A Note on Carleson-Hunt Type Theorems for Vilenkin-Fourier Series

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Abstract In this paper we discuss an analogy of the Carleson-Hunt theorem with respect to Vilenkin systems. In particular, we investigate the almost everywhere convergence of Vilenkin-Fourier series of $f \in L_p(G_m)$ for $p > 1$ in case the Vilenkin system is bounded. Moreover, we state an analogy of the Kolmogorov theorem for $p = 1$ and construct a function $f \in L_1(G_m)$ such that the partial sums with respect to Vilenkin systems diverge everywhere.

Keywords Fourier analysis · Vilenkin system · Vilenkin group · Vilenkin-Fourier series · Almost everywhere convergence · Carleson-Hunt theorem · Kolmogorov theorem

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1 Introduction

In 1947 Vilenkin [53, 54] investigated a group G_m, which is a direct product of the additive groups $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ of integers modulo m_k, where $m := (m_0, m_1, \ldots)$ are positive integers not less than 2, and introduced the Vilenkin systems $\{\psi_j\}_{j=0}^{\infty}$. These systems include as a special case the Walsh system, when $m \equiv 2$.

The classical theory of Hilbert spaces (see e.g the books [49] and [52]) certifies that if we consider the partial sums $S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\, \psi_k$, with respect to Vilenkin systems, then $\|S_n f\|_2 \leq \|f\|_2$. In the same year 1976 Schipp [37], Simon [43] and Young [58] (see also the book [41]) generalized this inequality for $1 < p < \infty$: there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f\|_p \leq c_p \|f\|_p , \quad \text{when} \quad f \in L_p(G_m).$$

It follows that for every $f \in L_p(G_m)$ with $1 < p < \infty$, $\|S_n f - f\|_p \to 0$, as $n \to \infty$. The boundedness does not hold for $p = 1$, but Watari [55] (see also Gosselin[18], Young[58]) proved that there exists an absolute constant c such that, for $n = 1, 2, \ldots,$ the weak type estimate $y\mu \{|S_n f| > y\} \leq c \|f\|_1 , \ f \in L_1(G_m), \ y > 0$ holds.

The almost-everywhere convergence of Fourier series for L_2 functions was postulated by Luzin [30] in 1915 and the problem was known as Luzin's conjecture. Carleson's theorem is a fundamental result in mathematical analysis establishing the pointwise (Lebesgue) almost everywhere convergence of Fourier series of L_2 functions, proved by Carleson [8] in 1966. The name is also often used to refer to the extension of the result by Hunt [20] which was given in 1968 to L_p functions for $p \in (1, \infty)$ (also known as the Carleson-Hunt theorem).

Carleson's original proof is exceptionally hard to read, and although several authors have simplified the arguments there are still no easy proofs of his theorem. Expositions of the original Carleson's paper were published by Kahane [22], Mozzochi [31], Jorsboe and Mejlbro [21] and Arias de Reyna [35]. Moreover, Fefferman [14] published a new proof of Hunt's extension, which was done by bounding a maximal operator S^* of partial sums, defined by $S^* f := \sup_{n \in \mathbb{N}} |S_n f|$. This, in its turn, inspired a much simplified proof of the L_2 result by Lacey and Thiele [28], explained in more detail in Lacey [26]. In the books Fremlin [15] and Grafakos [17] it was also given proofs of the Carleson's theorem. An interesting extension of Carleson-Hunt result much more closer to L_1 space then L_p for any $p > 1$ was done by Carleson's student Sjölin [47] and later on, by Antonov [2]. Already in 1923, Kolmogorov [24] showed that the analogue of Carleson's result for L_1 is false by finding such a function whose Fourier series diverges almost everywhere (improved slightly in 1926 to diverging everywhere). This result indeed inspired many authors after Carleson proved positive results in 1966. In 2000, Kolmogorov's result was improved by Konyagin [25], by finding functions with everywhere-divergent Fourier series in a space smaller than L_1, but the candidate for such a space that is consistent with the results of Antonov and Konyagin is still an open problem.

The famous Carleson theorem was very important and surprising when it was proved in 1966. Since then this interest has remained and a lot of related research has been done. In fact, in recent years this interest has even been increased because of the close connections to e.g. scattering theory [32], ergodic theory [12, 13], the theory of directional singular integrals in the plane [3, 9, 11, 27] and the theory of operators with quadratic modulations [29]. We refer to [26] for a more detailed description of this fact. These connections have been discovered from various new arguments and results related to Carleson's theorem, which have been found and discussed in the literature. We mean that these arguments share some similarities, but each of them has also a distinct new ideas behind, which can be further developed and applied. It is also interesting to note that, for almost every specific application of Carleson's theorem in the aforementioned fields, mainly only one of these new arguments was used.

The analogue of Carleson's theorem for Walsh system was proved by Billard [4] for $p = 2$ and by Sjölin [46] and Demeter [10] for $1 < p < \infty$, while for bounded Vilenkin systems by Gosselin [18]. Schipp [38, 39] (see also [40, 56]) investigated the so called tree martingales and generalized the results about maximal function, quadratic variation and martingale transforms to these martingales and also gave a proof of Carleson's theorem for Walsh-Fourier series. A similar proof for bounded Vilenkin systems can be found in Schipp and Weisz [40, 56]. In each proof, it was proved that the maximal operator of the partial sums is bounded on $L_p(G_m)$, i.e.,

$$\left\| S^* f \right\|_p \leq c_p \left\| f \right\|_p , \quad \text{as} \quad f \in L_p(G_m), \quad 1 < p < \infty.$$

A recent proof of almost everywhere convergence of Vilenkin-Fourier series was given by Persson, Schipp, Tephnadze and Weisz [33] (see also the book [34]) in 2022. Convergence of subsequences of Vilenkin-Fourier series were considered in [6, 7, 50, 51].

Stein [48] constructed an integrable function whose Walsh-Fourier series diverges almost everywhere. Later on Schipp [36, 41] proved that there exists an integrable function whose Walsh-Fourier series diverges everywhere. Kheladze [23] proved that for any set of measure zero there exists a function in $f \in L_p(G_m)$ $(1 < p < \infty)$ whose Vilenkin-Fourier series diverges on the set, while the result for continuous or bounded functions was proved by Harris [19] or Bitsadze [5]. Simon [44] constructed an integrable function such that its Vilenkin-Fourier series diverges everywhere. Generalization of results by Simon [44] and Kheladze [23] can be found in [33, 34].

2 Preliminaries

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of the positive integers not less than 2. Define the group G_m as the complete direct product of the the additive group $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ of integers modulo with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures $\mu_k(\{j\}) := 1/m_k (j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$. In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$. The elements of G_m are represented by sequences $x := \left(x_0, x_1, \ldots, x_j, \ldots\right) \left(x_j \in Z_{m_j}\right)$. It is easy to give a base for the neighborhood of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}_+$, and $\overline{I_n} := G_m \setminus I_n$.

If we define the so-called generalized number system based on m by

$$M_0 := 1, \; M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero.

We define the generalized Rademacher functions, by $r_k(x) : G_m \to \mathbb{C}$,

$$r_k(x) := \exp\left(2\pi \imath x_k / m_k\right), \quad \left(\imath^2 = -1, \;\; x \in G_m, \quad k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see e.g. [1]).

If $f \in L_1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system as:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \quad (n \in \mathbb{N}), \; S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \;\text{ and}$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+)$$

respectively. Recall that (see e.g. Simon [42, 45] and Golubov et al. [16])

$$\sum_{s=0}^{m_k-1} r_k^s(x) = \begin{cases} m_k, & \text{if } x_k = 0, \\ 0, & \text{if } x_k \neq 0, \end{cases} \quad \text{and} \quad D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases} \tag{1}$$

A function P is called Vilenkin polynomial if $P = \sum_{k=0}^{n} c_k \psi_k$.

3 On Martingale Inequalities

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbb{N})$. If $\mathcal{F}$ denotes the set of Haar measurable subsets of G_m, then obviously $\mathcal{F}_n \subset \mathcal{F}$. By a Vilenkin interval we mean one of the form $I_n(x)$, $n \in \mathbb{N}$, $x \in G_m$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. An integrable sequence $f = (f_n)_{n\in\mathbb{N}}$ is said to be a martingale if f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$ and $E_n f_m = f_n$ in the case $n \leq m$. We can see that if $f \in L_1(G_m)$, then $(E_n f)_{n\in\mathbb{N}}$ is a martingale. Martingales with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ are called Vilenkin martingales. It is easy to prove (see e.g. Weisz [56, p.11]) that the sequence $(\mathcal{F}_n, n \in \mathbb{N})$ is regular, i.e., for all non-negative Vilenkin martingales (f_n),

$$f_n \leq R f_{n-1} \qquad \text{where} \qquad R := \max_{n\in\mathbb{N}} m_n, \qquad n \in \mathbb{N}. \tag{2}$$

Using (1), we can prove that $E_n f = S_{M_n} f$ for all $f \in L_p(G_m)$ with $1 \leq p \leq \infty$ (see e.g. [56]). By the well known martingale theorems, this implies that

$$\left\| S_{M_n} f - f \right\|_p \to 0, \ \text{ as } \ n \to \infty \ \text{ for all } \ f \in L_p(G_m) \ \text{ when } \ p \geq 1. \tag{3}$$

For a Vilenkin martingale $f = (f_n)_{n\in\mathbb{N}}$, the maximal function f^* is defined by $f^* := \sup_{n\in\mathbb{N}} |f_n|$. For a martingale $f = (f_n)_{n\geq 0}$ let $d_n f = f_n - f_{n-1}$ $(n \geq 0)$ denote the martingale differences, where $f_{-1} := 0$. The square function and the conditional square function of f are defined by

$$S(f) := \left(\sum_{n=0}^{\infty} |d_n f|^2 \right)^{1/2} \qquad \text{and} \qquad s(f) := \left(|d_0 f|^2 + \sum_{n=0}^{\infty} E_n |d_{n+1} f|^2 \right)^{1/2}.$$

We have shown the following theorem in [56]:

Theorem 9 *If* $0 < p < \infty$, *then* $\|f^*\|_p \sim \|S(f)\|_p \sim \|s(f)\|_p$. *If in addition* $1 < p \leq \infty$, *then* $\|f^*\|_p \sim \|f\|_p$.

4 a.e. Convergence of Vilenkin-Fourier Series

We introduce some notations. For $j, k \in \mathbb{N}$ we define the following subsets of $\mathbb{N}$:

$$I_{jM_k}^k := [jM_k, jM_k + M_k) \cap \mathbb{N} \quad \text{and} \quad \mathcal{I} := \{I_{jM_k}^k : j, k \in \mathbb{N}\}.$$

We introduce also the partial sums taken in these intervals:

$$s_{I_{jM_k}^k} f := \sum_{i \in I_{jM_k}^k} \widehat{f}(i)\psi_i.$$

For simplicity, we suppose that $\widehat{f}(0) = 0$. In [57] was proved that, for an arbitrary $n \in I_{jM_k}^k$, $s_{I_{jM_k}^k} f = \psi_n E_k(f\overline{\psi}_n)$. For $n = \sum_{j=0}^{\infty} n_j M_j \;\; (0 \le n_j < m_j)$, we define

$$n(k) := \sum_{j=k}^{\infty} n_j M_j, \qquad I_{n(k)}^k = \big[n(k), n(k) + M_k\big) \qquad (n \in \mathbb{N}). \tag{4}$$

Let

$$T^I f := T^{I_{n(k)}^k} f := \sum_{\substack{[n(k+1), n(k)) \supset J \in \mathcal{I} \\ |J| = M_k}} s_J f, \quad \text{for} \quad I = I_{n(k)}^k.$$

Lemma 10 *For all $n \in \mathbb{N}$ and $I_{n(k)}^k$ defined in (4), we have that*

$$S_n f = \sum_{k=0}^{\infty} T^{I_{n(k)}^k} f = \psi_n \sum_{k=0}^{\infty} \sum_{l=0}^{n_k - 1} \overline{r}_k^{n_k - l} E_k \left(d_{k+1}(f\overline{\psi}_n) r_k^{n_k - l} \right),$$

Lemma 11 *For all $k, n \in \mathbb{N}$, the following inequality holds:*

$$|T^{I_{n(k)}^k} f| \le R E_k \left(|s_{I_{n(k+1)}^{k+1}} f - s_{I_{n(k)}^k} f| \right), \quad \textit{where} \quad R := \max(m_n, n \in \mathbb{N}).$$

Lemma 12 *For all $n \in \mathbb{N}$, $\left(\overline{\psi}_n T^{I_{n(k)}^k} f\right)_{k \in \mathbb{N}}$ is a martingale difference sequence with respect to $(\mathcal{F}_{k+1})_{k \in \mathbb{N}}$.*

Let I, J, K denote some elements of $\mathcal{I}$. Let $\mathcal{F}_K := \mathcal{F}_n$ and $E_K := E_n$ if $|K| = M_n$. Assume that $\epsilon = (\epsilon_K, K \in \mathcal{I})$ is a sequence of functions such that ϵ_K is $\mathcal{F}_K$ measurable. Set

$$T_{\epsilon;I,J} f := \sum_{I \subset K \subset J} \epsilon_K T^K f, \qquad T_{\epsilon;I}^* f := \sup_{I \subset J} |T_{\epsilon;I,J} f|, \qquad T_\epsilon^* f := \sup_{I \in \mathcal{I}} |T_{\epsilon;I}^* f|.$$

If $\epsilon_K(t) = 1$ for all $K \in \mathcal{I}$ and $t \in G_m$, then we omit the notation ϵ and write simply $T_{I,J} f$, $T_I^* f$ and $T^* f$. For $I \in \mathcal{I}$ with $|I| = M_n$, let $I^+ \in \mathcal{I}$ such that $I \subset I^+$ and $|I^+| = M_{n+1}$. Moreover, let $I^- \in \mathcal{I}$ denote one of the sets $I^- \subset I$ with $|I^-| = M_{n-1}$. Note that $\mathcal{F}_{I^-} = \mathcal{F}_{n-1}$ and $E_{I^-} = E_{n-1}$ are well defined. We introduce the maximal functions s_I^* and s^* by $s_I^* f := \sup_{K \subset I} E_{K^-} |s_K f|$ and $s^* f := \sup_{I \in \mathcal{I}} s_I^* f$. Since $|s_{I^+} f|$ is $\mathcal{F}_{I^+}$ measurable, by the regularity condition (2), we conclude that $|s_{I^+} f| \le R E_I |s_{I^+} f| \le R s_{I^+}^* f$.

Lemma 13 *For any real number $x > 0$ and $K \in \mathcal{I}$, let $\epsilon_K := \chi_{\{t \in G_m : x < s_{K^+}^* f(t) \le 2x\}}$ and $\alpha_K := \chi_{\{t \in G_m : s_K^* f(t) > x, s_I^* f(t) \le x, I \subset K\}}$. Then*

$$T_\epsilon^* f \le 2 \sup_{K \in \mathcal{I}} \alpha_K T_{\epsilon;K}^* f + 4R^2 x \chi_{\{t \in G_m : s^* f(t) > x\}}.$$

Now we introduce the quasi-norm $\|\cdot\|_{p,q}$ $(0 < p, q < \infty)$ by

$$\|f\|_{p,q} := \sup_{x>0} x \left(\int_{G_m} \left(\sum_{I \in \mathcal{I}} \alpha_I \right)^{p/q} d\mu \right)^{1/p},$$

where α_I is defined in Lemma 13. Observe that α_I can be rewritten as

$$\alpha_I := \chi_{\{t \in G_m : E_{I^-} |s_I f(t)| > x, E_{J^-} |s_J f(t)| \le x, J \subset I\}}. \tag{5}$$

Denote by $P^{p,q}$ the set of functions $f \in L_1$ which satisfy $\|f\|_{p,q} < \infty$. For $q = \infty$,

$$\|f\|_{p,\infty} := \sup_{x>0} x \left(\int_{G_m} \left(\sup_{I \in \mathcal{I}} \alpha_I \right)^p d\mu \right)^{1/p} \qquad (0 < p < \infty).$$

It is easy to see that

$$\|f\|_{p,\infty} \le \|f\|_{p,q} \quad (0 < q < \infty) \quad \text{and} \qquad \|f\|_{p,\infty} = \sup_{x>0} x \mu(s^* f > x)^{1/p}.$$

Lemma 14 *Let $\max(1, p) < q < \infty$, $f \in P^{p,q}$ and $x, z > 0$. Then*

$$\mu \left(\sup_{I \in \mathcal{I}} \alpha_I T_{\epsilon;I}^* f > zx \right) \le C_{p,q} z^{-q} x^{-p} \|f\|_{p,q}^p, \text{ where } \alpha_I \text{ is defined in Lemma 13.}$$

Lemma 15 *Let $\max(1, p) < q < \infty$ and $f \in P^{p,q}$. Then*

$$\sup_{y>0} y^p \mu\Big(T^* f > (2 + 8R^2) y \Big) \le C_{p,q} \|f\|_{p,q}.$$

Let Δ denote the closure of the triangle in $\mathbb{R}^2$ with vertices (0, 0), (1/2, 1/2) and (1, 0) except the points $(x, 1-x)$, $1/2 < x \leq 1$.

Lemma 16 *Suppose that* $1 < p, q < \infty$ *satisfy* $(1/p, 1/q) \in \Delta$. *Then, for all* $f \in L_p$, *we have* $\|f\|_{p,q} \leq C_{p,q}\|f\|_p$.

Now we are ready to formulate our first main result.

Theorem 17 *Let* $f \in L_p(G_m)$, *where* $1 < p < \infty$. *Then*

$$\left\|S^* f\right\|_p \leq c_p \left\|f\right\|_p, \quad where \quad S^* f := \sup_{n\in\mathbb{N}} \left|S_n f\right|.$$

The next norm convergence result follow from Theorem 17.

Theorem 18 *Let* $f \in L_p(G_m),\ 1 < p < \infty$. *Then* $\|S_n f - f\|_p \to 0, \quad as \ n \to \infty$.

Our announced Carleson-Hunt type theorem reads:

Theorem 19 *Let* $f \in L_p(G_m)$, *where* $p > 1$. *Then* $S_n f \to f, \quad a.e., \ as \ n \to \infty$.

5 Almost Everywhere Divergence of Vilenkin-Fourier Series

A set $E \subset G_m$ is called a set of divergence for $L_p(G_m)$ if there exists a function $f \in L_p(G_m)$ whose Vilenkin-Fourier series diverges on E.

Lemma 20 *If* E *is a set of divergence for* $L_1(G_m)$, *then there is a function* $f \in L_1(G_m)$ *such that* $S^* f = \infty$ *on* E.

Lemma 21 *A set* $E \subseteq G_m$ *is a set of divergence for* $L_1(G_m)$ *if and only if there exist Vilenkin polynomials* $P_1, P_2, \ldots,$ *such that* $\sum_{j=1}^{\infty} \|P_j\|_1 < \infty$ *and*

$$\sup_{j\in\mathbb{N}_+} S^* P_j(x) = \infty \ (x \in E).$$

Corollary 22 *If* $E_1, E_2, \ldots$ *are sets of divergence for* $L_1(G_m)$, *then* $E := \cup_{n=1}^{\infty} E_n$ *is also a set of divergence for* $L_1(G_m)$.

Theorem 23 *If* $1 \leq p < \infty$ *and* $E \subseteq G_m$ *is a set of Haar measure zero, then* E *is a set of divergence for* $L_p(G_m)$.

Theorem 24 *There is a function* $f \in L_1(G_m)$ *whose Vilenkin-Fourier series diverges everywhere.*

Remark 25 For details of the above statements we refer to [33, 34].

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