Self-Similar Gravitational Collapse for Polytropic Stars

Matthew Schrecker

Abstract We describe the construction of self-similar solutions to the gravitational Euler-Poisson equations for polytropic gases, providing exact self-similar profiles for the gravitational collapse of stars. These results are based on joint work with Guo et al. (Arch. Rat. Mech. Anal. 246:957–1066, 2022).

Keywords Singularity formation · Gravitational collapse · Euler-Poisson · Non-linear PDE

Mathematics Subject Classification 85A05, 35Q31, 35Q85

1 Introduction

The rigorous description of the collapse of a star under its own gravity is a fundamental mathematical and physical problem, described by the gravitational Euler-Poisson system. Stellar collapse is an important stage in understanding both the formation and the death of stars. The self-similarity hypothesis suggests that, in certain cases, on approach to collapse, the star should adopt an approximately self-similar form, with the intertwining of spatial and time scales dictated by the scaling symmetries of the underlying physical system (see, for example [9]). In recent work, jointly with Guo et al. [7], we have rigorously constructed exactly self-similar solutions to the Euler-Poisson system for the full range of supercriticial exponents.

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In spherical symmetry, in three spatial dimensions, the (isentropic) gravitational Euler-Poisson equations take the form

$$\partial_t \rho + \partial_r(\rho u) + \frac{2}{r}\rho u = 0, \tag{1}$$

$$\rho\big(\partial_t u + u\partial_r u\big) + \partial_r p + \frac{1}{r^2}\rho m = 0, \tag{2}$$

where the density ρ and radial velocity u are functions only of time t and radial distance $r = |x|$. The mass m is determined through the relation

$$m(t,r) = 4\pi \int_0^r s^2 \rho(t,s)\,\mathrm{d}s, \tag{3}$$

corresponding to the radial component of the gravitational force field $\nabla\phi$, where ϕ is the gravitational potential determined through the Poisson equation

$$\Delta\phi = 4\pi\rho, \qquad \lim_{|x|\to\infty} \phi(t,x) = 0.$$

To close the system of equations, we require an equation of state for the pressure p, which we choose using the usual polytropic relation

$$p = p(\rho) = \kappa\rho^\gamma, \qquad \gamma \in \Big(1, \frac{4}{3}\Big), \quad \kappa > 0. \tag{4}$$

The main result of the paper [7], roughly stated, is then

Theorem 1 (Main Theorem, Rough Version) *For all $\gamma \in (1, \frac{4}{3})$, there exists a smooth initial data pair $(\rho_0(r), u_0(r))$, defined on $[0,\infty)$, with $\rho_0(r) \to 0$ as $r \to \infty$ such that the system (1)–(2) with initial data $(\rho, u)|_{t=-1} = (\rho_0, u_0)$ has a smooth solution $(\rho(t,r), u(t,r))$ for $t \in (-1, 0)$ such that, at the spatial origin $r = 0$, the density $\rho(t,0) \to \infty$ as $t \to 0^-$. For all $r > 0$, the limits of $\rho(t,r)$ and $u(t,r)$ exist as $t \to 0^-$ and define smooth functions $\rho(0,r)$, $u(0,r)$ on $(0,\infty)$.*

As advertised above, we seek this claimed solution through self-similarity. To make this notion precise, we first observe that the system of Eqs. (1)–(2) is invariant under the scaling

$$\begin{aligned} \rho(t,r) &\mapsto \lambda^{-\frac{2}{2-\gamma}} \rho\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big), \\ u(t,r) &\mapsto \lambda^{-\frac{\gamma-1}{2-\gamma}} u\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big). \end{aligned} \tag{5}$$

This motivates the definition of a self-similar variable

$$y = \frac{r}{\sqrt{\kappa}(-t)^{2-\gamma}} \tag{6}$$

and the ansatz

$$\begin{aligned} \rho(t,r) &= (-t)^{-2}\tilde{\rho}(y), \\ u(t,r) &= \sqrt{\kappa}(-t)^{1-\gamma}\tilde{u}(y). \end{aligned} \tag{7}$$

Substituting this ansatz into the spherically symmetric Euler-Poisson system, defining a new unknown

$$\omega(y) = \frac{\tilde{u}(y)}{y} + 2 - \gamma, \tag{8}$$

and dropping the $\sim$ notation yields, after rearrangement, the self-similar system

$$\begin{aligned} \rho' &= \frac{y\rho\left(2\omega^2 + (\gamma-1)\omega - \frac{4\pi\rho\omega}{4-3\gamma} + (\gamma-1)(2-\gamma)\right)}{\gamma\rho^{\gamma-1} - y^2\omega^2}, \\ \omega' &= \frac{4-3\gamma-3\omega}{y} - \frac{y\omega\left(2\omega^2 + (\gamma-1)\omega - \frac{4\pi\rho\omega}{4-3\gamma} + (\gamma-1)(2-\gamma)\right)}{\gamma\rho^{\gamma-1} - y^2\omega^2}. \end{aligned} \tag{9}$$

It is then clear that any smooth solution of (9) with $\rho(0) > 0$ and $\rho(y) \to \infty$ as $y \to \infty$ gives a collapsing solution of the original system (1)–(2) with density blowup at the origin at time $t = 0$.

Seminal work of Larson and Penston [10, 14] offered a numerical solution to this system in the case $\gamma = 1$ (so-called isothermal stars), describing self-similar stellar collapse. However, two fundamental quantities formally conserved along solutions of the system are the total mass and energy, defined by

$$\begin{aligned} M[\rho] &= 4\pi \int_0^\infty \rho r^2 \,\mathrm{d}r, \\ E[\rho,u] &= 4\pi \int_0^\infty \left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}\rho^\gamma - \frac{1}{8\pi}|\partial_r\phi|^2\right) r^2 \,\mathrm{d}r. \end{aligned}$$

In the isothermal setting, the self-similar ansatz of Larson and Penston leads to solutions of infinite mass and energy, as can be seen either directly from the asymptotics of the Larson–Penston solution, or predicted from the scaling relation (5). Indeed, one checks easily that

$$M[\rho_\lambda] = \lambda^{\frac{4-3\gamma}{2-\gamma}} M[\rho], \quad E[\rho_\lambda, u_\lambda] = \lambda^{\frac{6-5\gamma}{2-\gamma}} E[\rho, u].$$

Thus $\gamma = \frac{4}{3}$ is the *mass-critical* exponent and $\gamma = \frac{6}{5}$ is the *energy-critical* exponent. This observation led Yahil [15] to construct finite energy numerical solutions in the range $\gamma \in (\frac{6}{5}, \frac{4}{3})$ and, for this range of γ, these are the solutions rigorously constructed in Theorem 1.

A key difficulty in solving this ODE system rigorously is the presence of singularities in the system. As well as the regular singular point at the origin (due to the radial symmetry assumption), there is an *a priori* unknown further singularity whenever $\gamma\rho^{\gamma-1} - y^2\omega^2 = 0$. At such a point, the relative speed $y\omega$ is exactly the speed of sound in the gas, $\sqrt{p'(\rho)}$, motivating the following definition.

Definitions 2 Let (ρ, ω) be a C^1 solution of system (9) on an open interval $I \subset (0, \infty)$. A point $y_* \in I$ is called a *sonic point* if

$$\gamma\rho(y_*)^{\gamma-1} - y_*^2\omega(y_*)^2 = 0.$$

A sonic point y_* for the self-similar system corresponds to a backwards acoustic cone emanating from the spatio-temporal origin $(t, r) = (0, 0)$ in physical variables. Although the location of a sonic point is *a priori* unknown, the necessity of the existence of at least one such point is given by the physical asymptotic and boundary conditions at infinity and the origin. As we are looking for smooth solutions with positive density at the origin and density tending to zero at infinity, a simple Taylor expansion shows that we require

$$\begin{aligned} \rho(0) > 0, \quad \omega(0) = \frac{4 - 3\gamma}{3}, \\ \rho(y) \sim y^{-\frac{2}{2-\gamma}} \text{ as } y \to \infty, \quad \lim_{y\to\infty} \omega(y) = 2 - \gamma. \end{aligned} \tag{10}$$

The intermediate value theorem then immediately gives the existence of at least one sonic point for any smooth solution.

We can now state the main result of [7] rigorously.

Theorem 3 ([7, Theorem 1.3]) *Let* $\gamma \in \left(1, \frac{4}{3}\right)$. *Then there exists a global, real-analytic solution* (ρ, ω) *of* (9), (10) *with a single sonic point* $y_* \in (0, \infty)$ *and satisfying the natural, physical conditions*

$$\rho(y) > 0 \textit{ for all } y \in [0, \infty), \quad -\frac{2}{3}y < u(y) < 0 \textit{ for all } y \in (0, \infty). \tag{11}$$

It should be noted that such collapse solutions are not expected for $\gamma > \frac{4}{3}$, the mass-subcritical regime. In this range, it has been shown that no collapsing solutions of finite mass and energy can exist, see [3]. In the mass-critical case, $\gamma = \frac{4}{3}$, there is a famous family of solutions due to Goldreich and Weber, [4], which can either collapse or expand. These solutions are found using an effective separation of variables, allowing for the solution to be found as a time-modulated spatial profile satisfying a Lane-Emden type equation. In contrast, the solutions found in this work involve the careful balancing of all three main forces in the system: inertia, pressure and gravity.

In the isothermal case, $\gamma = 1$, the problem of existence of the Larson–Penston collapsing solutions was solved by Guo–Hadžić–Jang [6], who developed a delicate shooting argument based on the existence of local, smooth solutions around a candidate sonic point. In this case, the system (9) simplifies, and a less refined analysis is required to demonstrate the existence of the solutions.

More recently, the same authors have constructed exact self-similar blowup solutions to the Einstein-Euler equations in general relativity, [8]. This work constructs a smooth, self-similar spacetime with singularity (in both curvature and fluid variables) at a centre of symmetry. From this singularity, a null geodesic emanates and escapes to null infinity. To ensure the spacetime is physically meaningful, the spacetime is flattened far from the centre of symmetry to ensure that it is asymptotically flat. This therefore gives an example of a naked singularity for the Einstein-Euler equations.

We mention also the existence of collapsing (or imploding) self-similar solutions of the Euler equations without gravity, which were found recently by Merle et al. [12]. These solutions were constructed using a careful phase portrait analysis, based on an autonomous self-similar ODE system. The same authors proved the finite co-dimension stability of these solutions within the class of radially symmetric solutions, [13]. Later numerical work, [1] suggests that the finite co-dimension is positive, i.e. these solutions are unstable to generic radial perturbations.

In contrast, it is widely expected that the Larson-Penston and Yahil solutions are in fact stable in the class of radial solutions, based on numerical investigations, see [11]. The smoothness of the underlying self-similar profile appears to be essential for the stability properties, both for the full stability of the gravitational collapse and the finite co-dimensional stability of the gas flows.

2 Strategy of Proof

Before offering an outline of the proof, we first observe that there are two explicit solutions to the system (9), the Friedmann solution

$$\omega_F = \frac{4-3\gamma}{3}, \quad \rho_F = \frac{1}{6\pi},$$

and the far-field solution

$$\omega_f = 2-\gamma, \quad \rho_f = k y^{-\frac{2}{2-\gamma}}, \text{ where } k = \left(\frac{\gamma(4-3\gamma)}{2\pi(2-\gamma)^2}\right)^{\frac{1}{2-\gamma}}.$$

The Friedmann solution satisfies the boundary condition at the origin, but not at infinity, and the far-field solution satisfies the asymptotic condition as $y \to \infty$, but fails to be regular at the origin, compare (10). We find that these solutions each have a unique sonic point, $y_F(\gamma)$, $y_f(\gamma)$, respectively, with $0 < y_f(\gamma) < y_F(\gamma) < \infty$, defined by

$$y_F(\gamma) = \frac{3}{4-3\gamma}\sqrt{\frac{\gamma}{(6\pi)^{(\gamma-1)}}}, \quad y_f(\gamma) = \frac{\sqrt{\gamma}}{2-\gamma}\Big(\frac{4-3\gamma}{2\pi}\Big)^{\frac{\gamma-1}{2}}. \tag{12}$$

Thus, for each $\gamma \in (1, \frac{4}{3})$, the interval $[y_f(\gamma), y_F(\gamma)]$ is compact and we search for our solution with a sonic point in the open range (y_f, y_F) (we henceforth drop explicit dependence on γ).

The proof broadly proceeds in four steps. The first step is to construct local solutions around any candidate sonic point $y_* \in [y_f, y_F]$. This is based on a Taylor expansion argument to solve for

$$\rho(y; y_*) = \sum_{n=0}^{\infty} \rho_n(y_*)(y-y_*)^n, \quad \omega(y; y_*) = \sum_{n=0}^{\infty} \omega_n(y_*)(y-y_*)^n. \tag{13}$$

The order zero coefficients are determined from (9) by solving the pair of nonlinear equations

$$\gamma\rho_0^{\gamma-1} - y_*^2\omega_0^2 = 0, \quad 2\omega_0^2 + (\gamma-1)\omega_0 - \frac{4\pi\rho_0\omega_0}{4-3\gamma} + (\gamma-1)(2-\gamma) = 0.$$

This gives, for each $y_* \in [y_f, y_F]$, a unique choice $(\rho_0(y_*), \omega_0(y_*))$, with $\omega_0(y_f) = 2-\gamma$ and $\omega_0(y_F) = \frac{4-3\gamma}{3}$. A selection principle is necessary to determine the first order coefficients (for which there are two possible choices for every $y_* \in [y_f, y_F]$) and then a recurrence relation is used to determine the higher order coefficients. Through combinatorial bounds, these formal series are shown to converge in some neighbourhood of y_* and give a local, real-analytic solution of (9).

The second step is to extend the local solution to the right on the interval (y_*, ∞). This can be done for all $y_* \in [y_f, y_F]$, and the argument is based on the construction of dynamical invariances for the flow, using the precise structure of the non-linearities. To close some of the estimates, it is necessary to verify the sign of certain explicit (but high order) polynomial functions of γ and ω. This is achieved via the use of *interval arithmetic*, a rigorous computer-assisted form of proof that has attracted increasing attention in the PDE community in recent years (see, for example, [2, 5]).

The third step is the most difficult, and contains the key new ideas of the paper. This is to extend the local solution to the left onto the interval $(0, y_*)$. For a general y_*, the solution will *not* extend smoothly all the way to the origin, and so we here develop a shooting argument in order to find a critical $\bar{y}_*$ for which the solution does connect. The central difficulty is that the invariant region arguments used in extending to the right all fail in this direction, which is an unstable direction for the flow. Instead, we rely on a new *monotonicity lemma*, which allows us to obtain the necessary control on the solutions.

To be more precise, we observe that the solution needs to connect to the value $\frac{4-3\gamma}{3}$ at the origin in order to extend smoothly, and, for y_* close to y_F, the solution obtained by Taylor expansion around y_* will quickly decrease below this value. We therefore define the shooting set Y to be the set of candidate sonic points y_* for which the solution to the left intersects this value:

$$Y = \Big\{ y_* \in (y_f, y_F) \mid \text{ for all } \tilde{y}_* \in [y_*, y_F), \\ \text{there exists } y \text{ such that } \omega(y; \tilde{y}_*) = \frac{4-3\gamma}{3} \Big\}$$

and search for the critical $\bar{y}_*$ as the infimum of Y.

Due to the instability of the flow to the left and the possibility of hitting a second sonic point, it is hard to achieve uniform convergence estimates. Instead, we prove the key monotonicity lemma. First, for each $y_* \in Y$, we define the critical time $y_c(y_*)$ as the first touching time

$$y_c(y_*) = \inf\Big\{ y \mid \omega(\tilde{y}; y_*) > \frac{4-3\gamma}{3} \text{ for all } \tilde{y} \in (y, y_*) \Big\}.$$

Lemma 4 *Let* $\gamma \in (1, \frac{4}{3})$. *For all* $y_* \in Y$, *the solution* $(\rho(\cdot; y_*), \omega(\cdot; y_*))$ *defined by the formal Taylor expansion* (13) *and extended to the left on the interval* $[y_c(y_*), y_*]$ *satisfies the strict monotonicity condition*

$$\omega'(y; y_*) > 0 \textit{ for all } y \in [y_c(y_*), y_*].$$

With this monotonicity, we achieve sufficient control of the flow in order to establish that the solution associated to $\bar{y}_*$ exists on the interval $(0, \bar{y}_*)$, in particular ruling out the existence of another sonic point on this interval.

The final step is to show that this solution connects *smoothly* to the origin. The monotonicity simplifies many arguments, avoiding the need for the complicated topological upper- and lower-solution arguments required in [6]. This allows us to show that $\omega(y; \bar{y}_*) \to \frac{4-3\gamma}{3}$ as $y \to 0$, while the density remains bounded. By a further Taylor expansion at the origin and a local uniqueness result, we show that the solution is in fact locally analytic, completing the proof of Theorem 3.

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