal assortments are sequential revenue-ordered assortments. Specifically, there exists a set of decreasing thresholds $\{t_1^*, t_2^*, \ldots, t_{m+1}^*\}$, such that $S_k^* = \{i \in \mathcal{N} : t_{k+1}^* \le r_i < t_k^*\}$ for $k \in \mathcal{M} = [1, 2, \ldots, m]$. The values of $\{t_i^*\}_{i=1}^{m+1}$ are given in the following lemma.

Lemma B.1 *[15, Theorem 3.1]. There exists an optimal solution $(S_1^*, \dots, S_m^*)$ such that for $i \in S_k^*$, we have $t_{k+1}^* \le r_i < t_k^*$. Let $R_k(S_1^*, \dots, S_m^*) = \frac{W(S_k^*)}{(1+\sum_{\ell=1}^{k-1} V(S_\ell^*))(1+\sum_{\ell=1}^{k} V(S_\ell^*))}$. The value of t_k^* can be chosen as follows:*

$$t_1^* = +\infty, \quad t_k^* = \frac{R_{k-1}(S_1^*, \dots, S_m^*) + R_k(S_1^*, \dots, S_m^*)}{\frac{1}{1+\sum_{\ell=1}^{k-2} V(S_\ell^*)} - \frac{1}{1+\sum_{\ell=1}^{k} V(S_\ell^*)}} \quad \forall k \in \mathcal{M}\backslash\{1\}, \quad t_{m+1}^* = \frac{R_m(S_1^*, \dots, S_m^*)}{\frac{1}{1+\sum_{\ell=1}^{m-1} V(S_\ell^*)}}.$$

Based on this lemma, we analyze the structure of the optimal assortments and the advertising strategy. We denote the optimal solution of B.2 as $\mathbf{u}^*$ and $\mathcal{S}^*$.

Lemma B.2. *For the optimal problem (B.2), we have $\frac{\partial R(\mathcal{S}^*, \mathbf{u}^*)}{\partial u_i^*} \ge \frac{\partial R(\mathcal{S}^*, \mathbf{u}^*)}{\partial u_j^*} \ge 0$ for all products $i, j \in \mathcal{N}$ and $i < j$.*

[4] showed that in the MNL choice model, the partial derivative $h_i^1 \ge 0$, indicating that the seller has no incentive to reduce the utilities of products in order to maximize their expected revenue. In Lemma B.2, we extend this result to the SMNL choice model. Moreover, due to the sequential revenue-ordered property stated in Lemma B.1 being maintained for any feasible set of products, this result remains valid even under capacity constraints, meaning that the seller has no incentive to decrease product utilities in the capacity-constrained scenario either. If the seller has the ability to enhance product utilities, the optimal advertising strategy would be to allocate the entire budget to the product that generates the highest revenue.

Lemma B.3. *Denote the optimal solution of the following optimization problem as $(\mathbf{x}^*, \mathcal{S}^*)$. $x_1^* = B$ and $x_i^* = 0$ for all $i \in \mathcal{N} \setminus \{1\}$.*

$$\begin{aligned} \max_{\mathbf{x}, \mathcal{S}} \quad & \sum_{k=1}^{m} \frac{\sum_{i \in S_k} r_i (v_i + x_i)}{(1 + \sum_{l=1}^{k-1} \sum_{i \in S_l} (v_i + x_i))(1 + \sum_{l=1}^{k} \sum_{i \in S_l} (v_i + x_i))} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i \le B \end{aligned} \tag{B.3}$$

In our setting, the allocation of budget x_i to product i increases its utility to $v_i e^{f(c_i x_i)}$, where f is the nonlinear response function. Due to the heterogeneous advertising effectiveness, utility and nonlinear response function, the optimal advertising strategy may be more complex than a single-product advertising strategy. Given a specific sequence of assortments, finding the optimal advertising strategy is equivalent to solving the following optimization problem.

$$\begin{aligned} \max_{\mathbf{u}} \quad & \sum_{k=1}^{m} \frac{\sum_{i \in S_k} r_i u_i}{(1 + \sum_{l=1}^{k-1} \sum_{i \in S_l} u_i)(1 + \sum_{l=1}^{k} \sum_{i \in S_l} u_i)} \\ \text{s.t.} \quad & \mathbf{u} \in \mathcal{U} \end{aligned} \tag{B.4}$$

where $\mathcal{U} = \{\mathbf{u} | \sum_{i=1}^{n} m_i(u_i) \leq B, u_i \geq v_i, i = 1, \ldots, n\}$ and $m_i(\cdot) = g^{-1}(\frac{\cdot}{v_i})/c_i$. When $m = 1$, this problem is a single-ratio FP problem, which can be solved efficiently. However, the sum-of-ratio problem is generally NP-complete [14]. Hence, even though the optimal assortments may be sequential revenue-ordered assortments, finding the optimal advertising strategy may not be straightforward. As a result, we propose a heuristic method as an alternative approach.

B.1 Heuristic Method

The design of our heuristic method (listed in Algorithm 4) is based on two key observations. Firstly, given an advertising strategy, the optimal sequence of assortments can be found efficiently in polynomial time. Secondly, given the set of products to be displayed, the single-stage optimal advertising strategy is computationally tractable. Specifically, Algorithm 4 iteratively updates the assortments and advertising strategy until the expected revenue cannot be improved any further.

Algorithm 4. Heuristic for Unconstrained Multi-stage JAAOP

Input: preference weight $\mathbf{v}$, revenue $\mathbf{r}$, advertising effectiveness $\mathbf{c}$, budget B
Output: approximate assortment $\mathcal{S}^*$, advertising strategy $\mathbf{x}^*$
1: $i = 0, rev^0 = 0$
2: Implement Algorithm 1 and obtain the advertising strategy $\mathbf{x}^0$
3: **repeat**
4: $\quad i = i + 1$
5: $\quad$ Find the optimal sequence of assortments $\mathcal{S}^i$ and expected revenue rev^i based on the current advertising strategy $\mathbf{x}^{i-1}$
6: $\quad$ Find the optimal advertising strategy for $S^i = \cup_{j=1}^{m} S_j$, denoted as $\mathbf{x}^i$
7: **until** $rev^i < rev^{i-1}$

By exploring the structure of the objective function in (B.2), we next show that our heuristic method achieves an approximation ratio of 50%.

Lemma B.4. *Let* $(\mathcal{S}^*, \mathbf{x}^*), (\mathcal{S}^h, \mathbf{x}^h)$ *be the optimal values of* (B.2) *and our heuristic method. We have* $R(\mathcal{S}^h, \mathbf{x}^h) \geq \frac{1}{2} R(\mathcal{S}^*, \mathbf{x}^*)$.

C Numerical Study

In this section, we explore the effect of advertising on assortment optimization and validate the superiority of our algorithms compared with several heuristic methods on randomly generated instances and different response functions. The revenue of each product is drawn uniformly from the interval $[1, 10]$. For the preference weight v_i of product i, we first sample γ_i uniformly from the interval $[1, 10]$ and then assign $v_i = \gamma_i / \Delta$, where $\Delta = P_0 \sum_{i \in \mathcal{N}} \gamma_i / (1 - P_0)$. In this case, we guarantee the no-purchase probability when providing all products is exactly P_0. We consider three types of response functions: $g_1(x) = \sqrt{x} + 1$, $g_2(x) = \log(x+1)+1$, $g_3(x) = 2 - e^{-x}$. For advertising effectiveness, we consider the following settings.

- Setting A: The advertising effectiveness c_i of each product $i \in \mathcal{N}$ is drawn uniformly from the interval $[0, 1]$.
- Setting B: The advertising effectiveness c_i of each product $i \in \mathcal{N}$ is drawn independently from a standard log-normal distribution and rescaled by a factor of $\frac{1}{2\sqrt{e}}$ to make sure the same mean as setting A. In this case, there is more dispersion in advertising effectiveness.

We choose the number of products from $\{50, 100, 200\}$, the cardinality constraint K from $\{5, 10, 20\}$, and the value of P_0 from $\{0.1, 0.3\}$. For the multi-stage problem, the stage m is chosen from $\{3, 5, 8\}$. For each setting, we randomly generate 10 instances and calculate the average percentage of improvement over the non-advertising strategy. Finally, we denote our heuristic algorithm as HA.

C.1 Compared Heuristics

For the cardinality-constrained single-stage problem, the main challenge lies in finding the optimal advertising strategy as the optimal assortment for a given advertising strategy can be found efficiently in polynomial time. In order to tackle this difficulty, we propose two practical advertising strategies.

- Uniform advertising (UA) strategy: for any assortment S, we have $x_i = B/|S|$ if $i \in S$.
- Revenue advertising (RA) strategy: for any assortment S, we have $x_i = B \cdot \frac{r_i}{\sum_{i\in S} r_i}$ if $i \in S$.

We start with the optimal assortment with no advertising strategy S^1. After allocating the budget according to the heuristic method, we recompute the optimal assortment S^2; if $S^1 \neq S^2$, then we reallocate the budget and compute the new assortment. This process continues until the assortment is unchanged with advertising (Table 2).

C.2 Performance Evaluation

Table 1 presents the average performance of three heuristic algorithms for the single-stage joint advertising and assortment problem, evaluated over 36 different parameter settings. Algorithm 3 demonstrates superior performance compared to the other heuristic algorithms, particularly when P_0 is large and the cardinality constraint is small. In most cases, the RA strategy performs slightly better than the UA strategy. The performance of each heuristic algorithm does not vary significantly with an increase in the dispersion of advertising effectiveness. When the set of products is less attractive and the cardinality constraint is small, advertising has a more significant impact, and the gap between our algorithm and the compared heuristic algorithms is even larger.

Table 1. Average Performance of Tested Heuristic Algorithm on Single Stage Problem

Parameters				$g_1(x)$			$g_2(x)$			$g_3(x)$		
Setting	n	K	P_0	HA	UA	RA	HA	UA	RA	HA	UA	RA
A	50.0	5.0	0.1	27.08	25.82	25.88	21.38	19.47	19.53	18.5	17.74	17.78
	50.0	10.0	0.1	14.45	13.27	13.34	10.56	8.49	8.55	9.56	8.19	8.24
	50.0	20.0	0.1	9.76	7.95	8.11	6.86	4.14	4.29	6.03	4.08	4.22
	50.0	5.0	0.3	57.26	53.02	53.26	43.78	38.05	38.29	36.98	34.1	34.18
	50.0	10.0	0.3	31.94	29.85	29.94	22.75	18.14	18.2	20.26	17.45	17.5
	50.0	20.0	0.3	19.48	17.28	17.57	12.88	8.69	8.94	11.65	8.57	8.81
	100.0	5.0	0.3	75.21	68.19	68.19	57.29	48.65	48.64	48.09	43.32	43.3
	100.0	10.0	0.3	44.16	42.09	42.14	30.21	26.13	26.17	27.55	25.0	25.03
	100.0	20.0	0.3	25.23	23.88	24.01	15.62	11.89	12.01	14.57	11.73	11.84
	100.0	5.0	0.1	41.13	39.28	39.32	32.01	29.57	29.59	27.79	26.66	26.66
	100.0	10.0	0.1	21.69	20.87	20.93	15.04	13.68	13.73	14.26	13.16	13.2
	100.0	20.0	0.1	11.2	10.72	10.8	7.48	5.63	5.7	6.94	5.56	5.62
	200.0	5.0	0.1	60.48	52.98	53.04	46.5	38.29	38.35	38.54	34.33	34.38
	200.0	10.0	0.1	32.24	30.17	30.22	22.58	19.1	19.14	20.42	18.3	18.34
	200.0	20.0	0.1	16.55	16.13	16.2	10.77	8.39	8.44	9.7	8.28	8.32
	200.0	5.0	0.3	90.46	77.76	77.78	67.21	54.17	54.2	57.44	47.98	48.03
	200.0	10.0	0.3	54.22	50.86	50.85	37.26	30.78	30.75	33.49	29.34	29.32
	200.0	20.0	0.3	32.7	31.11	31.19	19.69	15.3	15.37	18.44	15.07	15.13
B	50.0	5.0	0.1	28.7	25.83	25.96	22.67	19.26	19.38	18.77	16.83	16.84
	50.0	10.0	0.1	15.59	13.89	14.02	11.37	8.94	9.06	9.72	8.29	8.4
	50.0	20.0	0.1	9.3	7.42	7.6	6.35	3.6	3.74	5.51	3.51	3.64
	50.0	5.0	0.3	61.14	53.73	54.03	45.3	38.11	38.34	35.77	31.92	31.85
	50.0	10.0	0.3	32.77	29.65	29.83	22.76	17.71	17.85	18.64	16.4	16.44
	50.0	20.0	0.3	20.43	17.14	17.6	13.86	8.73	9.09	11.69	8.43	8.76
	100.0	5.0	0.1	43.47	36.75	36.84	33.61	26.12	26.2	25.99	22.66	22.66
	100.0	10.0	0.1	20.68	19.2	19.29	14.48	11.72	11.82	12.75	11.13	11.21
	100.0	20.0	0.1	11.96	10.65	10.75	8.4	5.64	5.7	7.3	5.47	5.52
	100.0	5.0	0.3	76.08	63.21	63.22	56.17	43.04	43.05	43.28	37.37	37.41
	100.0	10.0	0.3	43.16	38.01	38.12	31.0	22.11	22.22	25.18	20.68	20.8
	100.0	20.0	0.3	25.42	22.46	22.62	17.15	10.94	11.1	14.24	10.52	10.66
	200.0	5.0	0.1	67.19	52.12	52.16	52.04	37.22	37.27	38.03	32.2	32.23
	200.0	10.0	0.1	31.58	28.81	28.84	20.74	17.75	17.78	18.75	16.72	16.76
	200.0	20.0	0.1	17.15	15.66	15.72	11.61	8.28	8.33	9.91	7.92	7.98
	200.0	5.0	0.3	95.94	67.3	67.19	68.48	43.43	43.3	51.34	38.86	38.79
	200.0	10.0	0.3	56.04	47.21	47.27	37.58	27.28	27.35	31.49	25.82	25.86
	200.0	20.0	0.3	32.8	28.57	28.73	21.97	13.85	14.02	18.11	13.11	13.22

Table 2. Average Performance of Tested Heuristic Algorithm on Multiple Stage Problem

Parameters				$g_1(x)$			$g_2(x)$			$g_3(x)$		
Setting	n	m	P_0	HA	UA	RA	HA	UA	RA	HA	UA	RA
A	50.0	3.0	0.1	7.32	5.31	5.94	4.99	2.03	2.57	4.48	2.02	2.55
	50.0	5.0	0.1	6.83	4.85	5.58	4.61	1.75	2.34	4.14	1.74	2.33
	50.0	8.0	0.1	6.69	4.75	5.51	4.49	1.68	2.3	4.04	1.68	2.29
	50.0	3.0	0.3	14.48	10.83	12.21	8.88	3.81	4.86	8.14	3.8	4.84
	50.0	5.0	0.3	14.55	10.92	12.35	8.92	3.79	4.9	8.18	3.78	4.87
	50.0	8.0	0.3	14.48	10.96	12.37	8.86	3.81	4.89	8.14	3.8	4.87
	100.0	3.0	0.1	5.67	3.95	4.42	3.43	1.12	1.45	3.22	1.12	1.45
	100.0	5.0	0.1	5.42	3.65	4.25	3.29	0.98	1.37	3.07	0.98	1.36
	100.0	8.0	0.1	5.37	3.62	4.23	3.24	0.96	1.36	3.04	0.96	1.35
	100.0	3.0	0.3	11.32	8.17	9.31	6.09	2.12	2.83	5.73	2.11	2.82
	100.0	5.0	0.3	11.18	8.17	9.31	5.99	2.11	2.82	5.64	2.11	2.82
	100.0	8.0	0.3	11.17	8.2	9.33	5.98	2.12	2.82	5.63	2.12	2.82
	200.0	3.0	0.1	4.55	2.96	3.33	2.54	0.58	0.77	2.34	0.58	0.77
	200.0	5.0	0.1	4.1	2.56	3.02	2.24	0.46	0.66	2.07	0.46	0.65
	200.0	8.0	0.1	4.01	2.55	2.99	2.18	0.46	0.65	2.02	0.46	0.65
	200.0	3.0	0.3	8.27	5.72	6.55	3.87	1.0	1.35	3.66	1.0	1.35
	200.0	5.0	0.3	8.5	5.96	6.8	4.07	1.07	1.43	3.86	1.07	1.43
	200.0	8.0	0.3	6.87	5.19	6.08	4.7	0.92	1.45	3.51	0.97	1.05
B	50.0	3.0	0.1	9.03	5.65	6.3	6.4	2.32	3.0	5.0	2.23	2.84
	50.0	5.0	0.1	6.22	4.12	4.88	3.85	1.24	1.74	3.49	1.24	1.73
	50.0	8.0	0.1	6.07	4.2	4.91	3.97	1.37	1.89	3.61	1.36	1.87
	50.0	3.0	0.3	14.16	9.91	11.33	9.04	3.27	4.29	8.2	3.24	4.25
	50.0	5.0	0.3	14.6	10.14	11.71	9.17	3.2	4.36	8.08	3.19	4.33
	50.0	8.0	0.3	13.93	10.0	11.38	8.74	3.28	4.26	7.73	3.26	4.22
	100.0	3.0	0.1	6.36	4.05	4.54	4.21	1.16	1.53	3.58	1.16	1.52
	100.0	5.0	0.1	5.43	3.13	3.63	3.54	0.78	1.07	2.73	0.78	1.06
	100.0	8.0	0.1	4.88	3.25	3.75	2.97	0.82	1.11	2.73	0.82	1.11
	100.0	3.0	0.3	12.09	7.96	9.2	7.17	2.09	2.86	6.16	2.09	2.84
	100.0	5.0	0.3	10.48	7.28	8.2	6.18	1.91	2.46	5.34	1.9	2.44
	100.0	8.0	0.3	10.77	7.32	8.24	6.52	1.96	2.53	5.38	1.94	2.49
	200.0	3.0	0.1	4.02	2.7	3.01	2.16	0.53	0.67	1.99	0.53	0.67
	200.0	5.0	0.1	3.93	2.31	2.7	2.31	0.42	0.59	1.9	0.42	0.59
	200.0	8.0	0.1	4.54	2.53	2.99	2.99	0.5	0.72	2.39	0.5	0.72
	200.0	3.0	0.3	8.04	5.4	6.14	4.44	1.0	1.32	3.89	1.0	1.31
	200.0	5.0	0.3	8.33	5.54	6.33	4.61	1.02	1.37	4.11	1.02	1.37
	200.0	8.0	0.3	7.94	5.31	6.05	4.37	0.94	1.25	3.71	0.94	1.25

The multi-stage setting has a decreasing impact on advertising as the seller is given more stages. Even without a cardinality constraint, the revenue improvement can still be significant when the utility of the no-purchase option is relatively high. The improvement under the UA strategy can be less than 0.5%, while the improvement using the heuristic method is at least 2%. This shows the importance of advertising strategy on expected revenue (Fig. 2).

Finally, we evaluate the computational efficiency of our Algorithm 3. Table 3 shows its average running time for different parameters. Our algorithm has a low computational complexity as it only requires solving two linear programming problems to find the optimal assortment and a few convex optimization problems to find the corresponding advertising strategy. The results in Table 3 demonstrate that our algorithm has a running time of less than 2 s for all cases, making it highly efficient.

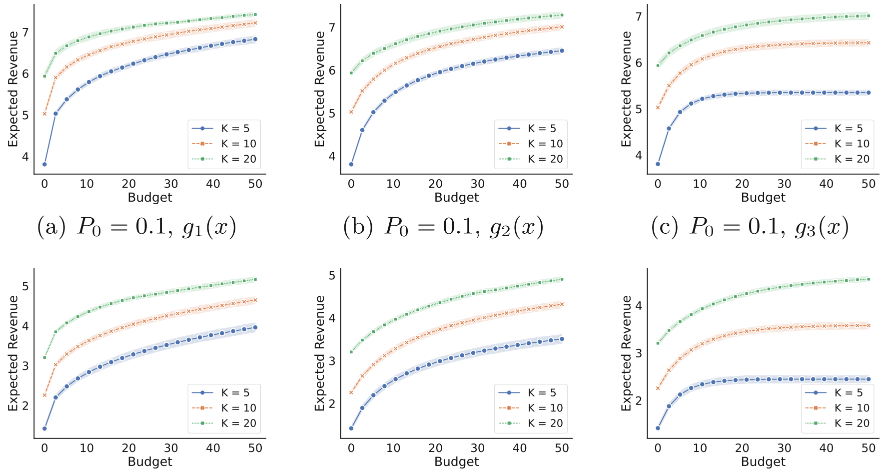
Table 3. Average Running Time of Algorithm 3

Parameters			$g_1(x)$		$g_2(x)$		$g_3(x)$	
n	K	P_0	A	B	A	B	A	B
50	5	0.1	0.076	0.065	1.499	1.533	1.131	1.1
50	10	0.1	0.074	0.075	1.543	1.452	1.454	1.445
50	20	0.1	0.072	0.129	1.342	0.74	1.651	1.472
50	5	0.3	0.111	0.136	0.625	0.562	0.986	0.778
50	10	0.3	0.131	0.122	0.57	0.57	0.677	0.671
50	20	0.3	0.127	0.154	0.764	0.721	0.96	0.87
100	5	0.1	0.09	0.142	1.042	1.129	1.046	0.916
100	10	0.1	0.128	0.115	0.83	0.775	1.28	0.768
100	20	0.1	0.165	0.131	0.749	0.818	1.305	1.105
100	5	0.3	0.167	0.193	0.615	0.744	0.935	0.739
100	10	0.3	0.196	0.197	0.749	0.742	0.751	0.744
100	20	0.3	0.163	0.174	0.826	0.827	0.932	0.851
200	5	0.1	0.219	0.166	0.764	1.06	1.593	1.474
200	10	0.1	0.169	0.226	0.895	0.838	1.179	1.033
200	20	0.1	0.165	0.166	1.015	1.136	0.807	0.948
200	5	0.3	0.368	0.289	0.962	1.086	1.039	0.986
200	10	0.3	0.289	0.238	1.054	1.106	0.967	1.019
200	20	0.3	0.215	0.235	1.084	1.059	1.141	1.125

C.3 Effect of Budget on Expected Revenue

In practicality, the seller must also decide on the advertising budget. Since the return per budget investment can reduce with increasing budget, this subsection examines the relationship between expected revenue and invested budget. The experiment has 100 products and the budget is varied from 0 to 50 while the revenue and advertising effectiveness are kept constant. 100 preference weights are sampled for each budget and response function. The results of the expected revenue for each of these settings are displayed in Fig. 1.

The expected revenue is shown to increase with an increase in advertising budget as illustrated by Fig. 1. For the first response function g_1, when the budget is adequate and $P_0 = 0.1$, the difference in revenue between the different cardinality constraints becomes small, as indicated by Fig. 1(a). Hence, when the seller has an adequate budget, limiting their focus to a small group of products does not result in a significant reduction in revenue. For the same response function, the trend of increasing expected revenue remains consistent across different values of P_0, with lower values leading to higher expected revenue. For the third response function, $g_3(x) = 2 - e^{-x}$, the increase in expected revenue becomes insignificant when more than 20 units of the budget are allocated to advertising.



(a) $P_0 = 0.1,\ g_1(x)$ (b) $P_0 = 0.1,\ g_2(x)$ (c) $P_0 = 0.1,\ g_3(x)$

(d) $P_0 = 0.3,\ g_1(x)$ (e) $P_0 = 0.3,\ g_2(x)$ (f) $P_0 = 0.3,\ g_3(x)$

Fig. 1. The relationship between budget and expected revenue for 100 products in different settings

D Omitted Proofs

Proof of Lemma 1: Consider an arbitrary advertising strategy $\mathbf{x}$ that satisfies $\sum_{i \in S^*} x_i = B_1 < B$, define $\Delta = B - B_1$ and $\tau = \arg\max_{i \in S^*} r_i$. To prove this lemma, we can increase the expected revenue by allocating the remaining budget Δ to the product τ. Let S^* denote the optimal assortment, and $\mathbf{x}'$ denote this new strategy. Because $\mathbf{x}'$ and $\mathbf{x}$ only differs in entry τ, we rewrite

$R(S^*, \mathbf{v}, \mathbf{x}')$ as $\frac{\beta + r_\tau v_\tau (g(c_\tau(x_\tau + \Delta)) - g(c_\tau x_\tau))}{\alpha + v_\tau (g(c_\tau(x_\tau + \Delta)) - g(c_\tau x_\tau))}$ and rewrite $R(S^*, \mathbf{v}, \mathbf{x})$ as $\frac{\beta}{\alpha}$, where $\beta = \sum_{i \in S^*} r_i v_i g(c_i x_i)$ and $\alpha = 1 + \sum_{i \in S^*} v_i g(c_i x_i)$. Notice that

$$R(S^*, \mathbf{v}, \mathbf{x}') - R(S^*, \mathbf{v}, \mathbf{x}) = \frac{(r_\tau \alpha - \beta) \cdot v_\tau (g(c_\tau(x_\tau + \Delta)) - g(c_\tau x_\tau))}{\alpha \cdot (\alpha + v_\tau (g(c_\tau(x_\tau + \Delta)) - g(c_\tau x_\tau)))}.$$

Because $\tau = \arg\max_{i \in S^*} r_i$, we have $r_\tau - r_i \geq 0, \forall i \in S^*$. Thus, $r_\tau + \sum_{i \in S^*} (r_\tau - r_i)(v_i g(c_i x_i)) > 0$, which is equivalent to $r_\tau (1 + \sum_{i \in S^*} v_i g(c_i x_i)) > \sum_{i \in S^*} r_i v_i g(c_i x_i)$. Hence, $r_\tau \alpha > \beta$. Because $g(\cdot)$ is an increasing function, we have $g(c_\tau(\Delta + x_\tau)) - g(c_\tau x_\tau) \geq 0$. Moreover, because both $r_\tau \alpha - \beta$ and $g(c_\tau(\Delta + x_\tau)) - g(c_\tau x_\tau)$ are non-negative, we obtain that $R(S^*, \mathbf{v}, \mathbf{x}') \geq R(S^*, \mathbf{v}, \mathbf{x})$. □

Proof of Lemma 3: *Proof:* Because $(S^*, \mathbf{x}^*)$ is optimal solution, we have $R(S^*, \mathbf{v}, \mathbf{x}^*) \geq R(S_\mathbf{v}, \mathbf{v})$. According to Lemma 2, there exists an S^* such that $S^* = \{i \in \mathcal{N} | r_i > R(S^*, \mathbf{v}, \mathbf{x}^*)\}$. The following chain proves this lemma: $S^* = \{i \in \mathcal{N} | r_i > R(S^*, \mathbf{v}, \mathbf{x}^*)\} \subseteq \{i \in \mathcal{N} | r_i > R(S_\mathbf{v}, \mathbf{v})\} = S_\mathbf{v}$. □

Proof of Lemma 4: *Proof:* Because $g(\cdot)$ is an increasing concave function, its inverse function $g^{-1}(x)$ is a convex function. Moreover, because $\frac{u}{v_i}$ is a linear function, its composition with $g^{-1}(\cdot)$ is also a convex function. Finally, because $\mathcal{U}_1 = \{\mathbf{u} | \sum_{i=1}^n m_i(u_i) \leq B\}$, which is the level set of $\sum_{i=1}^n m_i(u_i)$, is a convex set, its intersection with the convex set $\mathcal{U}_2 = \{\mathbf{u} | u_i \geq v_i, i = 1, \ldots, n\}$ is also a convex set. □

Proof of Lemma 5: *Proof:* For a given assortment S, we can represent any feasible advertising strategy $\mathbf{y}$ that satisfies $\sum_{i \in S} y_i = B$ as a convex combination of $\mathbf{x}_i$; that is, $\mathbf{y} = \sum_{i \in S} \lambda_i \mathbf{x}_i$, where $\lambda_i = y_i / B$ and $\sum_{i \in S} \lambda_i = 1$. Assume $k = \text{argmax}_{j \in S} L(S, \mathbf{x}_j)$. We have $L(S, \mathbf{y}) \leq L(S, \mathbf{x}_k)$ based on the following observation:

$$L(S, \mathbf{y}) = L(S, \sum_{i \in S} \lambda_i \mathbf{x}_i) = \frac{\beta + aB \sum_{i \in S} \lambda_i r_i v_i c_i}{\alpha + aB \sum_{i \in S} \lambda_i v_i c_i},$$

where $\alpha = 1 + \sum_{i \in S} v_i$ and $\beta = \sum_{i \in S} r_i v_i$. By the definition of k, we have $L(S, \mathbf{x}_k) \geq L(S, \mathbf{x}_j), \forall j \in S$. Moreover, $L(S, \mathbf{x}_k) \geq L(S, \mathbf{x}_j)$ is equivalent to

$$\alpha c_k v_k r_k - \beta c_k v_k \geq \alpha c_j v_j r_j - \beta c_j v_j + aBc_k v_k (r_j - r_k) c_j v_j, \tag{D.1}$$

based on the following observation:

$$\begin{aligned} L(S, \mathbf{x}_k) - L(S, \mathbf{x}_j) &= \frac{\beta + aBr_k c_k v_k}{\alpha + aBc_k v_k} - \frac{\beta + aBr_j c_j v_j}{\alpha + aBc_j v_j} \\ &= \frac{(\beta + aBr_k c_k v_k)(\alpha + aBc_j v_j) - (\beta + aBr_j c_j v_j)(\alpha + aBc_k v_k)}{(\alpha + aBc_k v_k)(\alpha + aBc_j v_j)} \\ &= aB \cdot \frac{\alpha(c_k v_k r_k - c_j v_j r_j) + \beta(c_j v_j - c_k v_k) + aBc_k v_k (r_k - r_j) c_j v_j}{(\alpha + aBc_k v_k)(\alpha + aBc_j v_j)}. \end{aligned}$$

By multiplying λ_j by both sides of (D.1) for all $j \in S$ and summing up all inequalities, we have

$$\alpha c_k v_k r_k - \beta c_k v_k \geq \alpha \sum_{j \in S} \lambda_j c_j v_j r_j - \beta \sum_{j \in S} \lambda_j c_j v_j + aBc_k v_k \sum_{j \in S} \lambda_j (r_j - r_k) c_j v_j. \tag{D.2}$$

Using a similar argument as the one used to prove the equivalence of $L(S, \mathbf{x}_k) \geq L(S, \mathbf{x}_j)$ and (D.1), we show that (D.2) is equivalent to $L(S, \mathbf{x}_k) \geq L(S, \sum_{i \in S} \lambda_i \mathbf{x}_i) = L(S, \mathbf{y})$. □

Proof of Lemma 6: *Proof:* Let $Z = R(S_{\mathbf{v}}, \mathbf{v})$ denote the expected revenue of $S_{\mathbf{v}}$ when $B = 0$, we have $\sum_{i \in S_{\mathbf{v}}} (r_i - Z) v_i = Z$. If there exists a product $i \in S_{\mathbf{v}}$ such that $r_i < Z$, then removing this product from $S_{\mathbf{v}}$ would increase the expected revenue, which contradicts the assumption that $S_{\mathbf{v}}$ is the optimal assortment when $B = 0$. Thus, we have $S_{\mathbf{v}} \subseteq \{1, \ldots, T\}$, where $T = \max\{i | i \in S_{\mathbf{v}}\}$. Similarly, let $Z^* = R(S^*, \mathbf{v}, \mathbf{x}^*)$, we have $S^* \subseteq \{1, \ldots T^*\}$ where $T^* = \max_i \{i | r_i \geq Z^*\}$. Since $Z^* \geq Z$, we conclude that $r_{T^*} \geq r_T$, otherwise we have $Z^* \leq r_{T^*} < Z$ or $Z \leq r_{T^*} < r_T$. Thus we have $S^* \subseteq \{1, \ldots T^*\} \subseteq \{1, \ldots, T\}$. □

Proof of Lemma A.2: *Proof:* When $g(x) = ax + 1$ for some $a \geq 0$, the objective function of (A.4) can be written as $\sum_{i=1}^n \alpha_i + a \sum_{i=1}^n \alpha_i c_i x_i$. Allocating the entire advertising budget to the product that has the largest $\alpha_i c_i$ maximizes $\sum_{i=1}^n \alpha_i + a \sum_{i=1}^n \alpha_i c_i x_i$. □

Proof of Lemma A.3: *Proof:* Consider a fixed feasible assortment S. If $f(x) = \log(ax+1)$ for some $a \geq 0$, then the objective function $R(S, \mathbf{p}, \mathbf{x})$ can be written as $\sum_{i \in S} \alpha_i + a \sum_{i \in S} \alpha_i c_i x_i$. It is easy to verify that the optimal advertising strategy for S must come from $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$, where $\mathbf{x}_0$ is an all-zero vector. Let $S(\mathbf{x})$ be the optimal assortment under the advertising strategy $\mathbf{x}$. Thus $S(\mathbf{x})$ contains the top K products that have the largest $\alpha_i(ac_i x_i + 1)$. Because $ac_i x_i \geq 0$ for all $i \in \{1, \ldots, n\}$, we have $S(\mathbf{x}_i) = S(\mathbf{x}_0) = \{1, \ldots, K\}$, and the expected revenue for $S(\mathbf{x}_i)$ is $W(\sum_{j=1}^K \alpha_j + a\alpha_i c_i B)$ for all $i \in \{1, ..., K\}$. When $j \in \{K+1, ..., n\}$, there are two possible cases: $S(\mathbf{x}_j) \setminus S(\mathbf{x}_0) = \{\emptyset\}$ or $S(\mathbf{x}_j) \setminus S(\mathbf{x}_0) = \{j\}$.

Case 1 : When $S(\mathbf{x}_j) \backslash S(\mathbf{x}_0) = \{\emptyset\}$ for all $j \in \{K+1, \ldots, n\}$, the optimal assortment is $\{1, \ldots, K\}$. Because the expected revenue for $S(\mathbf{x}_j)$ is $W(\sum_{j=1}^K \alpha_j)$ for all $j \in \{K+1, \ldots, n\}$, which is the same as $S(\mathbf{x}_0)$, and $t_1 = \operatorname{argmax}_{i \in \{1, \ldots, K\}} W(\sum_{j=1}^K \alpha_j + a\alpha_i c_i B)$, the optimal advertising strategy is $\mathbf{x}_{t_1}$.

Case 2 : When $S(\mathbf{x}_j) \backslash S(\mathbf{x}_0) = \{j\}$ for some $j \in \{K+1, ..., n\}$, the expected revenue for $S(\mathbf{x}_j)$ is $W(\sum_{i=1}^{K-1} \alpha_i + \alpha_j(ac_j B + 1))$. We denote this subset as S_c. Because $t_2 = \operatorname{argmax}_{j \in S_c} W(\sum_{i=1}^{K-1} \alpha_i + \alpha_j(ac_j B + 1))$, $\mathbf{x}_{t_2}$ is the best advertising strategy in $\{\mathbf{x}_{K+1}, \ldots, \mathbf{x}_n\}$. Moreover, because $\mathbf{x}_{t_1}$ is the best advertising strategy in $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_K\}$, the better strategy between $\mathbf{x}_{t_1}$ and $\mathbf{x}_{t_2}$ must be the optimal advertising strategy. □

Proof of Lemma A.4: When $c_i = c$ for all $i \in \mathcal{N}$, the objective function of (A.5) can be simplified to $h(\mathbf{x}, S) = \sum_{i \in S} \alpha_i g(cx_i)$. To prove the first part of this lemma, we show that for any optimal solution $(S, \mathbf{x})$, we can construct a new solution $(S', \mathbf{x}')$, where $S' = \{1, \ldots, K\}$, which is no worse than $(S, \mathbf{x})$. Due to the monotonicity of the objective function, $|S| = K$ can be assumed. We construct such $\mathbf{x}'$ as follows: for each $i \in \{1, \ldots, K\}$, let $x'_i = x_{L(i)}$, where $L(i)$ represents the product that has the i-th largest α_i in S. Therefore $h(\mathbf{x}', S') - h(\mathbf{x}, S) =$

$\sum_{i\in S'}(\alpha_i - \alpha_{L(i)})g(cx_i) \geq 0$; the inequality exists because S' contains the top K products that have the largest α_i. Hence, $(\mathbf{x}', S')$ is no worse than $(S, \mathbf{x})$.

We next prove that the optimal advertising strategy $\mathbf{x}^*$ satisfies $x_i^* \geq x_j^* \ \forall i \leq j$ through contradiction. Assume there exist two products $i, j \in S^*$ such that $x_i^* < x_j^*$ and $i < j$. We can construct a new advertising strategy $\mathbf{x}$ such that $x_k = x_k^*$ for $k \notin \{i, j\}$, and $x_i = x_j^*, x_j = x_i^*$. The following chain proves that $h(\mathbf{x}, S^*) - h(\mathbf{x}^*, S^*) = (\alpha_i - \alpha_j)\cdot(g(cx_j^*) - g(cx_i^*))$:

$$\begin{aligned} h(\mathbf{x}, S^*) - h(\mathbf{x}^*, S^*) &= \alpha_i g(cx_i) + \alpha_j g(cx_j) - \alpha_i g(cx_i^*) - \alpha_j g(cx_j^*) \\ &= \alpha_i(g(cx_j^*) - g(cx_i^*)) + \alpha_j(g(cx_i^*) - g(cx_j^*)) \\ &= (\alpha_i - \alpha_j)\cdot(g(cx_j^*) - g(cx_i^*)). \end{aligned}$$

Because $\alpha_i \geq \alpha_j$ and $x_i^* < x_j^*$, $\mathbf{x}$ is a better solution than $\mathbf{x}^*$ which contradicts to the assumption that $\mathbf{x}^*$ is the optimal solution. □

Proof of Lemma B.2: *Proof:* For simplicity, let $h_i^k = \frac{\partial R(\mathcal{S}^*, \mathbf{u}^*)}{\partial u_i^*}$ be the partial derivative for product i in assortment S_k^*, and denote $A_k = \sum_{l=1}^k V(S_l^*)$ and $B_k = \frac{W(S_k^*)}{V(S_k^*)}$. We have

$$h_i^k = \frac{(r_i - \frac{W(S_k^*)}{1+A_k})}{(1+A_{k-1})(1+A_k)} - \sum_{j=k+1}^{m} B_j \cdot \left(\frac{1}{(1+A_{j-1})^2} - \frac{1}{(1+A_j)^2}\right).$$

According to Lemma B.1, $S_k^* = \{i \in \mathcal{N} | t_{k+1}^* \leq r_i < t_k^*\}$. We first consider the products in the same stage, that is $i, j \in S_k^*$, and $i < j$. $h_i^k \geq h_j^k$ because $r_i \geq r_j$. Then, we consider the cases i and j in two different stages. The difference between h_i^k and h_j^{k+1} is

$$\begin{aligned} h_i^k - h_j^{k+1} &= \frac{(r_i - \frac{W(S_k^*)}{1+A_k})}{(1+A_{k-1})(1+A_k)} - B_{k+1}\cdot\left(\frac{1}{(1+A_k)^2} - \frac{1}{(1+A_{k+1})^2}\right) - \frac{(r_j - \frac{W(S_{k+1}^*)}{1+A_{k+1}})}{(1+A_k)(1+A_{k+1})} \\ &\geq \left[\frac{1}{(1+A_{k-1})(1+A_k)} - \frac{1}{(1+A_k)(1+A_{k+1})}\right] t_{k+1}^* - \frac{W(S_k^*)}{(1+A_{k-1})(1+A_k)^2} \\ &\quad - \frac{2(1+A_k) + V(S_{k+1}^*)}{(1+A_k)^2(1+A_{k+1})^2}\cdot W(S_{k+1}^*) + \frac{W(S_{k+1}^*)}{(1+A_k)(1+A_{k+1})^2} \\ &= \frac{W(S_k^*)}{(1+A_{k-1})(1+A_k)^2} + \frac{W(S_{k+1}^*)}{(1+A_k)^2(1+A_{k+1})} - \frac{W(S_k^*)}{(1+A_{k-1})(1+A_k)^2} \\ &\quad - \frac{2(1+A_k) + V(S_{k+1}^*)}{(1+A_k)^2(1+A_{k+1})^2}\cdot W(S_{k+1}^*) + \frac{W(S_{k+1}^*)}{(1+A_k)(1+A_{k+1})^2} \\ &= \frac{2 + A_k + A_{k+1} - 2(1+A_k) + V(S_{k+1}^*)}{(1+A_k)^2(1+A_{k+1})^2}\cdot W(S_{k+1}^*) \\ &= 0. \end{aligned}$$

The first inequality uses the fact that $r_i \geq t_{k+1}^* \geq r_j$. Lastly, for the product in the last assortment S_m, we have $h_i^m = \frac{r_i - \frac{W(S_m^*)}{1+A_m}}{(1+A_{m-1})(1+A_m)}$. In this case, $r_i \geq t_{m+1}^* = \frac{W(S_m^*)}{1+A_m}$, which means $h_i^m \geq 0$. □

Proof of Lemma B.3: *Proof:* Let $Q(\mathbf{x}) = R(\mathcal{S}^*, \mathbf{x})$. Based on the analysis in Lemma B.2, $\frac{\partial Q(\mathbf{x})}{\partial x_i} \geq \frac{\partial Q(\mathbf{x})}{\partial x_j} \geq 0$ for all $i < j$. For any $\mathbf{x}$ satisfying the budget constraint, through the mean value theorem, we have

$$\begin{aligned}
Q(\mathbf{x}) - Q(\mathbf{x}^*) &= \nabla Q(\mathbf{x} + (1-c)\mathbf{x}^*)^T \cdot (\mathbf{x} - \mathbf{x}^*) \\
&= Q(\mathbf{x}^c)^T \cdot (\mathbf{x} - \mathbf{x}^*) \\
&= \sum_{i=2}^{n} \frac{\partial Q(\mathbf{x}^c)}{\partial x_i} x_i^c + \frac{\partial Q(\mathbf{x}^c)}{\partial x_1}(x_1^c - B) \\
&\leq \frac{\partial Q(\mathbf{x}^c)}{\partial x_1} \sum_{i=2}^{n} x_i^c + \frac{\partial Q(\mathbf{x}^c)}{\partial x_1}(x_1^c - B) \\
&= \frac{\partial Q(\mathbf{x}^c)}{\partial x_1}(\sum_{i=1}^{n} x_i^c - B) \\
&\leq 0.
\end{aligned}$$

Here, $c \in (0, 1)$, and we denote $\mathbf{x} + (1-c)\mathbf{x}^*$ as $\mathbf{x}^c$. The first inequality exists because $\frac{\partial Q(\mathbf{x}^c)}{\partial x_i} \leq \frac{\partial Q(\mathbf{x}^c)}{\partial x_1}$, and the last inequality is due to the budget constraint. □

Proof of Lemma B.4: *Proof:* Let $T_k^* = \cup_{i=1}^k S_i^*$. We have

$$\begin{aligned}
R(\mathcal{S}^*, \mathbf{x}^*) &= \sum_{k=1}^{m} \cdot \frac{W(T_k^*, \mathbf{x}^*) - W(T_{k-1}^*, \mathbf{x}^*)}{(1 + V(T_{k-1}^*, \mathbf{x}^*))(1 + V(T_k^*, \mathbf{x}^*))} \\
&= \sum_{k=1}^{m-1} \frac{W(T_k^*, \mathbf{x}^*)}{1 + V(T_k^*, \mathbf{x}^*)} \left\{ \frac{1}{1 + V(T_{k-1}^*, \mathbf{x}^*)} - \frac{1}{1 + V(T_{k+1}^*, \mathbf{x}^*)} \right\} + \frac{W(T_m^*, \mathbf{x}^*)}{(1 + V(T_{m-1}^*, \mathbf{x}^*))(1 + V(T_m^*, \mathbf{x}^*))} \\
&\leq \max_{S, \mathbf{x}} \frac{W(S, \mathbf{x})}{1 + V(S, \mathbf{x})} \left[\sum_{k=1}^{m-1} \left\{ \frac{1}{1 + V(T_{k-1}^*, \mathbf{x}^*)} - \frac{1}{1 + V(T_{k+1}^*, \mathbf{x}^*)} \right\} + \frac{1}{1 + V(T_{m-1}^*, \mathbf{x}^*)} \right] \\
&= \max_{S, \mathbf{x}} \frac{W(S, \mathbf{x})}{1 + V(S, \mathbf{x})} \left[1 + \frac{1}{1 + V(T_1^*, \mathbf{x}^*)} - \frac{1}{1 + V(T_m^*, \mathbf{x}^*)} \right] \\
&\leq 2 \max_{S, \mathbf{x}} \frac{W(S, \mathbf{x})}{1 + V(S, \mathbf{x})} \\
&\leq 2R(\mathcal{S}^h, \mathbf{x}^h).
\end{aligned}$$

The last inequality holds because our heuristic method starts with the optimal solution of the single-stage problem and iteratively improves upon it. □

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