Weakly Nondominated Solutions of Set-Valued Optimization Problems with Variable Ordering Structures in Linear Spaces

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Abstract. In this paper, weakly nondominated solutions of set-valued optimization problems with variable ordering structures are investigated in linear spaces. Firstly, the notion of weakly nondominated element of a set with a variable ordering structure is introduced in linear spaces, and the relationship between weakly nondominated element and nondominated element is also given. Secondly, under the assumption of nearly $\mathcal{C}(y)$-subconvexlikeness of set-valued maps, scalarization theorems of weakly nondominated solutions for unconstrained set-valued optimization problems are established. Finally, two duality theorems of constrained set-valued optimization problems are obtained. Some examples are given to illustrate our results. The results obtained in this paper improve and generalize some known results in the literatures.

Keywords: Set-valued maps · Variable ordering structures · Weakly nondominated solution · Scalarization · Duality

1 Introduction

Set-valued analysis has become an important branch of nonlinear analysis since it is widely applied in various areas of the human real life. For example, Debreu [5] used the fixed point theorem of the set-valued map, which is an important mathematical tool, to prove the existence of the Walrasian equilibrium theorem. Some works about set-valued analysis can be founded in [2,3,10]. Recently, many researchers have paid attention to the set-valued optimization problem which is a kind of optimization problem with the objective map being a set-valued map. Yang et al. [16] introduced the nearly cone subconvexlike set-valued map and

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established optimality conditions involving Lagrangian multiplier and scalarization of set-valued optimization problems. Zhao et al. [19] used the improvement set E, which was introduced by Chicco et al. [4], to define nearly E-subconvexlike set-valued map and investigated the weak E-optimal solution of set-valued optimization problems. Zhou et al. [20] studied scalarizations and optimality of constrained set-valued optimization using improvement sets and image space analysis.

On the other hand, in order to compare different objective values of the optimization problem, we need to establish a partial ordering relation which is induced by a pointed closed convex cone. Generally speaking, the ordering relation involving optimization problems is determined by a fixed convex cone. However, in actual situations, different decision-makers have different preferences in different environments. Therefore, the partial ordering relation involved in optimization problems is no longer determined by a fixed convex cone. Instead, it is determined by a variable ordering cone related to environment, times, economy and other factors. This kind of optimization problems are called optimization problems with variable ordering structures. More general concepts of ordering structures were introduced by Yu [18] in terms of domination structures. Eichfelder and Kasimbeyli [6,7] studied optimal elements and proper optimal elements in vector optimization with variable ordering structures. Shahbeyk [13] investigated Hartley properly and super nondominated solutions in vector optimization with variable ordering structures. Further, approximate solutions of vector optimization problems with variable ordering structures were also studied in [12,14,17].

Recently, some researchers have studied optimization problems in linear spaces without any topology structure. Li [11] used the separation theorem of convex sets in a real linear space to establish a theorem of the alternative for cone subconvexlike set-valued maps and obtained optimality conditions for vector optimization of set-valued maps. In linear spaces, properly efficient solutions of set-valued optimization problems, including Benson properly efficient solution [8] and super efficient solution [21], also were introduced.

However, to the best of our knowledge, there are few literatures involving set-valued optimization with variable ordering structures in linear spaces. Therefore, how to generalize some results obtained by the above references from topological spaces to linear spaces is interesting.

Inspired by [6,11,16,18], we will research weakly nondominated solution of set-valued optimization problems with variable ordering structures in linear spaces. This paper is organized as follows. In Sect. 3, we give some preliminaries including some basic notions and lemmas. In Sects. 4, we establish scalarization characterizations of weakly nondominated solution of unconstrained set-valued optimization problems with variable ordering structures in linear spaces. In Sects. 5, we obtain two duality theorems of unconstrained set-valued optimization problems, including a weak dual theorem and a strong dual theorem.

2 Preliminaries and Lemmas

Throughout this paper, we suppose that X and Y are two real linear spaces. Let A and M be two nonempty sets in X and Y, respectively. 0 stands for the zero element in every space. The generated cone of M is defined as $\text{cone}M := \{\lambda m \mid m \in M,\ \lambda \geq 0\}$. M is called a convex cone iff

$$\lambda_1 m_1 + \lambda_2 m_2 \in M,\ \forall \lambda_1, \lambda_2 \geq 0,\ \ \forall m_1, m_2 \in M.$$

M is said to be pointed iff $M \cap (-M) = \{0\}$. M is said to be nontrivial iff $M \neq \{0\}$ and $M \neq Y$. The algebraic dual of Y is denoted by Y^*. Let C be a nontrivial, pointed and convex cone in Y. The algebraic dual cone C^+ of C is defined as $C^+ := \{y^* \in Y^* \mid \langle y,\ y^* \rangle \geq 0,\ \forall y \in C\}$, where $\langle y,\ y^* \rangle$ denotes the value of the linear functional y^* at the point y.

Definition 2.1 [9]**.** Let M be a nonempty subset in Y. The algebraic interior of M is the set $\text{cor}M := \{m \in M \mid \forall h \in Y,\ \exists \epsilon > 0,\ \forall \lambda \in [0, \epsilon], m + \lambda h \in M\}$.

Definition 2.2 [1]**.** Let M be a nonempty subset in Y. The vector closure of M is the set $\text{vcl}M := \{m \in Y \mid \exists h \in Y,\ \forall \epsilon > 0,\ \exists \lambda \in [0, \epsilon], m + \lambda h \in M\}$.

In this paper, we assume that the variable ordering structure is given by the set-valued map $\mathcal{C} : Y \rightrightarrows Y$ with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for any $y \in Y$. Let $F : X \rightrightarrows Y$ be a set-valued map on A. We write

$$\langle F(x),\ y^* \rangle := \{\langle y,\ y^* \rangle \mid y \in F(x)\}$$

and

$$F(A) := \bigcup_{x \in A} F(x).$$

Now, we give a new notion of generalized convexity with the variable ordering structure.

Definition 2.3. Let $F : X \rightrightarrows Y$ be a set-valued map on A, and $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone for all $y \in Y$. F is called near $\mathcal{C}(y)$-subconvexlike on A iff, for any $y \in Y$, $\text{vcl}(\text{cone}(F(A) + \mathcal{C}(y)))$ is a convex set in Y.

Remark 2.1. When Y is a topological space and $\mathcal{C}(y) = C$ for any $y \in Y$, Definition 2.3 reduces to Definition 2.2 in [16].

Definition 2.4. [18]**.** Let M be a nonempty subset of Y, and $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in M$. $\overline{y} \in M$ is called a nondominated element of M w.r.t. $\mathcal{C}$ (denoted by $\overline{y} \in \text{N}(M, \mathcal{C}(\cdot))$) iff there does not exist $y \in M$ such that $\overline{y} \in y + \mathcal{C}(y) \backslash \{0\}$. Equivalently, $\overline{y} \notin M + \mathcal{C}(y) \backslash \{0\}$ for any $y \in M$.

Remark 2.2. It follows from Definition 2.4 that $\overline{y} \in M$ is a nondominated element of M w.r.t. $\mathcal{C}$ iff there exists $\mathcal{C} : Y \rightrightarrows Y$ such that $M \cap (\overline{y} - \mathcal{C}(y)) = \{\overline{y}\}$ for any $y \in M$.

Definition 2.5 [6]**.** Let M be a nonempty subset of Y, and $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in M$. $\overline{y} \in M$ is called a weakly nondominated element of M w.r.t. $\mathcal{C}$ (denoted by $\overline{y} \in \text{WN}(M, \mathcal{C}(\cdot))$) iff there does not exist $y \in M$ such that $\overline{y} \in y + \text{cor}\mathcal{C}(y)$.

Remark 2.3. It follows from Definition 2.5 that $\overline{y} \in \text{WN}(M, \mathcal{C}(\cdot))$ iff there exists $\mathcal{C} : Y \rightrightarrows Y$ such that $(M - \overline{y}) \cap (-\text{cor}\mathcal{C}(y)) = \emptyset$ for any $y \in M$.

Remark 2.4. Clearly, $\text{N}(M, \mathcal{C}(\cdot)) \subseteq \text{WN}(M, \mathcal{C}(\cdot))$. However, the following example shows that $\text{WN}(M, \mathcal{C}(\cdot)) \nsubseteq \text{N}(M, \mathcal{C}(\cdot))$.

Example 2.1. Let $Y = \mathbb{R}^2, M = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1 - 1)^2 + (y_2 - 1)^2 \leq 1\} \cup \{(0, 0), (0, -1)\}$ and $\overline{y} = (0, 0)$. The set-valued map $\mathcal{C} : Y \rightrightarrows Y$ is defined as

$$\mathcal{C}(y) := \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 - y_1 \geq 0,\ y_2 \geq 0,\ y_1 \geq 0\}, & y \in Y \setminus \{(1, \frac{1}{2})\} \\ \{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 - y_1 \leq 0,\ y_2 \geq 0,\ y_1 \geq 0\}, & y = (1, \frac{1}{2}). \end{cases}$$

It is easy to check

$$(M - \overline{y}) \cap (-\text{cor}\mathcal{C}(y)) = \emptyset, \forall y \in M.$$

However, there exists $\tilde{y} = (0, -1) \in M$ such that $M \cap (\overline{y} - \mathcal{C}(\tilde{y})) = \{(0, -1), (0, 0)\} \neq \{(0, 0)\}$. Therefore, $\overline{y} \in \text{WN}(M, \mathcal{C}(\cdot))$ and $\overline{y} \notin \text{N}(M, \mathcal{C}(\cdot))$. Thus, $\text{WN}(M, \mathcal{C}(\cdot)) \nsubseteq \text{N}(M, \mathcal{C}(\cdot))$.

Definition 2.6 [7]**.** let M be a nonempty subset of Y, and $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in M$. $\overline{y} \in M$ is called a weakly max-nondominated element of M w.r.t. $\mathcal{C}$ (denoted by $\text{WMN}(M, \mathcal{C}(\cdot))$) iff there does not exist $y \in M$ such that $\overline{y} \in y - \text{cor}\mathcal{C}(y)$.

Let $F : A \rightrightarrows Y$ be a set-valued map with nonempty value. Consider the following unconstrained set-valued optimization problem:

$$(\text{SVOP}) \begin{cases} \text{Min } F(x) \\ x \in A, \end{cases}$$

where $A \subseteq X$.

Based on Definition 2.5, we introduce the concept of the weakly nondominated solution of (SVOP).

Definition 2.7. $\overline{x} \in A$ is called a weakly nondominated solution of (SVOP) w.r.t $\mathcal{C}$ iff there exist $\overline{x} \in A$, $\overline{y} \in F(\overline{x})$ and $\mathcal{C} : Y \rightrightarrows Y$ with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in F(A)$ such that $\overline{y} \in \text{WN}(F(A), \mathcal{C}(\cdot))$. $(\overline{x}, \overline{y})$ is called a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$.

Lemma 2.1 [15]**.** Let $P, Q \subseteq Y$ be two convex sets such that $P \neq \emptyset$, $\text{cor}Q \neq \emptyset$ and $P \cap \text{cor}Q = \emptyset$. Then, there exists a hyperplane separating P and Q in Y.

Similarly to Lemma 2.1 [11] and Lemma 3.21(b) [9], we have the following lemmas.

Lemma 2.2. Let $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in Y$. Then, $\mathcal{C}(y) + \text{cor}\mathcal{C}(y) = \text{cor}\mathcal{C}(y)$ for $y \in Y$.

Lemma 2.3. Let $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in Y$. Then,

$$\text{cor}\mathcal{C}(y) \subseteq \{b \in Y | \langle b, b^* \rangle > 0, \forall b^* \in \mathcal{C}(y)^+ \setminus \{0\}\}, \forall y \in Y.$$

3 Scalarization

In this section, we will establish scalarization theorems of an unconstrained set-valued optimization problem in the sense of weakly nondominated element. Now, we consider the following scalar problem of (SVOP):

$$(\text{SVOP})_\varphi \begin{cases} \text{Min } \langle F(x), \varphi \rangle \\ x \in A, \end{cases}$$

where $\varphi \in Y^* \setminus \{0\}$.

Definition 3.1 [11]**.** Let $\overline{x} \in A$ and $\overline{y} \in F(\overline{x})$. $\overline{x}$ is called an optimal solution of $(\text{SVOP})_\varphi$ iff

$$\langle \overline{y}, \varphi \rangle \leq \langle y, \varphi \rangle, \forall y \in F(A).$$

$(\overline{x}, \overline{y})$ is called an optimal element of $(\text{SVOP})_\varphi$.

Now, we give an optimality necessary condition of weakly nondominated element of (SVOP) under the suitable assumptions.

Theorem 3.1. Let $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for any $y \in F(A)$. Suppose that the following conditions hold.

(i) $(\overline{x}, \overline{y})$ is a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$;

(ii) $F - \overline{y}$ is nearly $\mathcal{C}(\cdot)$-subconvexlike on A.

Then, for any $y \in F(A)$, there exists $\varphi \in (\mathcal{C}(y))^+ \setminus \{0\}$ such that $(\overline{x}, \overline{y})$ is an optimal element of $(\text{SVOP})_\varphi$.

Proof. Since $(\overline{x}, \overline{y})$ is a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$, we have

$$(F(A) - \overline{y}) \cap (-\text{cor}\mathcal{C}(y)) = \emptyset, \forall y \in F(A). \tag{1}$$

We assert that

$$\text{cone}(F(A) + \mathcal{C}(y) - \overline{y}) \cap (-\text{cor}\mathcal{C}(y)) = \emptyset, \forall y \in F(A). \tag{2}$$

Otherwise, there exsists $y_0 \in F(A)$ such that

$$\text{cone}(F(A) + \mathcal{C}(y_0) - \overline{y}) \cap (-\text{cor}\mathcal{C}(y_0)) \neq \emptyset. \tag{3}$$

By (3), there exist $d > 0$, $y_1 \in F(A)$ and $c \in \mathcal{C}(y_0)$ such that $d(y_1 + c - \overline{y}) \in -\text{cor}\mathcal{C}(y_0)$. Hence,

$$y_1 + c - \overline{y} \in -\text{cor}\mathcal{C}(y_0). \tag{4}$$

It follows from (4) and Lemma 2.2 that

$$y_1 - \overline{y} \in -c - \text{cor}\mathcal{C}(y_0) \subseteq -\mathcal{C}(y_0) - \text{cor}\mathcal{C}(y_0) = -\text{cor}\mathcal{C}(y_0),$$

which contradicts (1). Hence, (2) holds. We again assert that

$$\text{vcl}(\text{cone}(F(A) + \mathcal{C}(y) - \overline{y})) \cap (-\text{cor}\mathcal{C}(y)) = \emptyset, \forall y \in F(A). \tag{5}$$

Otherwise, there exist $y_1 \in F(A)$ and $a \in \text{vcl}(\text{cone}(F(A) + \mathcal{C}(y_1) - \overline{y}))$ such that

$$a \in -\text{cor}\mathcal{C}(y_1). \tag{6}$$

Since $a \in \text{vcl}(\text{cone}(F(A) + \mathcal{C}(y_1) - \overline{y}))$, there exist $h \in Y$ and $\lambda_n > 0$ with $\lim\limits_{n\to\infty} \lambda_n = 0$ such that

$$a + \lambda_n h \in \text{cone}(F(A) + \mathcal{C}(y_1) - \overline{y}), \quad \forall n \in \mathbb{N}, \tag{7}$$

where $\mathbb{N}$ is the set of the natural numbers. It follows from (6) that $a \in \text{cor}(-\text{cor}\mathcal{C}(y_1))$. Therefore, for the above h, there exists $\lambda' > 0$ such that

$$a + \lambda h \in -\text{cor}\mathcal{C}(y_1), \forall \lambda \in [0, \lambda'].$$

Taking a sufficiently big $n' \in \mathbb{N}$ such that $\lambda_{n'} \in [0, \lambda']$, we have

$$a + \lambda_{n'} h \in -\text{cor}\mathcal{C}(y_1). \tag{8}$$

It follows from (7) and (8) that $a + \lambda_{n'} h \in \text{cone}(F(A) + \mathcal{C}(y_1) - \overline{y}) \cap (-\text{cor}\mathcal{C}(y_1))$, which contradicts (2). Therefore, (5) holds.

By Condition (ii), $\text{vcl}(\text{cone}(F(A) + \mathcal{C}(y) - \overline{y}))$ is a convex set for any $y \in F(A)$. Clearly, $\text{vcl}(\text{cone}(F(A) + \mathcal{C}(y) - \overline{y})) \neq \emptyset$ and $\text{cor}(\mathcal{C}(y)) \neq \emptyset$ for any $y \in F(A)$. Hence, it follows from Lemma 2.1 that there exists $\varphi \in Y^* \setminus \{0\}$ such that

$$\langle y_2, \varphi \rangle \geq \langle y_3, \varphi \rangle, \forall y \in F(A), \forall y_2 \in \text{vcl}(\text{cone}(F(A) + \mathcal{C}(y) - \overline{y})), \forall y_3 \in -\mathcal{C}(y). \tag{9}$$

By (9), we obtain

$$\langle y_2, \varphi \rangle \geq 0, \forall y \in F(A), \forall y_2 \in F(A) + \mathcal{C}(y) - \overline{y}. \tag{10}$$

Since $0 \in \mathcal{C}(y)$ for any $y \in F(A)$, it follows from (10) that

$$\langle \overline{y}, \varphi \rangle \leq \langle y, \varphi \rangle, \forall y_2 \in F(A). \tag{11}$$

We assert that $\varphi \in (\mathcal{C}(y))^+ \setminus \{0\}$ for all $y \in F(A)$. Otherwise, there exists $\tilde{y} \in F(A)$ such that $\varphi \notin (\mathcal{C}(\tilde{y}))^+ \setminus \{0\}$. Thus, there exists $c' \in \mathcal{C}(\tilde{y})$ such that

$$\langle c', \varphi \rangle < 0. \tag{12}$$

Since $\overline{y} \in F(A), \tilde{y} \in F(A)$ and $c' \in \mathcal{C}(\tilde{y})$, it follows from (10) that

$$\langle c', \varphi \rangle \geq 0,$$

which contradicts (12). Therefore, $\varphi \in (\mathcal{C}(y))^+ \setminus \{0\}$ for all $y \in F(A)$. (11) shows that $(\overline{x}, \overline{y})$ is an optimal element of $(\text{SVOP})_\varphi$. □

Remark 3.1. Theorem 3.1 improves the necessity of Theorem 3.1 [11] in the following two aspects. Firstly, the fixed ordering cone C in Theorem 3.1 [11] has been replaced by the variable ordering cone $\mathcal{C}(.)$ in Theorem 3.1. Secondly, the C-subconvexlikeness of F in Theorem 3.1 [11] has been replaced by the near $\mathcal{C}(\cdot)$-subconvexlikeness of F in Theorem 3.1.

The following example is used to illustrate to Theorem 3.1.

Example 3.1. Let $X = Y = \mathbb{R}^2$ and $A = [0, 2] \times \{0\} \subseteq \mathbb{R}^2$. The set-valued map $F : X \rightrightarrows Y$ on A is defined as follows:

$$F(x_1, x_2) := \{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = x_1, 1 - \sqrt{1 - (x_1 - 1)^2} \leq y_2 \leq 1 + \sqrt{1 - (x_1 - 1)^2}\} \cup \{(0, 0)\},$$

where $(x_1, x_2) \in A$. Let $\overline{x} = (0, 0)$ and $\overline{y} = (0, 0)$. The set-valued map $\mathcal{C} : Y \rightrightarrows Y$ is defined as

$$\mathcal{C}(y) := \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 - y_1 \geq 0,\ y_2 \geq 0,\ y_1 \geq 0\}, & y \in Y \setminus \{(1, \frac{1}{2})\} \\ \{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 - y_1 \leq 0,\ y_2 \geq 0,\ y_1 \geq 0\}, & y = (1, \frac{1}{2}). \end{cases}$$

It is easy to check that Conditions (i) and (ii) in Theorem 3.1 are satisfied. Therefore, for any $y \in F(A)$, there exists $\varphi = (1, 1) \in (\mathcal{C}(y))^+ \setminus \{(0, 0)\} = \{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 \geq 0,\ y_1 \geq 0\} \setminus \{(0, 0)\}$ such that

$$\langle (0, 0), \varphi \rangle = 0 \leq \langle y, \varphi \rangle = y_1 + y_2, \forall (y_1, y_2) \in F(A).$$

Hence, $((0, 0), (0, 0))$ is an optimal element of $(\text{SVOP})_\varphi$.

Theorem 3.2. Let $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued map with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\text{cor}\mathcal{C}(y) \neq \emptyset$ for any $y \in F(A)$. Let $\overline{x} \in A$ and $\overline{y} \in F(\overline{x})$. Suppose that the following conditions hold.
(i) $\varphi \in (\mathcal{C}(y))^+ \setminus \{0\}$ for any $y \in F(A)$;
(ii) $(\overline{x}, \overline{y})$ is an optimal element of $(\text{SVOP})_\varphi$.
Then, $(\overline{x}, \overline{y})$ is a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$.

Proof. By Condition (ii), we have

$$\langle \overline{y}, \varphi \rangle \leq \langle y, \varphi \rangle, \forall y \in F(A). \tag{13}$$

Suppose that $(\overline{x}, \overline{y})$ is not a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$. Then, there exists $\tilde{y} \in F(A)$ such that $(F(A) - \overline{y}) \cap (-\text{cor}\mathcal{C}(\tilde{y})) \neq \emptyset$. Let

$$a \in (F(A) - \overline{y}) \cap (-\text{cor}\mathcal{C}(\tilde{y})). \tag{14}$$

It follows from (14) that there exists $y_1 \in F(A)$ such that

$$a = y_1 - \overline{y} \in -\mathrm{cor}(\mathcal{C}(\tilde{y})). \tag{15}$$

By (15) and Lemma 2.3, we have

$$\langle y_1 - \overline{y}, \varphi \rangle < 0. \tag{16}$$

On the other hand, it follows from (13) that $\langle y_1 - \overline{y}, \varphi \rangle \geq 0$, which contradicts (16). Therefore, $(\overline{x}, \overline{y})$ is a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$. □

Remark 3.2. When $\mathcal{C}(y) = C$ for any $y \in F(A)$, Theorem 3.2 reduces to the sufficiency of Theorem 3.1 in [11].

The following example is used to illustrate to Theorem 3.2.

Example 3.2. In Example 3.1, let $\overline{x} = (0,0)$ and $\overline{y} = (0,0) \in F(0,0)$. There exists $\varphi = (1,1) \in (\mathcal{C}(y))^+ \setminus \{0\} = \{(y_1, y_2) \in \mathbb{R}^2 | \ y_2 \geq 0, \ y_1 \geq 0\} \setminus \{(0,0)\}$. Hence, Condition (i) in Theorem 3.2 holds. Clearly,

$$\langle (0,0), (1,1) \rangle = 0 \leq \langle (y_1, y_2), (1,1) \rangle = y_1 + y_2, \forall (y_1, y_2) \in F(A).$$

Therefore, Condition (ii) in Theorem 3.2 holds. It is easy to check

$$(F(A) - \overline{y}) \cap (-\mathrm{cor}\mathcal{C}(y)) = \emptyset, \forall y \in F(A).$$

Thus, $(\overline{x}, \overline{y})$ is a weakly nondominated element of (SVOP) w.r.t. $\mathcal{C}$.

4 Duality

In this section, we will consider the duality problem of the constrained set-valued optimization problem and present a weak and a strong duality theorem in sense of weakly nondominated element.

Let $\widehat{S} \neq \emptyset$ be a nonempty subset of X. Let $D \subseteq Z$ be a nontrivial pointed convex cone in Z. Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be two set-valued map on $\widehat{S}$. We consider the following constrainted set-valued optimization problem:

$$(\mathrm{CSVOP}) \begin{cases} \mathrm{Min}\ F(x) \\ G(x) \cap (-D) \neq \emptyset \\ x \in \widehat{S}. \end{cases}$$

The feasible set of (CSVOP) is denoted by $S := \{x \in \widehat{S} | \ G(x) \cap (-D) \neq \emptyset\}$.

Let $\mathcal{C} : Y \rightrightarrows Y$ be a set-valued maps with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\mathrm{cor}\mathcal{C}(y) \neq \emptyset$ for any $y \in Y$. We write

$$C_1 := \{y \in Y | \exists (\varphi, \mu) \in (\mathcal{C}(y)^+ \setminus \{0\}) \times D^+, \forall d \in \bigcup_{x \in \widehat{S}} (\langle F(x), \varphi \rangle + \langle G(x), \mu \rangle), d \geq \langle y, \varphi \rangle\}.$$

Now, we give the definition of the weakly nondominated element of (CSVOP).

Definition 4.1. Let $\overline{x} \in S$ and $\overline{y} \in F(\overline{x})$. $\overline{x} \in S$ is called a weakly nondominated solution of (CSVOP) w.r.t $\mathcal{C}$ iff there exists $\mathcal{C} : Y \rightrightarrows Y$ with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\mathrm{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in F(S)$ such that $\overline{y} \in \mathrm{WN}(F(S), \mathcal{C}(\cdot))$. $(\overline{x}, \overline{y})$ is called a weakly nondominated element of (CSVOP) w.r.t. $\mathcal{C}$.

Firsty, we present a weak duality theorem.

Theorem 4.1. For any $\overline{y} \in C_1$, there exists $\varphi \in (\mathcal{C}(\overline{y}))^+\backslash\{0\}$ such that

$$\langle \overline{y}, \varphi \rangle \leq \langle y, \varphi \rangle, \forall y \in F(S). \tag{17}$$

Proof. Since $\overline{y} \in C_1$, there exists $(\varphi, \mu) \in (\mathcal{C}(\overline{y})^+\backslash\{0\}) \times D^+$ such that

$$d \geq \langle \overline{y}, \varphi \rangle, \forall d \in \bigcup_{x \in \widehat{S}} (\langle F(x), \varphi \rangle + \langle G(x), \mu \rangle). \tag{18}$$

By (18), we have

$$\langle y, \varphi \rangle + \langle z, \mu \rangle \geq \langle \overline{y}, \varphi \rangle, \forall x \in S, \forall y \in F(x), \forall z \in G(x). \tag{19}$$

According to $\mu \in D^+$, we have

$$\langle z, \mu \rangle \leq 0, \forall z \in G(x) \cap (-D). \tag{20}$$

It follows from (19) and (20) that (17) holds. □

Remark 4.1. When set-valued maps $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ become vector-valued maps $f : X \to Y$ and $g : X \to G$, Theorem 4.1 reduces Theorem 4.5 in [6].

Next, we state the following strong duality theorem.

Theorem 4.2. $\mathcal{C} : Y \rightrightarrows Y$ with $\mathcal{C}(y)$ being a nontrivial pointed convex cone and $\mathrm{cor}\mathcal{C}(y) \neq \emptyset$ for all $y \in C_1$. Suppose that the following conditions hold:

(i) $(\overline{x}, \overline{y})$ is a weakly nondominated element of (CSVOP);

(ii) $F - \overline{y}$ is nearly $\mathcal{C}(\cdot)$-subconvexlike on S;

(iii) There exists $\varphi \in (\mathcal{C}(\overline{y}))^+\backslash\{0\}$ such that

$$\langle \overline{y}, \varphi \rangle \leq \langle y, \varphi \rangle, \forall y \in F(S); \tag{21}$$

(iv) For the above φ,

$$\inf \bigcup_{x \in S} \langle F(x), \varphi \rangle = \sup\{\inf \bigcup_{x \in \widehat{S}} (\langle F(x), \varphi \rangle + \langle G(x), \mu \rangle) | \mu \in D^+\}, \tag{22}$$

and

$$\sup\{\inf \bigcup_{x \in \widehat{S}} (\langle F(x), \varphi \rangle + \langle G(x), \mu \rangle) | \mu \in D^+\}$$

has at least one solution.

Then, $\overline{y}$ is a weakly max-nondominated element of C_1 w.r.t. $\mathcal{C}$.

Proof. Since $\sup\{\inf\bigcup_{x\in\widehat{S}}(\langle F(x),\varphi\rangle + \langle G(x),\mu\rangle)|\mu \in D^+\}$ has at least one solution, it follows from (22) that there exists $\overline{\mu}\in D^+$ such that

$$\inf\bigcup_{x\in S}\langle F(x),\varphi\rangle = \inf\bigcup_{x\in\widehat{S}}(\langle F(x),\varphi\rangle + \langle G(x),\overline{\mu}\rangle). \tag{23}$$

By (21) and (23), we obtain

$$d \geq \inf\bigcup_{x\in S}\langle F(x),\varphi\rangle = \langle\overline{y},\varphi\rangle, \forall d \in \bigcup_{x\in\widehat{S}}(\langle F(x),\varphi\rangle + \langle G(x),\overline{\mu}\rangle). \tag{24}$$

(24) shows that $\overline{y}\in C_1$. Therefore, $\overline{y}\in F(S)\cap C_1$.

We assert that

$$c - \overline{y} \notin \mathrm{cor}\mathcal{C}(c), \forall c \in C_1. \tag{25}$$

Otherwise, there exists $\overline{c}\in C_1$ such that

$$\overline{c} - \overline{y} \in \mathrm{cor}\mathcal{C}(\overline{c}). \tag{26}$$

According to (26) and Lemma 2.3, we have

$$\langle\overline{c} - \overline{y}, \varphi'\rangle > 0, \forall\varphi' \in (\mathcal{C}(\overline{c}))^+ \setminus \{0\}. \tag{27}$$

It follows from (27) that

$$\langle\overline{c}, \varphi'\rangle > \langle\overline{y}, \varphi'\rangle, \forall\varphi' \in (\mathcal{C}(\overline{c}))^+ \setminus \{0\}. \tag{28}$$

Since $\overline{y}\in F(S)$, (28) contradicts Theorem 4.1. Therefore, (25) holds. Thus, $\overline{y}$ is a weakly max-nondominated element of C_1 w.r.t. $\mathcal{C}$. □

Remark 4.2. It follows from Theorem 3.1 that Conditions (i) and (ii) ensure the existence of φ in Condition (iii).

Remark 4.3. Theorem 4.2 improves Theorem 4.6 [6] in the following two aspects. First, the $\mathcal{C}(\overline{y})$-convexity of f in Theorem 4.6 [6] has been replaced by the nearly $\mathcal{C}(\cdot)$-subconvexlikeness of $F - \overline{y}$ in Theorem 4.2 which is much weaker than the $\mathcal{C}(\overline{y})$-convexity of f. Secondly, we delete the convexity of $\widehat{S}$ and D-convexity of G which is need in Assumption 4.1 of Theorem 4.6 [6].

5 Conclusions

In this paper, we studied weakly nondominated solutions of set-valued optimization problems with variable ordering structures. We obtain some scalarization characterizations and dual theorems. Our results are obtained in linear spaces without any topological structure. In the future, we will investigate properly nondominated solutions of set optimization problems with variable ordering structures in linear spaces.

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