Deterministic Impartial Selection with Weights

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Abstract. In the impartial selection problem, a subset of agents up to a fixed size k among a group of n is to be chosen based on votes cast by the agents themselves. A selection mechanism is *impartial* if no agent can influence its own chance of being selected by changing its vote. It is *α-optimal* if, for every instance, the ratio between the votes received by the selected subset is at least a fraction of α of the votes received by the subset of size k with the highest number of votes. We study deterministic impartial mechanisms in a more general setting with arbitrarily weighted votes and provide the first approximation guarantee, roughly $1/\lceil 2n/k \rceil$. When the number of agents to select is large enough compared to the total number of agents, this yields an improvement on the previously best known approximation ratio of $1/k$ for the unweighted setting. We further show that our mechanism can be adapted to the impartial assignment problem, in which multiple sets of up to k agents are to be selected, with a loss in the approximation ratio of $1/2$.

Keywords: Impartial selection · Mechanism design · Social choice

1 Introduction

Votes and referrals are a key mechanism in the self-organization of communities: political parties elect their representatives, researchers review and rate each other's manuscripts, and hyperlinks on the web attribute topical relevance to an external resource. Oftentimes, the agents who give the recommendations are themselves interested in being within a top-rated fraction of their group: to occupy a prestigious position, be invited to a conference, or to have a website appear more prominently in search results. Objectives like these provide an incentive to deviate from a fair evaluation of one's peers. In particular, agents might omit a recommendation for an immediate contender in order to be ranked above them when the votes are counted.

In a seminal work, Alon et al. [1] initiated the search for impartial mechanisms to aggregate the votes cast by n agents who want to elect k individuals among them, which we refer to as the exact (n, k)-selection problem. The authors require that no agent is able to influence their own chance of being selected by

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adjusting the subset of peers that they vote for, while, at the same time, the agents selected by the mechanism should receive an expected sum of votes that is close to that of the highest voted subset of size k. We refer to the first condition as *impartiality* and to the second as *α-optimality*, where $\alpha \in [0, 1]$ denotes the performance guarantee. If the mechanism is allowed to make use of random choice and agents may vote for any subset of their peers, then the best known performance guarantee is $\frac{k}{k+1}\left(1 - \left(\frac{k-1}{k}\right)^{k+1}\right)$, which gives $1/2$ for the selection of a single agent and approaches $1 - 1/e$ as $k \to \infty$ [4]. It is further known that no impartial mechanism can be better than $k/(k+1)$-optimal, which is tight only for $k = 1$. We discuss variants with a limited number of votes per participant as related work.

The problem only becomes more difficult in the deterministic setting, where the mechanism is forced to choose one agent over another even for highly symmetric input. The instance in which two agents vote for each other and one of them shall be selected requires the mechanism to break the tie, based on an external preference list, in favor of one of the agents. Impartiality demands that the same agent must be selected also when the other agent withdraws its vote. But then, an agent with no votes is selected, even though the other agent still receives one. This yields a performance guarantee of zero for the selection of a single agent in the worst case. Even for $k > 1$, no positive performance guarantee is possible [1], unless, surprisingly, when the mechanism is allowed to select less than k agents in some instances. In this case an algorithm achieving $\alpha = 1/k$ is known [4]. We refer to this relaxation as the *inexact* (n, k)-selection problem. Since this insight, the gap towards the best known upper bound, which is $(k - 1)/k$ in the inexact selection setting, remained remarkably wide.

More generally, the selection problem allows for votes to be weighted: one then compares the total weight of the selected agents to that of the maximum-weight subset of size k. In a peer review setting, reviewers are often asked to rate the manuscript under consideration on a point scale that ranges from a recommendation to reject to a claim of excellence. An editor or program chair would then aggregate these scores and accept a limited number of highly rated submissions. While the established rule to disclose any conflicts of interest protects, if obeyed, against abuse based on personal ties, authors whose papers are on the verge of selection might still profit from giving ratings below their honest estimate, unless the selection mechanism is impartial. In this setting, although computational studies have been made [2], no deterministic mechanism providing a worst-case guarantee was known to date.

1.1 Our Contribution

We propose a deterministic impartial mechanism that can be applied in the weighted setting and which achieves a performance guarantee of $1/\lceil 2n/k \rceil$, for $k \geq 2\sqrt{n}$ even, and $(k - 1)/(k\lceil 2n/(k - 1)\rceil)$, for $k \geq 2\sqrt{n} + 1$ odd. In particular, it achieves asymptotically a guarantee of $\alpha = 1/4$ for selecting at most half and $\alpha = 1/3$ for selecting at most two thirds of the agents. These are the first lower bounds for deterministic selection with weights. In its applicable range, the mechanism further improves upon the previous best bound of $1/k$ in the unweighted setting. The improvement is most noticeable when k is large, where the gap between the previously best known lower and upper bounds of $1/k$ and $(k-1)/k$, respectively, has been widest. The construction is best behaved whenever $b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$: here a guarantee of $\alpha = 1/b$ is provided and the analysis of the mechanism is tight. The mechanism uses a well-structured set of partitions of the agents, whose existence we study in Sect. 3 using a connection to hypergraph theory and graph coloring. The mechanism itself and the proof of the approximation guarantee are presented in Sect. 4.

In Sect. 5, we show how the mechanism can be adapted to assign agents to multiple size-limited subsets, which may represent tasks to distribute or committees to form. In this setting, we lose only a factor of 1/2 in the performance guarantee, independent of the number of subsets.

1.2 Related Work

Impartiality as a desirable axiom in multi-agent problems was introduced by De Clippel et al. [11] and was first studied in the context of peer selection in parallel by both Holzman and Moulin [15] and Alon et al. [1]: The work by Holzman and Moulin studied the existence of impartial mechanisms satisfying further axioms such as unanimity and notions of monotonicity, while the research by Alon et al. showed that no deterministic impartial mechanism aiming to select exactly k agents can achieve any constant approximation ratio. In response, Bjelde et al. [4] showed that when fewer than k agents may be selected, $1/k$-optimality is guaranteed by the *bidirectional permutation* mechanism, which picks either one or two agents, depending on the instance. The authors further proved an upper bound of $(k-1)/k$ for any deterministic impartial mechanism.

Continuing the axiomatic line, Tamura and Ohseto [24] studied k-selection in the single-nomination setting and showed that impartiality is compatible with two natural notions of unanimity. Their mechanism was extended to the case of a higher, but constant, maximum number of nominations by Cembrano et al. [9]. Further, Aziz et al. [2] proposed a mechanism satisfying certain monotonicity properties and confirmed its performance in a computational study.

Several works have focused on randomized impartial selection. Alon et al. proposed a family of mechanisms based on a random partition of the agents that yield the first lower bounds on the approximation ratio for this setting, namely $1/4$ for $k = 1$ and $1 - O(1/\sqrt[3]{k})$ for general k. They also provided respective upper bounds of $1/2$ and $1 - \Omega(1/k^2)$. Fischer and Klimm [14] closed the gap for $k = 1$ by giving a $1/2$-approximation algorithm. Bousquet et al. [5] designed a mechanism with an approximation guarantee that goes to one as the maximum score of an agent goes to infinity. A restricted variant of particular importance, first studied in the work of Holzman and Moulin, arises when each agent can vote for exactly one other agent. Here, Fischer and Klimm provided both lower and upper bounds which were later improved by Cembrano et al. [10].

A setting closely related to the impartial selection of k agents is that of *peer review* in which, in contrast to the classic k-selection problem, the votes are weighted and represent a score assigned to a submission. Kurokawa et al. [18] studied a model where first a limited number of weighted votes is sampled and then the selection is performed. The authors proposed an impartial randomized mechanism providing a constant approximation ratio with respect to the (non-impartial) mechanism that randomly samples the votes and selects the best possible set of k agents given these votes. Mattei et al. [21] studied this problem from an axiomatic and experimental point of view, while Lev et al. [19] extended this work to the setting with noisy assessments. Dhull et al. [12] explored the scope and limitations of partition-based mechanisms for peer review in terms of approximating the selection of the best k papers.

Beyond multiplicative approximation, some works have studied the scope and limitations of impartial mechanisms in terms of additive guarantees [6–8] and additional economic axioms [13,20]. Impartiality has also been considered for the selection of agents where preferences come from correlated types [22], for the selection of vertices in graphs with maximal progeny [3,26,27], and for generating social rankings of agents who rank each other [16]. For a survey on incentive handling in peer mechanisms, see Olckers and Walsh [23].

2 Preliminaries

For $n \in \mathbb{N} := \mathbb{Z}_{\geq 1}$, we define the ranges $[n] := \{1, \dots, n\}$ and $[n]_0 := \{0, \dots, n-1\}$ and we write $\mathcal{A}_n$ for the set of non-negative $n \times n$ matrices with zero diagonal. An instance of the weighted selection problem is fully described by an integer k and a weight matrix $A \in \mathcal{A}_n$, where k is the number of agents to be selected and A_{ij} corresponds to the weight of the vote that agent i casts for agent j. For $A \in \mathcal{A}_n$, we write A_{-i} for the matrix obtained when removing the i-th row of A. Given $A \in \mathcal{A}_n$ and $R, S \subseteq [n]$, we write

$$\sigma_R(S; A) := \sum_{i \in R,\ j \in S} A_{ij}$$

for the score of the agents in S limited to R, and $\sigma(S; A)$ short for $\sigma_{[n]}(S; A)$. We omit the weight matrix A whenever it is clear from the context and we write j short for $S = \{j\}$ in the above definitions.

Let $n, k \in \mathbb{N}$ with $k < n$ in the following. For $A \in \mathcal{A}_n$, we let

$$\textsc{Opt}_k(A) := \underset{S \subseteq [n]:\ |S|=k}{\arg\max}\ \sigma(S; A)$$

denote an arbitrary set with the largest score among vertex subsets of size k. We write just $\textsc{Opt}_k$ when the weight matrix is clear.

An (n, k)-selection mechanism is a function $f : \mathcal{A}_n \to 2^{[n]}$ such that $|f(A)| \leq k$ for every $A \in \mathcal{A}_n$. Such a mechanism is *impartial* if, for every pair of instances

$A, A' \in \mathcal{A}_n$ and for all $i \in [n]$ such that $A_{-i} = A'_{-i}$, it holds that $f(A) \cap \{i\} = f(A') \cap \{i\}$. We further call an (n, k)-selection mechanism *α-optimal* if

$$\frac{\sigma(f(A); A)}{\sigma(\textsc{Opt}_k(A); A)} \geq \alpha$$

holds for all $A \in \mathcal{A}_n$ and some $\alpha \in [0, 1]$.

We write $E \dot{\cup} F$ for the disjoint union of sets E and F. For a multiset E, we write $\mu_E(e)$ for the multiplicity of $e \in E$ and $\mu(E)$ for the cardinality of E.

A hypergraph is a pair $H = (V, E)$ where V is a finite set of *vertices* and where $E \subseteq 2^V$ is a multiset of *(hyper-)edges*. We say that H is *d-regular* if each vertex is contained in exactly d edges, i.e., $\mu(\{e \in E \mid v \in e\}) = d$ for all $v \in V$; *b-uniform* if each edge contains exactly b vertices, i.e., $|e| = b$ for all $e \in E$; and *linear* if two distinct edges intersect in at most one vertex, i.e., $|e_1 \cap e_2| \leq 1$ for all $e_1, e_2 \in E$ with $\mu_E(e_1) > 1$ or $e_1 \neq e_2$. The *dual* of H is $H^* = (E, X)$ where $X := \{\{e \in E \mid v \in e\} \mid v \in V\}$ is a multiset of sets. One may think of the dual graph in terms of the vertex–edge incidence matrix, which is transposed when taking the dual graph. Note that the dual graph may have repeated edges and loops even if the original graph does not have either.

We call a 2-uniform hypergraph without repeated edges a (simple) graph. For a graph $G = (V, E)$, an edge b-coloring is a mapping $\pi \colon E \to [b]$. It is *feasible* if $\pi(e_1) \neq \pi(e_2)$ for all $e_1, e_2 \in E$ with $e_1 \cap e_2 \neq \emptyset$. Likewise, a vertex b-coloring is a mapping $\pi \colon V \to [b]$ that we call feasible if $\pi(u) \neq \pi(v)$ for all $u, v \in V$ such that $u, v \in e$ for some $e \in E$.

3 Partition Systems

The present work takes inspiration from the *partition mechanism*. This mechanism was first proposed by Alon et al. [1] for the setting of randomized $(n, 1)$-selection, and variants for selecting more than one agent have been studied by Bjelde et al. [4], Aziz et al. [2], and Xu et al. [25]. In its original formulation due to Alon et al., the partition mechanism assigns each agent into a *voter set* S_1 and a *candidate set* S_2 uniformly at random. It then considers only votes from agents in S_1 to agents in S_2 and selects an agent from S_2 with maximum score. This mechanism is impartial as it considers only votes of agents with no chance of being selected and it is 1/4-optimal, intuitively, as we see every fourth vote in expectation. The (n, k)-selection variant by Bjelde et al. [4] partitions the agents into k sets instead of two and selects one agent from each set that has the highest score from all other sets, additionally considering internal votes that are directed from left to right according to a random permutation of the agents. This variant preserves impartiality and provides a guarantee that varies from $1/2$ to $1 - 1/e$ as k grows from 1 to infinity.

The partition mechanism, although achieving a good ratio when randomization is possible, performs poorly in the deterministic setting. If agents are assigned in any fixed way, votes may be adversarially placed between agents in the same set (and opposite to the order given by the permutation of the agents if such a step is considered), so that the mechanism cannot do any better, in the worst case, than selecting agents with no votes, while the maximum score may be arbitrarily high.

In the following, we build the foundation for a partition-based (n, k)-selection mechanism that is robust against such adversarial placement of votes. To achieve this, agents appear in the candidate set of more than one partition and with a disjoint set of contenders each time. This way, votes not seen for a candidate agent in one partition will be seen in another partition wherein that agent reappears as a candidate. Of course, repeated candidacy may lead to the same agent being selected multiple times, at the expense of contenders with a high number of votes. To minimize this possibility, we let every agent contest just twice and we remove duplicate votes. As our goal is to select up to k agents, we define k such partitions. For now, we make also the simplifying assumption that n and k allow the candidate sets to have equal size b. This is without loss of generality as we may fill smaller partitions with dummy agents who cast and receive no votes and are disfavored when breaking ties. We call a collection of partitions meeting these requirements a *balanced partition system*.

A partition into voters and candidates is fully described by either set. A balanced partition system may thus be written as a family E of candidate subsets of the set of agents V or, in other words, as a hypergraph $H = (V, E)$ without repeated edges, where each $e \in E$ is the candidate set of a single partition. To fulfill the requirements of a balanced partition system, H has to be 2-regular, so that every agent appears in exactly two candidate sets, and b-uniform, so that all candidate sets $e \in E$ have the same size $|e| = b$. The remaining requirement that no two agents compete twice against each other, formally $|e_1 \cap e_2| \leq 1$ for all $e_1, e_2 \in E$ with $e_1 \neq e_2$, translates to H being linear. The following lemma, whose proof is omitted due to space constraints, implies that we can represent a partition system further by a simple graph.

Lemma 1. *A hypergraph is* 2*-regular and linear if and only if its dual is a simple graph.*

By Lemma 1 and the fact that order and size as well as degree and rank are dual for hypergraphs, there is a one-to-one correspondence between balanced partition systems where n agents are distributed among k candidate sets of size b on the one hand, and b-regular simple graphs of order k and size n on the other hand. In the simple graph representation, edges correspond to agents while incident vertices correspond to candidate sets that the agents appear in.

In the analysis of the mechanism, we will bound the weight selected by it by that of a subset U of top-voted agents that pairwise do not compete. More precisely, U will be a set of maximum weight among a partition of the k top-voted agents into b many subsets with this property. If the mechanism does not select some agent i from U, then only because it makes up for the agent's score in the two partitions that agent i appears in, and which are pairwise disjoint for the agents in U. This leads to a lower bound of $(k/b)/k = 1/b$, stated in Lemma 4. To ensure the existence of b such sets, we require that any subgraph of H induced by

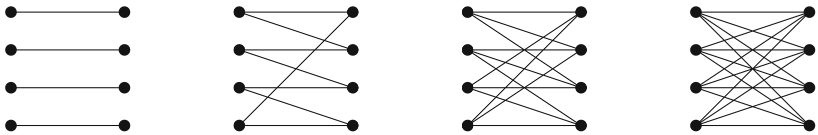

Fig. 1. The construction of Lemma 2 for $k = 8$ vertices and degree $b \in [4]$: (a) the $4P_2$ ($n = 4$ edges), (b) the cycle C_8 ($n = 8$), (c) the cube graph Q_3 ($n = 12$), and (d) the complete bipartite graph $K_{4,4}$ ($n = 16$). Every edge represents an agent and every vertex corresponds to a partition. A vertex and an edge are incident if the corresponding agent is in the corresponding candidate set.

k vertices can be partitioned into b many (internally) independent sets. We call a balanced partition system whose corresponding hypergraph has this property *robust*. In terms of the b-regular dual graph $G := H^*$, the condition is equivalent to the existence of an edge coloring with b colors for every subgraph induced by k edges: the edges of any one color do not share a vertex, which corresponds to vertices not sharing a hyperedge in H. By Kőnig's line coloring theorem [17], a sufficient condition for such a coloring to exist is that G is bipartite. The proofs of Lemma 3 and 4 will formalize these ideas.

Bipartite and b-regular graphs of even order k and size n exist for all $b = 2n/k$ with $b \leq k/2$. A simple construction is depicted in Fig. 1 and described by the following lemma, whose proof is omitted due to space constraints.

Lemma 2. *Let $b, k, n \in \mathbb{N}$ with $k' := k/2 \in \mathbb{N}$ and $b = 2n/k \leq k'$. Then, $G = (V, E)$ with $V := [k]_0$ and $E := \{\{i, k' + ((i + \ell) \bmod k')\} \mid i \in [k']_0, \ell \in [b]_0\}$ is a b-regular bipartite graph of order k and size n.*

We condense the findings of this section in the following lemma.

Lemma 3. *Let $n, k \in \mathbb{N}$ with $k < n$ be such that $b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$. Let further V with $|V| = n$ denote a set of agents. Then, one may form k partitions $S_1^p \mathbin{\dot\cup} S_2^p = V$, $p \in [k]$, such that*

(i) $|S_2^p| = b$ *for all* $p \in [k]$,
(ii) $|S_2^p \cap S_2^q| \leq 1$ *for all* $p, q \in [k]$ *with* $p \neq q$,
(iii) $|\{p \in [k] \mid v \in S_2^p\}| = 2$ *for all* $v \in V$, *and*
(iv) *for every* $U \subseteq V$, *there is a partition* $\dot\bigcup_{t \in [b]} U_t = U$ *with* $u \in S_2^p \Rightarrow v \notin S_2^p$ *for all* $t \in [b]$, $u, v \in U_t$ *with* $u \neq v$, *and* $p \in [k]$.

Proof. For n, k, and b as in the statement, Lemma 2 guarantees the existence of a b-regular bipartite graph $G = (X, V)$ of order $|X| = k$ and size $|V| = n$. Let $H := G^* = (V, E)$ be its dual graph. Note that H is b-uniform and has order n and size k. By Lemma 1, H is further 2-regular and linear. As $b \geq 2$ by definition, it follows from linearity that H has no repeated edges, i.e., E is a set.

We use H to form a system of partitions of V. First, enumerate E by an arbitrary but fixed bijection $\phi\colon [k] \to E$. Then, for every $p \in [k]$, define a candidate set $S_2^p := \phi(p)$ and the associated voter set $S_1^p := V \setminus \phi(p)$. As H is b-uniform, we have (i) by construction. As it is linear, (ii) follows. Since H is 2-regular, also (iii) holds.

It remains to show property (iv). By Kőnig's line coloring theorem [17], there exists a feasible edge b-coloring $\pi \colon V \to [b]$ of G. Let G' be the subgraph of G induced by an edge set $U \subseteq V$. Clearly, π restricted to U remains a feasible edge b-coloring. The dual $H' := (G')^*$ is the subgraph of H induced by the vertex set U. In terms of H', π assigns colors to vertices. Since π restricted to U is feasible for G', it follows from vertex–edge duality that vertices in H' are colored differently if they appear in a hyperedge together, i.e., π is a feasible vertex coloring for H. Define thus $U_t := \{v \in U \mid \pi(v) = t\}$ for each color $t \in [b]$. Then, the sets U_t are disjoint by definition and $\dot{\bigcup}_{t\in[b]} U_t = U$ as $\pi(U) \subseteq \pi(V) \subseteq [b]$. Let finally $t \in [b]$ and $u, v \in U_t$ with $u \neq v$ and assume towards a contradiction that $u, v \in S_2^p$ for some $p \in [k]$. Then, $u, v \in \phi(p) \in E$ and $\pi(u) = t = \pi(v)$ by construction of S_2^p and U_t, contradicting that π is a feasible vertex coloring for $H = (V, E)$. □

Formally, we write $\mathcal{S}(n, k)$ for an arbitrary but fixed sequence $((S_1^p, S_2^p))_{p\in[k]}$ with $S_1^p \,\dot\cup\, S_2^p = [n]$ for every $p \in [k]$ that fulfills the conditions of Lemma 3. We assume for technical reasons that $S_2^1 = [b]$.

4 Impartial Selection

We are prepared to construct a mechanism that provides the first approximation guarantee for deterministic impartial selection with weighted votes. Our main result is the following.

Theorem 1. *Let $n, k \in \mathbb{N}$ with $1 < k < n$ and $k - k \bmod 2 \geq 2\sqrt{n}$. Then, there exists an (n, k)-selection mechanism that is impartial and α-optimal with*

$$\alpha = \frac{k - k \bmod 2}{k \left\lceil \frac{2n}{k - k \bmod 2} \right\rceil}.$$

The performance guarantee of Theorem 1 is shown in Fig. 2. It starts from $2/k$ for $k - k \bmod 2 = 2\sqrt{n}$ and grows up to $1/3$ for $k - k \bmod 2 \in [2n/3, n-1]$.

The main idea of the algorithm is as follows. We construct a robust partition system of the set of agents, i.e., a set of k many partitions of the agents into voters and candidates such that each agent appears as a candidate twice and with disjoint sets of contenders. For the second candidacy, we remove votes that are already present in the first candidacy to avoid double-counting. Then, the mechanism selects the top scoring candidate from each partition, possibly selecting some agents twice. This mechanism is impartial as voters and candidates are disjoint in each partition. The performance guarantee stems mainly from the fact that every vote is counted exactly once.

In Sect. 3, we showed that a robust partition system is guaranteed to exist as long as n and k satisfy $k < n,\ b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$. In the following,

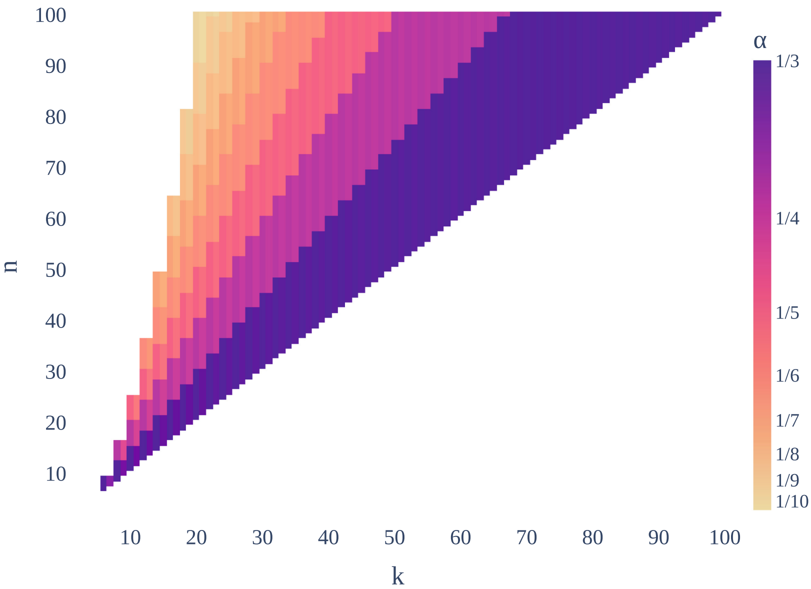

Fig. 2. The performance guarantee of Theorem 1 for permissible n and k.

we assume these conditions in order to define and analyze our mechanism; we lift them in the end to obtain the general result stated in Theorem 1.

Given n and k as in Lemma 3, our selection mechanism is formally described by Algorithm 1; we refer to it as SELECT_k and denote its output by $\text{SELECT}_k(A)$ for a given input matrix $A \in \mathcal{A}_n$. The procedure considers a partition system with the properties stated in Lemma 3 and performs two main steps. Recall that each agent $j \in [n]$ appears in two candidate sets; we denote their indices by $l(j) < r(j) \in [k]$ such that $j \in S_2^{l(j)} \cap S_2^{r(j)}$. The mechanism first computes the *modified score* $\hat{\sigma}_{S_1^p}(j)$ for each $j \in [n]$ and each $p \in \{l(j), r(j)\}$, which is simply the actual score $\sigma_{S_1^{l(j)}}(j)$ for $p = l(j)$. For $p = r(j)$, however, we omit the votes from agents $i \in S_1^{l(j)}$ in order to avoid double counting. The mechanism then selects the vertex with the highest modified score out of each candidate set, breaking ties in favor of the largest index.[1] Figure 3 illustrates a possible execution of SELECT_6 on an instance $A \in \mathcal{A}_9$.

Throughout this section, whenever n, k, and $A \in \mathcal{A}_n$ are fixed, we write $((S_1^1, S_2^1), \ldots, (S_1^k, S_2^k))$, $l(j)$, $r(j)$, $\hat{\sigma}_{S_1^p}(j)$, i^p, and X for each $p \in [k]$ and $j \in [n]$ to refer to the objects defined in SELECT_k. We only specify the input matrix A as an argument when it is not clear from the context. The following lemma constitutes the main technical ingredient for the proof of Theorem 1.

[1] We sometimes compare tuples, for example $(\sigma(j), j)$, in lexicographical order. We use standard inequality signs as well as the min and max operators for this purpose.

Algorithm 1: $\textsc{Select}_k(A)$

Input: weight matrix $A \in \mathcal{A}_n$
Output: set $X \subseteq [n]$ with $|X| \leq k$
let $((S_1^1, S_2^1), \ldots, (S_1^k, S_2^k)) = \mathcal{S}(n,k)$;
for $j \in [n]$ **do**
  let $\{l(j), r(j)\} = \{p \in [k] : j \in S_2^p\}$ with $l(j) < r(j)$;
  define $\hat{\sigma}_{S_1^{l(j)}}(j) \leftarrow \sigma_{S_1^{l(j)}}(j)$ and $\hat{\sigma}_{S_1^{r(j)}}(j) \leftarrow \sigma_{S_1^{r(j)} \setminus S_1^{l(j)}}(j)$;
end
initialize $X \leftarrow \emptyset$;
for $p \in [k]$ **do**
  take $i^p = \arg\max_{j \in S_2^p} (\hat{\sigma}_{S_1^p}(j), j)$ and update $X \leftarrow X \cup \{i^p\}$
end
return X

Lemma 4. *Let $n, k \in \mathbb{N}$ with $k < n$ be such that $b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$. Then, $\textsc{Select}_k$ is an impartial and $1/b$-optimal (n,k)-selection mechanism.*

Proof. We consider n and k as in the statement. We first note that $\textsc{Select}_k$ returns a subset of $[n]$ of size at most k and is well-defined as we have $|\{p \in [k] : j \in S_2^p\}| = 2$ for every $j \in [n]$. The former holds since i^p is a single vertex for every $p \in [k]$ and $X = \bigcup_{p \in [k]} \{i^p\}$; the latter follows from property (iii) of Lemma 3 since $b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$.

To see that $\textsc{Select}_k$ is impartial, let $A, A' \in \mathcal{A}_n$ and $j \in [n]$ such that $A_{-j} = A'_{-j}$. Suppose $j \in \textsc{Select}_k(A)$. From the definition of the mechanism, we have that there is $p \in [k]$ such that $j = \arg\max_{i \in S_2^p} (\hat{\sigma}_{S_1^p}(i; A), i)$. Since $j \in S_2^p$ and $A_{-j} = A'_{-j}$, we have both that $\hat{\sigma}_{S_1^p}(j; A) = \hat{\sigma}_{S_1^p}(j; A')$ and, for every $i \in S_2^p \setminus \{j\}$, that $\hat{\sigma}_{S_1^p}(i; A) = \hat{\sigma}_{S_1^p}(i; A')$. This yields $j = \arg\max_{i \in S_2^p} (\hat{\sigma}_{S_1^p}(i; A'), i)$. Thus, we obtain from the definition of the mechanism that $j \in \textsc{Select}_k(A')$. We conclude that $\textsc{Select}_k(A) \cap \{j\} = \textsc{Select}_k(A') \cap \{j\}$.

It remains to show that $\textsc{Select}_k$ has an approximation ratio of $1/b$. To this end, we let $A \in \mathcal{A}_n$ be an arbitrary weight matrix. First, observe that

$$\hat{\sigma}_{S_1^{l(j)}}(j) + \hat{\sigma}_{S_1^{r(j)}}(j) = \sigma_{S_1^{l(j)}}(j) + \sigma_{S_1^{r(j)} \setminus S_1^{l(j)}}(j) = \sigma(j) \tag{1}$$

for every $j \in [n]$, since property (ii) of Lemma 3 implies $S_1^{l(j)} \cup S_1^{r(j)} = [n] \setminus \{j\}$. Furthermore, the definition of i^p yields that

$$\hat{\sigma}_{S_1^p}(i^p) \geq \hat{\sigma}_{S_1^p}(j) \tag{2}$$

for every $p \in [k]$ and $j \in S_2^p$. Given these two facts, we claim that

$$\hat{\sigma}_{S_1^{l(j)}}(i^{l(j)}) + \hat{\sigma}_{S_1^{r(j)}}(i^{r(j)}) \geq \sigma(j) \tag{3}$$

for every $j \in [n]$. To see this, we fix $j \in [n]$. If $i^p = j$ for each $p \in \{l(j), r(j)\}$, inequality (3) follows immediately from equality (1). If $|\{j\} \cap \{i^p : p \in$

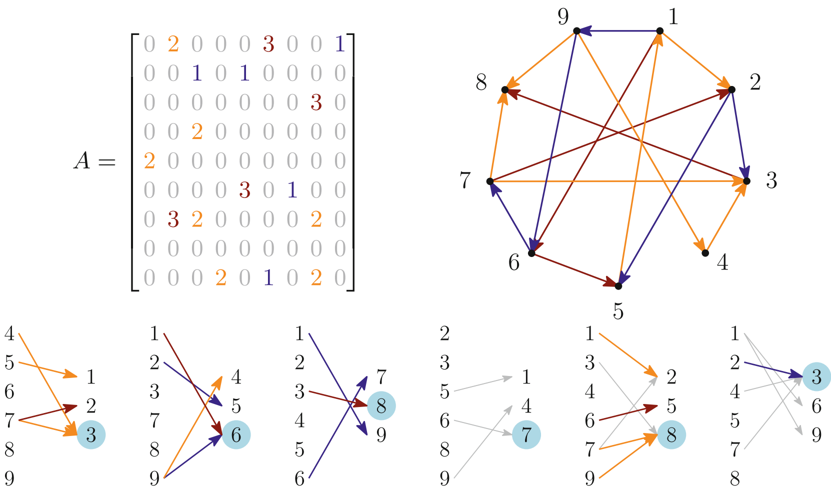

Fig. 3. Example of $\textsc{Select}_6(A)$ for $A \in \mathcal{A}_9$. The weight matrix A is shown alongside its graph representation, where edges of weight 1 are in blue, weight 2 are in orange, weight 3 are in red, and edges of weight 0 are not included. The partition system is given below, where omitted edges are shown in gray. For each partition, the selected vertex is highlighted in light blue. Observe that $\sigma(\textsc{Select}_6(A)) = 17$ and $\sigma(\textsc{Opt}_6(A)) = 27$; the multiplicative guarantee provided by Lemma 4 for this instance is 1/3. (Color figure online)

$\{l(j), r(j)\}\}| = 1$, say w.l.o.g. $i^{l(j)} = j$ and $i^{r(j)} = h \neq j$, we have that

$$\hat{\sigma}_{S_1^{r(j)}}(h) \geq \hat{\sigma}_{S_1^{r(j)}}(j) = \sigma(j) - \hat{\sigma}_{S_1^{l(j)}}(j),$$

where the inequality follows from (2) and the equality from (1). In this case, inequality (3) follows from $j = i^{l(j)}$ and $h = i^{r(j)}$. Finally, if $j \notin \{i^p : p \in \{l(j), r(j)\}\}$, we have from (2) that

$$\hat{\sigma}_{S_1^{l(j)}}(i^{l(j)}) \geq \hat{\sigma}_{S_1^{l(j)}}(j) \quad \text{and} \quad \hat{\sigma}_{S_1^{r(j)}}(i^{r(j)}) \geq \hat{\sigma}_{S_1^{r(j)}}(j)$$

so that inequality (3) follows from summing up these two inequalities and applying equality (1). This concludes the proof of inequality (3).

Letting χ denote the indicator function for logical propositions, we note that

$$\begin{aligned}
\sigma(\textsc{Select}_k(A)) = \sum_{j \in \textsc{Select}_k(A)} \sigma(j) &= \sum_{j \in \textsc{Select}_k(A)} \left(\hat{\sigma}_{S_1^{l(j)}}(j) + \hat{\sigma}_{S_1^{r(j)}}(j) \right) \\
&\geq \sum_{j \in \textsc{Select}_k(A)} \left(\hat{\sigma}_{S_1^{l(j)}}(j)\chi(j = i^{l(j)}) + \hat{\sigma}_{S_1^{r(j)}}(j)\chi(j = i^{r(j)}) \right) \\
&= \sum_{p \in [k]} \hat{\sigma}_{S_1^p}(i^p).
\end{aligned} \tag{4}$$

Indeed, the first equality follows from the definition of $\textsc{Select}_k(A)$, the second one from equality (1), the inequality simply from $\chi(\cdot) \leq 1$, and the last equal-

Algorithm 2: $\textsc{Gen}_{\textsc{Alg},k}(A)$

Input: weight matrix $A \in \mathcal{A}_n$
Output: set $X \subseteq [n]$ with $|X| \le k$
define $\tilde{A} \in \mathcal{A}_{\tilde{n}}$ as

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & \text{if } i, j \in [n], \\ 0 & \text{otherwise.} \end{cases}$$

return $\textsc{Alg}(\tilde{A})$

ity follows from the definition of i^p for each $p \in [k]$. We next use inequalities (3) and (4) to conclude the bound stated in the lemma.

For $b := 2n/k$, we know from property (iv) of Lemma 3 that there is a partition $\dot{\bigcup}_{t\in[b]} U_t = \textsc{Opt}_k(A)$ such that $i \in S_2^p$ implies $j \notin S_2^p$ for all $t \in [b]$, $i, j \in U_t$ with $i \neq j$, and $p \in [k]$. We obtain that, for every $t \in [b]$,

$$\sigma(\textsc{Select}_k(A)) \ge \sum_{p\in[k]} \hat{\sigma}_{S_1^p}(i^p) \ge \sum_{j\in U_t} \left(\hat{\sigma}_{S_1^{l(j)}}(i^{l(j)}) + \hat{\sigma}_{S_1^{r(j)}}(i^{r(j)})\right) \ge \sigma(U_t), \tag{5}$$

where the first inequality follows from inequality (4), the second one from the fact that $\{l(i), r(i)\} \cap \{l(j), r(j)\} = \emptyset$ for every $t \in [b]$ and every $i, j \in U_t$ with $i \neq j$, and the last one from inequality (3). This yields

$$\sigma(\textsc{Select}_k(A)) \ge \max_{t\in[b]} \sigma(U_t) \ge \frac{1}{b}\sum_{t\in[b]} \sigma(U_t) = \frac{1}{b}\sigma(\textsc{Opt}_k(A)).$$

Here, the first inequality follows from (5), the second one from the observation that the maximum of a set of values is at least as large as their average, and the equality from the fact that $\{U_t\}_{t\in[b]}$ is a partition of $\textsc{Opt}_k(A)$. Therefore, we obtain that $\textsc{Select}_k$ is α-optimal for

$$\frac{\sigma(\textsc{Select}_k(A))}{\sigma(\textsc{Opt}_k(A))} \ge \frac{1}{b} = \alpha. \qquad \square$$

In order to conclude our main result, it only remains to extend the bound given by Lemma 4 to the case where at least one of the conditions $b := 2n/k \in \mathbb{N}$ or $b \le k/2 \in \mathbb{N}$ is not satisfied. To this end, we show a general way to extend bounds on the approximation ratio for given values of $\tilde{n}$ and $\tilde{k}$ to other values n and k: whenever $n \le \tilde{n}$ and $k \ge \tilde{k}$, we can do so preserving impartiality and only losing a factor of $\tilde{k}/k$.

Given $k, \tilde{k}, \tilde{n}, n \in \mathbb{N}$ with $k \le \tilde{k} < \tilde{n} \le n$, and an $(\tilde{n}, \tilde{k})$-selection mechanism $\textsc{Alg}$, we can generalize $\textsc{Alg}$ to the (n, k)-selection mechanism $\textsc{Gen}_{\textsc{Alg},k}$. This is formally described by Algorithm 2, whose output is denoted by $\textsc{Gen}_{\textsc{Alg},k}(A)$ for an input matrix $A \in \mathcal{A}_n$. This algorithm simply extends A to the $\tilde{n} \times \tilde{n}$ matrix $\tilde{A}$ by adding $\tilde{n} - n$ many all-zero rows and columns to it, and then applies $\textsc{Alg}$ on $\tilde{A}$. As before, whenever $\tilde{n}$, n, k, $\textsc{Alg}$, and $A \in \mathcal{A}_n$ are fixed, we use $\tilde{A}$ to refer to the object defined in Algorithm 2 for this input. In a slight overload of notation, when we consider $A' \in \mathcal{A}_n$ as an input, we write simply $\tilde{A}'$ for the matrix defined in Algorithm 2 on input A'. We obtain the following lemma.

Lemma 5. *Let $\tilde{k}, k, n, \tilde{n} \in \mathbb{N}$ with $\tilde{k} \leq k < n \leq \tilde{n}$ be such that there exists an impartial and $\tilde{\alpha}$-optimal $(\tilde{n}, \tilde{k})$-selection mechanism* $\textsc{Alg}$*. Then* $\textsc{Gen}_{\textsc{Alg},k}$ *is an impartial and α-optimal (n, k)-selection mechanism with $\alpha = (\tilde{k}/k)\tilde{\alpha}$.*

Proof. Let n, k, $\tilde{n}$, and $\tilde{k}$ be as in the statement. Let also $\textsc{Alg}$ denote the impartial and $\tilde{\alpha}$-optimal $(\tilde{n}, \tilde{k})$-selection mechanism.

In order to see that $\textsc{Gen}_{\textsc{Alg},k}$ is impartial, let $A, A' \in \mathcal{A}_n$ and $i \in [n]$ such that $A_{-i} = A'_{-i}$. This implies $\tilde{A}_{-i} = \tilde{A}'_{-i}$, thus the impartiality of $\textsc{Alg}$ yields

$$\textsc{Gen}_{\textsc{Alg},k}(A) \cap \{i\} = \textsc{Alg}(\tilde{A}) \cap \{i\} = \textsc{Alg}(\tilde{A}') \cap \{i\} = \textsc{Gen}_{\textsc{Alg},k}(A') \cap \{i\}.$$

To prove the approximation guarantee, we let $A \in \mathcal{A}_n$ be an arbitrary weight matrix and observe that

$$\frac{\sigma(\textsc{Gen}_{\textsc{Alg},k}(A))}{\sigma(\textsc{Opt}_{\tilde{k}}(\tilde{A}))} = \frac{\sigma(\textsc{Alg}(\tilde{A}))}{\sigma(\textsc{Opt}_{\tilde{k}}(\tilde{A}))} \geq \tilde{\alpha}, \tag{6}$$

where the equality follows from the definition of $\textsc{Gen}_{\textsc{Alg},k}$ and the inequality follows from the $\tilde{\alpha}$-optimality of $\textsc{Alg}$. On the other hand, as $\tilde{k} \leq k$ and $\sigma(j, \tilde{A}) = 0$ for every $j \notin [n]$, we know that

$$\frac{\sigma(\textsc{Opt}_k(A))}{k} = \frac{1}{k} \max_{S \subseteq [n]\colon |S|=k} \sigma(S; A) \leq \frac{1}{\tilde{k}} \max_{S \subseteq [n]\colon |S|=\tilde{k}} \sigma(S; A) = \frac{\sigma(\textsc{Opt}_{\tilde{k}}(\tilde{A}))}{\tilde{k}},$$

i.e., the average score of the k top-voted agents of input A can be no larger than the average score of the $\tilde{k}$ top-voted agents of input $\tilde{A}$. Plugging this inequality into (6) concludes the proof as

$$\frac{\sigma(\textsc{Gen}_{\textsc{Alg},k}(A))}{\sigma(\textsc{Opt}_k(A))} \geq \frac{\tilde{k}}{k} \frac{\sigma(\textsc{Gen}_{\textsc{Alg},k}(A))}{\sigma(\textsc{Opt}_{\tilde{k}}(\tilde{A}))} \geq \frac{\tilde{k}}{k}\tilde{\alpha}.$$ □

Our main result now follows from the last two lemmas.

Proof of Theorem 1. Let n and k be as in the statement. We define

$$\tilde{k} := k - k \bmod 2 \quad \text{and} \quad \tilde{n} := \frac{k - k \bmod 2}{2} \left\lceil \frac{2n}{k - k \bmod 2} \right\rceil.$$

It is clear that $\tilde{n}, \tilde{k}$ are natural numbers with $\tilde{k} \leq k < n \leq \tilde{n}$ and that

$$b := \frac{2\tilde{n}}{\tilde{k}} = \left\lceil \frac{2n}{k - k \bmod 2} \right\rceil \in \mathbb{N}.$$

Moreover, we have that

$$\tilde{n} = \frac{k - k \bmod 2}{2} \left\lceil \frac{2n}{k - k \bmod 2} \right\rceil \leq \frac{k - k \bmod 2}{2} \left\lceil \frac{2\frac{(k-k \bmod 2)^2}{4}}{k - k \bmod 2} \right\rceil = \frac{\tilde{k}^2}{4},$$

where the inequality follows from the condition $k - k \bmod 2 \geq 2\sqrt{n}$ in the statement. This yields $b = 2\tilde{n}/\tilde{k} \leq \tilde{k}/2 \in \mathbb{N}$. By Lemma 4, this implies that $\text{SELECT}_{\tilde{k}}$ is an impartial and $\tilde{\alpha}$-optimal $(\tilde{n}, \tilde{k})$-selection mechanism with

$$\tilde{\alpha} = \frac{1}{b} = \frac{1}{\left\lceil \frac{2n}{k - k \bmod 2} \right\rceil}.$$

Since $\tilde{n}, \tilde{k} \in \mathbb{N}$ are such that $\tilde{k} \leq k$ and $\tilde{n} \geq n$, Lemma 5 implies that $\text{GEN}_{\text{SELECT}_{\tilde{k}}, k}$ is an impartial and α-optimal (n, k)-selection mechanism with

$$\alpha = \frac{\tilde{k}}{k}\tilde{\alpha} = \frac{k - k \bmod 2}{k\left\lceil \frac{2n}{k - k \bmod 2} \right\rceil}. \qquad \square$$

The mechanism and its approximation ratio naturally extend to the widely studied unweighted setting, where one restricts to matrices $A \in \mathcal{A}_n$ with $A_{ij} \in \{0, 1\}$ for every $i, j \in [n]$. This improves on the previous best lower bound of $1/k$ whenever the number of agents to select is high enough compared to n for Theorem 1 to be applicable: if $k - k \bmod 2 \geq 2\sqrt{n}$, the theorem guarantees the existence of an (n, k)-selection mechanism that is impartial and α-optimal with

$$\alpha = \frac{k - k \bmod 2}{k\left\lceil \frac{2n}{k - k \bmod 2} \right\rceil} \geq \frac{k - k \bmod 2}{k\left\lceil \frac{2(k - k \bmod 2)^2}{4(k - k \bmod 2)} \right\rceil} = \frac{2}{k}.$$

We end this section by showing that the analysis of our (n, k)-selection mechanism SELECT_k for n and k satisfying the conditions of Lemma 4 is tight.

Theorem 2. *Let $n, k \in \mathbb{N}$ with $k < n$ be such that $b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$. Then, for every $\varepsilon > 0$ we have that SELECT_k is not $(1/b + \varepsilon)$-optimal.*

Proof. Let n and k be as in the statement and consider the partition system $((S_1^1, S_2^1), \ldots, (S_1^k, S_2^k)) = \mathcal{S}(n, k)$. Recall that we defined $\mathcal{S}(n, k)$ such that $S_2^1 = [b]$. Considering $l(j)$ and $r(j)$ as defined in Algorithm 1 for every $j \in [n]$, we note that for each $j \in S_2^1$ we have $l(j) = 1$. For each $j \in S_2^1$, we let $h(j)$ be an arbitrary agent in S_1^1 such that $h(j) \in S_2^{r(j)}$. Such vertex is guaranteed to exist, since from property (ii) of Lemma 3 we know that $S_2^{l(j)} \cap S_2^{r(j)} = \{j\}$, and from property (i) we have that $|S_2^{r(j)}| = b > 1$.

We consider the instance given by $A \in \mathcal{A}_n$ with $A_{ij} = 1$, if $j \in S_2^1$ and $i = h(j)$, and $A_{ij} = 0$, otherwise. Intuitively, this construction aims to have $A_{ij} > 0$ for some $i \in S_1^p$ and $j \in S_2^p$ only if $p = 1$, so that the only agent with a strictly positive score selected by the mechanism, among b agents with a strictly positive score, is i^1. An example of this construction and the corresponding outcome of the mechanism is illustrated in Fig. 4. It is clear that $\text{OPT}_k(A) = [b]$ and $\sigma(\text{OPT}_k(A)) = b$. On the other hand, we have that $\sigma(i^1) = 1$ and, for every $p \in \{2, 3, \ldots, k\}$, that $\hat{\sigma}_{S_1^p}(j) = 0$ for every $j \in S_2^p$. This is because we have $\sigma(j) = 0$ for every $j \notin [b]$ and, whenever there is a $j \in [b] \cap S_p^2$, we also

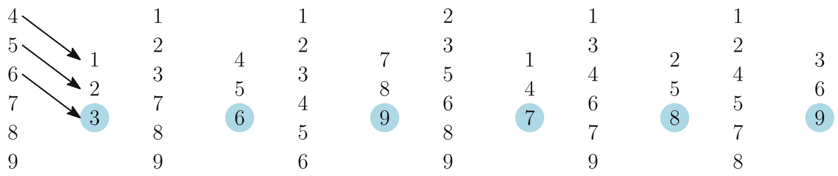

Fig. 4. Example of the construction of the proof of Theorem 2 for $n = 9$ and $k = 6$ with 3 votes of weight 1: agent 4 votes for agent 1, agent 5 votes for agent 2, and agent 6 votes for agent 3. All votes are only seen in the first partition. Since agents with positive scores have the smallest indices, they are not selected in their second candidate set.

have $h(j) \in S_p^2$. Moreover, for every $p \in \{2, 3, \dots, k\}$ such that there exists a $j \in [b] \cap S_p^2$, we have that $j \neq \max S_2^p$ since $h(j) \in S_2^p$ and $h(j) > j$. This yields $\sigma(i^p) = 0$ for every $p \in \{2, 3, \dots, k\}$, thus $\sigma(\text{SELECT}_k(A)) = 1$. This concludes the proof as

$$\frac{\sigma(\text{SELECT}_k(A))}{\sigma(\text{OPT}_k(A))} = \frac{1}{b}. \qquad \square$$

In terms of general upper bounds on the approximation ratio that an impartial mechanism can achieve, the best known is $(k-1)/k$ [4]. Even for the regime $k - k \bmod 2 \geq 2n/3$, in which our mechanism provides a lower bound of $1/3$ and considerably improves the previously best bound of $1/k$ [4], the gap remains large. Further improvements in either lower or upper bounds arise as the main direction for future work.

5 Impartial Assignment

In this section, we consider a generalization of the impartial selection problem in which agents are not selected into one but *assigned* to at most one of m many sets, which we refer to as *jobs*. Each job $\ell \in [m]$ can be assigned at most k agents, so that we obtain the impartial selection problem as the special case where $m = 1$. We first extend the notation from Sect. 2 to this new setting.

For $n, m \in \mathbb{N}$ with $m \leq n$, we consider m-tuples of weight matrices $\mathbf{A} = (A_1, A_2, \dots, A_m) \in \mathcal{A}_n^m$, each of them representing the weighted votes for one job. Let further $k < n$ in the following; an instance of the assignment problem is then given by the tuple $\mathbf{A}$ and the value k. We let

$$\mathcal{X}_k := \big\{\mathbf{X} = (X_1, X_2, \dots, X_m) \in \big(2^{[n]}\big)^m : |X_i| \leq k \text{ and } X_i \cap X_j = \emptyset \\ \text{for every } i, j \in [m] \text{ with } i \neq j\big\}$$

denote the set of feasible assignments, i.e., the set of tuples $\mathbf{X}$ containing m pairwise disjoint subsets of agents, each with cardinality at most k. In a slight overload of notation, for $\mathbf{X} \in \mathcal{X}_k$ and $\mathbf{A} \in \mathcal{A}_n^m$, we write

$$\sigma(\mathbf{X}; \mathbf{A}) := \sum_{\ell \in [m]} \sigma(X_\ell; A_\ell)$$

to refer to the sum, over the jobs, of the score of the set assigned to each job according to $\mathbf{X}$, and we simply write $\sigma(\mathbf{X})$ when the instance is clear from the context. Finally, for $\mathbf{A} \in \mathcal{A}_n^m$, we let

$$\textsc{Opt}_k(\mathbf{A}) := \arg\max_{\mathbf{X} \in \mathcal{X}_k} \sigma(\mathbf{X}; \mathbf{A})$$

denote an arbitrary assignment with the largest score among feasible assignments. We write just $\textsc{Opt}_k$ when the instance is clear.

An (n, m, k)-assignment mechanism is a function $f\colon \mathcal{A}_n^m \to \left(2^{[n]}\right)^m$ such that $f(\mathbf{A}) \in \mathcal{X}_k$ for every $\mathbf{A} \in \mathcal{A}_n^m$. Such a mechanism is *impartial* if, for every pair of instances $\mathbf{A} \in \mathcal{A}_n^m$ and $\mathbf{A}' \in \mathcal{A}_n^m$ and for all agents $i \in [n]$ such that $(A_\ell)_{-i} = (A'_\ell)_{-i}$ holds for each job $\ell \in [m]$, it also holds that $(f(\mathbf{A}))_\ell \cap \{i\} = (f(\mathbf{A}'))_\ell \cap \{i\}$ for every $\ell \in [m]$. We further call an (n, m, k)-assignment mechanism *α-optimal* if

$$\frac{\sigma(f(\mathbf{A}); \mathbf{A})}{\sigma(\textsc{Opt}_k(\mathbf{A}); \mathbf{A})} \geq \alpha$$

holds for all $\mathbf{A} \in \mathcal{A}_n^m$ and some $\alpha \in [0, 1]$.

We are prepared to state the main theorem of this section.

Theorem 3. *Let $n, m, k \in \mathbb{N}$ with $1 < k < n$, $mk \leq n$, and $k - k \bmod 2 \geq 2\sqrt{n}$. Then, there exists an (n, m, k)-assignment mechanism that is impartial and α-optimal with*

$$\alpha = \frac{k - k \bmod 2}{2k \left\lceil \frac{2n}{k - k \bmod 2} \right\rceil}.$$

The proof of this result is omitted due to space constraints. The main ingredient is an adaptation of our mechanism from Sect. 4 that selects from each partition not one but m many agents: one for each set $\ell \in [m]$. We leave the partitioning step unchanged and, for the second step, assign m agents from each candidate set to different jobs in a way that the score obtained for each partition is maximized. In case an agent is assigned to two different jobs, we assign it to the one for which it receives the highest number of votes.

Impartiality of this mechanism follows from a similar reasoning as in the proof of Theorem 1: whenever the vote of an agent is taken into account, the agent is not part of the candidate set. The approximation guarantee makes use of a detailed analysis of the case $b := 2n/k \in \mathbb{N}$ and $b \leq k/2 \in \mathbb{N}$, which is somewhat more intricate than the analysis in Sect. 4. We consider subsets of agents that are assigned to any job in the optimal assignment and are not mutual contenders. We then use the key fact that, when considering the two partitions in which some agent i is in the candidate set, the mechanism assigns agents in a way that the sum of votes of the assigned agents in both partitions is at least the number of votes that i receives for any job. Exploiting the robust partitioning structure as before allows us to take the best of these subsets and conclude via an averaging argument. Here we lose an additional factor of $1/2$ due to the possibility that an agent is initially assigned to two jobs. The extension to general values n, m, and k is then analogous to that of Sect. 4.

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