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Franz W. Peren · Thomas Neifer
Editors

Operations Research and Management

Quantitative Methods for Planning and Decision-Making in Business and Economics

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Reiner Clement

in memoriam

Editors

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Preface

The following book is based on the experiences of many master students who studied business or economics and participated at the editor's lectures over more than fifteen years. The co-authors of this book collected all these contributions by composing individual articles for which the publishers of this book are grateful.

The publishers would like to thank their academic colleagues who have contributed to this work and to many consulting projects with creativity, knowledge and dedication for more than 25 years. In particular, they would like to thank Steven Dyla, Michelle Jarsen & Nawid Schahab, who were instrumental in managing and creating this book.

Should any mistakes remain, such errors shall be exclusively at the expense of the publishers. The publishers are thankful in advance to all users of this book for any constructive comments or suggestions.

Bonn, August 2023

Franz W. Peren
Thomas Neifer

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The Decision Tree Procedure

Katharina Völker & Maresa Hümmer*

Abstract In an interconnected world, decision-making has become increasingly complex and critical. This complexity is driven by a multitude of interconnected decisions and the rapid evolution of technologies and markets. Managers must navigate this landscape by quickly evaluating numerous options and selecting the optimal course of action, particularly in areas such as investment, financing, and production processes. Decision theory, a subfield of operations research, offers valuable insights and tools to address these challenges. By providing the theoretical foundations and practical case studies, this work gives a deeper understanding of the decision tree procedure to support decision-making.

1 Introduction

In an increasingly fast-moving and interconnected world, there is not only the fact that more and more decisions must be made, but also a wealth of complex interrelationships exists between the individual facts [1].

The impacts of the rapidly changing surroundings also effect the global economy and the markets. Innovations that were brand new yesterday, are already obsolete today [2].

In order to keep up with the fast-growing competition that accompanies change, it is necessary that managers understand the needs of their business. This also includes considering all possible decisions quickly and selecting the best possible alternative from the spectrum of all potential approaches. Examples of important managerial

* The following presentation of the decision tree procedure has been developed at the Bonn-Rhein-Sieg University in Sankt Augustin, Germany, in 2020 under the leadership of Prof. Peren. It is based on scientific essays from Nihat Bucakli, Ute Faßbender, Nathanael Wilder, and Benjamin Hafemeister, which have been developed in the same course as years before.

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tasks that require complex decisions are the determination of investments, financing or production processes [3].

To achieve an overview of all the abundance of complex choices (multi-level decisions), multiple approaches have been researched in the context of the decision theory. This theory belongs to the scientific discipline of operations research, which mainly deals with the efficient and effective solution of problems [4].

One of the used methods under operations research is the so-called "decision tree procedure," which is described in the following explanations, both in its theoretical components and underpinned with practical applications by case studies to provide a better understanding.

1.1 Theoretical Classification of the Decision Tree Procedure

The decision tree procedure, as mentioned above, can be classified into the busi-ness decision theory. Günther Bamberg describes it as a "[. . .] logical and empiri-cal analysis of rational [. . .] decision behavior" [5].

There are four different areas under the decision theory: the descriptive, the normative or prescriptive, and the statistical direction (see Table 1) [5].

Table 1 Different Areas of Decision Theory

Name	**Objective**
Descriptive direction	Explains *how* decisions should be made in the most rational way, fitting to the given objective. → development of logical conclusions.
Normative / prescriptive direction	Explains *why* decisions are made and why they are made in a specific way. → formulates statements about reality.
Statistical direction	Mathematical , statistical approach

Source: Own representation according to Zimmermann (2008) [4]; Bamberg (2019) [5].

As one of the methods used in decision theory, the decision tree procedure can help find the most beneficial solution for a complex problem [6].

When it comes to a concatenation of multiple binary, yes-or-no decisions, this method also provides a good overview of all necessary information in order to finally reach a decision [7]. When using the decision tree procedure, information is classified into different categories, as described in the following section.

1.2 Basic Structure of a Decision Model

Generally, all decision models follow the same basic structure [8]. The main idea of decision theory is the decomposition of a complex problem into individual compo-

nents. These are compiled from the alternatives of action, environmental influences, and consequences of them, as well as the goals and preferences of the decision maker (see Figure 1) [7].

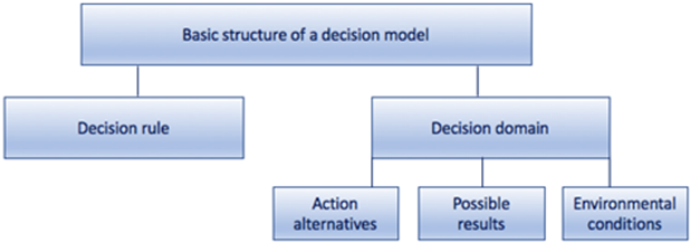

Fig. 1 Basic Structure of a Decision Problem. Source: Own representation according to Laux et al. (2014) [8].

1.2.1 Decision Rule

A rational decision should always lead to the selection of the option that results in the greatest possible satisfaction of needs [9].

The decision rule determines which action alternatives are selected from the entirety of possibilities to achieve the most favorable outcome. In order to set a target, it is necessary to have a precise vision of it, which means to be able to assess the different possibilities with regard to their consequences.

The decision rule consists of a preference function that assigns preference values to the existing alternatives. Also, an optimization criterion needs to be defined which specifies the characteristics of the preferences. The expression of these values of each action alternative indicates to which extent the alternative serves to achieve the goal.

Laux states that priority is always given to the alternative with the highest preference value. Therefore, the decision rule should be in general [8]:

$$\phi(A_a) \rightarrow \max! \tag{1}$$

1.2.2 Decision Domain

One speaks of a decision problem only when at least two possible alternatives for action are available to solve a situation. Those options are individual, self-selected values by the decision maker, the so-called variables or parameters of decision [9].

In addition, all possible consequences, the target variables, need to be considered. These can consist of single or multiple values. If there are multiple possible results,

one is speaking of a constellation. As there can be an abundance of different outcomes, the decision maker selects in advance only after the relevant consequences. For instance, the decision about a huge investment is made under the consideration of price, usefulness, amortization time, etc. and other conditions, such as color of the

Furthermore, environmental conditions, which are not influenceable by the decision-making person, are part of that kind of model. Whether those parameters, the so-called decision relevant data, are suggestible mainly depends on the situation in which the decision is made. Sometimes it is even possible that environmental conditions are not even known. Therefore, the exact impact of a choice cannot be predicted with certainty under unclear environmental conditions and multiple alternatives [8], [10].

A distinction is made between decision making under certainty and uncertain-ty. The last point can be separated into the uncertainty in the narrow sense and risk (see Figure 2).

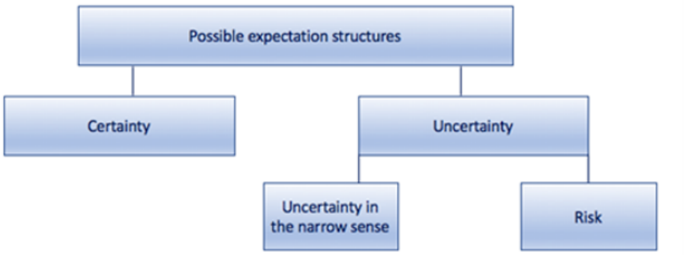

Fig. 2 Decisions under Certainty and Uncertainty. Source: Own representation according to on Laux et al. (2014) [8].

One speaks of an uncertain situation in the narrow sense when it is known that one of the environmental conditions will occur, but the probability of its occurring is not known exactly.

On the contrary, there is a risk if the probability, whether it is subjective or objective, of the occurrence of the respective conditions is known.

It is very unlikely that all external environmental conditions are well known, but if this is the case, it is a situation under certainty [4], [10].

1.3 The Decision Tree

There are various possibilities to visualize a decision process with its different action alternatives. One option is the representation in a decision matrix (see Figure 3).

	Environmental event 1 (s_1)	...	Environmental event i (s_i)	...	Environmental event n (s_n)
	$p(s_1)$	...	$p(s_2)$	...	$p(s_3)$
action alternative 1 (a)	a_1	...	a_i	...	a_n
action alternative 2 (b)	b_1	...	b_i	...	b_n
action alternative 3 (c)	c_1	...	c_i	...	c_n

Fig. 3 Decision Matrix with one Target. Source: Own representation according to Eisenführ and Weber (2013) [7].

A condition matrix consists of the combination of different alternative actions with diverse environmental states. Each combination of an alternative action with an external condition represents a possible consequence.

In the case of the existence of only one target, each outcome can be described exactly by a target variable, see Figure 3. In the case of several target possibilities, they are represented in a vector of all target variables [4], [7].

Each decision matrix can easily be transferred into a decision tree. Especially in multi-stage decision processes, it is advisable to use the decision tree method [7].

This method also represents possible alternatives for action, whereas a status tree only visualizes the current and future environmental conditions expected by the decision maker. Therefore, the decision tree can be seen as an extended form of the condition tree [8].

In literature, the components of a decision tree are described in the same way. It always consists of a decision node, one or multiple chance nodes, utility nodes and the branches (see Table 2) [7], [11]. The typical shape of a decision tree is shown in Figure 4.

1.4 Process of Decision-Making with the Decision Tree

As the structure of a decision tree was explained in the previous chapter, the following is an explanation of how to use and evaluate the decision tree. Essentially, there are five steps that are taken during a decision tree analysis (see Figure 5).

In order to better underpin this process, the example of a company acquisition decision is presented in Table 3 below.

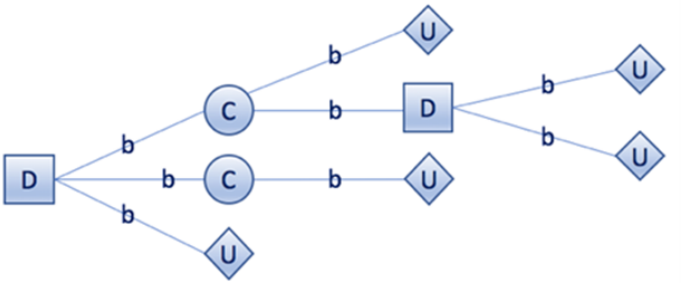

Fig. 4 Typical Shape of a Decision Tree. Source: Own representation according to Jensen and Nielsen (2007); Eisenführ and Weber (2013) [7], [11].

Table 2 Explanation of the Decision Tree Components

D → Decision Node	A decision node is the starting point of a decision tree. At this point, a decision must be made. Branches (b) beginning there, symbolize the possible action alternatives between which can be chosen.
C → Chance Node	A chance node represents an event or an environmental state. Branches (b) beginning there, represent further possible events or states with their respective probabilities. The sum of the probabilities at the chance node needs to equal one.
U → Utility Node	A utility node represents a consequence. Every branch of the decision tree needs to end in a final consequence.
B → Branch (with label)	Branches symbolize action alternatives or environmental events or states. They usually appear after decision and chance nodes or end up in consequences.

Source: Own representation according to Eisenführ and Weber (2013) [7]; Jensen and Nielsen (2007) [11].

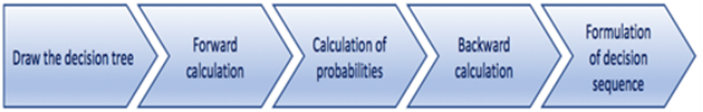

Fig. 5 Steps of a Decision Tree Analysis. Source: Own representation according to Zimmermann and Stache (2001) [12]; Hoffmeister (1997) [13].

Table 3 Process Steps of the Decision Tree Procedure

Process step	Action
Draw the Decision Tree	First, the decision tree needs to be drawn. Starting from the first decision node, the drawing process follows a top-down approach, ending with the utility nodes. *Example:* Decision whether another company should be acquired or not, with all possible action alternatives, environmental conditions and consequences.
Forward Calculation	In the process step of forward calculation, all information necessary for a decision is collected and noted along the respective branches of the decision tree. Finally, the total cash flow along all branches will be determined from the start decision node to every end node. *Example:* Notation of the costs and revenues at every chance node of the acquisition decision and determination of the total cashflow along the different branches.
Calculation of Probabilities	For every chance node a probability of occurrence will be determined at this process step. It is important that the sum of the probabilities of all possible situations, following a chance node, equals one.
Backward Calculation	At the process step of backward calculation, expected values of all chance nodes will be determined. These calculated values will be noted at the nodes. Beginning from the last chance node of every branch the expected values are compared. Only the node with the highest expected value will be left, all other branches will be cut off. *Example: The expected value of the branches that represent a positive acquisition decision of company A is significantly lower than the value of a decision for acquisition of company B → Cutting of the branches with the positive acquisition decision for company A.*
Formulation of Decision Sequence	After the unfavorable branches of the decision tree have been chopped off in the previous process step, the decision tree displays only the best decision sequences. On this basis, a final decision should be made.

Source: Own representation according to Zimmermann and Stache (2001) [12]; Hoffmeister (1997) [13].

2 Methods of Decision Tree Procedure

One can see from the analysis of the decision tree procedure described in the previous chapter that it is not a question of calculating all possible solutions down to exact detail [12].

In this context, various methods have been developed that deal with finding the optimal solution through a decision tree. These include the complete enumeration, the incomplete enumeration, the dynamic optimization and the branch and bound method [14].

2.1 Complete Enumeration

The method of complete enumeration includes the comparison of all conceivable solutions that fit to a decision problem. It contains the requirement of finding the optimal and most beneficial alternative to solve an issue. It is a fact that after the execution of the complete enumeration procedure one receives a widespread overview of possible action alternatives with the corresponding decision consequences (see Table 4) [15].

However, this method has the great disadvantage that it is hardly possible to keep track of an increasing number of possible solutions in an economical way [4]. In this context, companies need to ask themselves if and to what extent resources should be invested in the implementation of the decision-making process.

Table 4 Overview of Complete Enumeration

Main target	Identification and selection of the optimal solution of all possibilities that come into consideration
Approach	Detailed calculation of all existing solutions → accurate approach
Characteristics	→ Accurate method, resulting in finding the best possible solution → very high expenditure to prepare the necessary data and the conduction of the analysis

Source: Own representation according to Zimmermann and Stache (2001) [12].

2.2 Incomplete Enumeration

The method of incomplete enumeration (see Table 5), in contrast to complete enumeration, aims not at analyzing all possible action alternatives in detail. Already during the calculation of a new solution, it is compared to already analyzed ones. If it becomes clear that the target function value of the new possibility is lower than the value of the already calculated one, this alternative is discarded immediately [14].

It cannot be said with one hundred percent probability that the solution determined by this method is the most optimal. But in comparison to the complete enumeration, this approach requires much less effort for data acquisition and data analysis [12].

The method of enumeration belongs to the heuristic approaches, which includes the process of decision-making despite limited knowledge. With regard to incomplete enumeration, this means that only a part of all possible solutions is mapped and therefore only a subset of them is analyzed [16].

As the previous elaborations have already confirmed, both the method of complete and incomplete enumeration have some advantages as well as disadvantages (see Figure 6).

Table 5 Overview of Incomplete Enumeration

Main target	Identification and selection of the optimal solution of all possibilities that come into consideration
Approach	Restriction to the analysis of promising solution approaches → heuristic approach
Characteristics	→ comparison of target functions with already analyzed ones → solutions with lower target value are not further considered

Source: Own representation according to Zimmermann and Stache (2001) [12].

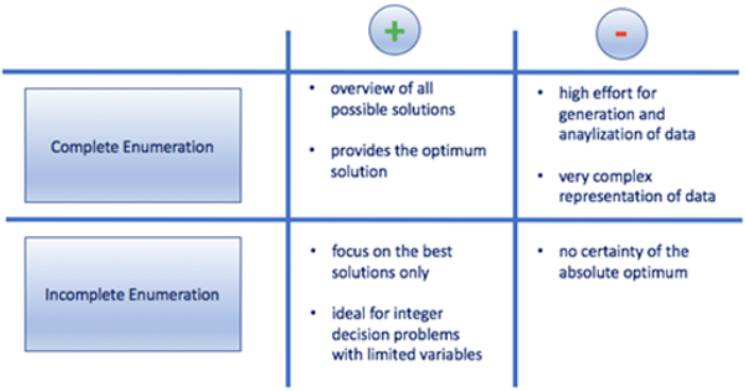

Fig. 6 Comparison of Complete and Incomplete Enumeration. Source: Own representation according to Zimmermann and Stache (2001) [12].

2.3 Dynamic Optimization

The method of dynamic optimization basically aims at breaking down a complex optimization problem into individual sub-problems. These sub-elements will then be examined step-by-step with regard to their solution. At each phase, only the decision alternatives that occur at the specific stage are analyzed [12].

This method is particularly recommended for optimization problems where many consecutive interdependent decisions have to be made to solve an overriding issue (see Table 6 [16].

The three different phases of dynamic optimization will be explained in the following paragraphs.

2.3.1 Decomposition

As a first step, the primary problem is decomposed into sub-elements. This decomposition can take place either on a temporal or spatial basis. After reduction of complexity, the individual sub-problems are analyzed and finally combined again.

2.3.2 Backward Calculation

The process of backward calculation has been previously explained in more detail in chapter 1.4. With regard to the method of dynamic optimization, this means to determine the optimal decision for the different action alternatives at each level starting with the last stage. Alternatively, phase 2 can also start with the forward calculation.

2.3.3 Forward Calculation

Finally, the forward calculation follows insofar as the backward calculation was conducted in phase 2.

Under this approach, the action alternatives at every stage are examined with regard to their target value beginning from the starting point [12].

Table 6 Overview of Dynamic Optimization

Main target	Identify the optimum overall solution
Approach	Decomposition of complex problems into trivial, interdependent sub-problems
Characteristics	→ application in dynamic processes based on time (problems of investment or warehousing problems) → dynamic model which includes chaining systems, parameters are dependent on current data

Source: Own representation according to Zimmermann and Stache (2001) [12]; Domschke et al. (2015) [16]; Grundmann (2002) [15].

2.4 Branching and Bounding

The branch and bound method works by breaking down the total set of acceptable resolutions into subsets. The target is to work out that specific subset which contains the optimum solution [12].

The visualization of the branch and bound method is done via a decision tree. The total number of all permissible solution possibilities forms the starting point of the tree. Alternatively, a suitable subset of the conceivable solutions can be selected as the initial point [15], [17].

Branching and bounding is performed in parallel along the branches of the decision tree.

2.4.1 Branching

Branching is essentially about breaking down the total quantity of all solutions into suitable subsets. Afterwards, these are gradually brought down further, creating a tree structure [12], [18].

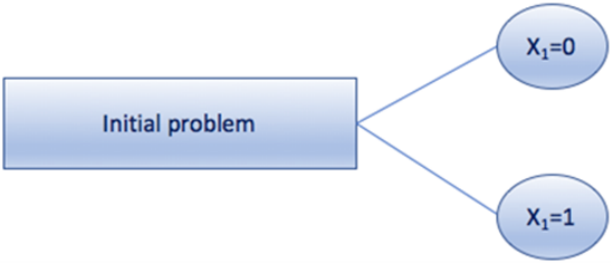

Fig. 7 Process of branching. Source: Own representation according to Stingl (2002) [18]; Zimmermann and Stache (2001) [12].

Each branch of the decision tree (see Figure 7 represents a solution strategy for the initial problem or one of the sub-problems.

2.4.2 Bounding

The next step of the procedure is to set upper and lower boundaries for the sub-problems. These limits provide information about whether a problem should be analyzed in more detail. Those limits are not rigid but are continuously adjusted during the analysis. As soon as a possible alternative is outside the specified tolerance range, this branch of the decision tree is not further taken into consideration and thus, cut off [4], [15].

By excluding irrelevant solution alternatives, it is ensured that not all non-target branches of the decision tree are analyzed, but only those that lie within the boundaries, and thus fulfill the target function, are further examined [15].

Example: In case of a maximum problem, the determined value of the sub-problem analysis must not undercut the lower limit. Otherwise, this alternative is regarded as not target-oriented and thus not analyzed further.

The consequence of this is cutting of the branch in the decision tree.

Table 7 Overview of the Branch and Bound Method

Main target	Identify the optimum overall solution
Approach	Set upper and lower boundaries for the target values of specified sub-problems
Characteristics	→ if the target value of a sub-problems lays outside the specified boundaries, option is not considered any more. → cut off of respective decision tree branch

Source: Own representation according to Zimmermann (2008) [4]; Grundmann (2002) [15].

3 Economic Relevance and Critical Appraisal

Finally, the economic relevance of the decision tree method is worked out in this section. In accordance, a number of practical fields of action are displayed. Furthermore, the advantages and disadvantages of this discipline of operations research are compared and critically evaluated [7].

3.1 Economic Relevance

In this fast-moving world, a manager must be able to react quickly to environmental conditions and make the best possible decisions accordingly. The decision-making process is an essential part of the management process. In addition, it consists of the following components: target detection, planning, realization and checking (see Figure 8) [19].

Fig. 8 The management process. Source: Own representation according to Graf and Sprengler (2013) [19].

One can clearly see in the figure above that the management process consists of repetitive successive tasks. The different steps of the management process are (see Table 8):

Table 8 Patterns of the management process

Target detection	Specification of a business strategy and formulation of objectives derived from it. Derived from these targets, subordinate business processes should be aligned.
Decision making	Selection of suitable action alternatives that contribute to achieving the specified strategic goals.
Planning	Planning of the action steps selected to achieve the specified goals.
Realization	Realization of planned measures.
Checking	Comparison of the actual situation with the target situation, if required adjustment measures must be taken.

Source: Own representation according to Graf and Sprengler (2013) [19]; Züger (2011) [20].

Decision-making plays an important role here. As already mentioned, managers in companies often have to deal with complex, multidimensional decision-making problems. This is where the decision tree method is particularly important, as it allows managers to keep track of these complex structures and to select that specific action alternative that offers an optimal solution. Entrepreneurial examples of a multi-level decision problem are:

- Credit scoring
- Examination of investments
- Analysis and categorization of customer groups
- Financial analysis and evaluation of attractiveness for investment forms
- Determining the optimal production program
- Aquisition of another company
- Stockholding decisions

3.2 Critical Appraisal

The method of the decision tree procedure serves, as described in the previous remarks, to present a decision problem in a clear and concise manner. In addition, the decision-making person is urged to formulate the objectives, action alternatives, environmental influences and the consequences resulting from these parameters in a precise and comprehensible form [7].

This leads to a clear overview of existing decision problems and acceptable action alternatives, which can contribute significantly to the solution identification process.

Nevertheless, this procedure also holds some disadvantages. Depending on the complexity and extent of the decision problem under uncertainty, i.e., when the development of environmental conditions is unknown, it is not always achievable to list and analyze all conceivable solutions. In these cases, it is not possible to make a one hundred percent correct statement with the decision tree.

The following Table 9 provides an overview of the advantages and disadvantages of the decision tree procedure.

Table 9 Advantages and Disadvantages of the Decision Tree Procedure

Advantages	**Disadvantages**
Transparent and clear form of visualization	Depending on the extent of the acceptable decision options, the decision tree may become too complex and inefficient.
Ideal for branched and multi-level decision problems	The process of problem solving is determined by action alternatives and environmental conditions, which are selected by the subjective view of the decision-making person. Therefore, objectivity is not always ensured.
Depending on the solution method used, only the relevant sub-problems are considered in more detail → efficiency gain	Limited presentation of all acceptable action alternatives and solutions approaches.
Less time for analyzing required compared to other methods	No guarantee of finding the optimal solution.
Possibility of combination with other decision techniques	
During the construction of the decision tree, even more possible action alternatives can be considered and integrated into the decision-making process	

Source: Own representation according to Eisenführ and Weber (2013) [7]; Dahan et al. (2014) [21].

4 Case Studies and Software

As already stated in the introduction, the explored theoretical knowledge will be applied to real cases, here for investment decisions. Following that line, the next chapter contains two case studies to show how the decision tree procedure can be used practically. In addition, a software example will be presented later in this chapter. The visualization of decision trees is facilitated with the utilization of this software.

4.1 Case Study I

Initial situation: Harold is planning on opening a sustainable fast-food restaurant, but he is unsure if he should make the investment or not. The decision tree procedure and roll-back process can be used for his decisive process.

Harold only has one alternative: to invest or not to invest. If the investment is made, the business could grow or could decrease. If the business development is not satisfying, an advertising campaign has the potential to change a less successful business development into a positive one [22].

The following data are known [22]:

- Investment costs: $20,000
- Sales of successful business development: $50,000
- Sales of unsuccessful business development: $10,000
- Costs of advertising: $21,000

The following probabilities can be assumed [22]:

- Success of business development: 20 %
- Failure of business development: 80 %
- Success of advertising campaign: 60 %
- Failure of advertising campaign: 40 %

Will the investment be viable with the measures of the decision tree?

As a first step, the decision tree has to be drawn. In preparation of that, the decision nodes, chance nodes and end nodes have to be defined [7], [22]. Regarding Harold's plan, the first decision node is characterized by the decision occurrence: to invest or not to invest. In the case of no investment, nothing occurs, and the branch is closed by an end node. In the case of investment, two opportunities can occur: the investment could be successful or not. If the investment is successful, nothing else occurs and the branch again ends with an end node. But if the investment is not successful, Harold has the chance to change the unsuccessful investment into a successful one by launching an advertising campaign. This is the second decision node: advertising or no advertising. In the case of no advertising, nothing will change, and the branch ends with an end node. In the case of advertising, another chance node will follow: the advertising can be successful or not [22]. With all this information, the decision tree can be drawn along the defined chance and end nodes. The corresponding decision tree is drawn in Figure 9 below:

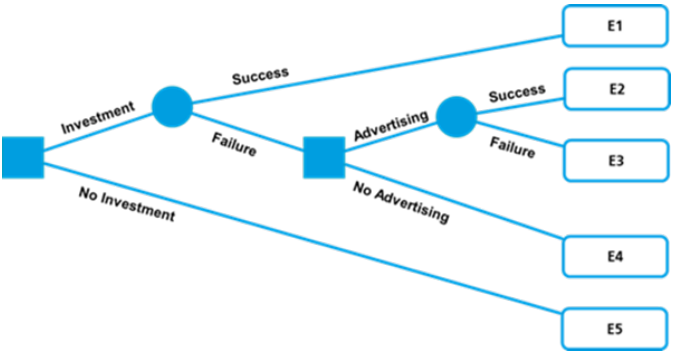

Fig. 9 Binary Decision Tree. Source: Own representation according to Jensen and Nielsen (2007) [11]; Eisenführ and Weber (2013) [7].

As the second step, the cash flows (end nodes) have to be calculated by forward calculation. In order to reach this, deposits and disbursements need to be determined along the branches [13], [22]. Based on the information from Harold's case, the following Figure 10 shows the decision tree including the deposits and disbursements along the branches and the end nodes (cash flows):

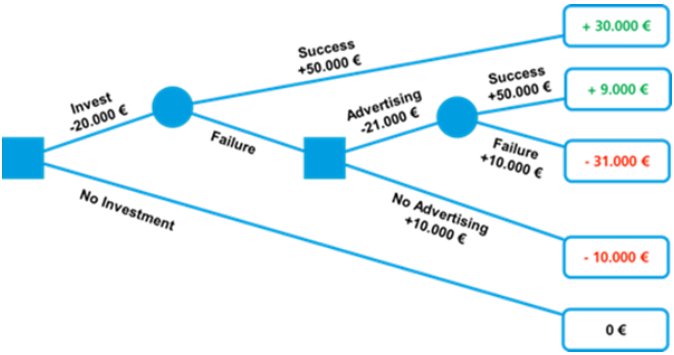

Fig. 10 Decision tree incl. deposits disbursements and cash flows. Source: Own representation according to Möhlmann (2009) [22].

With the utilization of the end notes, the process of forward calculation will be explained as follows:

E1: The branch starts with the investment (disbursement of $20,000), followed by the success of the investment (deposits of $50,000). By forward calculation the branch leads to a cash flow of $30,000 (E1) [22]:

$$-\$20,000 + \$50,000 = \$30,000 \text{ (E1)} \tag{2}$$

E2: The branch starts with the investment (disbursement of $20,000), followed by the failure of the investment. However, in this case, Harold launches an advertising campaign (disbursement of $21,000) to change the less successful investment into a positive one (deposits of $50,000). By forward calculation the branch leads to a cash flow of $9,000 (E2) [22]:

$$-\$20,000 - \$21,000 + \$50,000 = \$9,000 \text{ (E2)} \tag{3}$$

The forward calculations for the other end nodes are examined below [22]. The method of the calculation of the cash flows remains unaltered. That is why no detailed explanation is needed at this point.

E3:

$$\begin{aligned} &-\$20,000 \text{ (investment)} \\ &-\$21,000 \text{ (advertising)} \\ &+\$10,000 \text{ (sales of unsuccessful business)} \\ &= -\$31,000 \end{aligned} \tag{4}$$

E4:

$$\begin{aligned} &-\$20,000 \text{ (investment)} \\ &+\$10,000 \text{ (sales of unsuccessful business)} \\ &= -\$10,000 \end{aligned} \tag{5}$$

E5:

$$\$0 \text{ (no investment)} \tag{6}$$

To assess the investment decision, the profit expectation is decisive. To achieve this, the decision tree includes the probabilities of risks and chances for every logical path of occurrences and future decisions as a first step. In the second step, the probabilities have to be determined along the branches. This is the basis for the following backward calculation [13]. The next Figure 11 illustrates the decision tree including the probabilities along the branches:

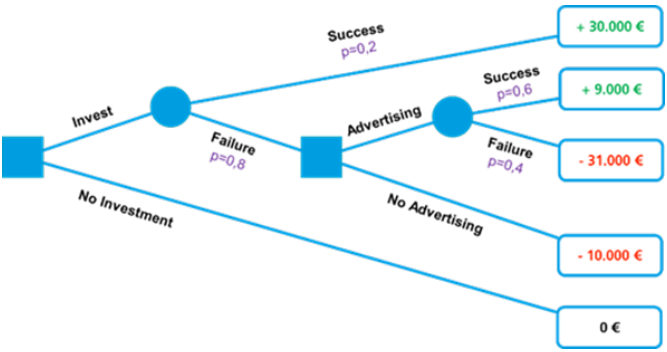

Fig. 11 Decision tree incl. probabilities. Source: Own representation according to Möhlmann (2009) [22].

With the information from Harold's case, the excepted profit can be calculated with the help of the so-called "roll-back calculation." This method requires multi-

plying the cash flows with the given probabilities along the branches and afterwards adding them together path by path. The non-optimal branches get separated (cropped) [13], [22]. The calculations are explained as follows [22]:

1. Expected profit of promotional activities:
   $\mu = 0.6 * \$9,000 + 0.4 * \$3,000 = -\$7,000$
2. Excepted profit of the investment:
   $\mu = 0.2 * \$30,000 + 0.8 * (-\$7,000) = \$400$
3. Separating the non-optimal branches. These branches of the tree will not be considered. In this case, two branches have to be separated:
   - $-\$7,000$ (expected profit of promotional activities) > $-\$10,000$ (expected profit of no promotional activities)
     Result: The expected profit of promotional activities is higher than the expected profit of no promotional activities. Thus, the branch of no advertising has to be separated.
   - \$400 (expected profit of the investment) > \$0 (expected profit of no investment)
     Result: The expected profit of the investment is higher than the expected profit of no investment. Hence the branch of no investment has to be separated.

Figure 12 illustrates the decision tree including the results.

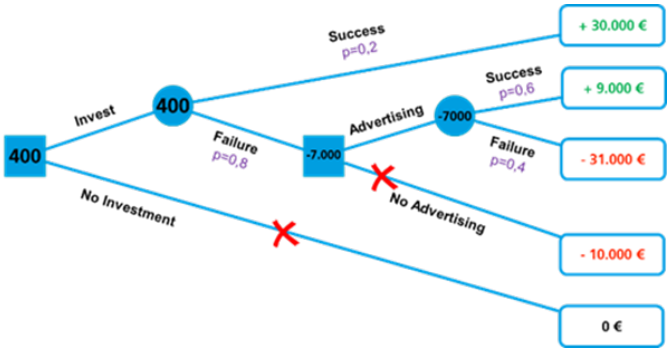

Fig. 12 Final Decision Tree. Source: Own representation according to Möhlmann (2009) [22].

The decision tree can also be drawn differently [22]. Figure 13 shows the different illustrations of the decision tree including the results from above.

In this perspective, the deposits represent the end nodes and the disbursements are noted along the branches. The expected profit can be calculated as well with the roll-back calculation [22]:

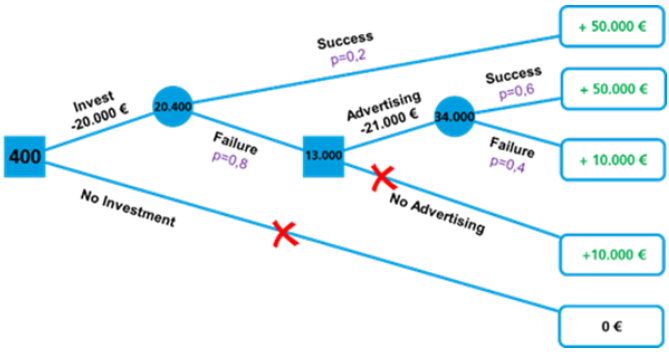

Fig. 13 Final Decision Tree, second version. Source: Own representation according to Möhlmann (2009) [22].

1. Expected value of advertising campaign:
   $\mu = 0.6 * \$50,000 + 0.4 * \$10,000 - \$21,000 = \$13,000$
2. Excepted profit of the investment:
   $\mu = 0.2 * \$50,000 + 0.8 * \$13,000 - \$20,000 = \$400$
3. Separating the non-optimal branches (the same procedure as above):
   - $\$13,000$ (expected profit of promotional activities) > $\$10,000$ (expected profit of no promotional activities)
     Result: An advertising campaign is superior to no advertising campaign. That is why the no advertising branch has to be separated.
   - \$400 (expected profit of the investment) > \$0 (expected profit of no investment)
     Result: The investment is superior to no investment. Consequentially, the no-investment branch has to be separated.

With respect to the information of the decision tree, Harold´s profit will increase by \$400. Therefore, Harold should open a sustainable fast-food restaurant. If the business development is not satisfying after the investment, Harold should additionally launch an advertising campaign to change a less successful business development into a more successful one [22].

4.2 Case Study 2

Initial situation:[2] Harold has opened a sustainable fast-food restaurant and the business is going well. Some years later, Harold would like to buy a coffee machine to

[2] This case study was created by the authors of the handout.

offer fair-trade coffee. With this additional investment, he would like to gain new customers. Again, he is unsure whether he should make the investment or not. The decision tree procedure and the roll-back process can be used also for this decision-making.

Harold has one alternative: investment or no investment. If the investment is made, the number of customers could increase sharply, moderately, or remain the same. If the number of customers is not higher than before, Harold would like to separately conduct a special event to attract new customers and eventually increase the total amount of customers in his business.

The following data are known:

- Investment expenses: $5,000
- Profit (sharp increase in customer numbers): $12,000
- Profit (moderate increase in customer numbers): $9,000
- Profit (same number of customers): $8,000
- Costs of special event: $500

The following probabilities can be assumed:

- Success of investment with sharp increase in customer numbers: 10%
- Success of investment with moderate increase in customers numbers: 20%
- Failure of investment (here unaltered customers): 70%
- Success of special event with sharp increase in customer numbers: 10%
- Success of special event with moderate increase in customer numbers: 30%
- Failure of special event (here unaltered customers): 60%

Will the investment be viable?

With the information given from the initial situation, the decision tree can be drawn. For preparation purposes, the decision nodes and the chance nodes have to be defined [7]. The first decision node is characterized by the decision occurrence: investment or no investment. If Harold decides not to invest, nothing will happen, and the branch is closed by an end node. If he decides to invest, three possibilities can occur: a sharp or moderate increase in customer numbers or a same number of customers, which stands for the failure of the investment. If customers increase sharply or moderately, the branches end with end nodes. But if the number of customers remains the same, Harold can start his additional special event to attract new customers. Thus, the next decision node is characterized by the decision occurrence: launching the special event or not launching it. In the case of no special event, nothing will happen; the branch will be closed by an end node. In the case of a special event, another chance node will follow a sharp or moderate increase in customer numbers or an exact number of customers. Figure 14 illustrates the corresponding decision tree.

The next step is characterized by applying a forward calculation to the cash flows (end nodes). Therefore, deposits and disbursements need to be determined along the branches [13]. Referring to the information given from the initial situation, Figure 15 shows the decision tree including the deposits and disbursements along the branches and the end nodes (cash flows).

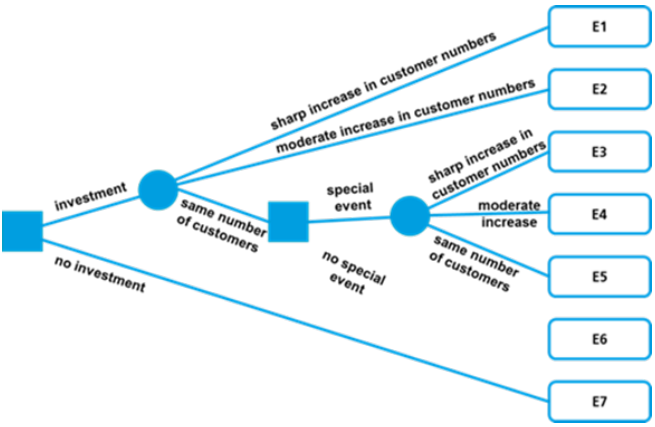

Fig. 14 Binary decision tree. Source: Own representation according to Möhlmann (2009) [22].

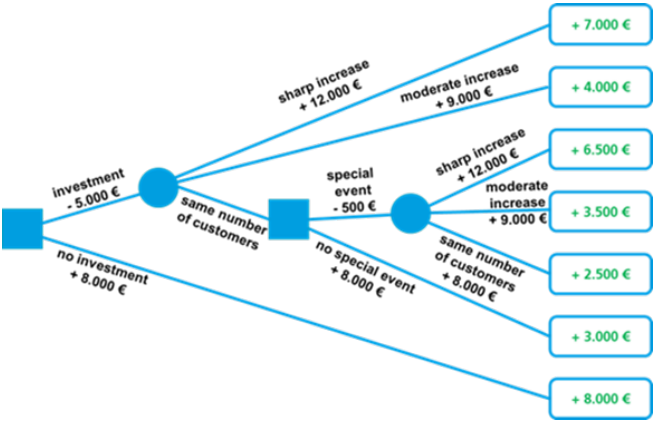

Fig. 15 Decision Tree incl. deposits, disbursements, and cash flows. Source: Own representation according to Möhlmann (2009) [22].

The process of forward calculation will be explained in the following via the utilization of the end notes:

E1: The branch starts with the investment (disbursement of $\$5,000$), followed by a sharp increase in customer numbers (deposits of $\$12,000$). By forward calculation the branch leads to a cash flow of $\$7,000$ (E1) [22]:

$$-\$5,000 + \$12,000 = \$7,000 \text{ (E1)} \tag{7}$$

E3: The branch starts with the investment (disbursement of $\$5,000$), followed by the failure of the investment. But in this case, a special event takes place (disbursement of €500) to attract more customers (deposits of $\$12,000$). By forward calculation the branch leads to a cash flow of $\$6,500$ (E3):

$$-\$5,000 - \$500 + \$12,000 = \$6,500 \text{ (E3)} \tag{8}$$

The forward calculations for the other options (end nodes) are examined below [22]. Due to the fact that the method of the calculation of the cash flows remains the same, no detailed explanation is needed at this point of examination.

E2:

$$\begin{aligned} &-\$5,000 \text{ (investment)} \\ &\underline{+\$9,000 \text{ (moderate increase in customer numbers)}} \\ &= \$4,000 \end{aligned} \tag{9}$$

E4:

$$\begin{aligned} &-\$5,000 \text{ (investment)} \\ &-\$500 \text{ (special event)} \\ &\underline{+\$9,000 \text{ (moderate increase in customer numbers)}} \\ &= \$3,500 \end{aligned} \tag{10}$$

E5:

$$\begin{aligned} &-\$5,000 \text{ (investment)} \\ &-\$500 \text{ (special event)} \\ &\underline{+\$8,000 \text{ (same number of customers)}} \\ &= \$2,500 \end{aligned} \tag{11}$$

E6:

$$\begin{aligned} &-\$5,000 \text{ (investment)} \\ &\underline{+\$8,000 \text{ (same number of customers)}} \\ &= \$3,000 \end{aligned} \tag{12}$$

E7:

$$\$8,000 \text{ (same number of customers)} \tag{13}$$

Secondly, the expected profit has to be calculated by backward calculation [13]. For preparation, Figure 16 includes the probabilities along the branches.

With the information provided above, the expected profit can be calculated with the roll-back calculation. The roll-back calculation multiplies the cash flows with the probabilities along the branches, and afterward, they have to be added together.

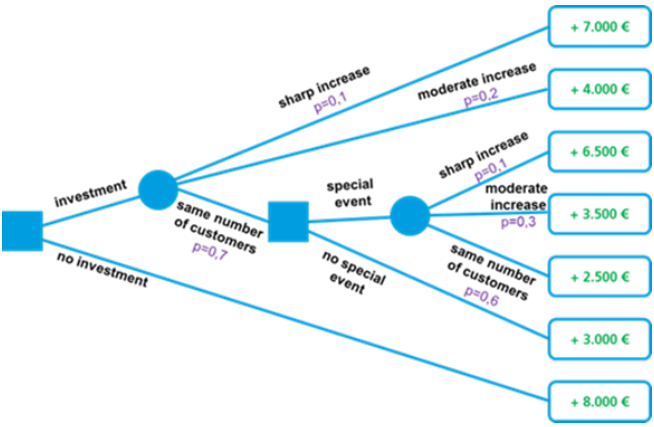

Fig. 16 Decision Tree incl. probabilities. Source: Own representation according to Möhlmann (2009) [22].

The non-optimal branches have to be separated (cropped) [13]. The calculations are explained below:

1. Expected profit of special event:
   $\mu = 0.1 * \$6,500 + 0.3 * \$3,500 + 0.6 * \$2,500 = \$3,200$
2. Expected profit of the investment:
   $\mu = 0.1 * \$7,000 + 0.2 * \$4,000 + 0.7 * \$3,200 = \$3,740$
3. Separating the non-optimal branches: These branches of the tree will not be considered. In this case, two branches have to be separated:
   - $\$3,200$ (expected profit of special event) > $\$3,000$ (expected profit of no special event)
     Result: The expected profit of the special event is higher than the expected profit of no special event. With respect to this, the branch of no special event has to be separated.
   - $\$3,740$ (expected profit of the investment) < $\$8,000$ (expected profit of no investment)
     Result: The expected profit of the investment is lower than the expected profit of no investment. Referring to that the branch of investment has to be separated.

Figure 17 shows the decision tree including the results.

Considering the information shown by the decision tree, Harold should not make the investment. The expected profit will be lower than the profit before. Harold would not increase his profit with the offer of fair-trade coffee.

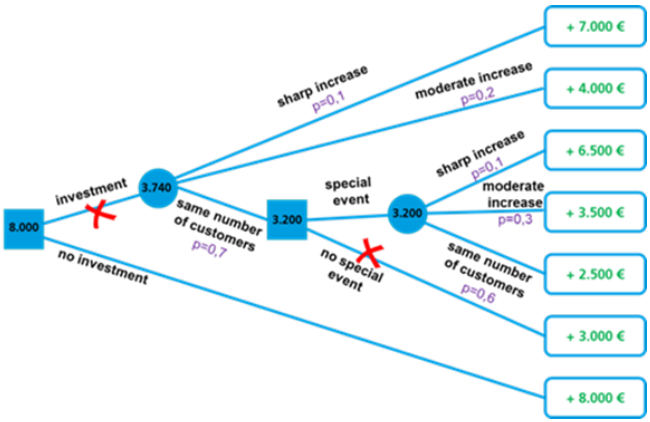

Fig. 17 Final Decision Tree. Source: Own representation according to Möhlmann (2009) [22].

4.3 Example from Practice

A product of the Palisade Corporation, the PrecisionTree, illustrates how the decision tree can be used in practice. The PrecisionTree is a Microsoft Excel tool that allows enterprises to visualize, organize and analyze their complex, sequential decisions on a quantitative basis in Microsoft Excel with a decision tree. It helps companies cope with complex, multi-stage decisions, like making an investment in a new production machine or buying a used one [23]. Decision problems or strategies often contain several more branches, decision-, chance- and end-nodes in real life than previously illustrated in the case studies. Therefore, it is almost unavoidable to refer to technical support in order to make a proper decision. The PrecisionTree from the Palisade Corporation is able to provide this support when it comes to greater complexity. Figure 18 illustrates a decision tree that was created with the help of this tool and simultaneously underlines the greater capacity that is related to technical support.

The Palisade Corporation offers several video tutorials to provide an overview of the possibilities of the tool. The corresponding hyperlinks are listed below:[3]

- http://www.palisade.com/QuickStart/DE/PrecisionTree/Step1.htm
- http://www.palisade.com/QuickStart/DE/PrecisionTree/Step2.htm
- http://www.palisade.com/QuickStart/DE/PrecisionTree/Step3.htm

The tools of the Palisade Corporation are used all over the world. The most widespread tools by Palisade Corporation are @RISK and the DecisionTools Suite [25]. More than 90% of the Fortune Global 500 companies are customers of the Palisade Corporation. To refer just to some big players within the customers of the

[3] Accessed Oct. 24, 2021.

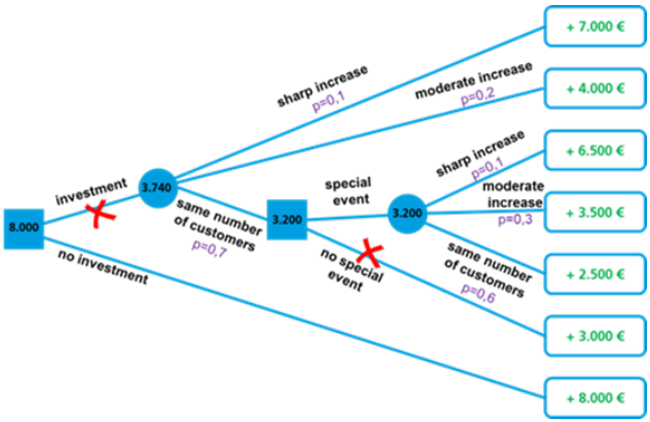

Fig. 18 PrecisionTree utilization/ example. Source: Own representation according to Palisade Corporation (Ed.) (2017) [24].

Palisade Corporation, they include: Royal Dutch Shell, Procter & Gamble, Mitsubishi, and Allianz [26].

5 Conclusion

Above all, the unique feature of the decision tree is that it enables management to combine analytical techniques with a clear picture of the impact of future decision alternatives and events. By using the decision tree, management can consider various courses of action with clarity and a more profound basis. The interdependencies between present decision alternatives, uncertainty, and future choices plus their results become more visible. Nevertheless, it should be mentioned that the decision tree concept does not offer final answers to (investment) decisions in the face of uncertainty. Additionally, extensive decision trees could become very complex. However, the concept is valuable for illustrating the structure of investment decisions. By that, it can provide excellent support in the evaluation of capital investment opportunities.

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Linear Optimization

Franz W. Peren

Abstract Linear optimization, a fundamental technique of operations research, plays a central role in the optimization of decision processes. This work gives an overview of linear programming and highlights its importance in solving complex problems by optimizing linear models with constraints.

1 Introduction

Linear optimization or *linear programming* (linear planning) is to be used for the determination of extreme values if the constraints of a linear mathematical model are in the form of inequalities and/or equations.

2 The Linear Programming Approach

(1) Target function
$z = z(x_1, x_2, \ldots, x_n) \Rightarrow$ opt!
opt = optimisation = maximisation or minimisation

(2) Constraints
$\phi_j = \phi_j(x_1, x_2, \ldots, x_n) \leq c_j \quad \text{with} \quad j = 1, \ldots, m$
Note: The $\leq$ - restrictions can also be represented as $\geq$ - restrictions when multiplied with -1 and vice versa.

(3) Non-negativity conditions
$x_i \geq 0 \quad \text{with} \quad i = 1, \ldots, n$

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Springer Texts in Business and Economics,
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3 Graphical Solution

For clearer demonstration, the model discussed below will first be limited to two variables, x_1 and x_2, and two restrictions. The linear programming approach is demonstrated below:

(1) Target function
$z = z(x_1, x_2, \ldots, x_n) \Rightarrow$ opt!
(2) Constraints
$a_{11}x_1 + a_{12}x_2 \leq a_1$ $\qquad a_{ij} \in \mathbb{R};\ \ i;j = 1;2 = constant$
$a_{22}x_1 + a_{22}x_2 \leq a_2$
(3) Non-negativity condition
$x_1, x_2 \geq 0$

The solution set is the set of all ordered pairs of variates $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ that satisfy the above-mentioned constraints:

$$\mathbb{S}_S = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R} \times \mathbb{R} \wedge a_{11}x_1 + a_{12}x_2 \leq a_1 \wedge a_{21}x_1 + a_{22}x_2 \leq a_2\} \quad (1)$$

Each restriction divides the coordinate system into the relevant half-plane. Due to the non-negativity condition, only the fourth quadrant of the coordinate system is relevant. The possible solutions are shown in the following example by the hatched area. The *target function* marks the corresponding straight line (contour line) $z = z(x_1, x_2)$. The optimum (= the searched value pair), (x_1^{opt}, x_2^{opt}), is obtained, depending on the task, by parallel shifting the target function $z = z(x_1, x_2)$. In the case of maximisation (minimisation), the target function is shifted outwards (inwards) - away from (towards) the origin - until it reaches the limited plane due to the restrictions (hatched area).

The optimum is unique if the target function meets a corner of the relevant area. The solution is ambiguous if the target function is parallel (congruent) to one of the constraint lines.

Example:

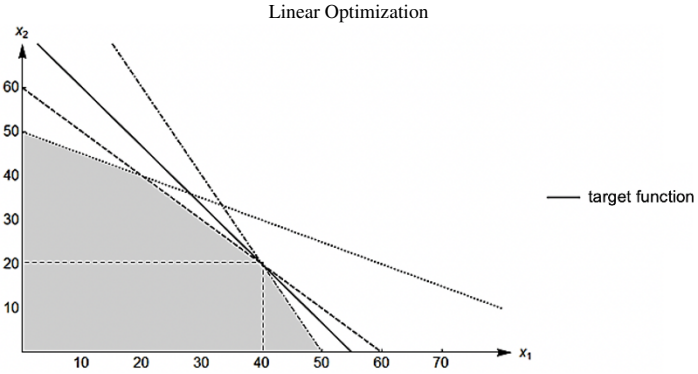
Two products are manufactured with three different machines, whereby the capacities of the machines are limited. x_1 has a contribution margin of \$150 per unit. x_2 has a contribution margin of \$100 per unit. The objective is to maximize the contribution margin (CM), under consideration of the limited machines capacities.

Decision variables:	x_1, x_2 = quantities of products produced
Target function:	$CM(x_1, x_2) = 150x_1 + 100x_2 \Rightarrow$ max!
Constraints:	(1) $4x_1 + 2x_2 \leq 200$ with $x_1, x_2 \leq 0$
	(2) $2x_1 + 4x_2 \leq 200$
	(3) $2x_1 + 2x_2 \leq 120$

Interpretation of the constraint (1):

$4\ \text{TU/QU} \cdot x_1\ \text{QU} + 2\ \text{TU/QU} \cdot x_2\ \text{QU} \leq 200\ \text{TU}$ QU = quantity units
TU = time units

with $4; 2 \mathrel{\widehat{=}}$ production coefficients
$200 \mathrel{\widehat{=}}$ maximum capacity of the first machine



Calculation of the axis intercepts:

(1) $4x_1 + 2x_2 \leq 200\ \text{TU}$

$$x_1 = 0 \Rightarrow x_2^{max} = 100\ \text{QU}$$
$$x_2 = 0 \Rightarrow x_1^{max} = 50\ \text{QU}$$

(2) $2x_1 + 4x_2 \leq 200\ \text{TU}$

$$x_1 = 0 \Rightarrow x_2^{max} = 50\ \text{QU}$$
$$x_2 = 0 \Rightarrow x_1^{max} = 100\ \text{QU}$$

(3) $2x_1 + 2x_2 \leq 120\ \text{TU}$

$$x_1 = 0 \Rightarrow x_2^{max} = 60\ \text{QU}$$
$$x_2 = 0 \Rightarrow x_1^{max} = 60\ \text{QU}$$

target function $150x_1 + 100x_2$ = e.g. $\$1,500$

$\Rightarrow$ $x_1^{max} = 10$ QU
$\Rightarrow$ $x_2^{max} = 15$ QU
$\Rightarrow$ Shift of this straight line until it contacts the boundary surface defined by the constraints.

The optimal distribution of the goods x_1 and x_2 can now be observed on the axes. To maximise the total contribution margin, 40 units for x_1 and 20 units for x_2 are produced.

The following solution is obtained for the target function:

$150x_1 + 100x_2 \qquad \Rightarrow \max!$
$150 \cdot 40 + 100 \cdot 20 \qquad = \$8,000$

If the optimal quantities of both products are selected, the amount covered is $\$8,000$.

4 Primal Simplex Algorithm

The *primal (= original) simplex algorithm* is an *iteration method* for the gradual approach to the optimum. It is valid for at least two variables. The maximum value (or minimum value) z_{opt} of the target function $z = z(x_i)$ with $i = 1, \ldots, n$, is given if the underlying *simplex tableau* is in canonical form and the coefficients of all non-basic variables ≥ 0 (≤ 0). The form of a linear mathematical model is canonical if one variable (= base variable) has the coefficient 1 in each constraint and this variable does not arise in all other constraints.

5 Simplex Tableau (Basic Structure)

The simplex tableau is based on the following (basic) linear programming approach:

Primal Simplex Algorithm | Linear Programming Approach

(1) Target function
$z = z(x_i) = b_1x_1 + b_2x_2, \ldots, + b_nx_n \quad \Rightarrow \quad \max! \text{ or } \min!$
$b_i \in \mathbb{R} = constant$
$i = 1, \ldots, n$

(2) Constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq a_1 \qquad a_{ij} \in \mathbb{R} = constant$$
$$\vdots \qquad \text{with } j = 1, \ldots, m$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq a_m$$

(3) Non-negativity condition

$$x_i \geq 0 \qquad \text{with } i = 1, \ldots, n$$

If a simplex tableau exists in canonical form and all coefficients of the non-basic variable ≥ 0 (≤ 0) are within the target function, then z_0 corresponds to the minimum (maximum) value of the function.

Example:

A company manufactures two goods of the quantities x_1 and x_2 [measured in quantity units (QU)] using three machines, A, B and C, which are limited in their capacities [time units/month (TU/month)]. The maximum available capacities [measured in TU/month], as well as the production coefficients for the goods quantities [measured in TU/1 QU], are listed in the following table:

machine	**production coefficients [TU/QU]**		**max. capacities [TU/month]**
A	2 *TU* / 1 *QU* x_1	1 *TU* / 1 *QU* x_2	200
B	1 *TU* / 1 *QU* X_1	1 *TU* / 1 *QU* x_2	120
C	1 *TU* / 1 *QU* x_1	3 *TU* / 1 *QU* x_2	240
profit per unit	\$2 / 1 *QU* x_1	\$3 / 1 *QU* x_2	

The total profit in \$ has to be maximized.

Target function: $P(x_1, x_2) = 2x_1 + 3x_2 \Rightarrow \max!$

Constraints: $2x_1 + x_2 \leq 200$ with $x_1, x_2 \geq 0$

$x_1 + x_2 \leq 200$

$x_1 + 3x_2 \leq 120$

To set up the *simplex tableau*, *auxiliary variables* are introduced for the unused machine capacities (y_1, y_2, y_3) and the target function is set to zero. The subsequent equations are as follows:

(I) $\Rightarrow \quad 2x_1 + x_2 + y_1 = 200$

(II) $\Rightarrow \quad x_1 + x_2 + y_2 = 120$

(III) $\Rightarrow \quad x_1 + 3x_2 + y_3 = 240$

(IV) $\Rightarrow \quad -2x_1 - 3x_2 + P = 0$

From these equations, more precisely from the coefficients of the variables, the *simplex tableau* is created in the next step.

x_1	x_2	y_1	y_2	y_3	P	available capacities
2	1	1	0	0	0	200
1	1	0	1	0	0	120
1	3	0	0	1	0	240
−2	−3	0	0	0	1	0

non-basic variables *basic variables*

Since the first two columns in the 4th row contain negative values, the simplex tableau or the "solution" shown here is not optimal. For the first improvement step, the second column is selected as *pivot column*, because −3 represents the smallest value of all negative values within the bottom row. To select the *pivot row*, the value of the pivot column for each row is divided by the machine capacity. This identifies the (current) bottleneck capacities of the machines. The following is obtained:

Machine A:
$$\frac{200\ \frac{\text{h}}{\text{month}}}{1\ \frac{\text{h}}{\text{QU}}} = 200\ \frac{\text{QU}}{\text{month}}$$

Machine B:
$$\frac{120\ \frac{\text{h}}{\text{month}}}{1\ \frac{\text{h}}{\text{QU}}} = 120\ \frac{\text{QU}}{\text{month}}$$

Machine C:
$$\frac{240\ \frac{\text{h}}{\text{month}}}{3\ \frac{\text{h}}{\text{QU}}} = 80\ \frac{\text{QU}}{\text{month}}$$

⇒ Machine C is the first to reach its capacity limit, so the 3rd row is our pivot row. To get a 1 at the position of the 3rd row (pivot row) and 2nd column (pivot column), the third row is multiplied by the corresponding inverse value (here $\frac{1}{3}$).

It follows:

x_1	x_2	y_1	y_2	y_3	P	available capacities
2	1	1	0	0	0	200
1	1	0	1	0	0	120
1/3	1	0	0	1/3	0	80
−2	−3	0	0	0	1	0

The next step is to create the unit vector for x_2 so that the pivot element (3rd row/2nd column) equals one (and all other column elements equal zero). This is achieved by adding the (-1)fold of the 3rd row to rows 1 and 2 and adding the 3fold of the 3rd row to the last row. The result is:

x_1	x_2	y_1	y_2	y_3	P	available capacities
5/3	0	1	0	−1/3	0	120
2/3	0	0	1	−1/3	0	40
1/3	1	0	0	1/3	0	80
−1	0	0	0	1	1	240

basic variables

There is still a negative value in row 4/column 1, therefore this simplex tableau or the solution shown here is not optimal. For the second improvement step, the 1st column is the *pivot column*, because −1 represents the last negative value within the bottom row. To select the *pivot row*, the value of the pivot column is divided by the machine capacity for each row again. The result is:

Machine A:
$$\frac{120\ \frac{\text{h}}{\text{month}}}{\frac{5}{3}\ \frac{\text{h}}{\text{QU}}} = 72\ \frac{\text{QU}}{\text{month}}$$

Machine B:
$$\frac{40\ \frac{\text{h}}{\text{month}}}{\frac{2}{3}\ \frac{\text{h}}{\text{QU}}} = 60\ \frac{\text{QU}}{\text{month}}$$

Machine C:
$$\frac{80\ \frac{\text{h}}{\text{month}}}{\frac{1}{3}\ \frac{\text{h}}{\text{QU}}} = 240\ \frac{\text{QU}}{\text{month}}$$

$\Rightarrow$ Machine B is the first to reach its capacity limit, so the 2nd row is our current pivot row. To get a 1 at the position of the 2nd row (pivot row) and 1st column (pivot column), the second column must be multiplied by the corresponding inverse value (here $\frac{3}{2}$).

It follows:

x_1	x_2	y_1	y_2	y_3	P	available capacities
5/3	0	1	0	−1/3	0	120
1	0	0	3/2	−1/2	0	60
1/3	1	0	0	1/3	0	80
−1	0	0	0	1	1	240

To create the unit vector for x_1 so that the pivot element 1st row/2nd column equals one (and all other column elements equal zero):

- add $(-\frac{5}{3})$fold of the 2nd row to the 1st row,
- add $(-\frac{1}{3})$fold of the 1st row to the 3rd row,
- add the 2nd row to the last row.

It follows:

x_1	x_2	y_1	y_2	y_3	P	available capacities
0	0	1	−2.5	0.5	0	20
1	0	0	1.5	−0.5	0	60
0	1	0	−0.5	0.5	0	60
0	0	0	1.5	0.5	1	300

basic variables

There are no more negative elements in the 4th row of the matrix. Now the optimal solution is reached. To optimize this problem discussed here, 60 units of x_1 and 60 units of x_2 have to be produced, whereby a capacity of 20 hours remains unused with machine A. The maximum profit is $300.

6 Dual Simplex Algorithm

The *dual simplex algorithm* is used if negative values exist in the far right column of the simplex tableau. The aim is to replace all negative values in the right column by iteration with positive values so that in a first step a feasible solution is found. Afterwards, the primal (= original) simplex method can be used to find the optimal solution.

The dual simplex algorithm starts with the definition of a minimization problem:[1]

Minimise	$z = \sum_{j=1}^{n} c_j \cdot x_j$
by considering the constraints	$\sum_{j=1}^{n} d_{ij} \cdot x_j \geq b_j$
	$x_j \geq 0 \qquad c_j \geq 0$

(1) Target function

$z = z(x_i) = \sum_{i=1}^{n} c_i \cdot x_i \qquad \Rightarrow$ min!

$c_i \in \mathbb{R}$ = constant with $i = 1, \ldots, n$

(2) Constraints

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m$

$x_i \geq 0$ and $a_{ij} \in \mathbb{R}$ = constant with $j = 1, \ldots, m$

(3) Non-negativity condition

$x_i \geq 0$ with $i = 1, \ldots, n$

After transforming into a maximisation problem and the introduction of slack variables y_1, ..., y_m the initial simplex tableau is generally represented as follows:

x_1	$\cdots$	x_n	y_1	$\cdots$	y_m	z	b
a_{11}	$\cdots$	a_{1n}	1	$\cdots$	0	0	$-b_1$
$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a_{m1}	$\cdots$	a_{mn}	0	$\cdots$	1	0	$-b_m$
c_1	$\cdots$	c_n	0	$\cdots$	0	1	z-value

[1] If the values in the right column of the simplex tableau are negative, then the quantities shown there are missing, i.e. they would have to be procured. Therefore, it inevitably follows that this is a minimization problem.

Example:[2]
A firm produces three goods in quantities x_1, x_2, and x_3 [QU] using two machines, A and B, which are subject to minimum economic or technical utilization [ZE/day]. The production coefficients for the quantities of goods x_1, x_2, and x_3, measured in [TU/ 1 QU], are summarized in the table below:

machine	**production coefficients [TU/QU]**			**min. utilization [TU/day]**
A	4 TU / 1 QU x_1	2 TU / 1 QU x_2	5 TU / 1 QU x_3	12
B	2 TU / 1 QU x_1	3 TU / 1 QU x_2	1 TU / 1 QU x_3	8
unit costs in $100	$ 0.8 / 1 QU x_1	$ 1.0 / 1 QU x_2	$ 0.75 / 1 QU x_3	

The total costs in $ are to be minimised.

Target function: $K(x_1, x_2, x_3) = 0.8x_1 + x_2 + 0.75x_3 \Rightarrow \min!$
Constraints: $4x_1 + 2x_2 + 5x_3 \geq 12$
$2x_1 + 3x_2 + x_3 \geq 8$
with $x_i \geq 0 \qquad i = 1, 2, 3$

First the $\geq$ constraints are to be transformed into $\leq$ constraints by multiplication with -1. Furthermore, the minimisation problem has to be transformed into a maximisation problem. The non-negativity conditions remain. Now the standard form, as known from the *Linear Programming Approach*, exists:

$$-4x_1 - 2x_2 - 5x_3 \leq -12$$
$$-2x_1 - 3x_2 - x_3 \leq -8$$
$$-0.8x_1 - x_2 - 0.75x_3 = -K \Rightarrow \max!$$

with $x_i \geq 0 \qquad i = 1, 2, 3$

Slack variables y_j are to be integrated so that equations can be achieved:

$$-4x_1 - 2x_2 - 5x_3 + y_1 = -12$$
$$-2x_1 - 3x_2 - x_3 + y_2 = -8$$
$$-0.8x_1 - x_2 - 0.75x_3 = -K = k \Rightarrow \max!$$

with $x_i \geq 0 \qquad i = 1, 2, 3$
and $y_j \geq 0 \qquad j = 1, 2$

[2] The following example is based on Zimmermann, H.-J. (2005), p. 102 et sqq.

That equation system can be transformed into the following simplex tableau:

	x_1	x_2	x_3	y_1	y_2	b_i
I	−4	−2	−5	1	0	−12
II	−2	−3	−1	0	1	−8
III	0.8	1	0.75	0	0	0

The upper simplex tableau shows the non-basic variables x_1, x_2 and x_3 and the basic variables y_1 and y_2. The only possible solution on a mathematical point of view could be x_1, x_2, $x_3 = 0$ and $y_1 = -12$, $y_2 = -8$. It is dual, but not primal feasible. Since there are negative values at the right column the primal simplex algorithm cannot be used. The negative values have to be eliminated first.
In contrast to the *primal simplex algorithm*, first the pivot row is selected instead of the pivot column. For this, the row with the smallest negative value in the b_i column must be identified. If there are several rows with the same smallest value, one of them can be chosen.
In the table below, this row is marked in grey. −12 is the smallest negative value. In consequence the first row is the pivot row.

	x_1	x_2	x_3	y_1	y_2	b_i
I	−4	−2	−5	1	0	−12
II	−2	−3	−1	0	1	−8
III	0.8	1	0.75	0	0	0

Now each value from the target function row is to divide by the respective value in the pivot row, as illustrated at the following table:

	target function	row I	quotient
x_1	0.8	−4	−0.2
x_2	1	−2	−0.5
x_3	0.75	−5	−0.15

The column with the maximum negative value is determined as the pivot column. The pivot element is located where the pivot row crosses the pivot column. The pivot element is marked in dark grey at the table below.

	x_1	x_2	x_3	y_1	y_2	b_i
I	−4	−2	−5	1	0	−12
II	−2	−3	−1	0	1	−8
III	0.8	1	0.75	0	0	0

To get a 1 at the position of the 1st row (pivot row) and the 3rd column (pivot column), the 1st row is multiplied with the corresponding inverse value ($-\frac{1}{5}$).

It follows:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8 = $-4 \cdot (-\frac{1}{5})$	0.4	1	−0.2	0	2.4
II	−2	−3	−1	0	1	−8
III	0.8	1	0.75	0	0	0

The next step is to create the unit vector for x_3. This is achieved by adding the 1st row to the 2nd row and adding the (-0.75)fold of the 1st row to the 3rd row. The result is:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	−0.2	0	2.4
II	−1.2	−2.6	0	−0.2	1	−5.6
III	0.8 · (−0.75) + 0.8 = 0.2	0.4 · (−0.75) + 1 = 0.7	1 · (−0.75) + 0.75 = 0	−0.2 · (−0.75) + 0 = 0.15	0 · (−0.75) + 0 = 0	2.4 · (−0.75) + 0 = −1.8

The right column still contains a negative value within the top two rows.

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	−0.2	0	2.4
II	−1.2	−2.6	0	−0.2	1	−5.6
III	0.2	0.7	0	0.15	0	−1.8

In consequence the current status cannot be the optimal solution.
The following iteration starts with the identification of the next pivot element. −5.6 is now the smallest negative value of the first two rows. The second row is the pivot row.

	target function	row II	quotient
x_1	0.2	−1.2	−0.167
x_2	0.7	−2.6	−0.269
x_3	0	0	0

The column with the maximum negative value (= the minimum positive absolute value) is determined as the pivot column. The pivot element is located where the pivot row crosses the pivot column. The pivot element is marked in dark grey at the table below.

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	−0.2	0	2.4
II	−1.2	−2.6	0	−0.2	1	−5.6
III	0.2	0.7	0	0.15	0	−1.8

To get a 1 at the position of the 2nd row (pivot row) and the 1st column (pivot column), the 2nd row is multiplied with the corresponding inverse value $(-\frac{1}{1.2})$.

It follows:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0.8	0.4	1	−0.2	0	2.4
II	1	2.1667	0	0.1667	−0.8333	4.6667
III	0.2	0.7	0	0.15	0	−1.8

The next step is to create the unit vector for x_1. This is achieved by adding the (-0.8)fold of the 2nd row to the 1st row and adding the (-0.2)fold of the 2nd row to the 3rd row. The result is:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	−1.3333	1	−0.3333	0.6666	−1.3333
II	1	2.1667	0	0.1667	−0.8333	4.6667
III	0	0.2667	0	0.1167	0.1667	−2.7333

The right column still contains a negative value within the top two rows. In consequence the current status cannot be the optimal solution.
The following iteration starts with the identification of the next pivot element. $-1,333$ is now the smallest negative value at the first two rows. The first row is the pivot row.

	target function	row I	quotient
x_1	0	0	0
x_2	0.2667	−1.3333	−0.2
x_3	0	1	0

The second column is determined as the pivot column. The pivot element is located where the pivot row crosses the pivot column. The pivot element is marked in dark grey at the table below.

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	−1.3333	1	−0.3333	0.6666	−1.3333
II	1	2.1667	0	0.1667	−0.8333	4.6667
III	0	0.2667	0	0.1167	0.1667	−2.7333

To get a 1 at the position of the 1st row (pivot row) and the 2nd column (pivot column), the 1st row is multiplied with the corresponding inverse value ($-\frac{1}{1.3333}$).

It follows:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	1	−0.75	0.25	−0.5	1
II	1	2.1667	0	0.1667	−0.8333	4.6667
III	0	0.2667	0	0.1167	0.1667	−2.7333

The next step is to create the unit vector for x_2. This is achieved by adding the (-2.1667)fold of the 1st row to the 2nd row and adding the (-0.2667)fold of the 1st row to the 3rd row. The result is:

	x_1	x_2	x_3	y_1	y_2	b_i
I	0	1	−0.75	0.25	−0.5	1
II	1	0	1.625	−0.375	0.25	2.5
III	0	0	0.2	0.05	0.3	−3

The right column contains no more negative values at the first to rows. So the optimal solution has been identified.

$x_1 = 2.5$
$x_2 = 1$
$k = -K = -3 \quad \Rightarrow \quad K = 3 \cdot \$100 = \$300$

To minimise the total costs, 2.5 QU of x_1, 1 QU of x_2 and no QU of x_3 are to be produced per day. The resulting minimum costs are \$300 per day.

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Cutting and Packaging Optimization

Kalvin M. Kroth

Abstract Cutting and packing problems, common in manufacturing and logistics, demand systematic solutions to minimize costs and optimize resources. This paper explores the utility of typologies such as Dyckhoff's and Wäscher et al.'s in simplifying complex problem types. It presents an operational research workflow for companies to economically address these challenges.

1 Introduction

Cutting and packing problems can be found in many different forms and variants in the business context. Manufacturing industries in textile, furniture or automotive industries, for example, cut the forms of their finished products from raw material, and companies specialized in transporting goods then have to pack them into containers, airplanes or trucks. But companies in other industries also often have to deal with challenges of this nature in the context of internal logistics.

In order to minimize material or transport costs, for example, or to maximize the output of production, it is important that companies approach the present cutting and packing problems (or C&P problems) as methodically and economically as possible. In addition, it is important that human resources are used in a goal-oriented manner when looking for an optimal solution to save time and personnel costs.

Since cutting and packing problems can occur in very different and highly complex forms, it is not easy to find the optimal solution for a given situation. To reduce the complexity of those optimization problems, it is useful for companies to utilize an appropriate typology. A typology is a methodical tool with which real phenomena are ordered and made manageable by expressing what is considered to be essential. This way, the highly heterogeneous complexity of business reality can be reduced to a few homogeneous problem types [1]. As soon as a company has clearly identified and described a concrete C&P problem at hand based on such a typology, it can look specifically for solutions.

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In the field of cutting and packing optimization, several of such typologies are available in the literature. Dyckhoff already provided a first approach in 1991 [2]. Based on this, Wäscher et al. have developed their improved typology [3]. The aim of this term paper is to elaborate an operational research workflow using these two typologies, which can be utilized by companies to solve internal cutting and packing optimization problems economically.

2 Basics of C&P Problems

Cutting and packing problems belong to 'geometric combinatorics, 'which is a branch of mathematics. The term describes 'the study of geometrics and their combinatorial structure' [4]. At first glance, the similarities between cutting prob-lems and packing problems may not be immediately apparent. But the strong relationship between the process of cutting objects into smaller items on the one hand and the process of packing smaller items into larger objects on the other has already been realized in scientific articles since the seventies [2].

2.1 General Structure of C&P Problems

Both processes have a common logical structure that can be determined as follows:

1. In both cases there are two groups of basic data. Those consist of the group of so-called small items and the group of large objects. The elements of each group can be defined as geometric entities with a fixed shape. Often, the group of small objects is also referred to as 'the demand' or 'the order book of small items 'and the large objects are referred to as 'the stock.'
2. Both in the cutting process and in the packing process, the group of smaller items has to be assigned to the group of larger objects. This act of assignment realizes specific geometrical patterns. Geometrical figures within those patterns that don't belong to small items are unused space and are called 'trim loss' [2].

To clarify the nature of the general logical structure of cutting and packing problems, two examples are shown below:

Example 1: A tube manufacturer receives an order from a customer with an order quantity of 4 tubes with a length of 20 cm, 1 tube with a length of 25 cm, 1 tube with a length of 36 cm and one tube with a length of 46 cm. The manufacturer produces his products by cutting them from raw material in the form of large tubes with a standard length of 100 cm, which he himself purchases from his supplier. In this case, the 100 cm tube is the large object, and the customer's order quantities represent the group of small items. The way the manufacturer cuts the large tube into the smaller ones results in the cutting patterns (see Figure 1).

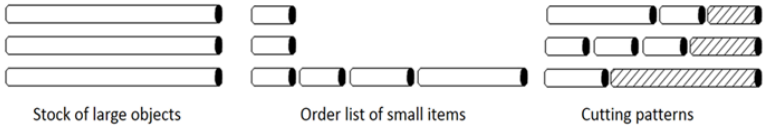

Fig. 1 Group of Large Objects, Group of Small Items and Resulting Cutting Patterns. Source: Own representation according to Dyckhoff (1990) [2].

Example 2: A regional parcel service has numerous identical delivery vehicles. In order to successfully manage the daily operations, the parcel service has to efficiently pack the delivery vehicles. The storage of the vehicle is the large object, and the packages are the small items. The objective is to ensure that after the packages are loaded, the empty space is minimized. In this case, the unused space within the delivery vehicle is considered as trim loss (see Figure 2).

Fig. 2 Group of Large Objects, Group of Small Items and Resulting Packing Patterns - Example 2. Source: Own representation according to Dyckhoff (1990) [2].

2.2 Characteristics of C&P Problems

In his attempt to create a general topology for C&P problems, Dyckhoff identified a list of different characteristics that can be used to describe every cutting and packing problem. This list consists of 9 entries [2]. The role of those characteristics for classifying a given problem may differ. Other researchers like Wäscher et al. and even Dyckhoff himself only used a subset of those characteristics to create a system of classification for C&P problems. Those features are considered particularly important for the classification of C&P problems because they have a decisive impact on the choice and the complexity of the solution approaches. Those four characteristics are named ‘dimensionality,’ ‘assignment restrictions,’ ‘assortment of small items / large objects’ and ‘shape of the small items’ [2].

2.2.1 Dimensionality

According to Dyckhoff, dimensionality is the most important characteristic of every C&P problem. The term describes the minimum number of partial dimensions that is needed to define the geometrical patterns, in which the small items are assigned to the large objects. In the literature, a distinction is made between one-dimensional, two-dimensional, three-dimensional and n-dimensional problems, where n > 3.

While the first three types only include spatial dimensions, n-dimensional cutting and packing problems also include aspects such as time [3]. This is the case, for example, when in a bakery, the baked goods have to be placed in an oven and the respective baking time has to be taken into account in the creation of the packing patterns.

2.2.2 Assignment Restrictions

Another important characteristic of C&P problems is restrictions regarding the kind of assignment of the small items to the large objects. Here, two important scenarios can be identified. These vary with respect to the relative number of small items and large objects that have to be used in the C&P process:

1. In the first scenario, all the large objects have to be used, but only a subset of the small items has to be assigned to them. In the literature there are different names for this case. Dyckhoff uses the German term "Beladeproblem" in his typology [2], while Wäscher et al. refer to such kind of assignment as output maximization problems [3].
2. In the opposite scenario, all the small items have to be assigned to only a subset of the available large objects. Here the German term "Verladeproblem" is used by Dyckhoff [2], whilst Wäscher et al. name those scenarios input minimization problems [3].

2.2.3 Type of Assortment

The next important aspect of cutting and packing problems refers to the available assortment of the large objects on the one hand and the assortment of the small items on the other. An assortment is characterized by the shape and permitted number of objects (or items respectively). Here, three different kinds can be distinguished for the group of the large objects (see Figure 3):

1. The assortment consists of one single object only.
2. The assortment consists of several large objects that are identical to each other, regarding their shape. These assortments are called homogeneous.
3. The assortment consists of several large objects that differ in size and shape. These assortments are called heterogeneous [3].

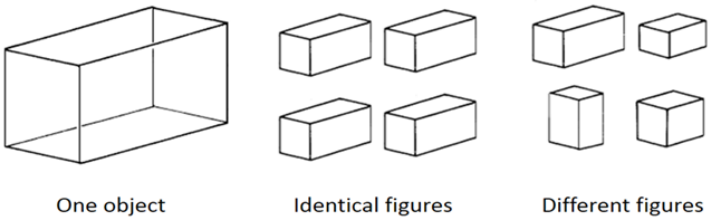

Fig. 3 Assortment Types for Large Objects. Source: Own representation according to Dyckhoff (1990) [2].

Just as is the case with the assortment of large items, the assortments of small items can be grouped into several distinct categories (see Figure 4):

1. The assortment consists of only a few items with different figures.
2. The assortment consists of many small items, which highly differentiate regarding their shape and size. Only a few or none of the items can be grouped together. Those assortments are called strongly heterogeneous.
3. The assortment consists of many small items. Those can be grouped into a few groups, in which they are identical to each other regarding their shape and size. Those assortments are called weakly heterogeneous
4. All small items have exactly the same shape and size regarding their problem-relevant dimensions. The assortment can be described as congruent [3].

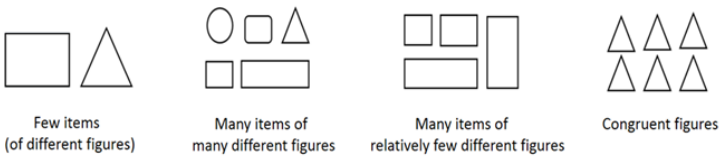

Fig. 4 Assortment Types for Small Items. Source: Own representation according to Dyckhoff (1990) [2].

2.2.4 Shape of Small Items

Directly related to the dimensionality and the assortment of the small items is the shape of the figure of the individual elements. The term figure can be described as the geometric representation of an item in the space of relevant dimensions. The figure of an element can be determined by three different aspects, namely the form of the figure, its size and its orientation. While the basic typology presented by Dyckhoff

does not make use of this characteristic when identifying different problem types, Wäscher et al. distinguish between small items with a regular form (like squares or rectangles) and small items with an irregular form (like concave and convex figures) [3].

The size of an element can be measured by its length, volume or area. Lastly, there are three different possible scenarios when it comes to the orientation of a figure: any orientation is allowed, only 90-degree turns are allowed, or the orientation is fixed and can't be changed at all [2].

2.2.5 Other Characteristics of C&P Problems

As mentioned above, mainly those four criteria are used to assign a concrete cutting and packing problem to a theoretical problem type. Beyond those, there are a number of other aspects that can be used to describe a given problem in the context of cutting and packing processes.

Availability: While the criteria 'type of assortment' gives information about the heterogeneity or homogeneity of the small items and large objects, it does not specify the available quantity of both data groups. Therefore, the characteristic of the availability captures that information. There might be C&P scenarios with a finite or infinite number of small items and large objects [2].

Pattern restrictions: Beside from restrictions regarding the assignment of items to objects, there might be rules for the construction of the specific patterns in which the items are placed which may lead to geometrical and/or combinatorial restrictions [2]. For example, there might be a minimal distance to be considered when placing small items next to each other. This could be the case, for example, when placing articles and advertisements on a newspaper page.

Quantity measurement: The quantity measurement describes the way in which both the small items and the large objects are measured. There is the possibility of discrete measurement and continuous measurement. With discrete measurement, the number of objects or items is expressed in real numbers (e.g., 13 packages or 4 bars of steel). With continuous measurement, properties such as the length, density or weight of objects and items are expressed in fractional numbers (e.g., a package with a length of 17.5 cm or a steel bar with a length of 5.50 m) [2].

Objectives: Often, the assignment of the small items to the large objects must follow certain objectives, which result from the business context in which the cutting and packing problems occur. These objectives can be considered restrictions, which re-fer either to the group of small items, the group of large objects or to the geometric patterns resulting from the assignment. Such a restriction can be, for example, that when positioning goods within a shelf layout, only those products that come from the same manufacturer or have the same target group may be grouped together [2].

Status of information: The last characteristic described here concerns not only cutting and packing problems, but also optimization problems in general. It answers the question whether all decision-relevant parameters are known for a given problem or whether there is uncertainty with respect to some of the quantities. With respect to

the resulting degree of information, a distinction is made between deterministic and stochastic problems. With deterministic problems, the parameters of the objective function as well as the constraints are assumed to be known. However, if at least one parameter is to be interpreted as a random number, the problem is of a stochastic nature [5].

3 Dyckhoff's Classification of C&P Problems

The properties described in Chapter 2 can now be used to generate a classification system of C&P problems. It should be noted that when generating such a system, the individual characteristics are not considered individually. Due to the numerous overlaps and interdependencies between them, combined types are generated to group C&P problems. As already mentioned, dimensionality, the kind of assignment and the types of assortments in particular have a special role in Dyckhoff's typology. He broke down the four properties as shown in Figure 5 [2].

1. Dimensionality		**3. Assortment of large objects**	
1	One-dimensional	O	One object
2	Two-dimensional	I	Identical figure
3	Three-dimensional	D	Different figures
N	N-dimensional with N > 3	**4. Assortment of large objects**	
2. Kind of assignment		F	Few items (of different figures)
B	All objects and a selection of items	M	Many items of many different figures
V	A selection of objects and all items	R	Many items of relatively few different (non-congruent) figures
		C	Congruent figures

Fig. 5 Categorization Criteria and Corresponding Values by Dyckhoff. Source: Own representation according to Dyckhoff (1990) [2].

Through the combination of the different values of the main characteristics, a total number of $4x2x3x4 = 96$ different types of C&P problems can be distinguished. Those types can then be related to specific notions known from literature and, more importantly, related to specific solution approaches for an optimized handling of the specific cutting and packing problem. Figure 6 shows Dyckhoff's approach to create such a classification of C&P problems. The symbols on the left-hand side of every characteristic value (shown in Figure 5) are used to denote every combination by a fourth-tuple (e.g., a two-dimensional C&P problem where a selection of a few items

of different figures must be assigned to one large object can be denoted as 2/B/O/F.) [2].

Type	Notion
1/B/O/	Knapsack problem (classical)
2/B/O/C	Pallet loading problem
/B/O/	More-dimensional knapsack problem
1/B/O/M	Dual bin packing problem
1/V/1/F, or 1/V/1/M	Vehicle loading problem
3/V/1/, or 3/B/O/	Container loading problem
1/V/1/M	Bin packing problem (classical)
1/V/1/R	Cutting stock problem (classical)
2/V/D/M	Twodimensional bin packing problem
2/V/1/R	Usual twodimensional cutting stock problem
1/ / /, 2/ / /, or 3/ / /	General cutting stock or trim loss problem
1/V/1/M	Assembly line balancing problem
1/V/1/M	Multiprocessor scheduling problem
1/B/O/R	Change making problem
n/B/O/	Multi-period capital budgeting problem

Fig. 6 Problem Types according to Dyckhoff's Typology. Source: Own representation according to Dyckhoff (1990) [2].

For a deeper understanding, a few of these problem types are briefly explained below:

Knapsack problem (1/B/O/): The knapsack problem describes a one-dimensional cutting and packing problem (1), where only a subset of the total number of available small items must be assigned to all of the available large objects (B). The assortment of the large objects only consists of one single element (O). In this case, the fourth symbol of the fourth-tuple denotation is missing because every single characteristic value is possible here. Whether the assortment of small items consists of a few or many items of different figures, of many items of non-congruent figures or of congruent figures, this has no impact on the underlying logic of the problem and therefore its classification [2].

Practical examples of knapsack problems can occur in the shipping industry: companies ship their packages through cargo vehicles with a fixed capacity. These packages differ in respect to their weight. Since dispatchers get paid on contract, the costs of sending those packages don't depend on the individual weight of each pack-

age. Therefore, the dispatcher tries to maximize his profit by packing the maximum weight of packages in a fixed volume [6].

2-dimensional bin packing problem (2/V/D/M): This class describes a one-dimensional C&P problem (1). In this case, all the available small items must be assigned to a subset of the large objects (V). The assortment of the large objects consists of elements with different figures (D) while the assortment of small items consists of many elements with different figures (M) [2].

These problems can occur, for example, in the textile industry: a tailor makes pants and jackets from silk. He obtains the raw material from his supplier in the form of 4 m x 10 m and 6 x 6 m long pieces of fabric. In order to buy as little silk as possible, he has to arrange the shapes for both garments optimally on the silk strip before cutting out the shapes. This way, he can minimize leftovers and thus keep waste to a minimum.

Container loading problem (3/V/I/) or (3/B/O/): The class of container loading problems describes cases of three dimensions (3) where two different scenarios can occur: in the first scenario, all small items have to be assigned to a few large objects (V), which are identical regarding their figure (I). In the second scenario, only a subset of the small items has to be assigned to one large object (B/O). In both cases, the characteristics of the assortment of the small items are indifferent to the general structure of the optimization problem [2].

As the name of the problem suggests, those problems often occur in the field of logistics when loading containers (but also trucks and planes) with packages: a furniture manufacturer wants to deliver a fixed order quantity to a customer by truck. The customer has ordered a total of 40 tables and 80 chairs. Because the goods have a length, height and width, all three dimensions have to be considered when trying to find an optimal pattern for placing them within the trucks. Although the manufacturer has enough trucks to transport all small items, to minimize delivery costs he wants to load the furniture in such a way that as few trucks as possible have to be used (3/V/I/-scenario).

4 Improved Typology

While Dyckhoff's approach to devising a system for classifying C&P problems is seen as a valuable contribution to the respective scientific discipline, it nevertheless has shortcomings and is therefore not suitable for an unambiguous classification of problems at hand.

Wäscher et al. list the following criticisms:

1. Due to linguistic incomprehensibilities, some of the terms used by Dyckhoff are not intuitively understandable in the international language area.
2. Not all C&P problems can be assigned to existing problem types.
3. Dyckhoff's typology is partly inconsistent and can lead to confusing results.

4. The described problem groups are not necessarily homogeneous. However, homogeneity is a key criterion for problem types if they are to be used to develop methods for problem solving or process optimization [3].

Therefore, an improved typology was developed which is based on Dyckhoff's work but eradicates the weaknesses described above.

As already mentioned in Chapter 2.2.4, in addition to the criteria used by Dyckhoff to classify C&P problems (dimensionality, kind of assignment, assortment of small items and assortment of large objects), Wäscher et al. use an additional criterion. This is called 'shape of small items' and differentiates between items with a regular shape (e.g., boxes, spheres and cylinders) and items with an irregular shape (e.g., convex or concave shapes) [3].

Furthermore, their typology includes a hierarchy of different problem types. A distinction is made between:

1. basic problem types. These are generated by the combination of the two criteria 'kind of assignment' and 'assortment of small items.' The basic type, as the name implies, acts as the basis for the other types.
2. intermediate problem types. In order to refine the classification of the different problem types and to increase the degree of homogeneity, the assortment of large objects is additionally used as a criterion to generate these types.
3. refined problem types. In a final step, the criterion of dimensionality is additionally used for classification, as well as the shape of the small items [3].

This paper does not claim to provide a detailed explanation of the different classification systems. Therefore, only an overview of the basic problem types is given (see Figure 7).

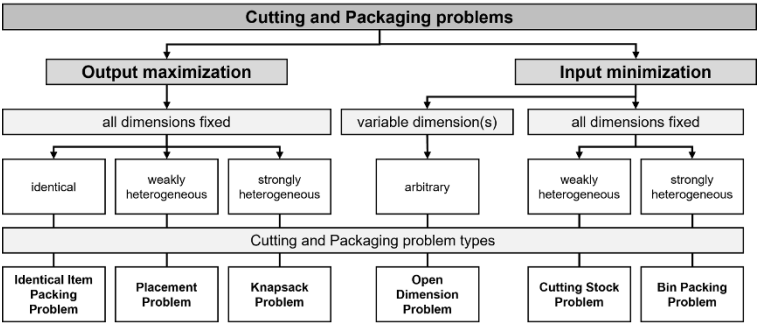

Fig. 7 Overview of Basic Problem Types by Wäscher et al. Source: Own representation according to Wäscher et al. (2007) [3].

In the first step, the 'kind of assignment' criterion is used to determine whether an optimization problem is of the output maximization type or of the input minimization

type. The former describes C&P problems where a limited set of large objects is available. As a result, it is not possible to allocate the entire range of value-bearing small items. Optimization problems of this type are therefore concerned with the question of how to fill the large objects in such a way that the value is maximized. All large objects must be used. This type is equivalent to Dyckhoff's case of the 'Beladeproblem.' The latter describe situations in which a finite set of small items must be allocated to an unbounded set of large objects. The goal here is to minimize the number of large objects required. This type is equivalent to Dyckhoff's case of the 'Verladeproblem' [3].

The next step is to answer the question whether the dimensions of the involved elements are fixed or variable. In optimization problems of the type "output maximization," it is generally assumed that all dimensions are fixed. In the case of input minimization, there are also cases where they can be variable.

This two-step classification scheme results in six different basic problem types, namely identical item packing problems, placement problems, knapsack problems, open dimension problems, cutting stock problems and bin packing problems [3].

5 Solution Approach and Resulting Workflow

The typology described above systematically aggregates the highly complex variety of different cutting and packing problems into a manageable number of homogeneous problem types. In the context of operational research, this enables companies to define their own optimization challenges in the C&P area based on the characteristics described in chapter 2.2 and to assign them to a corresponding theoretical problem type based on this typology. The company can then focus on researching and applying (computer-aided) appropriate mathematical methods for solving the corresponding problem in order to achieve an optimum planning of the company's internal cutting and packing processes. To demonstrate how effective those problem times can be linked to existing algorithms or mathematical approaches to solve those optimization problems, Figure 8 shows an extension of the improved typology created by Wäscher et al. The added row shows some, but not all, algorithms that can be used to optimize the individual cutting and packing problems.

Note that there are now algorithms listed for the identical item packing problem type. Since in this case all small items have the same shape, neither a real selection problem nor a grouping or allocation problem arises. Optimization problems of this kind are so easy to solve that no complex algorithms are needed.

The resulting operational research workflow is shown in Figure 9.

To demonstrate how that workflow can be used in business reality, a case study is presented in the following chapter.

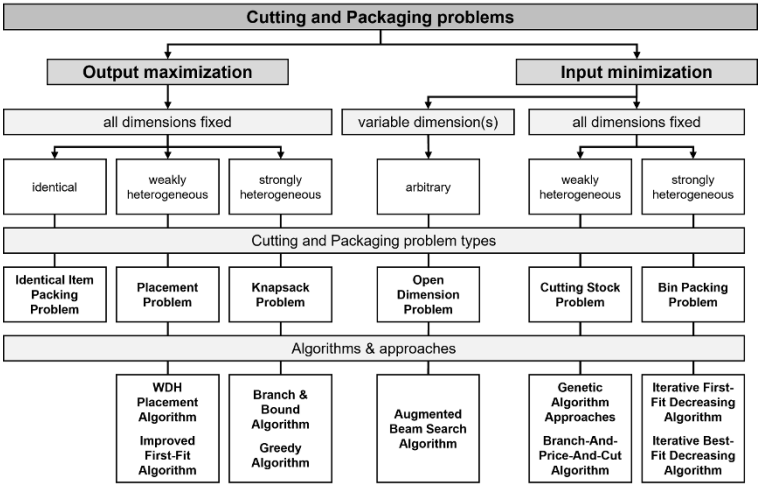

Fig. 8 Basic Problem Types and Corresponding Algorithms. Source: Own representation according to Wäscher et al. (2007) [3]; Kang and Park (2003) [7]; Akeb et al. (2011) [8]; Onwubolu and Mutingi (2003) [9]; Kellerer et al. (2004) [10]; Pál (2006) [11]; Domschke et al. (2015) [5].

6 Case Study: Packaging Optimization

An art dealer wants to sell his art collection at an annual sales exhibition. For this purpose, he rents a truck with a load capacity of 2,500 kg. The day before the exhibition when he wants to load his sculptures and artworks into the truck, he realizes that he can't take all the items with him because their total weight would exceed the maximum load capacity of the truck. The dealer now has to decide which items he wants to load into the truck and take with him. His artworks all have a different weight and value that he could achieve if he sold them.

First, the art dealer must accurately characterize his packing optimization problem at hand using the relevant criteria. The **dimensionality** is not relevant in this case study, since the bottleneck does not arise from the spatial dimensions but from the maximum load capacity of the large object. Thus, the criteria can be left out.

Regarding the **kind of assignment**, all available large objects (the single truck he rented) must be used in the allocation of the small items, but only a subset of them can be loaded.

The **assortment of large objects** in this case consists of only one large object (the truck). Since the small items (i.e., the individual pieces of art) all differ vastly in weight and value, it can be assumed that the **assortment of small items** is highly heterogeneous. The individual exhibits are items with an irregular **shape**.

Now these characteristics can be used to assign the optimization problem at hand to a theoretical problem type. In this case, the classification system of Wäscher et al.

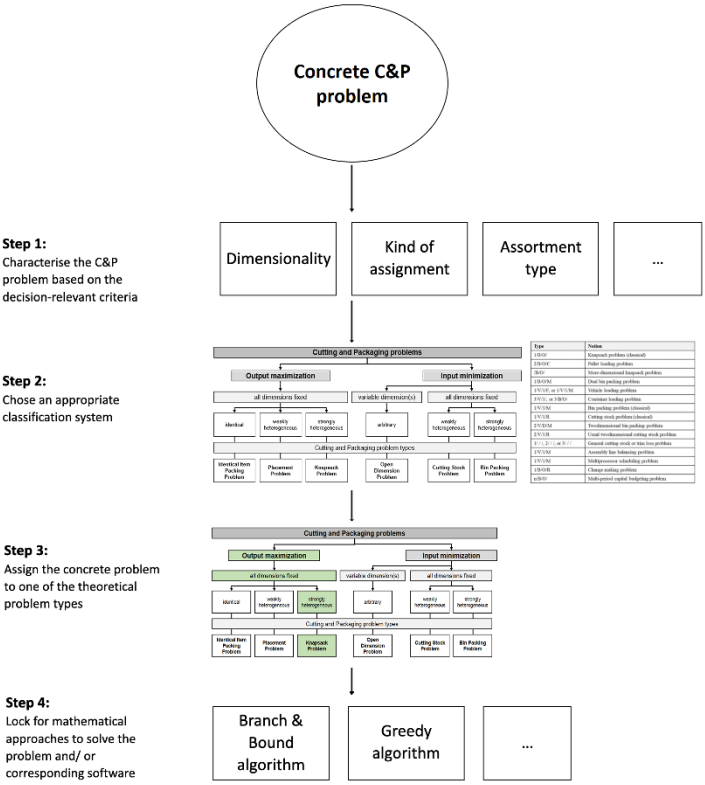

Fig. 9 Workflow for C&P Optimization Problems. Own representation according to Wäscher et al. (2007) [3]; Dyckhoff (1990) [2].

is utilized to determine a basic problem type. First the criterion "kind of assignment" is used to check whether the problem is an output maximization problem or an input minimization problem. Then it is checked whether the assortment of small items is homogeneous or heterogeneous. In the case study, we have an output maximization problem with a highly heterogeneous assortment of small items. These problems are called knapsack problems in the literature.

By assigning his concrete problem (see Figure 10 to a theoretical problem type, the retailer is now able to search for specific solutions. The problem can also be further specified: since the dealer can only load his artworks as a whole into the truck, this is a binary knapsack problem.

	item 1	item 2	item 3	item 4	item 5
value	1000	900	1200	500	900
weight	500	600	1200	1000	1200

Fig. 10 Order Book of Small Items with Corresponding Values and Weights. Own representation.

There are many different algorithms and methods that can be used to solve binary knapsack problems. One possibility is the so-called branch & bound method. As the name implies, the method is based on two different solution principles:

1. Branching: an initial maximization problem P_0 is divided into k sub-problems, where the union of the solution sets of the k problems results in that of P_0. Each individual sub-problem $P_1, ..., P_k$ can also be broken down into further sub-problems. The result of the division is a solution tree.
2. Bounding: this solution principle serves to restrict the branching process described above. Bounds for target function values are calculated, with the help of which one can decide whether sub-problems must be branched any further or not [5].

The maximization problem at hand can be formulated as a target function as follows:

$$\text{Maximize } F(x) = 1000x_1 + 900x_2 + 1200x_3 + 500x_4 + 900x_5 \quad (1)$$

taking into account the restriction:

$$500x_1 + 600x_2 + 1200x_3 + 1000x_4 + 1200x_5 \leq 2500; x_j \in \{0, 1\} \text{ for } j = 1, ..., 5 \quad (2)$$

The first step is to determine the upper global bound. To do this, one uses the method of LP relaxation. It is created by relaxing the condition $x_j \in \{0, 1\}$ for $j = 1, \ldots, 5$ to $0 \leq j \leq 1$. In other terms: in order to create the upper bound, one ignores the binary characteristic of the binary knapsack problem at hand. This enables one to calculate the value of the assigned small items that would be reached if the complete load capacity of the truck (the large object) would be utilized. The bound is calculated by dividing the value c_j of every small item by its weight w_j (resulting in the utility per weight unit) and sorting the resulting quotients in a decreasing order. The associated variables receive the value 1 in the order of this sorting as long as the sum of the respective weights does not exceed the maximum allowed weight b (load capacity of the truck). The first small item that can no longer be fully taken up is considered proportionally (so that exactly the maximum weight b is reached). In Figure 11 the utility (or value) per unit is formed for every small item [5].

Item 1 has the highest value (highest quotient), therefore it is placed first, followed by item 2, item 3, item 5 and item 4. After that, the accumulated weight for every item is calculated (see Figure 12). The cells are colored according to whether the

item	value	weight	quotient
x_1	1000	500	2
x_2	900	600	1,5
x_3	1200	1200	1
x_4	500	1000	0,5
x_5	900	1200	0,75

Fig. 11 Value per Weight Unit for Every Small Item. Own representation.

accumulated weight can be carried by the truck. If the cell is colored green, the item gets assigned to the truck. The accumulated weight does not exceed the maximum load capacity of the truck until item 5. Only 200 kg of that item can be loaded into the truck (about 16%). Therefore, the cell is colored yellow. Item 4 does not fit into the truck at all; thus it is colored red [5].

item	quotient	weight	cumulated weight
x_1	2	500	500
x_2	1,5	600	1100
x_3	1	1200	2300
x_5	0,75	1200	3500
x_4	0,5	1000	4500

max capacity of the truck

Fig. 12 Accumulated Weight and Loading Capacity of the Large Object. Own representation according to Domschke et al. (2015) [5].

Figure 13 shows the resulting values for every x_j. Items 1 to 3 are assigned to the truck completely, thus the respective x_j values equal 1. Item 4 is not assigned; its corresponding x value is equal to 0. Only 16.66% of item 5 is loaded into the truck, therefore its value is equal to 0.16666667.

item	value for xj
x_1	1
x_2	1
x_3	1
x_4	0
x_5	0,16666667

Fig. 13 Resulting x Values for Every Small Item. Own representation according to Domschke et al. (2015) [5].

The calculated values now can be used to solve the target function:

$$F(x) = 1000 * 1 + 900 * 1 + 1200 * 1 + 500 * 0 + 900 * 0.16666667 = 3250 \quad (3)$$

The global upper bound equals 3250.

In a next step, the lower bound is calculated. This is accomplished by simply removing the fractions from item 5 from the large object. Thus, x_5 equals 0. The new x values are used to solve the target function again:

$$F(x) = 1000 * 1 + 900 * 1 + 1200 * 1 + 500 * 0 + 900 * 0 = 3100 \quad (4)$$

The lower bound equals 3100. Unlike the upper bound, this value was calculated in compliance with the binary condition $x_j \in \{0, 1\}$. Therefore, it could be used as a permissible packing pattern for the knapsack problem at hand. However, at this point one can't be certain that this solution generates the maximal possible output for the art dealer [5].

The root P_0 of the solution tree mentioned above can now be drawn (see Figure 14). At this point the first branch is created by dividing the root problem into two sub-problems: in the first one, the upper bound and local lower bound have to be calculated again under the condition, that the item that was partly assigned before (x_5) is not loaded into the truck at all ($x_5 = 0$), resulting in sub-problem P_1. The other branch of the solution tree is solved under the condition that $x_5 = 1$ (resulting in sub-problem P_2) [5].

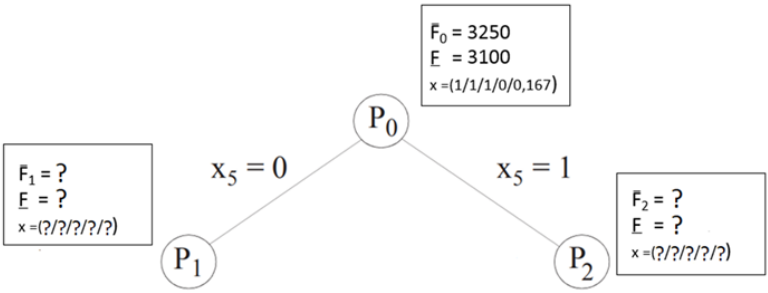

Fig. 14 Root and First Two Branches of the Knapsack Problem Tree. Source: Own representation according to Domschke et al. (2015) [5].

For each sub-problem, the procedures described above for calculating the upper and (local) lower bounds are carried out. Then the calculated bounds must be examined more closely, and it must be checked whether one of the following three cases is at hand:

1. The upper bound of the sub-problem, i.e., the optimal solution, is less than or equal to the best-known solution. In this case, the sub-problem and its sub-branches cannot lead to an even higher output.
2. The upper bound of the sub-problem P_i (i.e., the optimal solution) is larger than the best-known solution and the optimal solution of the relaxation of P_i' is also admissible for P_i. The result is therefore a new best solution.

3. The relaxation of the sub-problem has no admissible solutions. Or in other words: even if one ignores the binary condition of the problem and loads percentages of the small items, one does not arrive at a permissible result [5].

If one of these mutually exclusive cases has occurred, the corresponding branch does not need to be examined any further. This procedure is carried out until no more branches are left.

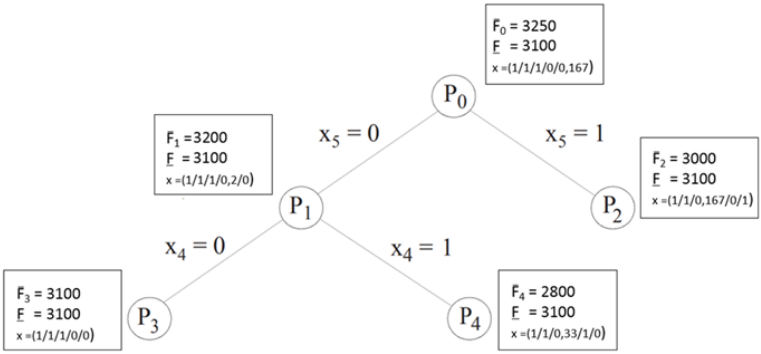

Fig. 15 Completed Knapsack Problem Tree. Source: Own representation according to Domschke et al. (2015) [5].

Figure 15 shows the complete knapsack problem tree for the packing problem at hand. For sub-problem P_2 as well as for sup-problems P_3 and P_4, case a is present: the calculated upper bound in each case is smaller than or equal to the already known optimal solution.

Accordingly, none of the present sub-problems need to be further subdivided. The optimal solution has already been calculated in the root of the problem tree at P_0.

Applied to the case study, the dealer at the sales exhibition can make the most profit ($\$3,100$) if he loads items 1, 2 and 3 into the truck. In doing so he utilizes 92% of the truck's maximum load capacity. Items 4 and 5 cannot be taken along.

7 Available Software

The example of solving an optimization problem in the cutting and packing context described in Chapter 5 could be solved quickly with a simple algorithm, but it was a very simplified problem. In business reality, companies are confronted with much more complex problems. As the complexity of a packing and cutting problem increases, the time needed to find the best solution grows exponentially. In that

case, the use of optimization software is recommended and often necessary to solve the problem in a reasonable time. The corresponding market offers companies a wide range of different software solutions that have been specially developed for optimizing cutting or packing problems.

7.1 Software for Packing Problems

There are many different computer programs on the market that enable users to solve cutting problems. The target groups of these products include, for example, companies from the textile industry, but also manufacturers of cars or wood-based products.

Figure 16 shows a selection of different cutting software, including the corresponding providers and the areas of application. This is not a complete list of all available options.

Company	Software	Industry / used for
Cargoload	Cargoload	Storage optimization
Fraunhofer SCAI	PackAssistant	For container load planning (with an homogeneous assortments of small items with an irregular shape)
Logivations GmbH	W2MO Case Optimization	Warehouse and distribution logistics
MultiScience GmbH	MultiPack	Use of storage place on pallets and returnable containers as well as the storage place in containers, railway wagons, HGVs and ULDs (with an homogeneous assortment of small items)
	Multimix	Optimization the storage of pallets and lorries (with an heterogeneous assortment of small items)
Simplan AG	MaxLoad Pro	Packing, palletizing and loading
	TOPS Pro	Packing, palletizing and loading

Fig. 16 Overview of Available Packing Software. Source: Own representation.

7.1.1 PackAssistant

PackAssistant supports the planning of container packing with an assortment of homogeneous small items using 3D data sets. Due to the complete consideration of the component geometries, even highly irregular parts can be packed in the best possible space-saving patterns. In addition, the software offers further functions for customer-specific packaging solutions:

- Users can define minimum distances between small items within packing patterns.
- Additional conditions such as maximum permissible weights of the large objects can also be considered when calculating the packing pattern.
- To ensure the stability of the container, stable layers can be defined for the parts [12].

7.1.2 MultiMix Modules

The MultiMix software (see Figure 17 of MultiScience GmbH consists of different modules for a variety of application scenarios. Like the PackAssistant software, it can be used to solve optimization problems with a homogeneous assortment of small items. In addition, the so-called MultiMix module enables the solution of packing problems with a heterogeneous assortment. According to the respective loading strategy different optimization algorithms are available [13].

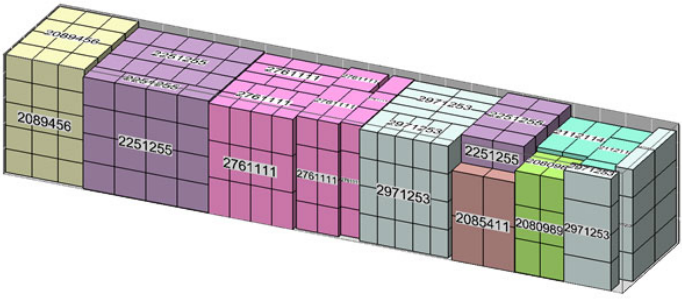

Fig. 17 User Interface MultiMix Module. Source: MultiScience GmbH (2023) [13].

7.2 Software for Cutting Problems

Just as for cutting problems, there are many different solutions that companies can use to solve optimization problems in the packing context. Figure 18 shows an overview of possible products. This list does not contain all the application programs available on the market, but only serves the purpose of providing the reader with a brief overview of possible alternatives.

Company	Software	Industry / used for
Boole & Partners	OptiCut	Panels and profiles cutting optimization
Fraunhofer SCAI	AutoCompactor	Used for the compaction of two-dimensional nests
	Autonester-L/-T	Leather and car manufacturing industries
	CutPlanner	Textile manufacturing industry
L & P Software GmbH	1aOpt	Cutting of acrylic, glued windows, pallets and railway materials
	Zuschnitt24	Manufacturing of rectangular glass, plastic, steel and wood
Schneeberger Optimierungen	BestOpt / SwitftOpt	Carpenters and for plastic, facade and metal processing

Fig. 18 Overview of Available Cutting Software. Source: Own representation.

7.2.1 CutPlanner

CutPlanner is a software that is used in the textile industry. It automatically creates a so-called cut plan from a customer's order. This cut plan is an assignment of sizes (small items) and fabric types to a template, which can then be used to produce the corresponding garments. The program uses algorithms from the field of linear programming in combination with techniques of pattern recognition, fast local search algorithms and heuristics which are adapted to the problem characteristics of C&P problems. In the user interface, the demand is displayed in the form of a table. Here the fabric, shape, and quantity of the required garments (order book of small items) are specified. The calculated output is displayed in the form of colored templates. The color-coding indicates which of the fabrics and which of the sizes are involved. A special feature of CutPlanner is the possibility to choose between two different modes for solving existing optimization problems. The so-called conventional mode

provides a quick solution to the problem, but the calculated results are only estimates. For an exact problem solution, users can use the exact mode, which requires more time [14].

7.2.2 OptiCut

OptiCut is a cutting optimization software for plates and bars based on algorithms for different calculation modes, formats and materials. Like the CutPlanner software, here users have different modes that can be used to perform optimization calculations. Besides a quick run and a standard mode, users can also use an advanced mode that allows changing the main cutting direction (orientation of the figure of the small items) during a run. Although this feature results in an additional work step, it can increase the reusability of the remnants and, in some cases, reduce the number of necessary plates (stock of large objects) [15].

8 Conclusion

Cutting and packing problems present many different companies with the challenge of making the best possible use of their resources. Particularly in logistics, but also in the value creation process of companies in the manufacturing sector, those responsible are confronted with many different optimization problems. To solve these problems as efficiently and purposefully as possible, it makes sense to follow an operational research approach and to first systematically analyze the issue at hand, classify it based on a typology and then research existing solution approaches.

Both cutting problems and packing problems are based on the same logical structure: in both cases there is a group of small items that must be assigned to a group of large objects. This results in geometric patterns; unused space is considered as trim loss. C&P problems can be described using a variety of different characteristics. Of particular importance is the dimensionality of a problem, i.e., the number of dimensions required to describe the problem. Also, the restrictions that have to be considered when assigning the small items to the large objects are an important characteristic, as well as the properties of both existing assortments.

After a given C&P problem has been evaluated based on these criteria, it is possible to assign the issue at hand to a superior problem type by means of a classification system. Based on the typology of Dyckhoff, Wäscher et al. have developed and optimized such a system which allows a clear and consistent assignment to homogeneous problem types. These include output maximization problems such as the knapsack problem and placement problems with a weakly heterogeneous assortment, and input minimization problems such as cutting stock problems or bin packing problems.

Once a concrete, real-world problem has been assigned to one of these problem types, companies can search specifically for approaches and algorithms. In the literature, a variety of different possible solutions can be found for each individual

problem type. As shown in the example presented in Chapter 5, C&P optimization problems can thus be solved relatively quickly and easily. However, since the corresponding issues in business reality are in most cases much more complex, it makes sense to acquire appropriate software for automated solutions. Those responsible in the company therefore do not need to be familiar with the algorithms and can focus on their core business.

There are a number of different software solutions for both cutting problems and packing problems. These are specialized for different application scenarios and offer the possibility to find the optimal solution for a business. One of these applications is OptiCut, which is used for cutting problems in the textile manufacturing industry. For packing problems, for example, companies can use the products of the German company MultiScience GmbH. Here, customers can choose between different modules that are adapted to optimization problems with an assortment of heterogeneous small items as well as homogeneous small items.

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Queueing Theory

Maximilian Adolphs, Sascha Feistner & Violetta Jahnke*

Abstract As soon as more orders are received than can be processed within a period, queues occur. This affects the company's performance and represents a challenge that has to be dealt with. Hence, the understanding and management of queues is a fundamental task for the operations management. In summary, it can be said that the queueing theory provides only the possibility to support planning functions. However, a certain behaviour of the system has to be assumed by the adoption of certain distributions (exponential function). In reality, several simulations are necessary otherwise the results may not be meaningful because they could offer a high spread. In some cases, adjustments of the system may be necessary. All in all, the use of the queueing theory theoretically enables managers to calculate and simulate a queueing model in a short time.

1 Overview

1.1 Fundamentals of the Queueing Theory

The queueing theory serves as a basis for planning schedules, inventory levels, etc. Waiting lines occur in each economic sector (primary, secondary, tertiary, quaternary sector) (see Table 1).

As customers, we come directly into contact with queues in the secondary sector. As workers, we encounter waiting lines at factories where jobs wait in lines in order to be worked on at different machines, and machines themselves wait to be overhauled (see Figure 2) [1].

* The following presentation of the queueing theory has been developed at the Bonn-Rhein-Sieg University in Sankt Augustin, Germany, in 2020 under the leadership of Prof. Peren. It is based on an unpublished manuscript by Sina Giesen, Bianca Schrandt, Melanie Drägestein, and Sascha Feistner, which has been developed in the same course years before.

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Table 1 Queues Per Economic Sector

economic sector	examples of queues
primary	miliking parlors
secondary	supermarket checkout, traffic lights
teritiary	production process, maintenance of machines
quaternary	data processing (process control)

Source: Own representation.

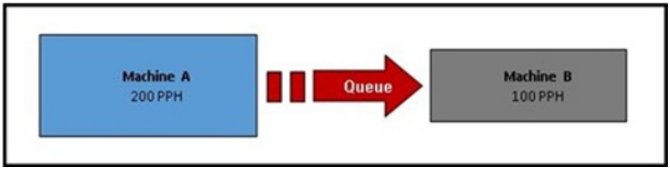

Fig. 1 Machine Queueing. Source: Own representation.

Waiting lines occur when the demand exceeds the capacity for request processing [2]. The wait time and the length of a queue are influenced by many factors. These questions will be more closely looked at and expanded upon in the course of this paper.

1.2 Economic Relevance

The central problem in operations management is the conflict between costs and velocity. The velocity reflects, inter alia, customer satisfaction and the service / process quality [3]. Equally, service employees could use the time saved by not waiting more wisely and effectively [4]. Waiting time and capacity utilization are elements of service quality as well. The queueing theory serves as an instrument that helps managers design the queue networks and make trade-off decisions. It can be used to evaluate additional costs of a faster service (more traffic lanes, additional landing strips, more checkout stands, etc.) and the inherent cost of waiting.

When designing a queueing system, it is also about avoiding fluctuations in capacity utilization, which are created by fluctuations in demand. Excessive machine capacity leads to excessive capital charges on fixed assets [1]. It should be noted that the waiting time accounts for the majority of the cycle time [5]. Moreover, as has already been mentioned, from a business point of view, waiting time is seen as unproductive time, which has the following impact:

Customers bail out when having to wait too long [5], long production times reduce the reaction times to competitors and inventory levels increase as semi-finished products must wait for available machines, leading to storage costs. The

time spent waiting at the checkout point is used by the so-called "checkout line candy" in a supermarket or by advertisements at bus stops and train stations. The trade-off between cost of waiting and cost of service is illustrated in Figure X.

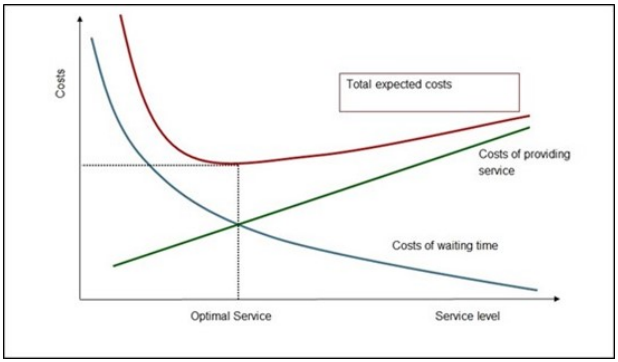

Fig. 2 Queueing Cost and Service Level. Source: Own representation according to Render et al. (2015) [4].

The waiting time and cost of waiting decrease with an increasing service level. Whereas the higher the service level (and so, the shorter the queues) (for example by more servers to serve customers or by faster service rates), the higher the cost of service.

In managing a queueing system, it is often desirable to determine an appropriate number of servers c for the system. A larger number of servers improves the service quality [6].

So, the operations management is responsible for finding the apex of this cost function [4] – and not the option with the shortest waiting time, etc. This apex is located at the point of intersection of the cost of waiting function and the cost of service function.

2 Queueing Model

2.1 Elements of the Queueing Model

Each system where the arrivals demand limited resources can be seen as a queueing system [3]. They do not differ in their general working method and consist of the following components: arrivals (arriving elements), queue, server and departures (served arrivals) (see Figure 3).

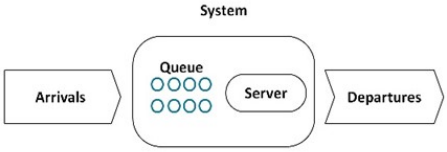

Fig. 3 The Queueing Model. Source: Own representation according to Zimmermann (2008) [3].

The queue and server form the actual queueing system, while the arrivals and departures represent the system's input and output.

The different versions of queues can be compared in the following picture. One queue ends either in a single server, respectively multiple servers, or multiple queues lead into a single, respectively multiple servers (see Figure 4) [3].

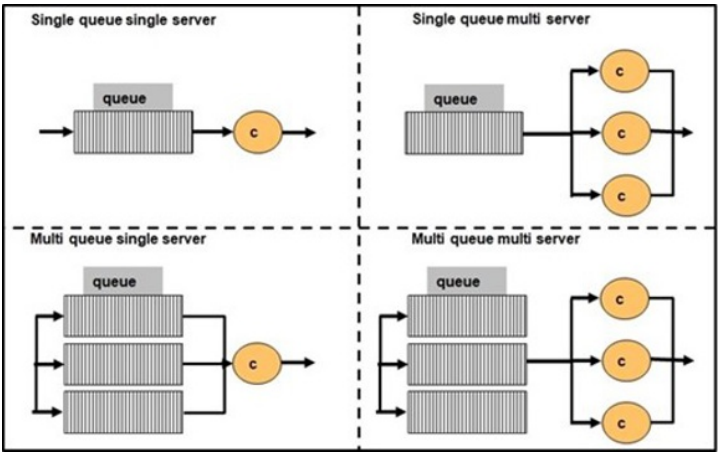

Fig. 4 Different Kinds of Queues. Source: Own representation according to Zimmermann (2008) [3].

Overall, a queueing system can be specified by many characteristics, which can be assigned to the different components. These characteristics and their assignment to the components will be more closely looked at in chapter 3.1. Instead, we will now take a closer look at the four central components of a queueing system.

2.1.1 Arrivals

First of all, it is unsure whether the elements want to get inside the system (in the queue), so they arrive randomly [3]. The arrival rate λ is used to describe the arriving

elements/arrivals. λ describes the average number of arriving elements per time unit It can be expressed depending on / subject to the expected value E of the arrival time t_a:

$$\lambda = \frac{1}{E(t_a)}$$

For example: On average, an order is received every three minutes. Therefore, lambda is one third per minute or 20 orders per hour [5].

To simplify, it can be assumed that the arrival rate is equal for each arrival. However, it is possible to differ two or more costumer groups which are arriving with different/separate frequency. The way a queueing system is designed (and the way people behave) can be influenced whether the arriving elements do in fact enter the system or whether they bail out because the queue is too long, etc. In these cases, the adjusted effective arrival rate has to be calculated.

2.1.2 Queue

In the queue are those customers / elements waiting for service / processing at the server (by a worker, machine, etc.). The queue appears when no free server is available. It ends with the beginning of the service. Consequently, the person(s) at the server being served right now is not (are not) a part of the queue anymore but are still part of the system. Similar to the arrival (entrance in the queue or not), the customer behavior within the queue is unsure (cf. bulking, jockeying, reneging) [3].

2.1.3 Server

Servers are the place of service provision. A queueing system can have one or more servers. Servers can either work on just one single job type (due to authorizations, education/training etc.) or several job types (single queueing system vs. multiprocessor system). Equally, either one customer or numerous customers (for example bus and train) can be served per service procedure at a time. Unless otherwise indicated, it is always assumed that only one customer can be served. The likelihood of service is equal at any time [3]. The speed in which the customers are being served by/at the server is called service rate μ (average number of served customers per time unit) and depends on the expected value E of the service time t_b:

$$\mu = \frac{1}{E(t_b)}$$

For example: On average, a worker at the server needs five minutes to process an order. Therefore: $\mu = 1/5$ orders per minute or 12 orders per hour. As $\lambda = 20 > \mu = 12$, a queue occurs [5]. Similar to the arrival rate, a constant service rate is assumed. On average, the worker's pace of work stays the same, no matter how long the queue in front of the server is, etc. Based on the two input parameters λ and μ, the server's utilization rate ρ is calculated as follows:

$$\rho = \frac{\text{arrival rate}}{\text{server rate}} = \frac{\lambda}{\mu}$$

It describes the probability that a server is in its operating period.

Consequently, the idle rate stands for the probability that no person is within the system and is calculated as: $P_0 = 1 - \rho$. If < 1 or $\lambda < \mu$ or $t_a > t_b$, then the system is stable. The stability decreases with increasing utilization ($\rho > 1$)" [5].

This theoretical assumption of stability is not always confirmed at high utilization levels (e.g. 90%) in practice or in simulations. This is caused by the statistical variations in arrival times and service times, which are approximately Poisson or exponentially distributed. Especially with high standard deviations, the stability often only sets in with very high case numbers [7].

Stability in the context of a queuing system does not mean that there are no queues, but only that the queuing length does not increase on average after settling at a certain level of stability [7]. The difference between differently distributed intermediate arrival times and service times for the existence of queues is clearly illustrated by the following comparative example (see Figure 5).

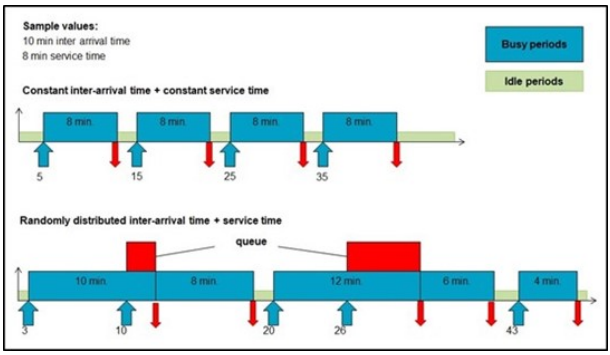

Fig. 5 Comparison of Constant vs. Randomly Distributed Time. Source: Own representation according to Helber (2015) [8].

While there are no queues for constant inter-arrival times and service times under 80% utilization, a queue is probable if the distribution of the arrival time or service time is a stochastic process.

In this context, it must also be pointed out that an occupancy rate of almost 100%, contrary to the colloquial expectation attitude, does not necessarily represent the optimal economic situation [5].

An improvement of the utilization from 80% to 95% leads, in the example above, to an extension of the average queue length from 3 units to 18 units.

2.1.4 Departures

After being served, the elements leave the system as departures. In a queueing network, viz. at successive queueing systems like a production with different manufacturing steps (viz. queuing systems), the departures of the preceding/ previous system describe and influence the arrivals of the subsequent / following system. By means of the queueing theory, it will be the task to identify bottlenecks and to avoid gridlocks [3].

Table 2 shall give examples of queueing systems broken down to their components.

Table 2 Different Queueing Systems

Arrivals	Queue	Server	System	Departures
customers	cashier line	service at the counter	shopping center	leaving the shopping center
damaged car	waiting area	car repair by mechanic	car service station	leaving the car service station
patients	waiting room	check-up by doctor	medical office	leaving the medical office
calls	queue at the phone line	consulting on the phone line	call center	hanging up the phone

Source: Own representation.

2.2 Structures and Queue Disciplines

The queue's length is not only influenced by the service rate, but also by the queue / scheduling discipline, meaning the order in which the waiting customers are being served. Generally, this happens according to the FIFO principle (First-In-First-Out), where the customers are served in the order of their arrival [9]. Another possibility is the LIFO principle (Last-In-First-Out), where the newer arrivals that join the queue are served first, and the customers already in the queue have to wait longer. LIFO examples are: elevators, full busses [9].

Further disciplines are random selection /order (for example random selection of a student in a course) and assigning priorities. Then, customers with a higher priority are moved forwards and served next [10]. It's further differentiated whether the person being served at the moment of a higher priority's arrival will be fully finished first and then the higher priority customer will be served next (non-preemptive) or whether the service will be interrupted, the higher priority customer jumps ahead and is served, and after that the previous customer's service will be continued (preemptive) (for example health insurance). In case it takes the same amount of time to serve each priority, the inter-priority average waiting time does not change

due to the assignment of priorities. Nonetheless, the waiting time for higher priorities decreases respectively, whereas the lower priorities' waiting time increases.

Further scheduling disciplines are thinkable as well, but are not described in the context of this paper [3], [10].

3 Description of Queueing Systems

3.1 Kendall Notation

In order to fully describe the respective characteristics and its specificity, in 1953, David George Kendall developed the Kendall notation, which is commonly used today [9]. Each characteristic is assigned a letter and is written in the following formalised order: A/S/c/K/P/Q (see Figure 6).

Fig. 6 Kendall Notation. Source: Own representation according to Baum (2013) [9].

Other notations (with different parameters) are possible as well [9]. Each characteristic can have different values according to its specificity (see Table 3).

Table 3 Different Queueing Systems

KENDALL notation	Description	Example
A	Distribution of interarrival time	M = Markov process (Poisson distribution) G = General distribution D = Deterministic
S	Distribution of service time	M = Markov process (exponential distribution) G = General distribution D = Deterministic
c	Number of servers	1, 2, …, ∞
K	Maximum number places in the queue system	1, 2, …, ∞ standard: ∞
P	Maximum population size entering the system	1, 2, …, ∞ standard: ∞
Q	Queueing dicipline	FIFO, LIFO, PRI, SIRO, ... standard: FIFO

Source: Own representation according to Baum (2013) [9].

The first parameter A represents the probability distribution of the arrival times, describing the distribution of the intervals between the arrivals. Normally it seen as a Markov process (M) because it follows the Poisson distribution [3]. Other distributions are the general distribution (i.e., normal distribution; G) or the deterministic distribution (i.e., scheduled arrivals; D).

The parameter S stands for the probability distribution of the service time, which can have the same specificities as parameter A. Having said this, it is seen as a Markov process (M), which does not follow a Poisson distribution, but an exponential distribution. The reason for this and an overview of the Markov process will be expanded upon in the following chapter / excursus. The service time represents the time needed by an element (i.e., waiting customer) until being served before leaving the queueing system.

The number of servers C describes how many customers can be served at the same time (if each server can only serve one customer at a time). In the theoretical case of an unlimited/ endless number of servers, there will be an available server for every arriving customer. Consequently, the waiting time is zero in this case. Using queueing theory, you can, for instance, determine how many servers have to be open during peak times / rush hour when the average waiting time shall stay constant.

When given a constant utilization rate c, the number of needed servers is calculated as follows:

$$c = \frac{\lambda}{\rho * \mu_{\text{per server}}}$$

This means that if the traffic level quadruples, the number of servers should also quadruple.

The parameter K describes the capacity of the queue. If all places in the queue are filled, no new customer will be able to become part of the system. In the case of a limited queue, in order to calculate the queue length and waiting time, the effective arrival rate (arriving elements being able to enter the system) has to first be determined.

Another limiting factor can be the population size P. This parameter considers the maximum number of elements possibly entering the queueing system (e.g. all possible customers of a bank who may be entering the bank branch → access only with a bank card.) [3]. The last parameter of the described Kendall notation stands for the queueing discipline Q determining in which order the waiting customers will be served. Normally, which means 'unless otherwise stated', the parameters K and P have the value ∞ (unlimited) [9] and the server pursuits the scheduling discipline FIFO. Therefore, the most common queueing system M/M/1/∞/∞/FIFO is abbreviated with M/M/1 [2].

The arrival times of an M/M/1-queue are Poisson and its service times are exponentially distributed. If there is only one server, the queue length is unlimited as well as the population size (No restrictions given to enter the queueing system). The waiting customers are served following the FIFO principle [9].

Within the scope of this paper, only the Markov queues will be discussed.

3.2 Excursion: Markov Process

The Markov process is named after Andrej Andrejewitch Markov, who was a Russian mathematician [9]. His intention was to determine the likelihood of future incidents. It is used for modelling random changes of status in a system assuming that different changes of status are conditioned by other statuses only for a terminated time. This assumption allows the forecast of a development in the future without knowing the complete history of changes of status in the past [2].

3.3 Parameters

Concluding the described parameters of queueing systems, the order to calculate them has the following structure (see Figure 7).

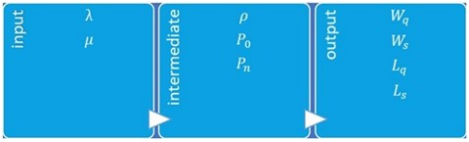

Fig. 7 Order of Parameter Calculation. Source: Own representation according to Srinivasan (2010) [11].

Using the input parameters λ and μ, the values of ρ und P_0 can be calculated. In order to answer the addressed question about the waiting time (pure waiting time W_q, time spent in the system W_s) and the number of waiting customers/elements in the whole system (L_q, L_s), the output parameters W_q, W_s, L_q and L_s are determined. In order to do so, it is important to define the queue according to the Kendall notation first, as the different variations of queues require different calculations / formulas.

Through using the distribution of inter-arrival time and the distribution of service time, calculations are possible without knowing the exact inter-arrival time or service time in each case [2]. The formulas are shown in Table4 and 5 [1], [5].

Depending on the characteristics of a given queue, different formulas have to be used to calculate the output parameters. The following formulas are used when analyzing a M/M/1 queue:

Looking back at the trade-off between cost of service and cost of waiting, these parameters can be used to calculate the average cost of waiting and cost of service. Setting them in relation to each other, you can determine the optimal number of servers based on this cost structure [5]. The optimal number of servers is given when the total expected costs (value of lost time waiting respectively cost of waiting + salary respectively cost of service) are the lowest.

In Table 6, the use of two serves (here: teams) neither has the lowest cost of waiting nor the lowest cost of service, but has the lowest cost in total, meaning that two servers are optimal.

Table 4 Parameters Part 1

Rate of arrivals	
$\lambda = \frac{1}{E(t_a)}$	t_a = Arrival time (randomly) $E(t_a)$ = Expectancy value
Rate of services	
$\mu = \frac{1}{E(t_b)}$	t_b = Serving time (randomly) $E(t_b)$ = Expectancy value
Rate of utilization	
$\rho = \frac{\lambda}{\mu}$ $\rho_0 = 1 - p$	Percentage of time a server is: being utilized by a customer (ρ) or not being utilized by a customer (ρ_0)
Number of servers	
$c = \frac{\lambda}{\rho * \mu_{\text{per server}}}$	

Source: Own representation according to Shortle et al. (2018) [6].

Table 5 Parameters Part 2

Length of queue	
$L_q = \frac{\rho^2}{1-\rho}$	L_q = average number of elements in the queue
Length of the system	
$L_s = \frac{\lambda}{\mu - \lambda}$	L_s = average number of elements the whole queue system
Waiting time in the queue	
$W_q = \frac{L_q}{\lambda}$	
Waiting time in the system	
$W_s = \frac{1}{\mu - \lambda}$	

Source: Own representation according to Shortle et al. (2018) [6].

Table 6 Decision-Making Based on Combined Queue Cost

Number of Teams working	**1**	**2**	**3**	**4**
Average number of ships arriving per shift	5	5	5	5
Average time each ship waits to be unloaded (hours)	7	4	3	2
Total ship hours lost per shift	35	20	15	10
Estimated cost per hour of idle ship time	$ 1,000	$ 1,000	$ 1,000	$ 1,000
Value of ship's lost time or waiting cost	$ 35,000	$ 20,000	$ 15,000	$ 10,000
Team salary (or service cost)	$ 6,000	$ 12,000	$ 18,000	$ 24,000
Total expected cost	$ 41,000	**$ 32,000**	$ 33,000	$ 34,000

Source: Own representation according to Render et al. (2012) [4].

4 Example

To demonstrate the utilization of the parameters explained in the chapters before, they are used to determine given indicators.

Case Study:

"At the counter of a bank, a new customer requests a withdrawal every 2.5 minutes (on average). The employee needs 2 minutes (on average) to distribute the cash."[2]

Task - Part I:

"Determine the following indicators and interpret them:

a) Rate of arrival
b) Rate of service
c) Rate of utilization
d) Likelihood that there is no queue in the bank
e) Likelihood that there are two or more customers in the bank"

Formulas - Part I:

a) Rate of arrival:

$$\lambda = \frac{1}{E(t_a)}$$

b) Rate of service:

$$\mu = \frac{1}{E(t_b)}$$

c) Rate of utilization:

$$\rho = \frac{\lambda}{\mu}$$

d) Likelihood that there is no queue in the bank:

$$P_0 = 1 - \rho$$

[2] Own representation according to Hedstück (2013) [12].

e) Likelihood for customers in the bank:

$$P_n = p^n(1-\rho)$$

Solution - Part I:

a) Rate of arrival:

$$\lambda = \frac{1}{E(t_a)} = \frac{1}{2.5} = 0.4 \text{ customers per min}$$

b) Rate of service:

$$\mu = \frac{1}{E(t_b)} = \frac{1}{2} = 0.5 \text{ customers per min}$$

c) Rate of utilization:

$$\rho = \frac{\lambda}{\mu} = \frac{0.4}{0.5} \approx 0.8 \approx 80\%$$

d) Likelihood that there is no queue in the bank:

$$P_0 = 1 - \rho = 1 - 0.8 = 0.2 \approx 20\%$$

e) Likelihood that there are two or more customers in the bank:

$$\begin{aligned} P(n>2) &= 1 - P_0 - P_1 \\ P_0 &\approx 0.2 \\ P_1 &\approx 0.8^1(1-0.8) \approx 0.16 \\ P(n>2) &\approx 1 - 0.2 - 0.16 \approx 0.64 \approx 64\% \end{aligned}$$

Interpretation - Part I:

Critical appraisal of the capacity level: Although the utilization rate is 80%, the utilization may differ substantially during a period of time (cf. Figure 8). While there is an under capacity from 8 to 10 o'clock, the utilization at 11 o'clock is more than 120%. Therefore, the utilization rate is only a statement for a period (for example all day) and not for a specific moment (see Figure 8).

Task - Part II:

"Determine the following indicators under the assumption that it is a M|M|1-Queueing system:

a) Average number of customers in the queue
b) Average number of customers in the bank
c) Average waiting time in the queue (in min.)
d) Average time of stay in the bank (in min.)"

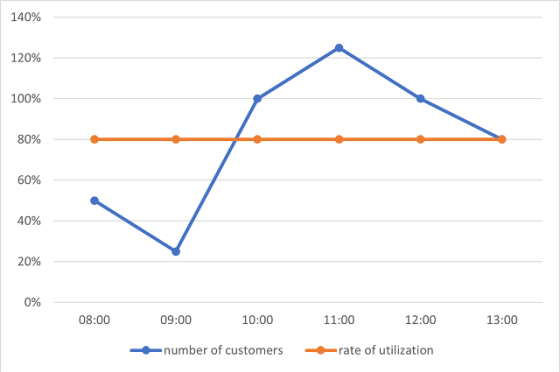

Fig. 8 Graphical Illustration of a Fictive Period. Source: Own representation.

Forumlas - Part II:

a) Average number of customers in the queue:

$$L_q = \frac{\rho^2}{1-\rho}$$

b) Average number of customers in the bank:

$$L_s = \frac{\lambda}{\mu - \lambda}$$

c) Average waiting time in the queue (in min.):

$$W_q = \frac{L_q}{\lambda}$$

d) Average time of stay in the bank (in min.):

$$W_s = \frac{1}{\mu - \lambda}$$

Solution - Part II:

a) Average number of customers in the queue:

$$L_q = \frac{\rho^2}{1-\rho} \approx \frac{0.8^2}{1-0.8} = 3.2 \text{ customers}$$

b) Average number of customers in the bank:

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{0.4}{0.5 - 0.4} = 4 \text{ customers}$$

c) Average waiting time in the queue:

$$W_q = \frac{L_q}{\lambda} \approx \frac{3.2}{0.4} \approx 8 \text{ min.}$$

d) Average time of stay in the bank:

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{0.5 - 0.4} = 10 \text{ min.}$$

5 Exercise

For further practice, the following exercise can be solved. The solution can be found in the appendix.

Case Study:

"Dr. Dent is a dentist who sees his patients in the order in which the patients arrive in his waiting room. In his practice, a new patient arrives every 12 minutes (on average). He needs 10 minutes (on average) for one patient. He is interested in knowing the average number of patients in the practice and in the waiting room.

He asked the receptionist to provide written records about the number of patients over a certain time. He found that the arrivals of his patients are approximately exponentially distributed, and the treatment times can also be approximately described by an exponential distribution. He decides to use an M|M|1 model since it assumes that patients who arrive to wait will stay standing until they can sit down when all seats are occupied in the waiting room" [3].

Questions:

1. Calculate λ, μ and ρ.
2. Calculate the average rate of patients in the practice (system), plus the average rate of the patients in the queue.
3. Calculate the average rate of a waiting patient until he is seen, plus the average rate of the total time of stay in the practice (in minutes).
4. What is the likelihood that the patient does not have to wait for treatment after he arrives?

Exercise Solution:

1.

$$\lambda = \frac{1}{E(t_a)} = \frac{1}{12} \approx 0.083334 \text{ patients per min.}$$
$$\mu = \frac{1}{E(t_b)} = \frac{1}{10} = 0.1 \text{ patients per min.}$$
$$\rho = \frac{\lambda}{\mu} = \frac{0.083334}{0.1} \approx 0.83334 \approx 83\%$$

2.

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{0.08334}{0.1 - 0.08334} \approx 4.8824 \approx 5 \text{ patients}$$
$$L_q = \frac{\rho^2}{1 - \rho} = \frac{0.8334}{1 - 0.8334} \approx 4.1667 \approx 4 \text{ patients}$$

3.

$$W_q = \frac{L_q}{\lambda} = \frac{4.1667}{0.08334} \approx 50.0012 \approx 50 \text{ min.}$$
$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{0.1 - 0.08334} \approx 59.9880 \approx 60 \text{ min.}$$

4. Likelihood that a patient does not have to wait:

$$P_0 = 1 - \rho = 1 - 0.8334 \approx 0.1667 \approx 17\%$$

6 Summary

The queueing theory is a mathematical approach to understanding the emergence of the queues in processing orders, which are part of a company's daily tasks and occur in every economic sector. The whole model is called the queueing system and consists of the arrival, queue, server and the departure. To describe such systems, we learned about the Kendall notation, which is the most known mathematical approach. With the Kendall notation, we can model a queueing system with different distributions of arrival and service time, different numbers of servers or a variety of queueing disciplines, and so on. Using the given formulas, we can calculate, for example, the average number of customers in the queue or the average waiting time in the queue. The outcomes are of incalculable value for any company. There are certainly numerous programs and tools that can simplify the calculation of the queueing parameters.

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Sequencing Problems

Thomas Neifer & Franz W. Peren*

Abstract Sequencing problems, which involve determining the optimal sequence of process steps, are critical in manufacturing, service, and distribution. These problems aim to achieve uniform resource utilization and minimize machine setup costs. Solutions can be approached using mathematical optimization methods for exact solutions or heuristic methods for good but possibly non-optimal solutions. In this work, sequencing problems are presented using examples such as the traveling salesman problem and the flow shop problem.

1 Introduction

The operational problem of sequencing describes the decision on the best possible sequence of process steps, primarily in the area of manufacturing production orders. Other affected areas are found in service as well as in sales. Basically, the sequencing problem describes the problem that a production order is processed in an unfavorable way according to its arrival, since this usually results in an uneven utilization of personnel and machines. Instead, the aim is to achieve this uniform utilization or to minimize the setup costs of machines [1].

The solution of sequencing problems can be done on two different levels: While the mathematical optimization methods represent mathematically exact (optimal) solutions, the heuristic methods are dedicated to the determination of a good, but possibly not optimal, solution [2].

In recent years, car sequencing in particular has attracted a lot of attention. Among other things, it emerged in the context of the so-called ROADEF challenge and attempted to solve a special problem of the automobile manufacturer Renault.

* The following article is based in part on two essays which have been written by the students Hannah Bodemann, and Sarah Jungheim in the winter term 2019/2020 [15].

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Compared to classical solution methods, such as constraint programming, powerful meta-heuristics and exact branch and bound methods were used [3].

As an introduction to the solution of sequencing problems, this paper focuses on the example solution of the Traveling Salesman Problem as well as the Flow Shop Problem.

2 Definition of Terms

For general understanding, the terminology of the sequencing problem shall be introduced at the beginning. Therefore, the definition of the terms sequencing, scheduling, sequencing planning as well as a classification of the relevance for companies follows.

Sequencing: Sequencing refers to the order in which orders are processed. In most cases, the allocation within the framework of this sequence is based on priorities to be specified by the company [2].

Scheduling: Scheduling aims at maximizing the efficiency of a company and minimizing costs. To this end, work processes and their workloads are arranged, controlled and optimized by efficiently planning and allocating machine and plant resources, personnel planning, production processes and material procurement [4], [5].

Increasing competitive pressure in an increasingly complex and dynamic market environment means that companies need to demonstrate a unique selling proposition in order to position themselves optimally in the market. Particularly in the area of production and logistics processes, scheduling ensures not only that costs are reduced, but also that it is possible to respond better to customer needs. Scheduling is of crucial importance, as setup times can account for up to 80% of cycle times [6].

Scheduling is part of time and capacity planning, the task of which is to coordinate and schedule the available capacities with regard to the due date of the in-coming order. Start and finish dates must be calculated in order to meet production deadlines and lead times. This requires the creation of routings. These plans contain information on the time, tooling, personnel and space required to complete the order. In addition, the specifications required for the requested qualities and their due dates must be determined [7].

Sequence Planning: Sequencing allows the optimal determination of order sequences on machines, which ensures that a smooth and timely production process can take place. The company can set individual preferences with regard to the goals to be achieved. Thus, among other things, a maximum degree of activation, minimization of throughput times, and on-time delivery can be defined as goals. However, there is a conflict of objectives between the minimization of lead times and the maximization of capacity utilization, which is expressed in the dilemma of sequencing.

Currently, there is no unified solution model for the sequencing problem. Therefore, one possibility to solve complex sequencing problems is the use of heuristics, which gain a high ranking by priority rules within the implementation of an approx-

imation method. Criteria are defined which determine the order in which the orders are to be created [7].

3 Solving Sequencing Problems

To illustrate the difficulties in solving sequencing problems, the Traveling Salesman Problem and the Flow Shop Scheduling Problem will be used in this paper. Figure 1 shows the possible solutions of sequencing problems.

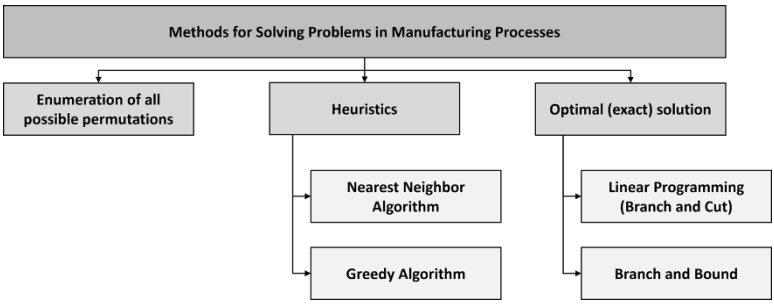

Fig. 1 Methods for Solving Problems in Manufacturing Processes. Source: Own representation according to Domschke (2015) [2].

3.1 Traveling Salesman Problem

The origin of the Salesman Problem is unclear. It is already mentioned in a German handbook from 1832. However, the problem is not discussed in mathematical details [8].

The Traveling Salesman Problem is a combined optimization; the problem itself is very complex. Therefore, many heuristic and exact methods have been developed to solve the problem.

In part, it is a problem of the whole practical vehicle routing problem. For better understanding, we focus on the symmetric Traveling Salesman Problem, i.e., undirected graphs. There are several methods of operational research to solve the Traveling Salesman Problem.

3.1.1 Heuristics

In the context of heuristics, a differentiation can be made between the heuristic opening procedure and the optimization procedure. The best results can be achieved with the genetic algorithm [2].

Deterministic heuristic opening procedures:

The Nearest Neighbor Algorithm belongs to the class of heuristic opening methods and is used, among other things, to solve the Traveling Salesman Problem. The course is essentially determined by the definition of the starting point, since at the beginning those nodes are included in the tour, which have the smallest distance to the starting point. It belongs to the greedy algorithms, because in the course of the process the best method is selected. This method is suitable only for a few cases of the Traveling Salesman Problem. Therefore, in some cases, the method leads to a rather worse solution.

Under the successive inclusion method, a circle with two random nodes is designed. At each repetition, a new node is added to the tour. The new nodes are added in such a way that the circle itself does not increase in size. The nodes with the largest distances are selected. From the starting point, one tries to reach the next node by the shortest path. The process is repeated from node to node until you reach the furthest node that was previously selected. It is important to note that the shortest distance can only be searched for the node that is not yet on the route [2].

Deterministic optimization methods:

Two subordinated methods can be distinguished. On the one hand the r-optimum method and on the other hand the 2-optimum method. In the context of the r-optimum one tries to improve a solution by changing two knots, which are located externally (see Figure 2) [2].

The 2-optimum method reviews improvement opportunities by exchanging two knots against two other ones (see Figure 3). There will only be an exchange if the length of the tour will be shortened through this procedure. The exchange, however, only takes place until every exchanging method has been reviewed [2].

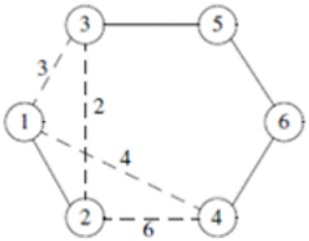

Fig. 2 Best Follower According to the r-Optimum Method. Source: Own representation according to Domschke (2015) [2].

Stochastic Methods:

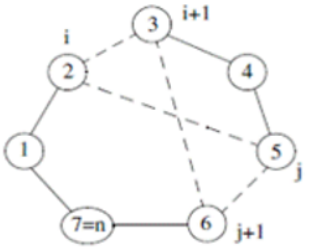

Fig. 3 Review of Exchanging Possibilities (2-Optimum Method). Source: Own representation according to Domschke (2015) [2].

At the beginning a selective optimization method will be executed, improvements will not be implemented immediately. Depending which extent, the worsening will be, it will be accepted. The decision this is made stochastically regarding the probabilities. The probability of decision is reduced in ongoing procedures [2].

3.1.2 Exact Solution

There are two methods which will be described here. For a symmetrical view of the problem, the Lagrange Relaxation is the method to choose and for the asymmetrical view of the problem the branch and bound method is applied [2].

Lagrange Relaxation:

The Lagrange relaxation of the problem is reached by the cancellation of particular constraints and a cost addition subsequently to the target function. This "new" target function is called Lagrange multiplicator or node weight. As soon as the shortest tour is getting bigger or the same as the new target function, this is called the lower bound. The upper bound is illustrated by the length of the shortest bound [2].

Ascent Method:

The ascent method is the method used for determination of the node weight. Furthermore the method provides lower bounds. When it comes to the Lagrange problem, this is really significant. As soon as the fitting node weight is found and the graph is depicting the minimal traveling tour, the solution of the problem is there. The ascent method is an iterative process solving nonlinear optimization problems. As the ascent method does not leads to a traveling tour, the solution for the problem is not necessarily granted. If this is the case, the lower bound is smaller as the upper one [2].

Branch and Bound Method:

This method is applied to solve a symmetrical traveling salesman problem. Here, no traveling tour is required, the node weight, however, has to be bigger or at least two. With the aid of the branch and bound method, problems are chosen from of a list of candidates with the help of the last-in-first-out (LIFO) method. A prohibited border is a variable, not taking the value of 0 or 1 [2].

The branch and bound method consists of the following components:

1. Upper Bound: As soon as heuristic methods are used, a traveling tour is tried to be built the length of the tour is the upper bound. A further opportunity is not to build a traveling tour.
2. Determination of the lower bound: As for the second component, the lower bound will be determined by using the ascent method. In order to establish a branch of the nodes, if necessary, prohibited bounds are not to be included.
3. Solve the problem: The problem becomes apparent if the lower bound is bigger than the upper bound, as described within the ascent methods. It appears additionally through logical tests. These logical tests are to be executed before including them in the candidate list. A result can be an increase in borders or, as desired, the solution of the problem.
4. Selection of problems: Not every problem is detected as described above. Then, the candidate list will be reviewed and the LIFO-rule is applied as mentioned above. The problem last listed will be reviewed first. It is reviewed if the lower bound is bigger than the lower one. If not, the problem can be deleted from the candidate list.
5. Branching problem: The problem is split in two or three partial problems in order to solve the problem [2].

3.1.3 Case Study

The following business case should illustrate the problem and show how the Traveling Salesman Problem can be solved. For convenience, the distances between cities will be rounded to 50 km.

Initial Situation:

- Visit $n - 1$ cities directly one after another
- Every city needs to be visited once (see Figure 4)
- Use shortest distance between the cities (see Table 1)
- Starting point is Bonn
- Come back to Bonn without visiting a city twice

Table 1 Distances of Cities

Distance [km]	j = Bonn (Bo)	Kiel (K)	Berlin (B)	Munich (M)
i = Bonn (Bo)	∞	500	600	550
Kiel (K)	500	∞	400	850
Berlin (B)	600	400	∞	550
Munich (M)	550	850	550	∞

Source: Own representation according to Domschke et al. (2015) [2].

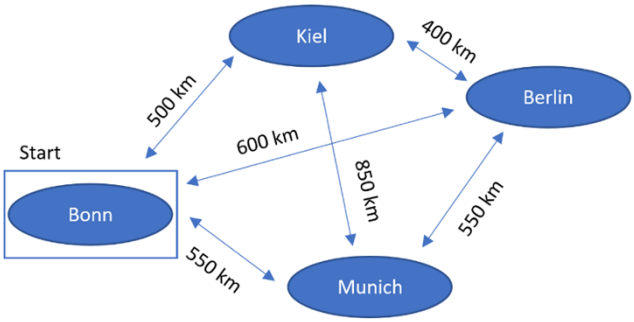

Fig. 4 Initial Situation of the Assigned Nodes. Source: Own representation according to Domschke (2015) [2].

Possible approach 1: Calculate all possible routes

Step 1: How many routes are possible?

Solution: $(n-1)! = 3! = 1 * 2 * 3 = 6$ ways

Step 2: Calculation of the shortest ways (see Table 2)

Table 2 Distances of cities

possible route	Bo, K, B, M, Bo	= 500 + 400 + 550 + 550	= 2, 000
possible route	Bo, K, B, M, Bo	= 550 + 850 + 550 + 600	= 2, 550
possible route	Bo, K, B, M, Bo	= 550 + 550 + 400 + 500	= 2, 000
possible route	Bo, K, B, M, Bo	= 550 + 850 + 400 + 600	= 2, 400
possible route	Bo, K, B, M, Bo	= 600 + 400 + 850 + 550	= 2, 400
possible route	Bo, K, B, M, Bo	= 600 + 550 + 850 + 550	= 2, 550

Source: Own research, following Domschke et al. (2015) [2].

Possible approach 2: Nearest Neighbor Algorithm

This intuitive algorithm can be used through the iterative visit of the last neighbor, the last point which is added to the journey by the peaks of all categories which are not visited yet, based on the starting point Bonn (see Table 3 and Figure 5).

Table 3 Coloured Highlighting of the Optimum Travel Times to the Cities.

Distance [km]	j = Bonn (Bo)	Kiel (K)	Berlin (B)	Munich (M)
i = Bonn (Bo)	∞	500	600	550
Kiel (K)	500	∞	400	850
Berlin (B)	600	400	∞	550
Munich (M)	550	850	550	∞

Source: Own research, following Domschke et al. (2015) [2].

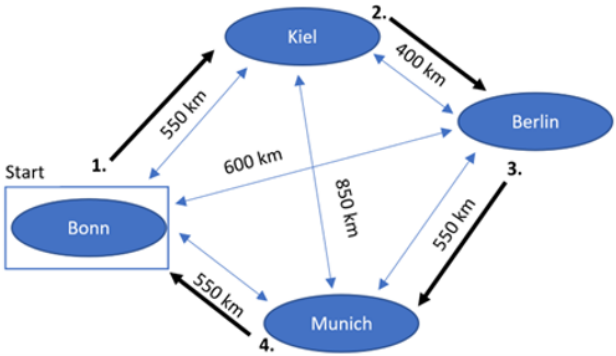

Fig. 5 Changed Routes. Source: Own representation according to Domschke (2015) [2].

Result: Route [Bo, K, N, M, Bo] = 500 + 400 + 550 + 550 = 2000

Comparison of the two approaches:

The differences between the calculation of all possible routes and the Nearest Neighbor Algorithm can be found in Table 4.

Table 4 Comparison of the Solution Possibilities.

	1. **Calculate all permutation**	2. **Nearest Neighbor Algorithm**
Methodology	All possible routes are calculated consecutively.	The routes are sorted by the shortest distances.
Solutions	2 routes, both 2, 000 km.	1 route, 2, 000 km.
Calculations	$n - 1!$	1
Pros	All possibilities are considered, and the optimum will be found.	Easy and fast solution.
Cons	Long calcualtion time.	Overlooking important alternatives or local optimum.

Source: Own research, following Domschke et al. (2015) [2].

3.2 Flow Shop Scheduling Problem

Flow Shop scheduling is part of the product planning and the machine capacity planning process. Flow Shop scheduling is a special case of the Job Shop planning in which all processes are strictly arranged [9]. The following chapter focuses on the production process and on the differences between Flow Shop and Job Shop scheduling.

3.2.1 Terms and Definitions of Flow Shop and Job Shop

Production sequencing is the main topic of production planning and production controlling. It helps to find the optimal sequence for processing n orders on m machines and enables to establish a plan for the machine capacity utilization. By scheduling, companies can shorten the cycle times of each order on all machines. Thus, they can shorten the total production time. Another objective of production scheduling is the maximal capacity utilization to minimize the time of machines being unused as well as underutilized [10].

This approach is also supported from the customer's perspective because by reaching a maximum in capacity utilization, it is far likely to deliver incoming orders in time. Therefore, scheduling contributes to better customer-supplier relationships [10].

There are two production types that must be distinguished in the context of solving scheduling problems – flow shop and job shop (see Table 5).

Table 5 Characteristics of Flow Shop and Job Shop

Characteristics	Flow Shop	Job Shop
Principle	Product oriented	Operation oriented
Application	Mass production	Small batch production
Workflow	Strict order for operations (for all jobs)	Jobs may have different order for operations
Processing time	Processing time of each operation is known and cannot be interrupted	
Overtaking orders	Not possible	Possible

Source: Own research, following Zimmermann and Stache (2001) [11].

Flow shop is a continuous manufacturing process in which all orders are running through the same sequence of machining steps. The sequence of operations is the same on every working station. Flow shop is product oriented and suitable for mass production. Regarding the flow shop operating activities, overtaking orders is not possible [11].

Job shop is a discontinuous and operation-oriented manufacturing process in which – compared to flow shop – the sequence of working stations is not necessarily identical. This means that every operation is assigned with an own sequence.

Often, several orders are waiting in line for a specific operation at a specific machine. Job shop production is flexible and does not operate the "first come, first serve" principle. Therefore, it is also possible to overtake orders and manage orders in different sequences. The sequence of one order is not fixed in the beginning of the production; it will be redefined after every station. This flexibility is not suitable for mass production; therefore, job shop is commonly used for small batch productions [11].

Figure 6 visualizes the different sequences of flow shop (production in a row) and job shop (production with more flexibility and different sequences).

Scheduling aims to determine the optimum machine utilization. Regarding the cycle times of every order, the purpose of scheduling is to find the optimal sequence

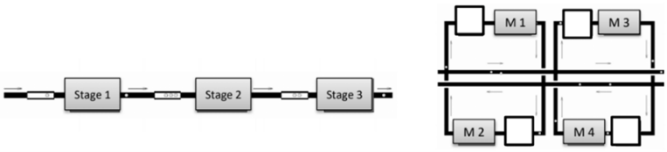

Fig. 6 Comparison of Flow and Job Shop. Source: Own representation according to Zimmermann and Stache (2001) [11].

of order production. In order to avoid the enumeration of each possible combination of n machine sequences, there are some suitable calculation methods for solving the problem of sequencing problems (see Figure 7).

In 1954, Johnson established "Johnson's rule", a method for solving the sequencing problem in the context of flow shop [12]. In the first instance, he developed a method for solving the sequencing problem for two machine production. Subsequently, he extended his solution to make it also applicable for productions with three machines.

Researchers have searched for similar solutions in solving the sequencing problems for job shop. The heuristic methods are either using reasonable priority rules or they try a certain amount of maximum possible combinations of machine utilization and generate a machine production plan after the best results gained.

Heuristic methods are a systematic search technique that quickly leads to a good approximation) [12].

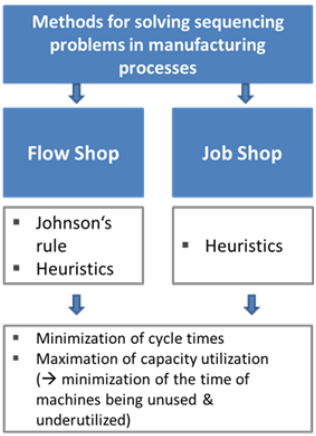

Fig. 7 Methods for Solving Problems in Manufacturing Processes. Source: Own representation according to Vahrenkamp (2008) [12].

3.2.2 Premises Using Johnson's Rule

For using Johnson's rule, there are some premises to consider. These premises can be differentiated into order-related, machine- and sequence-related [13].

1. Order-related premises: The first premise is that there must be a constant order volume so that the scheduling can be planned and controlled. Furthermore, the processing times must be known, constant and should include setup times (times when working systems are prepared and aligned so that they can process orders effectively). In Johnson´s rule transport times are not part of the scheduling planning.
2. Machine-related premises: In the context of Johnson's rule, the number of machines is known, whereas machine breakdowns are out of the calculation model. Orders are processed immediately. In Johnson's rule orders must not be separated, every order must be processed as a whole. Orders are processed one after another, the first order goes through a specific machine, then the next order and so on. Johnson's rule applies the FIFO (First in – First out) principle.
3. Sequence processing premises: Johnson's rule is only used for flow shop, it does not apply to job shop. For job shop, one must use specific heuristic methods – as mentioned above. In addition, Johnson's rule prohibits overtaking orders [13].

3.2.3 Case Study

The business case consists of two different scenarios using Johnson's rule.

The following points form the basis in both scenarios.

- The order O_j $(j = 1, 2, \ldots, 6)$ consists of 6 different product variations
- The production times t_{ij} are known
- There are no sequencing dependencies between the product variants
- The products are manufactured on a two-stage production process
- on the machines M_i $(i = 1, 2)$
- Each product must first pass M_1

The goal is to minimize the cycle time by using Johnson's rule to find an optimal sequence for all available orders.

The first scenario discusses a proposed solution for optimizing a time cycle of two machines using this rule (see Table 6).

There are 6 different product variants with a deposited time and different lengths.

The following three steps illustrate the solution to the sequencing problem using Johnson's rule. The first step determines the smallest element of all t_{ij}.

In Table 7, order 6 (O6) is the smallest element with a duration of 1 on machine 2. The second step is to arrange the orders in a new sequencing process. If the smallest element of all t_{ij} is an element of the processing time of machine 1, the order is placed at the first position of the new sequencing process. If the smallest element of all t_{ij} is an element of the processing time of machine 2, the order is placed at the

Table 6 Production Time of Two Machines with Different Product Variants

	Order O_j						
Machine M_i	O_1	O_2	O_3	O_4	O_5	O_6	Duration
Machine 1	16	5	4	14	12	5	56
Machine 2	6	12	7	9	15	1	50
Total	22	17	11	23	27	6	106

Source: Own research, following Zimmermann (1977) [14].

Table 7 Order with the Shortest Duration

	Order O_j						
Machine M_i	O_1	O_2	O_3	O_4	O_5	O6	Duration
Machine 1	16	5	4	14	12	5	56
Machine 2	6	12	7	9	15	1	50
Total	22	17	11	23	27	6	106

Source: Own research, following Zimmermann (1977) [14].

last position of the new sequencing process. If the smallest element of all t_{ij} are the same on machine 1 and machine 2, both orders can be arranged arbitrarily. If the currently smallest element of all t_{ij} is chosen, it can be canceled from the catalog. In this case, O6 with a duration of 1 is scheduled on machine 2. As this order is already placed in the last position, nothing changes at first (see Table 8).

Table 8 Start of the New Production Sequence

		Production sequence					
		1	2	3	4	5	6
Flow Step	1						**O_6**
	2						
	3						
	4						
	5						
	6						

Source: Own research, following Zimmermann (1977) [14].

After that, the next smaller order is considered and O6 will no longer be considered.

The next smaller element of all t_{ij} is O3 with a duration of 4. As this order is placed on machine 1, it is now placed at the first position of the new sequencing process (see Table 9).

The next order that is considered is O2 with a duration of five which is again an element of the processing time on machine 1. Therefore, O2 is placed at the second position of the new sequencing process (see Table 10).

Table 9 Flow Step Two of the New Production Sequence by Using Johnson's Rule

		Production sequence					
		1	2	3	4	5	6
Flow Step	1						**O_6**
	2	**O_3**					
	3						
	4						
	5						
	6						

Source: Own research, following Zimmermann (1977) [14].

Table 10 Flow Step Three of the New Production Sequence by Using Johnson's Rule

		Production sequence					
		1	2	3	4	5	6
Flow Step	1						**O_6**
	2	**O_3**					
	3		**O_2**				
	4						
	5						
	6						

Source: Own research, following Zimmermann (1977) [14].

After repeating this procedure for all orders, Table 11 shows the finished new production sequence.

Table 11 End of the New Production Sequence by Using Johnson's Rule

		Production sequence					
		1	2	3	4	5	6
Flow Step	1						**O_6**
	2	**O_3**					
	3		**O_2**				
	4					**O_1**	
	5				**O_4**		
	6			**O_5**			

Source: Own research, following Zimmermann (1977) [14].

O5 has been considered in the last flow step as it is the largest element of all t_{ij}.

Table 12 illustrates the new sequence of order processing after using Johnson's rule:

To figure out the savings after using Johnson's rule, there needs to be a comparison between the original sequence and the new production sequence. The following Table 13 depicts the development of start and end time of both machines relating to the orders.

The original production sequence has a cycle time of 66-time units and a waiting time of 16-time units. In comparison to this, the new production sequence has a cycle time of 58-time units and a waiting time of 4-time units.

Table 12 New Sequence of Order Processing after Using Johnson's Rule (Two Machines)

	Order O_j						
Machine M_i	O_3	O_2	O_5	O_4	O_1	O_6	Duration
Machine 1	4	5	12	14	16	5	56
Machine 2	7	12	15	9	6	1	50
Total	11	17	27	23	22	6	106

Source: Own research, following Zimmermann (1977) [14].

Table 13 Original and New Production Sequence Cycle Time (Two Machines)

Original production sequence							
	Order O_j						
Machine M_i	O_1	O_2	O_3	O_4	O_5	O_6	Duration
Machine 1	16	5	4	14	12	5	56
Machine 2	6	12	7	9	15	1	50
Machine 1 (start)	0	16	21	25	39	51	
Machine 1 (end)	16	21	25	39	51	56	
Machine 2 (start)	16	22	34	41	50	65	
Machine 2 (end)	22	34	41	50	65	66	
New production sequence							
	Order O_j						
Machine M_i	O_1	O_2	O_3	O_4	O_5	O_6	Duration
Machine 1	4	5	12	14	16	5	56
Machine 2	7	12	15	9	6	1	50
Machine 1 (start)	0	4	9	21	35	51	
Machine 1 (end)	4	9	21	35	51	56	
Machine 2 (start)	4	11	23	38	51	57	
Machine 2 (end)	11	23	38	47	57	58	

Source: Own research, following Zimmermann (1977) [14].

As a result, the cycle time could be reduced by 8-time units and the waiting time could be reduced by 12-time units by using Johnson's rule.

The second scenario contains a suggested solution for optimizing a time cycle of three machines. It is like a flow shop with two machines and contains further restrictive terms. There are two different restrictive terms (one of these restrictive terms must be applicable):

- The minimum processing time of machine 1 min (t_{1j}) must be at least as long as the maximum processing time of machine 2 max (t_{2j})
- $\min(t_{1j}) \geq \max(t_{2j})$

- The minimum processing time of machine 3 min (t_{3j}) must be at least as long as the maximum processing time of machine 2 max (t_{2j})
- $\min(t_{3j}) \geq \max(t_{2j})$

The following points form the basis in the second scenario:

- The order O_j ($j = 1, 2, \ldots, 5$) consists of 5 different product variants
- The production times t_{ij} are known
- There are no sequencing dependencies between the different product variants

The order consists of 5 different product variants with a deposited time and different lengths (see Table 14).

Table 14 Cycle Time of Three Machines with Different Product Variants

	Order O_j					
Machine M_i	O_1	O_2	O_3	O_4	O_5	Duration
Machine 1	6	10	7	6	5	34
Machine 2	5	4	2	4	2	17
Machine 3	3	11	6	7	2	39

$min(i_{1j}) \geq max(t_{2j})$ $M_1\, min = 5 \geq M_2\, max = 5$ ✓

$min(i_{3j}) \geq max(t_{2j})$ $M_3\, min = 2 \geq M_2\, max = 5$

Source: Own research, following Zimmermann (1977) [14].

The first step for minimizing the cycle time is to transform the three-stage production into a two-stage production. The cycle time of machine 1 and 2 as well as the cycle time of machine 2 and 3 are added. The following Table 15 illustrates this step.

Table 15 Cycle Time of Three Machines with Different Product Variants

			Order O_j					
Machine	M_i		O_1	O_2	O_3	O_4	O_5	Duration
M_1	⊕	Machine 1	6	10	7	6	5	34
M_2		Machine 2	5	4	2	4	2	17
M_2	⊕	Machine 2	5	4	2	4	2	17
M_3		Machine 3	3	11	6	7	2	29

		Order O_j					
Machine M_i		O_1	O_2	O_3	O_4	O_5	Duration
M_1	Machine 1	11	14	9	10	7	51
M_2	Machine 2	8	15	8	11	4	46
	Cycle Time	19	29	17	21	11	97

Source: Own research, following Zimmermann (1977) [14].

The next step is to determine the minimum order and enter the results into a new order matrix. O5 has the shortest cycle time with a duration of 11-time units, so it is put in the first position of the new ranking. Ranking of the new order processing:

- 1: O5 with 11 time units
- 2: O3 with 17 time units
- 3: O1 with 19 time units
- 4: O4 with 21 time units
- 5: O2 with 29 time units

In terms of ranking, the new processing order after using Johnson's rule can be seen in Table 16.

Table 16 New Sequence of Order Processing

		Production sequence				
		1	2	3	4	5
Flow Step	1					O_5
	2				O_3	
	3			O_1		
	4	O_4				
	5		O_2			

Source: Own research, following Zimmermann (1977) [14].

The next step is to establish the new solution and to compare it to the original flow step. The following Table 17 illustrates the duration of both sequences before and after using Johnson's rule.

Table 17 Previous and New Production Sequence Cycle Time

Production Sequence	Duration M_1	Duration M_2	Duration M_3
1	08:00 – 08:06	08:06 – 08:11	08:11 – 08:14
2	08:06 – 08:16	08:16 – 08:20	08:20 – 08:31
3	08:16 – 08:23	08:23 – 08:25	08:31 – 08:37
4	08:23 – 08:29	08:29 – 08:33	08:37 – 08:44
5	08:29 – 08:34	08:34 – 08:36	08:44 – 08:46
Production time in total			**46**
Production Sequence	Duration M_1	Duration M_2	Duration M_3
1	08:00 – 08:06	08:06 – 08:10	08:10 – 08:17
2	08:06 – 08:16	08:16 – 08:20	08:20 – 08:31
3	08:16 – 08:22	08:22 – 08:27	08:31 – 08:34
4	08:22 – 08:29	08:29 – 08:31	08:34 – 08:40
5	08:29 – 08:34	08:34 – 08:36	08:40 – 08:42
Production time in total Reduction			**42** **4**

Source: Own research, following Zimmermann (1977) [14].

The original one has a cycle time of about 46-time units and the new sequence has a cycle time of about 42-time units. In summary, the reduction time is set to 4-time units.

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Regression Analysis Using Dummy Variables

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Abstract Regression analysis is a versatile method for the analysis and description of business problems. It is based on the development of a model that allows a forecast into the future using historical data. Dummy coding is required when categorically independent variables are to be included in a multiple regression analysis. In the business context, for example, different customer groups can be differentiated based on categorical characteristics.

1 Introduction

Regression analysis is used to formalize the linear relationship between two or more characteristics using a regression function. In the context of OLS (Ordinary Least Squares) estimation, it assumes that all relevant influencing variables are considered to determine the optimal regression function. Otherwise, there is a misspecified regression function, whereby reality is distortedly modelled and wrong conclusions are drawn from it [1]–[4]. Due to their non-metric characteristics, however, categorical variables cannot easily be integrated into the regression function as explanatory variables. Nevertheless, they play an important role in many social science questions, so that a regression model that meets the requirements of OLS estimation can only be estimated by considering these variables. It should be noted that both the regressor and the regressand of the function can be nominally scaled [5]. If the dependent variable is of a categorical nature, the instruments of logistic regression analysis are used. However, if the independent variables are affected, dummy variables are used.

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2 Methodology

2.1 Derivation of the Simple and Multiple Regression Function

The classic simple regression function is:

$$\hat{Y} = b_0 + b_1 X_i \tag{1}$$

with:
$\hat{Y}$: regression function
b_0: level parameter
b_1: gradient parameter
X_i: $(i = 1, ..., n)$ values of X

If this is considered under multiple influencing variables, it becomes:

$$\hat{Y} = b_0 + b_1 X_i + ... + b_j X_j \tag{2}$$

with:
b_j: $(j = 1, ..., J)$ regression coefficients
J: number of independent variables

By including the residuals (e) one obtains:

$$Y = b_0 + b_1 X_1 + e \tag{3}$$

or in the case of multiple regression:

$$Y = b_0 + b_1 X_1 + ... + b_j X_j + e \tag{4}$$

This can also be expressed in matrix form to simplify the later derivation of the regression parameters for k different observations [6]:

$$Y = Xb + e \tag{5}$$

with:
Y: k-vector of the observed values of the dependent variable $(k = 1, 2, \ldots, K)$
X: $(k(j + 1))$-Matrix with the observed values of the regressors
b: $(j + 1)$-Vector of the regression coefficients including the level parameter
e: k-Vector of residuals
K: number of observations

2.2 Estimation of the Regression Parameters according to the Least Squares Method

2.2.1 Simple Regression

The regression coefficients are determined according to the least squares method (OLS) by minimizing the sum of squared residuals (SSR) between actual values and values estimated by the regression function as follows [6]:

$$SSR = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} e^2 \rightarrow min! \tag{6}$$

The minimum of the SSR is realized by the partial derivation according to both regression parameters and their respective equation with zero.

$$\frac{\partial SSR}{\partial b_0} = -2 \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i) = 0 \rightarrow \sum_{i=1}^{n} e_i = 0 \tag{7}$$

From the partial derivative after b_0 and its zeroing follows the first normal equation of the OLS, which shows the requirement fulfilled by the optimization of the summation of the residuals over all observed values to zero. Accordingly, the regression line is laid through the point cloud in such a way that the positive and negative deviations between estimated and actual values add up to zero [1].

$$\frac{\partial SSR}{\partial b_1} = -2 \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i) X_i = 0 \rightarrow \sum_{i=1}^{n} e_i X_i = 0 \tag{8}$$

Furthermore, the second normal equation of the OLS results from the partial differentiation according to the parameter b_1, which describes that the residuals weighted with the characteristic values also add up to zero [1]. By transforming the two normal equations, the regression parameters can be determined as [6]:

$$b_0 = \frac{\sum_{i=1}^{n} X_i^2 \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i Y_i}{n \sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2} \tag{9}$$

$$b_1 = \frac{n \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n \sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2} \tag{10}$$

2.2.2 Multiple Regression

Based on optimization within the framework of simple regression analysis, the target criterion is multiple regression:

$$SSR = \sum_{i=1}^{n} e_k^2 = \sum_{i=1}^{n} (Y_k - (b_0 + b_1 X_{1k} + ... + b_j X_{jk} + ... + b_j X_J k))^2 \rightarrow min! \quad (11)$$

Considering the matrix notation, this can also be formulated as [6]:

$$SSR = e'e = (Y - Xb)'(Y - Xb) \rightarrow min! \quad (12)$$

The regression parameters b can be estimated by partial differentiation:

$$b = (X'X)^{-1} X'Y \quad (13)$$

Since $X'X$ represents a square matrix with $J + 1$ rows or columns, its size is independent of the number of observed values K and can therefore also be used here. Furthermore, it can only be inverted if it has the complete rank $J + 1$. This implies that the number of observations K, must be greater than the number of explanatory variables X_j [6]. Furthermore, the independent variables shall not be correlated with each other. This is called multicollinearity and leads to a distortion of the slope parameters of the regression function due to the fact that it is no longer possible to assign the influence to a certain exogenous variable without any doubt [1]. These implications must be considered and are taken up again when the dummy variables are included. After the derivation of the regression parameters has been described, the categorical variables are now integrated into the regression function.

2.3 Integration of Categorial Influences via Dummy Variables

First, it should be noted that in the case of categorical influences, there are two basic ways of including them: On one hand, specific regression functions can be estimated for the respective categorial groupings. This partial consideration, however, results in a certain loss of information, since the respective influence may no longer meet with a sufficiently large dispersion of the characteristic values to be able to formulate this as precisely as possible [1]. On the other hand, dummy variables integrate (δ) categorical influences via a binary coding, where "1" indicates a presence and "0" indicates no presence of the respective property. Thus, these characteristics can also be treated metrically and integrated into the regression function and the following, it applies [7]:

$$\delta = \begin{cases} 1, & \text{property fulfilled} \\ 0, & \text{otherwise} \end{cases}$$

This already shows that several dummy variables must also be used in the case of several values of a nominally scaled characteristic. For k different characteristic values, $k - 1$ dummy variables are therefore required, which is why this technique can only be used for independent variables that are not limited in number [6]. However, the use of dummy variables is limited by the fact that the above mentioned

prerequisites for deriving an optimal regression function stipulate that the number of explanatory variables must be smaller than the number of observations.[1] However, this method has the great advantage that more observation values are available due to the joint estimation in a regression function [1].[2] A regression function with an integrated dummy variable looks like this:

$$\hat{Y} = b_0 + b_1 X_1 + b_2 \delta_1 \tag{14}$$

The reference always represents the "0" value of the dummy variable where the property in question is not fulfilled. If the property is present ($\delta_1 = 1$), in this case, the absolute element increases by b_2, which leads to two different regression functions that differ in the level parameter [1]. Furthermore, dummy variables can be differentiated into level and slope dummies. While level dummies assume that the categorical properties only influence the absolute element of the regression function, slope dummies allow the influence of the categorical variables on the respective slope parameters of the function to be taken into account [8].[3] This is illustrated below:

$$\hat{Y} = b_0 + b_1 X_1 + b_2 \delta_1 + b_3 \delta_2 \tag{15}$$

While the dummy variable δ_1 represents the influence of a nominally scaled characteristic on the absolute element b_0, δ_2 should consider the influence of this characteristic on the slope b_1 [1]. Therefore, the following applies for the ascent dummy δ_2:

$$\delta_2 = \begin{cases} X_1 \delta_1, & \text{property fulfilled} \\ 0, & \text{otherwise} \end{cases}$$

If the categorial property is fulfilled ($\delta_1 = 1; \delta_2 = X_1 \delta_1 = X_1 \cdot 1 = X_1$), this means for the regression function:

$$\hat{Y} = b_0 + b_1 X_1 + b_2 \delta_1 + b_3 \delta_2 = b_0 + b_1 X_1 + b_2 \cdot 1 + b_3 X_1 = b_0 + b_2 + (b_1 + b_3) X_1 \tag{16}$$

Accordingly, both the level parameter $b_0 + b_2$ and the slope parameter $(b_1 + b_3)$ are changed by the existence of the categorical value. If the nominal property is not given ($\delta_1 = 0; \delta_2 = X_1 \delta_1 = X_1 \cdot 0 = 0$), then the regression function is not influenced by it [8].

$$\hat{Y} = b_0 + b_1 X_1 + b_2 \cdot 0 + b_3 \cdot 0 = b_0 + b_1 X_1 \tag{17}$$

---

[1] It is recommended that the number of observed values be twice as high as the number of variables [6].

[2] This also has a positive effect on the number of degrees of freedom and thus the freely variable values, which do not affect a specific parameter (here, for example, the regression coefficient) [1].

[3] However, the correlation between the gradient dummy and the level dummy must be evaluated, especially since the gradient dummy represents a multiple of the level dummy [1].

Once the regression function has been estimated, its quality must be evaluated. This is described below.

2.4 Quality Measures

In principle, the quality check is carried out on two different levels. At the global level, the quality with which the regression function explains the variable Y in the sense that goodness of fit is checked. If the assessment is inadequate, the basic model must be reconsidered. The measures used for the evaluation on a global level represent the coefficient of determination R^2, the F-statistics or the p-value and the standard error S. The quality of the regression coefficients deals with the contribution of the individual parameters to the quality of the explanation of Y. The t-value of a t-test is available as a measure for this [1].

2.4.1 Goodness of Fit (Global Quality Criteria)

Starting from the optimization according to the OLS, all the following global quality criteria consider the SSR. However, their sole consideration is not sufficient, as they are not sufficiently meaningful. It must therefore be set in relation to a reference quantity [6].

Coefficient of Determination: The coefficient of determination considers the ratio of the declared scatter to the total scatter via the decomposition of the scatter into explained sum of squares (SSE), sum of squares residuals (SSR) and total sum of squares (SST) [9]:

$$R^2 = \frac{SSE}{SST} = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2} = \frac{SSE}{SSE + SSR} = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2}{\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2 + \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2} \tag{18}$$

The quotient moves between the two extreme cases $0 \leq R^2 \leq 1$, with:
$R^2 = 0$ in the absence of an explanation of the dispersion, and
$R^2 = 1$ in the event of a full explanation

In the case of multiple regression, the coefficient of determination is determined according to the above matrix representation as follows [7]:

$$R^2 = \frac{SSE}{SST} = \frac{b^T X^T Y - n\overline{Y}^2}{Y^T Y - n\overline{Y}^2} \tag{19}$$

The result of the coefficient of determination indicates how much percent of the variance of Y is explained by the dispersion of the independent characteristics and thus the regression function, while $(1 - R^2)$ remains unexplained [1].

However, the coefficient of determination is not sufficient to assess the quality. Apart from the fact that in the case of only two observation values within the framework of a single regression, a coefficient of determination of 1 always results and such a model cannot be trusted, neither the number of observations nor the number of variables and therefore, the complexity of the model are taken into account [6]. [4] This problem was taken up and solved by the corrected coefficient of determination (R^2_{corr}). In addition to the coefficient of determination, this coefficient has a penalty term, which increases with an increasing number of independent variables and a decreasing number of degrees of freedom and reduces the result of the coefficient of determination according to the formula [1].

$$R^2_{corr} = R^2 - \frac{J(1 - R^2)}{K - J - 1} \tag{20}$$

F-Statistic: In addition to the quality of adaptation of the model to the data on the coefficient of determination, the significance of the model and thus of the coefficient of determination also plays a role due to the sample data available in most cases. This checks whether the model can also be assumed for the underlying population beyond the sample. The starting point here is the assumption that the regression function $\hat{Y}$ estimated from the sample data represents a realization of a function Y present in the population [6]. This "actual" function Y possesses the unknown, "true" regression coefficients $\beta_0, \beta_1, \ldots, \beta_J$ and contains a stochastic (random) disturbance u, which is reflected in the residuals e and on the basis of which the function is also called a stochastic model of regression analysis [1]. As a stochastic model, Y is also regarded as a random variable and thus also the estimated regression parameters resulting from the realizations of Y. Thus, for repeated sampling, it can be assumed that the estimated values b_j vary by the actual parameters β_j. The "actual" regression function of the population is:

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_j X_j + \ldots + \beta_J X_J + u \tag{21}$$

If there is now a relationship between Y and the explanatory variables, then the slope parameters β_j of the population must also differentiate from zero, which is why the following null hypothesis is made [6]:

$$H_0 : \beta_1 = \beta_2 = \ldots = \beta_J = 0 \tag{22}$$

For their verification, an F-test is used which compares the test variable F_e with the critical and the F-distribution originating variable F_c (cf. F-distribution with the respective degrees of freedom present, and in the case of $F_e > F_c$ rejects the

[4] An increased number of explanatory parameters usually provides a better explanation of the actual values, but this does not necessarily have to be accompanied by good estimates. This is due to the decreasing number of degrees of freedom with additional explanatory variables [6].

null hypothesis and thus certifies a significant relationship [10]. The empirical test variable is calculated as:

$$F_e = \frac{\frac{SSE}{J}}{\frac{SSR}{K-J-1}} = \frac{\frac{R^2}{J}}{\frac{1-R^2}{K-J-1}} \tag{23}$$

This size is also based on the SSE. In addition, however, the number of degrees of freedom is put in relation. Whether or not the null hypothesis can be rejected can, however, be answered in a simplified way by means of the p-value which is based on the F-statistics and is available in the common statistics software. The p-value corresponds to the significance level at which the null hypothesis would just be rejected due to the given data situation. It describes the probability that an F-distributed random variable F^*, considering its degrees of freedom, is greater than the empirically available F-value F_e:

$$p\text{-value} = P(F^* > F_e) \tag{24}$$

If the p-value is smaller than the required significance level α, the null hypothesis is rejected and the model premises show a significant correlation [6].

Standard Error of Estimation: It is also possible to use the SSR to calculate the average error caused by the estimation of Y in relation to the arithmetic mean of Y. This quantity is referred to as the standard error of the estimate or disturbance term and is calculated as follows [10]:

$$S = \sqrt{\frac{1}{K-J-1} SSR} \tag{25}$$

The result indicates how much the standard error represents of the mean value. In principle, the lower the standard error, the more reliable the results of the regression estimate [10].

2.4.2 Quality Criteria of the Regression Coefficients

If the global quality criteria confirm an appropriate fit and thus a correlation in the population, the individual regression parameters should then be analyzed.[5] A suitable null hypothesis for the statistical significance test is [6]:

$$H_0 : \beta_j = 0 \tag{26}$$

In a further step, a confidence interval is calculated to determine the range of values within which the unknown, actual regression coefficients move [10].

[5] In the case of single regression with only one independent variable, it is not necessary to check the individual regression parameter, since the F statistics of the global analysis already take into account the separate influence of this one variable [6].

t-Test: The null hypothesis is tested within the framework of the t-test analogously to the procedure of the F-test. However, the test variable of the t-distribution is calculated as follows:

$$t_e = \frac{b_j - \beta_j}{S_{b_j}} \tag{27}$$

Since the null hypothesis assumes that the regression coefficient β_j corresponds to zero, this formula is simplified:

$$t_e = \frac{b_j}{S_{b_j}} \tag{28}$$

The standard error of the regression coefficient S_{b_j} can be determined for a single regression with only one independent variable using the standard error of the regression S as follows [1]:

$$S_{b_j} = S\sqrt{\frac{1}{\sum(X_{jk} - \overline{X}_j)^2}} \tag{29}$$

The standard error of the level parameter b_0 is calculated as:

$$S_{b_0} = S\sqrt{\frac{1}{K} + \frac{\overline{X}^2}{\sum(X_k - \overline{X})^2}} \tag{30}$$

However, if the regression is multiple, the following applies [9]:

$$S_{b_j} = S\sqrt{\frac{1}{\sum(X_{jk} - \overline{X}_j)^2(1 - R_j^2)}} \tag{31}$$

As will be explained later, R_j^2 describes the coefficient of determination for the regression of the explanatory variable X_j to the remaining variables of the regression function (multicollinearity). The following applies: The larger R_j^2, the higher the standard error ($R_j^2 \rightarrow 1$). Thus, the standard error for the regression coefficients in multiple regression also depends on the correlation between the explanatory variables considered and those still present [6].

The comparison between the absolute value of the empirical t-value t_e and the critical t-value t_c (value of the t-distribution for a given probability of safety $(1 - \alpha)$ and the number of degrees of freedom $= K - J - 1$) provides the answer to whether the null hypothesis can be rejected ($|t_e| > t_c$) or not ($|t_e| < t_c$) [10]. Analogous to the previous F-test, the p-value can also be used here for evaluation.

Confidence Interval of the Regression Coefficient: The confidence interval of the regression coefficient is derived from the factors of the t-test [10]:

$$b_j - tS_{b_j} \leq \beta_j \leq b_j + tS_{b_j} \tag{32}$$

The confidence interval is to be interpreted as the range of values of the true regression coefficient β_j for a given confidence probability. A relatively large confidence interval indicates a relatively high uncertainty and unreliability of the regression function [6].

2.4.3 Checking the Model Assumptions

The stochastic model of regression analysis, which is characterized by the disturbance parameter u, has already been referred to in the context of the examination of the global quality criteria. This disturbance variable expresses the existing uncertainty about the completeness and error susceptibility of the regression function used to explain the actual variable Y [9]. In order to counter this problem, various assumptions are made for the regression model, which should be subsequently introduced and always evaluated within the framework of model building [6].

Model Assumptions: As already described in the introduction, the model must be specified correctly. For a linear regression model, this means that all relevant[6] regressors are taken into account which, in the context of multiple regression analysis, are fulfilled by the condition of a higher number of observations (K) than the parameters to be estimated ($J + 1$) and that it is linearly expressed in the parameters β_0 and β_j. Furthermore, u_k shall be valid for different disturbance variables [10]:

- The expected value of the disturbance variables is zero: $E(u_k) = 0$
- The covariance of the regressors and disturbances and thus also their correlation coefficient is zero, whereby a connection between them can be excluded: $Cov(u_k, X_{jk}) = 0, r(u_k, X_{jk}) = 0$
- There is also no correlation between the respective interfering variables (autocorrelation): $Cov(u_k, u_{k+r}) = 0$
- There is a constant variance of the disturbance variables (homoscedasticity): $Var(u_k) = \sigma^2$
- The disturbing variables u_k are normally distributed

In addition, there should be no multicollinearity or linear dependency between the explanatory variables X_j [6].

These assumptions (apart from the specification of the normally distributed disturbance variable) are also referred to as Best Linear Unibased Estimators (BLUE), since among them the estimation according to the OLS shows all the required characteristics in terms of expected fidelity and efficiency [9].

Measurement and Impact of Model Assumption Violations: Most of the following infringements can be evaluated graphically. This will be discussed in more detail in the case study.

Nonlinearity: If there is a nonlinear relationship between the dependent and independent variables and this is not considered, then the model is misspecified by

[6] Regarding the relevance of regression factors, reference is already made to the overfitting or underfitting of a regression function.

distorted estimates of the regression parameters. The form of the relationship can be easily recognized by the scatter plot of X and Y. However, the above assumption only refers to a linearity in the regression parameters, which makes linearization of nonlinear relationships possible. If f is any nonlinear function, the nonlinear variable X can be replaced via $X' = f(X)$ and thus linearized [1]. The model

$$Y = \beta_0 + \beta_1 X' + u \quad \text{with} \quad X' = f(X) \tag{33}$$

is therefore linear in the coefficients β_0 and β_1 as well as in X', but not in X [6]. In addition, one reason for nonlinearity in the case of several explanatory variables can be seen in the fact that they are not additively linked to each other but, for example, multiplicatively, and this has effects on the respective regression coefficient (in the form of a slope that actually turns out differently) [6]. This is also referred to as the interaction effect, which, as already indicated under section 2.3, can be included with the help of a gradient dummy [1].

Expected Value u unequal to 0: A systematic error in the measurement of Y leads to the inclusion of this error in the absolute element b_0, since the disturbance variable thereby receives a systematic effect. This results in a distortion, which in most cases is only a secondary problem, since the level parameter is usually of secondary interest [6].

Incorrect Selection of Regressors: One requirement of a suitable regression function is that it takes all relevant influences into account (cf. section 2.4.3) or that there is no underfitting [9]. A distortion of the estimated parameters caused by this is, however, partially negligible if the covariance between the estimated regression parameters and the disturbance quantity containing the missing influences is zero as required ($Cov(u_k, X_{jk}) = 0$). Then it behaves similarly to the position with an expected value of the disturbance u deviating from zero, whereby "only" b_0 is subject to a distortion. However, if this assumption is violated ($Cov(u_k, X_{jk}) > 0$, positive correlation between u_k and X_{jk}), the part of the variation of Y triggered by u is incorrectly classified as the variation of X_j, whereby the estimator for b_j is distorted [6]. By the additional possibility of the integration of categorial characteristics, dummy variables therefore support the avoidance of underfitting [1]. On the other hand, the regression function may overfit if too many parameters are included in the equation. This can lead both to the fact that, as a result of an increasing number of variables, an influence that is actually present does not appear to be significant due to insufficient, distorted appreciation, and to the fact that a variable that is not relevant for the model proves to be significant [6]. To deal adequately with these problems, a sound theoretical knowledge of the facts is recommended.

Heteroscedasticity: A violation of the homogeneity of the k-variances of the error variable u_k by a non-constant variance leads to inefficient estimates as well as a distorted standard error of the regression coefficients S_{b_j}, whereby the confidence interval is also incorrectly determined. To identify heteroscedasticity, a scatter plot of the residuals against the estimated values of Y (and thus) should be performed. If heteroskeletal elasticity is present, a triangular pattern can often be observed (cf. Fig. 1) [6].

In practice, however, the heteroskeleton elasticity test is often dealt with using the Goldfeld/Quandt test, which assumes identical variances of two proportional sub-samples of the sample variants of the residuals (S_1^2, S_2^2) as a null hypothesis [11]:

$$H_0 : S_1^2 = S_2^2 \text{ (homoscedasticity)} \tag{34}$$

S_1^2 and S_2^2 are determined as follows:

$$S_1^2 = \frac{1}{K_1 - J - 1} \sum_{k=1}^{K_1} e_k^2 \tag{35}$$

$$S_2^2 = \frac{1}{K_2 - J - 1} \sum_{k=1}^{K_2} e_k^2 \tag{36}$$

The empirical test variable of the F-distribution is calculated as:

$$F_e = \frac{S_1^2}{S_2^2} \tag{37}$$

If this is compared with the critical F-value at $v_1 = K_1 - J - 1$ and $v_2 = K_2 - J - 1$ degrees of freedom and significance level α, the rejection of the null hypothesis leads to the existence of heteroskedasticity and thus an erroneous estimate [10].

Autocorrelation: Particularly in time series data, the disturbance variables in the population are correlated with each other. This is called autocorrelation and leads to a distortion of standard errors of the regression coefficients and thus of the confidence interval to be determined for the actual regression parameters. Here, too, a graphical evaluation provides initial insights into the possible existence of autocorrelation through the analysis of residuals. If the autocorrelation is positive, successive

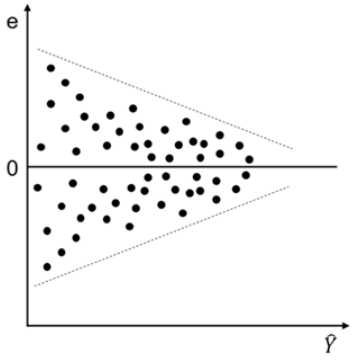

Fig. 1 Heteroscedasticity. Source: Own representation according to Backhaus et al. (2016) [5].

residuals are close to each other, whereas, in the case of negative autocorrelation, they fluctuate more strongly (cf. Fig. 2) [6].

However, this can also be checked using the so-called Durbin/Watson test, which considers a non-existent autocorrelation as a null hypothesis. The test statistic d is determined by means of:

$$d = \frac{\sum_{k=2}^{K}(e_k - e_{k-1})^2}{\sum_{k=1}^{K} e_k^2} \tag{38}$$

The result moves in the interval $0 \leq d \leq 4$ and can be interpreted by means of the following Figure 3 and Table 10 [6].

The critical values $d_u^+, d_o^+, 4 - d_o^+$ and $4 - d_u^+$, which can be taken in connection with the significance level from the Durbin/Watson table, separate the interval into 5 ranges [8].

For the attestation of a non-existent autocorrelation, it is therefore necessary that the value of the test statistic d is close to two.

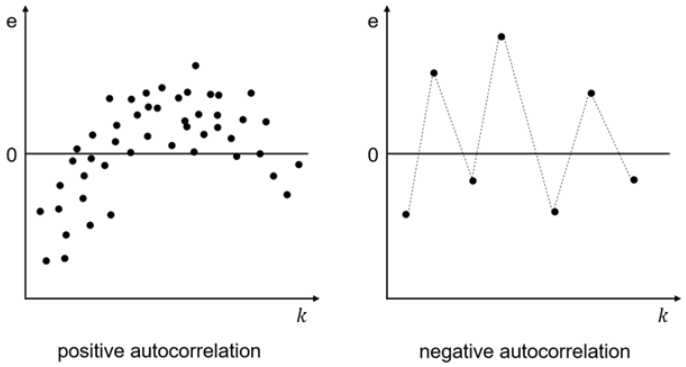

Fig. 2 Autocorrelation. Source: Own representation according to Backhaus et al. (2016) [6].

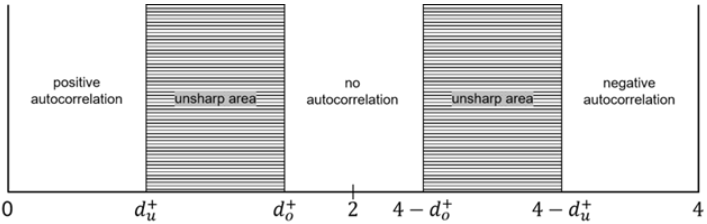

Fig. 3 Durbin-Watson-Test. Source: Own representation according to Backhaus et al. (2016); Dreger et al. (2014) [6], [8].

Table 1 Durbin-Watson test statistics

Test statistics	Decision
$d_o^+; \alpha/2 \leq d \leq 4 - d_o^+; \alpha/2$	do not reject H_0, no autocorrelation prevails
$d \leq d_u^+; \alpha/2$ or $d \geq 4 - d_u^+; \alpha/2$	reject H_0, there is an autocorrelation between the disturbance variables
$d_o^+; \alpha/2$: Upper limit value of the DW table for the level $\alpha/2$	
$d_u^+; \alpha/2$: Lower limit value of the DW table for level $\alpha/2$	

Source: Own representation according to Backhaus et al. (2016) [6].

Multicollinearity: To avoid distortions of the slope parameters, a visual examination of the correla-tion matrices of the independent variables already provides first impressions of the presence of multicollinearity. If high correlation coefficients are found, multi-collinearity can be assumed [1]. However, since only pairwise comparisons are possible, a high multicollinearity may exist despite low correlation coefficients of the correlation matrices [6]. By means of a collinearity diagnosis via the variance inflation factor (VIF), the assessment can be made more precise [1]:

$$VIF = \frac{1}{1 - R_j^2} \tag{39}$$

As already indicated under section 2.4.2 in the context of the t-test, the coefficient of determination R_j^2 reflects the regression of the independent variable X_j to the other variables of the regression function [10]. Results of the $VIF > 5$ can be assumed to show a strong multicollinearity [12].

Non-normal distribution of the disturbance: Although a non-normally distributed disturbance has no influence on the assessment of the estimators of the regression parameters according to the OLS method as BLUE (Best Linear Unbiased Estimators), this is relevant for the test statistics. A missing normal distribution within the framework of the disturbance causes via $u = (Y - \hat{Y}) = (Y - b_0 - b_1 X_1 - ... - b_j X_j)$ that the regression coefficients b_j must also be regarded as non-normal distribution. For the test procedures, however, it is necessary that the parameters to be estimated are regarded as normally distributed [6]. Here, however, the central limit value theorem helps with a sufficiently high ($K > 40$) observation number K, because then the estimators of the OLS method are nevertheless considered asymptotically normally distributed [11].

3 Examples

After the theoretical consideration of the concepts of dummy variables, some application areas of dummy variables are now explained. The advantage of dummies is that they can also be used to process exogenous influencing variables in the form of qualitative characteristics in a regression analysis. For this reason, they are used in numerous analyses, some of which are described below.

Rental price / m² by living space and location [1]: By means of a regression analysis, the net rent price for students in a medium-sized city is to be analyzed as a function of the two stochastically independent variables: apartment size and location. Size is a metric variable and location is a categorical variable. The analysis refers to apartments that are between 20 and 55 square meters in size. Since the residential area is ordinally scaled, it is coded as a dummy variable. Assuming there were four different residential areas, $((4 - 1) = 3)$ dummy variables are used (see Table 2). The fourth expression as a reference group opens when all other three categories are coded with "0". The concrete characteristic of the residential area represents the change of the level parameter b_1 of the regression function. The regression function and its subfunctions resulting from the coding look as follows:

$$Y_i = b_1 + b_2 X_{2i} + b_3 X_{3i} + b_4 X_{4_i} + b_5 X_{5i} \tag{40}$$

$$Y_i = b_1 + b_2 X_{2i} \quad \text{(state 1)} \tag{41}$$

$$Y_i = b_1 + b_3 + b_2 X_{2i} \quad \text{(state 2)} \tag{42}$$

$$Y_i = b_1 + b_4 + b_2 X_{2i} \quad \text{(state 3)} \tag{43}$$

$$Y_i = b_1 + b_5 + b_2 X_{2i} \quad \text{(state 4)} \tag{44}$$

with:
Y: Rental price/m²,
X_{2i}: residential space,
X_{3-5i}: dummy codings

A regression analysis with position 1 as the reference position can be performed using a sample in the form shown in Table 3.

The estimated net cold rent results for the case described would now be calculated using the regression model. To do this, the values b_1 to b_5 must be known. A calculation for this example is expressly not carried out here, since no real data sample is available.

Estimation of electricity consumption [1]: Another example for the use of dummy variables can be the estimation of the electricity consumption of a producing

Table 2 Dummy Variables for Rental price / m² by living space and location

	x_{3i}	x_{4i}	x_i
State 1	0	0	0
State 2	1	0	0
State 3	0	1	0
State 4	0	0	1

Source: Own representation according to Natrop (2015) [1].

Table 3 Dummy variables to estimate net cold rent

Index	Net cold rent in $/m²	Size	Dummy Location 2	Dummy Location 3	Dummy Location 4	Location
i	Y_i	X_{2i}	X_{3i}	X_{4i}	X_{5i}	
1	...	22	0	0	0	1
2	...	25	0	1	0	3
3	...	30	1	0	0	2
4	...	24	0	0	1	4

Source: Own representation according to Natrop (2015) [1].

company. For example, several dummy variables can be used here for different exogenous influencing parameters. It makes sense to use dichotomous variables for several dummies. In the given example, the electricity consumption in MhW is to be estimated in tons under consideration of the produced quantity of a good. As dummy variables, for example, phases of short-time work (KA), company holidays (BF), machine damage (MS) and additional plants (ZA) could be used. All dummies are coded with "0" for situation does not exist or "1" for situation exists. The following regression function is an example for this producing company:

$$Y = 48.76 + 0.56X - 7.9KA - 6.15BF + 5.7MS + 6.7ZA \tag{45}$$

The production-independent portion of electricity consumption is 48.76 MhW and is supplemented by the production-dependent portion, which is multiplied by a factor of 0.56. In the case of shortened work hours or company holidays, electricity consumption decreases because of fewer than usual available employee resources. On the other hand, machine damage and additional equipment increase consumption.

Other fields of application: The examples shown clearly show that dummy variables are frequently used in business and economic questions. Equally popular is the use of dummies in political questions, for example, to estimate the effect of various factors on the affinity for a party. There are also several papers on migration that deal with dummy variables. A current example is the analysis of the extent to which immigration changes labor supply [13]. Social dummies can also be used to answer sociological questions, such as the influence of social status on donation behavior [14]. In the context of economic analyses, dummy variables can be used to optimize the measurement of the influence of capital intensity and total factor

productivity on labor productivity: Dummies are used here to divide a sample into different time periods to reduce the influence of e.g. structural breaks [15].

4 Case Study

Following a description of the possible uses of dummy variables, the theoretical principles will now be illustrated in a case study. The case study deals with the analysis of the functional relationship between life expectancy (in the form of age) as a dependent variable Y and the explanatory variables weight (metric), gender (categorical) and smoker (categorical). The underlying data set can be found in Table 4. The free statistical programming language R is used here. The program packages used are tidyverse, gvlma, lmtest and car.

Table 4 Life Expectancy Dataset

Index	Age	Weigt	Smoker	Sex
1	88.77763	55	no	female
2	82.12195	56	no	male
3	83.24057	57	yes	female
4	81.06262	58	no	male
5	83.54351	59	yes	female
6	76.05518	60	yes	male
7	88.28971	61	no	female
8	81.27626	62	no	male
9	76.33333	63	yes	male
10	80.47571	64	no	male
11	75.08134	65	yes	male
12	80.30476	66	no	male
13	75.09165	67	yes	male
14	74.67315	68	yes	male
15	74.60952	69	yes	male
16	85.91038	70	no	female
17	74.25755	71	yes	male
18	80.71731	72	yes	female
19	85.59726	73	no	female
20	80.57857	74	yes	female
21	73.80317	75	yes	male
22	80.36661	76	yes	female
23	78.14371	77	no	male
24	85.50610	78	no	female
25	72.71530	79	yes	male
26	79.55561	80	yes	female

Source: The dataset originates from Voß (2000) [16].

Table 5 Data Frame Head

Index	Weight	Age	Smoker	Sex
1	88.77763	55	no	f
2	82.12195	56	no	m
3	83.24057	57	yes	f
4	81.06262	58	no	m
5	83.54351	59	yes	f
6	76.05518	60	yes	m

Source: Own calculation.

4.1 Data Preparation

Libraries & Data Import

```
library(tidyverse)
library(car)
library(gvlma)
library(lmtest)

# load data:
lifeexpectancy <- read.csv('.../lifeexpectancy.csv',
header = TRUE, sep = ",", stringsAsFactors = FALSE)
```

Subsequently, an initial assessment of the data structure takes place via:

Initial Assessment

```
head(lifeexpectancy)
```

The first 6 lines of the dataframe are displayed (see Table 5).

It is noticeable that the two categorical variables (smoker and sex) have not yet been binarily coded in the sense of the dummy variables. Since, in both cases, characteristics with only two expressions are involved, a dummy variable is required for each of the characteristics ($k - 1 = 2 - 1 = 1$). The following applies:

Table 6 Descriptive Statistics

Index	Weight	Age	Smoker	Sex
Min.: 1.00	Min.: 72.72	Min: 55.00	Length: 26	Length: 26
1st Qu.: 7.25	1st Qu.: 75.33	1st Qu.: 61.25	Class: character	Class: character
Median: 13.50	Median: 80.42	Median: 67.50	Mode: character	Mode: character
Mean: 13.50	Mean: 79.93	Mean: 67.50		
3rd Qu.: 19.75	3rd Qu.: 82.96	3rd Qu.: 73.75		
Max.: 26.00	Max.: 88.78	Max.: 80.00		

Source: Own calculation.

$$\delta_{smoker} = \begin{cases} 1 & \text{yes} \\ 0, & \text{no} \end{cases}$$

$$\delta_{sex} = \begin{cases} 1 & \text{male} \\ 0, & \text{female} \end{cases}$$

Accordingly, the references are non-smokers and female persons. The dummy variables are coded in R via:

Dummy Coding

```
# sex-dummy:
dsex <- model.matrix(~sex, data = lifeexpectancy)
dsex <- dsex[,-1]

# smoker-dummy:
dsmoker <- model.matrix(~smoker, data = lifeexpectancy)
dsmoker <- dsmoker[,-1]
```

The summary() function also provides an overview of the descriptive statistics for identifying discrepancies such as outliers or missing values, which are not present here (see Table 6).

Descriptive Statistics

```
summary(lifeexpectancy)
```

Table 7 Single Regression Model Summary

Call:				
lm(formula = age ~weight, data = lifeexpectancy)				
Residuals:				
Min	1Q	Median	3Q	Max
-5.3261	-4.8684	0.0673	2.0209	7.5996
Coefficients:				
	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	92.9123	7.9628	11.668	2.23e-11 ***
weight	-0.1924	0.1172	-1.641	0.114
—				
Signif. codes:				
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1				
Residual standard error: 4.484 on 24 degrees of freedom				
Multiple R-squared: 0.1009			Adjusted R-squared: 0.0634	
F-statistic: 2.692 on 1 and 24 DF			p-value: 0.1139	

Source: Own calculation.

4.2 Conducting the Regression Analysis

4.2.1 Single Regression

The function lm() is available in R for estimating the regression model. At the beginning, there is only a single regression with the only metric explanatory variable "weight". The function summary() provides the model summary here (see Table 7).

Single Regression

```
# single regression model:
srm <- lm(age ~ weight, data = lifeexpectancy)
summary(srm)
```

The estimated regression function is therefore:

$$\hat{Y} = 92.9123 - 0.1924 \cdot \text{weight} \tag{46}$$

The weight has a negative effect on life expectancy (in years) as a gradient parameter. If the weight increases by 1 kg, life expectancy decreases by 0.1924 years. However, this model has a very low coefficient of determination ($R^2 = 0.1009$), whereby only 10.1% of the variance of Y is explained by the dispersion of the weight and the fit of the regression function can therefore be judged as poor. Also, the

F statistics with a p-value of 0.1139 does not have a significantly different actual regression coefficient from zero β_1 and therefore, no significant correlation in the population can be assumed. This is also shown by the plot of the scatter diagram (see Figure 4):

Plot Single Regression Model

```
# plt srm
plot(lifeexpectancy$weight, lifeexpectancy $age)
abline(srm, col = "red")
```

4.2.2 Multiple Regression with Dummy Variables

The estimation of the regression function with the previously coded dummy variables is done by extending the above lm() function. The results are shown in Table 8.

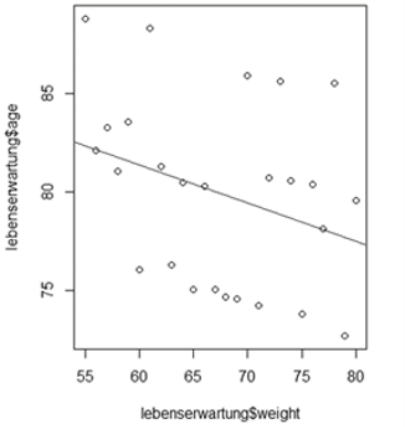

Fig. 4 Scatterplot. Source: Own calculation.

Table 8 Multiple Regression Model Summary

Call:				
lm(lm(formula = age ~weight + dsex + dsmoke, data = lifeexpectancy)				
Residuals:				
Min	1Q	Median	3Q	Max
-0.53167	-0.18957	-0.03841	0.16329	0.64975
Coefficients:				
	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	98.458108	0.573232	171.76	< 2e-16 ***
weight	-0.173092	0.008427	-20.54	7.63e-16 ***
dsex	-6.824481	0.124575	-54.78	< 2e-16 ***
dsmoke	-5.045249	0.127035	-39.72	< 2e-16 ***
—				
Signif. codes:				
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '·' 1				
Residual standard error: 0.31 on 22 degrees of freedom				
Multiple R-squared: 0.9961			Adjusted R-squared: 0.9955	
F-statistic: 1.854 on 3 and 22 DF			p-value: < 2.2e-16	

Source: Own calculation.

Multiple Regression Model

```
# regression model with dummys:
rmd <- lm(age ~ weight + dsex + dsmoke,
data = lifeexpectancy)
summary(rmd)
```

The regression function is estimated as:

$$\hat{Y} = 98.458108 - 0.173092 \cdot \text{weight} - 6.824481 \cdot \delta_{sex} - 5.045249 \cdot \delta_{smoker} \quad (47)$$

If the person is a smoker ($\delta_{smoker} = 1$), life expectancy decreases by 5.05 years. It is also worth noting that the male sex ($\delta_{sex} = 1$) reduces life expectancy even more than the smoker status. So, if the person is a man, life expectancy drops by 6.82 years.

The adjusted coefficient of determination $R^2_{adj} = 0.9955$ shows the increased goodness of fit compared to the simple regression model. Here 99.6% of the variance of Y is explained by the regression function. The F-statistics with a p-value < 5% can also be used to assume a significant correlation in the population. The examination of the individual regression coefficients for a significant difference of zero is also positive in all cases based on the p-values (Pr(>|t|)).

Table 9 Goldfeld-Quandt Test

Goldfeld-Quandt test				
data: rmd				
GQ = 0.047523	df1 = 9	df2 = 9	p-value = 0.8585	
alternative hypothesis: variance increases from segment 1 to 2				height

Source: Own calculation.

Table 10 Durbin-Watson Test

Lag	Autocorrelation	D-W Statistic	p-value
1	0.01160936	1.96456	0.874

Source: Own calculation.

The Goldfeld-Quandt test is part of the library(lmtest) and gives the following result (see Table 9):

Goldfeld-Quandt Test

```
gqtest(rmd)
```

Accordingly, there is no homoskedasticity and it may be an erroneous estimate. As required, the value of the D-W statistics here is close to 2, but this is also not significant, so a misspecification may have occurred (see Table 10):

Durbin-Watson Test

```
durbinWatsonTest(rmd)
```

However, a strong multicollinearity cannot be assumed here, as the results of the VIF are all far below 5 (see Table 11):

Variance Inflation Factor

```
# VIF:
vif(rmd)
```

Table 11 Variance Inflation Factor

weight	dsex	dsmoke
1.080975	1.025062	1.065948

Source: Own calculation.

5 Conclusion

Dummy variables help to optimize regression estimation using a relatively simple methodology. There are many areas of application, especially in the socio-economic field. In the context of big data and various forms of unstructured data, dummy variables also gain relevance in order to optimize models and to include other influences in the analysis.

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Heuristic Methods

Laura Schwarzbach & Ramona Schmitt

Abstract The term "heuristics" is derived from the ancient Greek word "heuriskein," which means to discover and explore. In the context of problem solving, heuristics are rules of thought that speed up the solution of complex problems, even though they often lead to satisfactory rather than optimal results. They are particularly valuable in solving optimization problems in the context of operations research. Heuristics are guided by principles such as factorization, modeling, well-defined goals, generation and testing, and bounded rationality. Although they cannot guarantee optimality, they serve as practical tools in various fields, including economics, sports, environment, medicine, and engineering.

1 Introduction

To get into the topic of Heuristic methods, let us start with a short example:

The Miller family takes a walk on a Sunday. They plan to finish their 16-km long journey in approximately 4 hours. This results in a walking speed of 4 km per hour (= 16 km / 4 hours). The family starts at 1 pm. Mr. Anderson, a neighbor of the Miller's, goes running on the same route at 3 pm with a faster speed of 8 km per hour [1].

At which time (x) and distance (y) do the two parties meet? This fairly easy problem can be solved either mathematically or graphically.

In order to determine at which time (x) they meet, we have to put up the fol-lowing equation:

Kilometer travelled by the Miller's = Kilometer travelled by Mr. Anderson

Or mathematically:

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$$\begin{aligned} \text{speed}_{\text{Mr. Anderson}} * (x - 2h) &= \text{speed}_{\text{Miller's}} * x \\ 8\frac{km}{h} * (x - 2h) &= 4\frac{km}{h} * x \\ x &= 4h \end{aligned}$$

To calculate the distance (y), we have to put (x) into the formula for the Miller's or for Mr. Anderson.

Mr. Anderson:

$$\begin{aligned} y &= \text{speed}_{\text{Mr. Anderson}} * (x - 2h) \\ y &= 8\frac{km}{h} * 2h \\ y &= 16km \end{aligned}$$

The Miller's:

$$\begin{aligned} y &= \text{speed}_{\text{Miller's}} * x \\ y &= 4\frac{km}{h} * 4h \\ y &= 16km \end{aligned} \tag{1}$$

The result is that they meet 16 km and 4 hours after the Miller Family started. So, they meet at 5 pm after their walk/run, because both parties come back home at the same time.

For a graphical solution, the formula for both parties can be put into a chart. The intercept between both functions shows the same result: they meet 4 hours after the Miller's started respectively 2 hours after Mr. Anderson started (x) and after 16 kilometers (y) at the end of their route (see Figure 1).

This example shows the mathematically optimal solution to a fairly easy problem. However, finding the optimal solution to more complex problems can be very time consuming and may require a disproportionate effort. In these cases, it becomes necessary to solve problems with approximations. Methods to find such approximate solutions to decision and optimization problems are called heuristics [2].

1.1 Definition and Characteristics

The word 'heuristic' can be traced back to the ancient Greek word 'heuriskein' which means discover and explore. A heuristic is a rule of thought that can be used to reduce the effort required to find a solution to complex problems. Besides the advantage of a reduced effort is the disadvantage that heuristics offer just a satisfactory and not the optimal solution. It may even happen that no usable solution results from the application of heuristic methods [3].

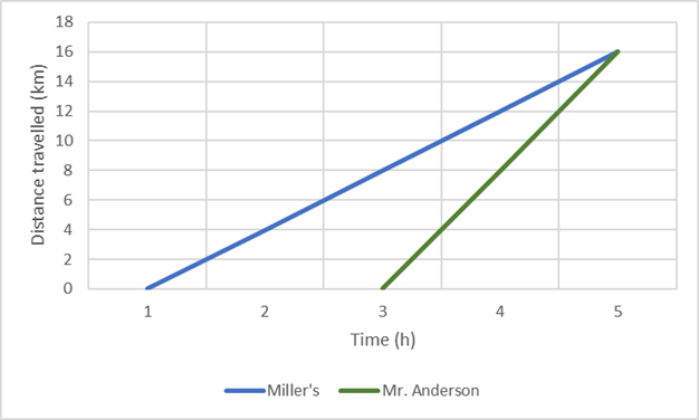


Fig. 1 Graphical Solution for the Time-Distance Example. Source: Own representation according to Bayerischer Rundfunk (Hrsg.) (2012) [1].

In the field of operations research, heuristic methods solve optimization problems that are too complex and multi-dimensional to be solved by other methods. The models are easy to understand, and they provide satisfying solutions with a reasonable expenditure of time and cost. Heuristics do not always find a solution and if they find a solution, it is normally not the optimum. In contrast to the exact methods based on mathematical optimization, heuristic research includes besides mathematical logic, also incalculable factors like experiences and insights. This basically applies to the order, allocation, grouping, and selection problems of combinatorial optimization [3].

Based on this logic, heuristics cannot determine the optimality of a solution and thus cannot guarantee it. But what some heuristic methods can do, is to show a worst-case scenario: it is possible to determine a factor for the deviation of the worst-case solution from the optimum value [4].

The usage of heuristic methods in real life is widespread: they get used to solving problems in different areas and branches like economics, sports, the environment, medicine, and engineering [5].

Although there are many different heuristic methods, they are all (more or less) based on the same heuristic principles as follows:

- Principle of factorization,
- Principle of modeling,
- Principle of well-defined targets,
- Principle of generate-and-test, and
- Principle of restricted rationality [6].

The *principle of factorization* describes the division of a complex problem into separate sub-problems to facilitate the solution of the overall problem. The problem is broken down into sub-problems that can be solved successively or in parallel [7].

The *principle of modeling* indicates that the sub-problems are defined in a way that allows solving them by known methods [8].

The *principle of well-defined targets* says that the decision criteria for evaluating the identified solutions must be clear and easily applicable [9].

The *principle of generate-and-test* recommends the development of one 'good' solution instead of several solutions. If this solution meets the minimum standards, it will be accepted as a problem solution – otherwise, the search will continue. This process is repeated until an acceptable solution is found [10].

The last principle (*Principle of restricted rationality*) refers to the nature of heuristic methods by recommending to pass on the optimum solution and just strive for a satisfying solution [11].

1.2 Analytical vs. Heuristic Methods

Analytical and heuristic methods differ in their approach and the resulting solutions. Heuristic methods brow the solution space, whereas analytical methods use the value range of a given target function. Therefore, analytical methods lead to exact results. In contrast, heuristic methods do not guarantee a solution and – if a solution is found – it is only about a local optimum [12].

The lack of the optimum solution is one of the main criticism of heuristics. However, no method for solving such complex problems to optimality is known. An advantage of the heuristic methods is that they are more flexible and less elaborate than the exact methods. Heuristic methods allow, e.g., the consideration of conditions that are difficult to model [6].

The decision between using an analytical or heuristic method depends on the solution's requirements in terms of applicability and quality. Analytical methods provide optimal solutions – heuristic methods provide satisfying solutions. Some famous analytical methods are the Branch-and-Bound (BB) and Branch-and-Cut methods. The heuristic methods will be presented later in this essay [6].

Figure 2 places both methods in an XY-chart. The x-axis shows the solution quality: the quality increases from just satisfying solutions on the left side to optimal solutions on the right side. The y-axis shows the applicability of the method to find a solution: this axis runs from bottom to top with decreasing restrictions and requirements as well as decreasing efforts [6].

Heuristic methods should be used if a satisfactory but possibly non-optimal solution is acceptable. The solution-finding process requires rarely any formal requirements and only limited efforts. In contrast, analytical methods guarantee optimal solutions but usually have formal restrictions and require high efforts in the solution-finding process. A method that guarantees optimal solutions, requires only limited

time to find a solution, and rarely has formal requirements, sounds optimal but does not exist [6].

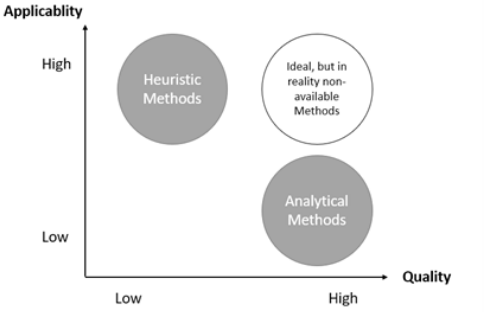


Fig. 2 Analytical vs. Heuristic Methods. Source: Own representation according to Grünig; Kühn (2017) [6].

Grünig and Kühn have also developed a step-by-step questionnaire to decide whether a heuristic or analytical method is applicable to a certain problem. If one of the following questions can be answered with 'No', then the heuristic method is more suitable than the analytical method:

1. Does the problem only concern quantity aspects?
2. Are there any standards to decide whether a solution is valid or not?
3. Is there a suitable analytical method available or can such a method be developed at reasonable time and cost? [6]

From these questions, it can be deduced that the application of heuristic methods makes sense if a problem also has qualitative aspects, if no requirements for the solution exist, or if no analytical method is available. In these cases, the problem is 'ill-defined' and/ or 'ill-structured' [11].

Heuristics are also used as part of the optimization methods to find and develop initial solutions. In the context of Branch-and-Bound methods (BB methods), heuristic solutions can provide a global upper or lower limit for the maximum or minimum target function value.

1.3 Heuristic Procedures

Operations Research differentiates between the four following heuristic procedures:

1. Opening procedure,
2. Local search and improvement procedure,
3. Incomplete exact procedure and
4. Combination of (1) – (3) [4].

Opening procedures are used to identify the first valid solution to the problem. A common approach is the step-by-step inclusion of elements to the solution. The quality of the solution depends on the design of the method and the invested (computing) time [4].

Local search and improvement procedures are based on an existing solution to the problem that was either identified randomly or developed through the opening procedure. A step-by-step approach compares systematically the current solution with new ones to find a better one. In each iteration, the neighborhood NB (x) of a current solution x is scanned and the solution progresses from the current solution to a solution of the neighborhood. This transformation is called a 'move'. The accepted solution is a local optimum [4].

Local search procedures and improvement procedures can be distinguished as follows: Improvement procedures end as soon as no better solution is found. This solution is a local optimum. Local procedures also allow steps back that result in an intermediate degradation of the target value, for example, Simulated Annealing and Tabu Search [4].

Incomplete exact procedures provide the best solution found in an aborted ex-act procedure. For example, when executing a Brand-and-Bound search, it is possible to stop them before an optimal solution has been found or when the quality of the solution is enough [4].

The last procedure is a *combination* of the first three procedures, and it is also called metaheuristic. Their aim is to overcome the issue of selecting a local optimum by adding elements to the existing solution [2]. Depending on the information input, heuristics can also be divided into deterministic and stochastic methods: Deterministic methods lead to the same solution in case of multiple applications to the same problem because of constant conditions. Stochastic methods, on the other hand, contain a random component that leads to different solutions when applied several times to the same problem [4].

2 Selected Heuristic Problems

A lot of existing problems of combinatorial optimization can be solved with heuristic methods. The problems can be divided into the following categories:

- Order problems,
- Allocation problems,
- Grouping problems and
- Selection problems [4].

Three of the most common problems in these categories are presented below:

1. Traveling-Salesman Problem,
2. Dynamic Warehousing, and
3. Knapsack Problem.

2.1 Traveling-Salesman Problem

A typical combinational optimization is the Traveling-Salesman Problem (short: TSP). A variety of analytical and heuristic solution methods have been developed for this problem. The TSP can be considered an example of general tour planning problems.

The Traveling-Salesman Problem describes an order problem. The problem is to identify the shortest route of a traveling salesman based on the order of the cities visited. When the salesman starts his tour in City 0, the TSP consists of finding the shortest route, so he visits a finite number of cities n (1,2,. . . ,n) exactly once and then returns to City 0 where he started. The respective travel distance between each pair of cities is given [5].

To solve the traveling salesman problem, it must be first represented by a model. The graph $G = (V, E)$ contains nodes (V) and edges (E): the nodes represent the cities and the edges describe the connection between these two cities. Each edge has a certain length. An edge is weighted if costs are added to it and unweighted if not. A graph is complete if there is always an edge between every two nodes. This can be achieved by inserting an artificial edge if there is no edge. A route in this graph is a circle that contains each node exactly once. The resulting graph is also called a Hamilton graph [13].

Furthermore, it must be determined whether it is a symmetrical or asymmetrical TSP: In the asymmetric TSP, the edge between two nodes can have different lengths depending on the direction. An asymmetric TSP is usually modeled using a directed graph. If the salesman visits each city once, there exists $(n-1)!$ possible tours. That the route takes longer in one direction than the other is often due to construction sites or one-way streets, as well as different means of transportation.

In symmetric TSP, however, each tour has the same length in both directions. Therefore, the symmetry halves the number of possible tours (n-1)!/2. A symmetric TSP is usually modeled using an undirected graph, i.e. the direction between two nodes is not specified [14].

2.1.1 Opening Procedures for the TSP

The first step to handling the Traveling-Salesman Problem is to adopt an opening procedure. In the context of deterministic opening procedures, it can be distinguished between many different heuristic methods. In the following, we will take a closer look at the neighbor and insertion heuristics.

The *Nearest-Neighbor heuristic* always looks for the nearest neighbor as the name suggests and selects the currently best alternative in each iteration. The procedure starts the route with any node t_0V. In the iteration process, it adds the node with the shortest distance to t_0 to the route. Afterward, the iteration i $(i = 1, \ldots n-1)$ extends the route by the node t_i which has not yet been scheduled and has the shortest distance to node t_{i-1}. Node t_i is therefore the follower of node t_{i-1}. The Farthest-Neighbor heuristic works in the opposite way and looks for the most distant

node in each iteration. However, the heuristic of the Nearest- and Farthest-Neighbor generally provides sub-optimal solutions or even very poor solutions because not all aspects are considered, e.g., the distance between the start node and the last node visited [4].

In the following, both heuristics are applied to the following example. Table 1 shows the distance between six German cities. In the case of an asymmetric TSP with 6 cities, there are $(6 - 1)! = 120$ possibilities to build the route. A symmetric TSP halves the possibilities to 60. Because of the increasing possibilities with an increasing number of cities, an attempt is not efficient and is no longer possible. Heuristic methods are used to reduce the possibilities and to find a more efficient way.

Table 1 Distances of Cities

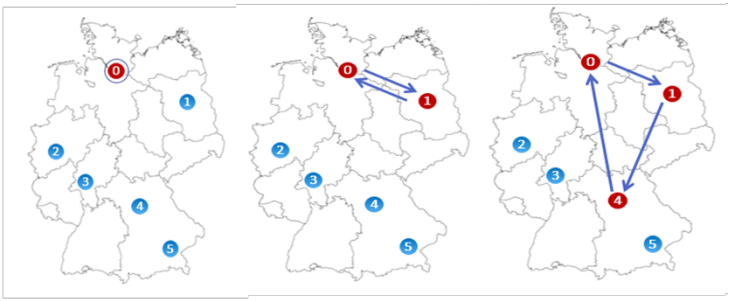
From/To	Hamburg(0)	Berlin(1)	Nuremberg(4)	Munich(5)	Frankfurt(3)	Cologne(2)
Hamburg(0)	–	288	610	775	497	431
Berlin(1)	288	–	445	584	551	572
Nuremberg(4)	610	445	–	165	226	409
Munich(5)	775	584	165	–	392	575
Frankfurt(3)	497	551	226	392	–	192
Cologne(2)	431	572	409	575	192	–

Source: Own representation.

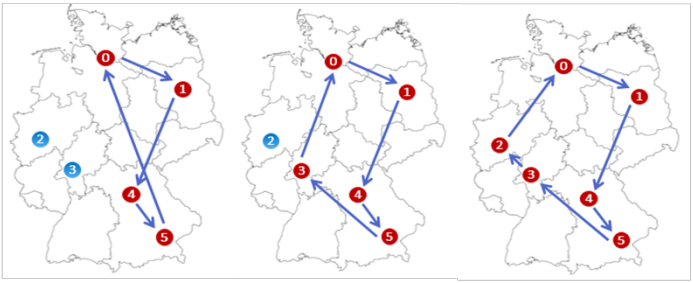
The starting point can be chosen randomly. In the example, the starting point is Hamburg (0). In the case of the Nearest-Neighbor heuristic, the tour will be developed in the following iteration steps:

The nearest neighbor of the starting point in Hamburg is Berlin with 288 kilometers. In the first iteration, Berlin (1) is added to the route. The route between Ham-burg and Berlin is now 576 kilometers long. The second iteration looks for the nearest neighbor to Berlin which is not yet integrated into the route. The added city is Nuremberg (4) with a distance of 445 kilometers from Berlin. The route now leads from Hamburg to Berlin to Nuremberg and back to Hamburg and is 1.343 kilometers long. In the following iteration processes, these steps are repeated and the city with the shortest distance to the last added city which is not yet included in the route is added. This process continues until all cities have been included in the route. The route resulting from this heuristic method is Hamburg – Berlin – Nuremberg – Munich – Frankfurt – Cologne – Hamburg with 1.913 kilometers. Figure 3 illustrates the iteration steps just described.

But the randomly chosen starting point has an impact on the solution. If the route had started in Frankfurt, the order of the cities visited would have changed to Frankfurt – Cologne – Nuremberg – Munich – Berlin – Hamburg – Frankfurt, and the distance would have been 2,135 kilometers. Consequently, the same heuristic method can lead to different solutions depending on the starting point. This leads to inefficiency: the optimal route cannot be ensured and there is a risk that the edges cross.



Phase	Starting Point	1. Iteration	2. Iteration
Tour	Hamburg	Hamburg - Berlin - Hamburg	Hamburg - Berlin - Nuremberg - Hamburg
Distance		576 km	1.343 km



Phase	3. Iteration	4. Iteration	5. Iteration
Tour	Hamburg - Berlin - Nuremberg - Munich - Hamburg	Hamburg - Berlin - Nuremberg - Munich - Frankfurt - Hamburg	Hamburg - Berlin - Nuremberg - Munich - Frankfurt - Cologne - Hamburg
Distance	1.673 km	1.787 km	1.913 km

Fig. 3 TSP Nearest-Neighbor Heuristic. Source: Own representation.

In the case of the *Farthest-Neighbor heuristic*, the route would be Hamburg – Munich – Berlin – Cologne – Nuremberg – Frankfurt – Hamburg with 3,063 kilometers. If the starting point would change to Frankfurt, the distance would be 3,121 kilometers. The results of the Farthest-Neighbor heuristic are generally worse than the results of the Nearest-Neighbor heuristic. In practice, this heuristic plays an underpart and is here only listed for completeness.

Another method to find the best solution is the *Insertion Heuristic*. This method starts with two arbitrary nodes t_0 and t_1 of V and the circle $r = [t_0, t_1, t_0]$. If a node is added in the iteration process which increases the length of the circle r least, it is called the *Nearest-Insertion heuristic*. Another option, called the *Farthest-Insertion*

heuristic, is to insert the farthest node to the circle. As a result, the structure of the route is determined at an early stage of the procedure [4].

If we apply the Nearest-Insertion heuristic to the example above, the tour begins with two arbitrary nodes and inserts in the following iteration steps the city with the smallest distance to the existing route. As a rule, the tour starts with a randomly chosen starting point (here: Hamburg) and the closest city to the starting point (here: Berlin). In the following steps, the closest city to the current route (circle r) is determined and it is calculated at which point this city will be integrated into the existing route so that the length of the round trip increases least. In the case of the example above, the shortest route using the Nearest-Insertion heuristic is 1,886 kilometers long and runs through the six cities in the following order: Ham-burg – Berlin – Munich – Nuremberg – Frankfurt – Cologne – Hamburg.

The results of the iteration process to solve this problem by the Nearest-Insertion heuristic are illustrated in Figure 4.

When using the Farthest-Insertion heuristic, the route starts with the two furthermost cities and the successive integration of other cities into the route. In the example, the route starts with Hamburg – Munich – Hamburg. After all, cities have been integrated into the route, it has a length of also 1,886 kilometers and runs through the cities in the same route as the Nearest-Insertion heuristic but in reverse order: Hamburg – Cologne – Frankfurt – Nuremberg – Munich – Berlin – Hamburg.

In order to compare the results, Figure 5 finally presents the results for the solution of the TSP according to the presented heuristic procedures.

Comparing the two opening procedures presented, it is shown that the insertion heuristics lead to better outcomes than the neighborhood heuristics. Nevertheless, both methods are suitable for solving the Traveling-Salesman Problem, but you should be aware that the solution is always a local optimum, that other heuristic methods can lead to other solutions, and that the quality of the solution strongly depends on the city in which the route starts.

2.1.2 Improvement Procedures for the TSP

The Traveling-Salesman Problem can also be solved by improvement procedures. In contrast to the opening procedures, improvement procedures are used if a solution already exists. In the following, a distinction is made between deterministic and stochastic improvement procedures.

In the context of deterministic improvement procedures, the r-optimal methods are particularly worth mentioning. They assume a valid solution and try to improve it by interchanging r of the edges with r other edges. Consequently, a 2-optimal (short: 2-opt) method checks all possible interchanges of 2 edges of the route against 2 other edges. If the length of the route can be reduced, the swap takes place and the search for an improvement starts again. The procedure is over as soon as no improvement can be achieved by checking all possible interchanges [4].

In the following example (see Figure 6), an initial tour (a) should be improved by a 2-opt method.

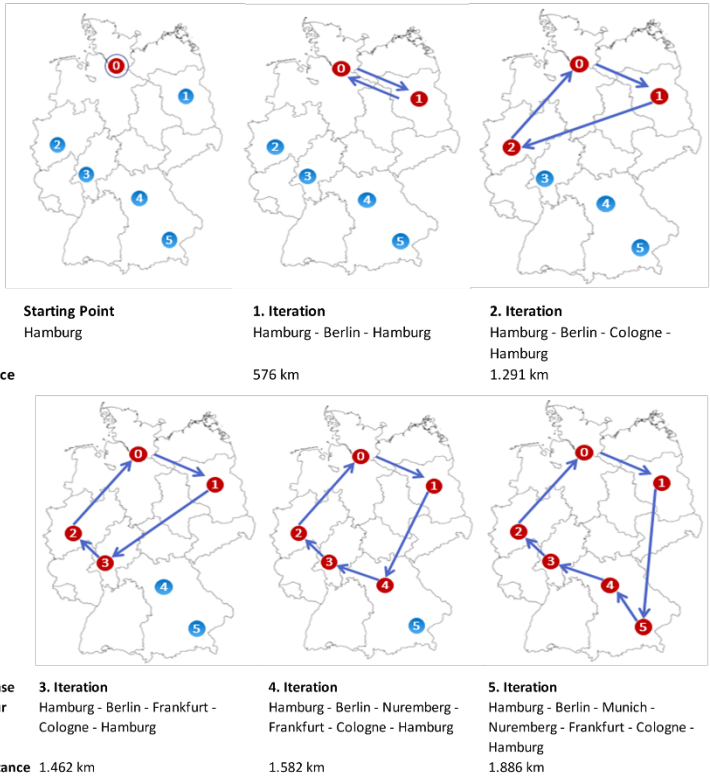
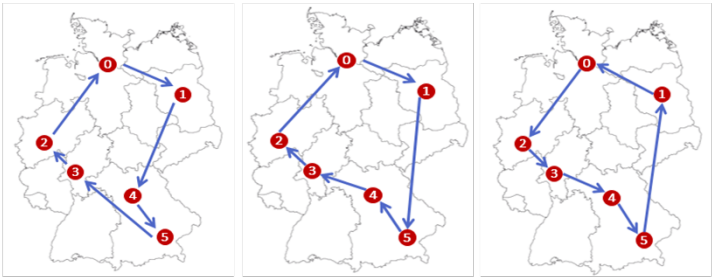


Fig. 4 TSP Nearest-Insertion Heuristic. Source: Own representation.

The first improvement (b) is the exchange of the edges AB and CD for AC and BD. An even shorter route (c) results if the edge EF is replaced by EA and the edge AC by FC. The second improvement (d) is the replacement of the edge Depot-E for Depot-A and the edge AF for EF. It is not possible to improve this route anymore because the distance can no longer be shortened by exchanging edges. If no further improvement of the route is possible, a local optimum is found [5].

The TSP Problem can also be solved with a stochastic heuristic as the Simulated Annealing. The Simulated Annealing looks for a random solution. If this solution leads to an improvement of the existing solution, it is adopted. In contrast to the deterministic improvement procedures, a temporary degradation is also accepted. In the event of degradation, the solution is only adopted with a certain probability. This probability depends on the extent of the degradation. In the further problem-



Heuristic	Nearest-Neighbour Heuristic	Nearest-Insertion Heuristic	Farthest-Insertion Heuristic
Tour	Hamburg - Berlin - Nuremberg - Munich - Frankfurt - Cologne - Hamburg	Hamburg - Berlin - Munich - Nuremberg - Frankfurt - Cologne - Hamburg	Hamburg - Cologne - Frankfurt - Nuremberg - Munich - Berlin - Hamburg
Distance	1.913 km	1.886 km	1.886 km

Fig. 5 TSP – Comparison of the Represented Opening Procedures. Source: Own representation.

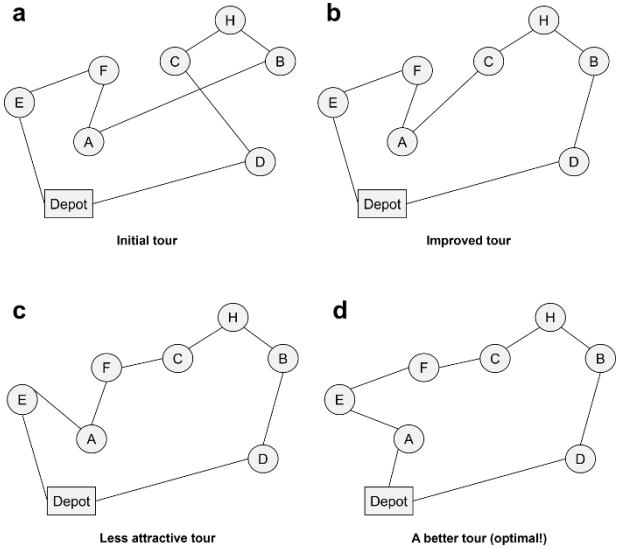


Fig. 6 TSP Possible Tour Improvement with the 2-Opt Method. Source: Own representation according to Salhi (2017) [5].

solving process, the probability is gradually reduced ("annealed") by using a so-called temperature parameter [4].

2.2 Dynamic Warehousing

Many businesses in the production industry and some other branches have the problem of an ever-increasing volume of orders and therefore a high variability in the number of those. To face the issue of optimal production quantities to meet the requirement of operations at minimal effort there is the possibility to use another heuristic method. The so-called "Dynamic Warehousing" is defined as follows:

> "Dynamic warehousing is a warehousing strategy consisting of purchasing warehousing services in a pay-per-use model, on an (electronic) marketplace. This is also called on-demand warehousing." [15]

To describe the procedure of dynamic warehousing the example below is given.

The fictitious production company "What Ever Ltd." gives an estimation of the demand for material (r) for their production for the subsequent period (t) of six months. To facilitate the analyses the following assumptions were made:

1. Consideration of only one product/material
2. No lead time (e.g. for delivery)
3. No stock at the end of the last period
4. Inventory carrying cost rate and order costs remain stable [16]

The company needs to identify a forward method to determine the average costs per period as a function of the number of periods. The goal is to find the optimal order quantity and the optimal point of time to minimize the inventory carrying costs and order costs. An overview of the values can be found in Table 2.

Table 2 Parameters of the "What Ever Ltd." Example

Month (i)	1	2	3	4	5	6
Demand (d_i)	80	80	90	100	100	120
Order Cost (c_{order})	80 € / order					
Inventory carrying cost rate ($r_{inventory}$)	0.60 € / part					
Inventory carrying Factor ($f_{inventory}$)	0.5 for the 1st month 1.5 for the 2nd month 2.5 for the 3rd month 3.5 for the 4th month 4.5 for the 5th month 5.5 for the 6th month					

Source: Own representation according to Weller et al. (2017) [17].

The table shows a slight rise in the demand over the period of six months from 80 to 120. The order costs remain stable at 80 € per order and the inventory carrying cost

rate for each item in the order amounts to 0.60 €. Furthermore, it shall be assumed that the warehousing factor rises steadily with the following values: 0.5, 1.5, 2.5, 3.5, 4.5, and 5.5.

To meet the requirement of operations at minimum costs there are several approaches. The problem can be solved analytically or even by a heuristic instrument. Figure 7 gives an insight into potential solution approaches within lot sizing techniques. To give an exemplary illustration of problem-solving through heuristic methods two approaches are chosen to solve the situation: Least-Unit-Cost-Method (LUC) and Part-Period-Balancing (PPB) [16].

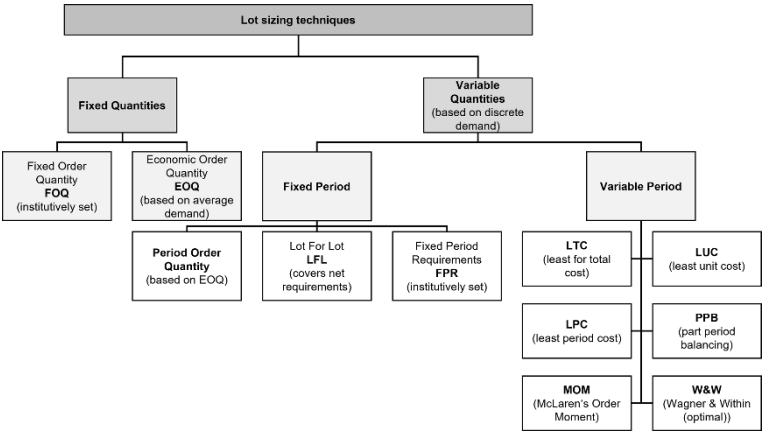


Fig. 7 Differentiation of Lot Sizing Techniques. Source: Own representation according to Ribo Auto Parts Co., Ltd. (ed.) (2016) [18].

The Least-Unit-Cost-Method focuses on the minimization of the costs per unit. The costs per unit are defined by the following mathematical formula:

$$\frac{c_{\text{inventory}i} + c_{\text{order}}}{d_i}, \tag{2}$$

whereas $c_{\text{inventory}_i}$ consists of each order volume v_i multiplied by the inventory carrying factor $f_{\text{inventory}}$ and the inventory carrying cost rate $r_{\text{inventory}}$. The order costs are defined by c_{order} whilst d_i constitutes the demand as a function of the period.

For orders stretching over multiple periods, for instance, if an order is placed in period $i = 1$ for periods one, two, and three, the formula must be modified to:

$$\frac{(\sum_{k=1}^{j} f_{\text{inventory}} * v_{i+k-1}) * r_{\text{inventory}} + c_{\text{order}}}{\sum_{h=1}^{i+j-1} d_h}, \tag{3}$$

whereas j is the number of periods for which the order is placed. If an order is placed in period two for the following periods two, three, and four, then $i = 2$ and $j = 3$ [16].

This compares to the Part-Period-Balancing-Method (PPB) which has the purpose that the inventory carrying cost $c_{\text{inventory}}$ have to be lower than the order costs c_{order}. Inventory costs for a certain period $c_{\text{inventory}_i}$ are still calculated as in the LUC method. In contrast to this method, the PPB method compares the value of the inventory carrying cost with the order cost. The different methods are illustrated in Figure 8.

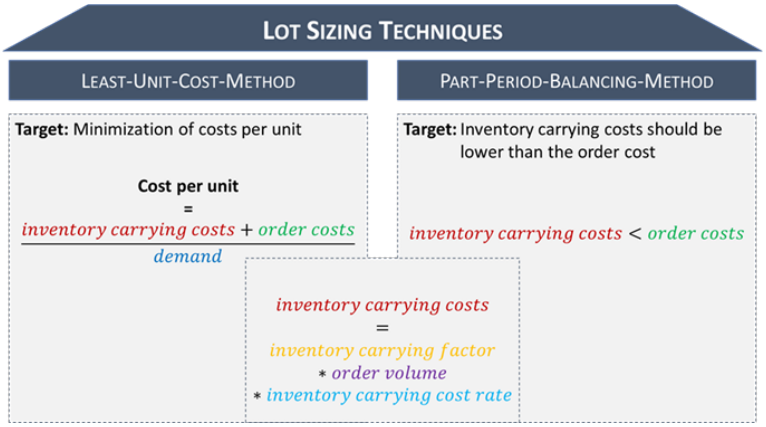


Fig. 8 Comparison of LUC and PPB Approaches. Source: Own representation based on Weller et al. (2017) [17].

To get a better understanding of the theoretical principles the following example gives a short insight into practical implementation. In the first period "What Ever Ltd." decides whether they want to order for just the first month or beyond that. The calculation steps are shown in Figure 9. If the company would only order for the first period, they would have to pay 1.30 € per unit. In case they stretch the order over the first and second month the costs per unit would be reduced by 0.20 € at a level of 1.10 € per unit. Adding an additional month and raising the order to 250 would result in a decrease of 1.24 € per unit. Within the focus to reduce the costs per unit at a minimal level, the company should decide to order 160 units per 1.10 € in the first month. An order of 80 or 250 units in the first month would lead to higher costs per unit. This procedure has then to be calculated for the leftover periods three to six [16].

As a result of applying the LUC method, the company should buy their orders in a 2-0-2-0-2-0 order pattern. Thus, they will reduce their costs per unit at a minimum

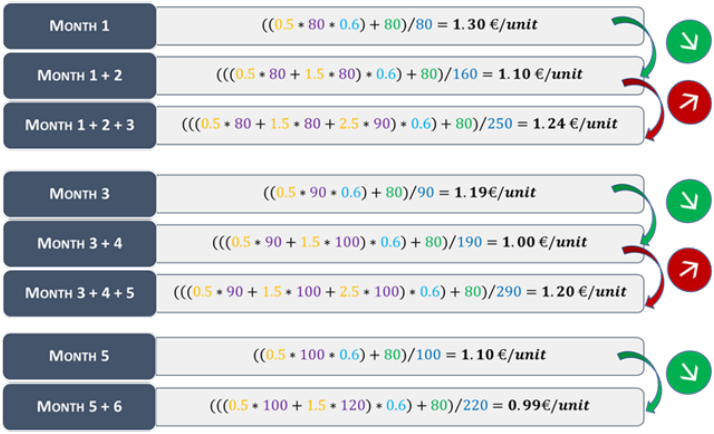


Fig. 9 Calculation Overview for the LUC Method. Source: Own representation based on Weller et al. (2017) [17].

and the amount of the total costs to 591 Euros. Table 3 shows the outcome of the previous calculations.

Table 3 Result Overview for the LUC Method

Month (i)	1	2	3	4	5	6
Demand (d_i)	80	80	90	100	100	120
Order Cost (c_{order})	80 € / order					
Inventory carrying cost rate ($r_{inventory}$)	0.60 / part					
Inventory carrying Factor ($f_{inventory}$)	0.5 for the 1st month 1.5 for the 2nd month 2.5 for the 3rd month 3.5 for the 4th month 4.5 for the 5th month 5.5 for the 6th month					

Source: Own representation.

The PPB method deals with a comparison of costs. As in the prior section already mentioned is the goal of this method to keep the inventory carrying costs smaller than the order costs.

An order decision will be reached under the condition that the following formula is fulfilled:

$$\left(\sum_{k=1}^{j} f_{\text{inventory}_k} * v_{i+k-1}\right) * r_{\text{inventory}} < c_{\text{order}} \tag{4}$$

This shall be made clear using the same example as for the LUC method. Figure 10 gives an overview of the calculation steps. In the first month the inventory carrying costs amount to 24 Euro. This is less than the order costs, which amount to 80 Euro. If the order is stretched and the second month is also taken into account the inventory carrying costs increase to a value of 96 Euro. This means that the inventory carrying costs exceed the order costs by 16 Euro, making an order for 2 months or more no longer profitable. This will lead to the decision of ordering just 80 units in the first month. This process has to be repeated for the remaining month two to six.

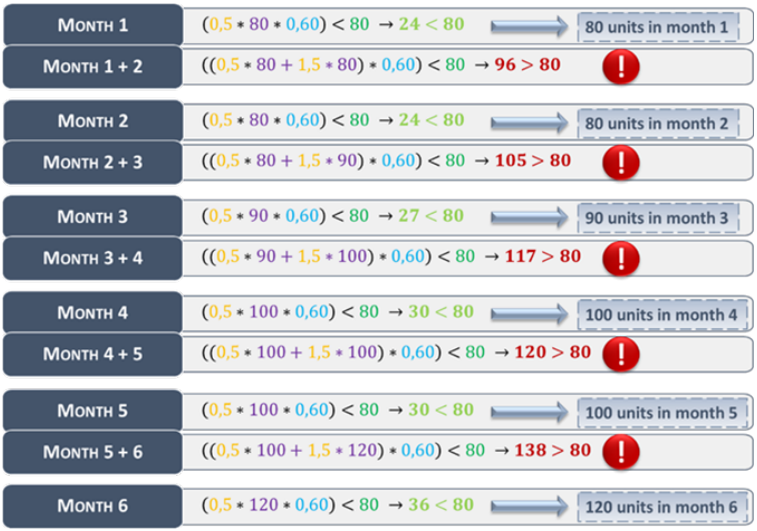


Fig. 10 Calculation Overview for the PPB Method. Source: Own representation according to Weller et al. (2017) [17].

Applying the PPB method, the table shows that it is not cost-effective to stretch the orders over several months. Instead, the method shows that a 1-1-1-1-1-1 ordering pattern is the one with the highest effectiveness. In this case the amount of total cost to 651 Euro. The costs are shown in Table 4.

The inventory carrying costs rise with increasing order quantity, while the order costs remain stable at the same level of 80 Euros. The decisive factor is, therefore, $c_{\text{inventory}}$.

Comparing the LUC method with the PPB method, it becomes clear that they produce different results. While the PPB method aims to ensure that the optimum

Table 4 Result Overview for the PPB Method

Month (i)	1	2	3	4	5	6
Demand (d_i)	80	80	90	100	100	120
Order Cost (c_{order})	80 € / order					
Inventory carrying cost rate ($r_{inventory}$)	0.60 € / part					
Inventory carrying Factor ($f_{inventory}$)	0.5 for the 1st month 1.5 for the 2nd month 2.5 for the 3rd month 3.5 for the 4th month 4.5 for the 5th month 5.5 for the 6th month					

Source: Own representation according to Weller et al. (2017) [17].

order costs and storage costs are approximately the same, the LUC method has the intention to minimize the cost per unit. Both heuristic methods yield different results. Conclusively, total costs are different between the two methods (see Table 5): 591 Euros for the LUC and 652 Euros for the PPB. On the basis of this result and if the cost factor is the determining factor, the LUC method should be chosen [16].

Table 5 Comparison of LUC and PPB Method with Wagner-Within-Algorithm

	Least-Unit-Cost Method (LUC)	**Part-Period-Balancing Method (PPB)**	**Wagner-Within-Algorithm**
Order	2 – 0 – 2 – 0 – 2 – 0	1 – 1 – 1 – 1 – 1 – 1	2 – 0 – 2 – 0 – 2 – 0
Total Costs	591.00 €	651.00 €	591.00 €
Type of Method	heuristic	heuristic	exact(optimum)

Source: Own representation according to Weller et al. (2017) [17].

In conclusion, both heuristic methods offer a first approximate solution. However, they cannot necessarily provide the optimal result. This can be precisely determined by the Wagner-Within algorithm. This is a procedure for determining the optimum lot size for a product with dynamic demand in single-stage production without consideration of capacity restrictions [19].

Without extensively covering this particular approach in detail, using the latter leads to a 2-0-2-0-2-0 ordering pattern as a reference. With this approach, an optimal cost calculation can be made, which produces the same result as the LUC method. A comparison of the results can be found in Table 5.

2.3 Knapsack Problem

The Knapsack Problem describes a selection problem more specifically a packing problem. Only a limited number of items can be packed into the knapsack from a certain quantity with an assigned weight and value, as it has a capacity limit. The

question that arises here is according to which preference the individual articles are selected. This problem is to be explained using the Knapsack method with its various approaches [20].

Consider Jack, a bank robber who has made it his goal to break into a store and steal jewelry. For this he brought a knapsack with him, which can carry a maximum weight (c) von 2,000g. Every single item he steals has a certain weight (w_i) and a corresponding value (v_i). His goal is to make a profit of 40,000 Euros. When he breaks into the shop, he finds items listed in Table 6.

Table 6 Items of the Knapsack Problem

Item	1	2	3	4	5	6	7	8	9	10
Weigth in $g(w_i)$	200	135	655	180	80	520	700	130	210	680
Value in Euro (v_i)	7,000	3,780	15,065	4,320	3,200	14,040	25,900	2,600	7,560	14,960

Source: Own representation according to Honerlage et al. (2017) [21].

Now Jack has to decide whether to take the item or not (yes or no). This can al-so be shown in the following notation:

{0,1}

This binary decision basis results in Jack having 210 = 1,024 possibilities to decide between the individual items. To make a good decision in a reasonable time, a heuristic method should be used. Considering choosing between 60 items, the possibilities increase to 260 = 1,152,921,521,504,606,846,976. Assuming that a computer has a computing power of 1.000.000.000 possibilities per second, it would need 1,152,921,504.61 seconds or 36.56 years to calculate all possibilities. To address this problem, two approaches are described below [16].

For these restrictions some data need to be taken into consideration:

- Weight ≤ 3,000 g
- Profit ≥ 40,000 €

2.3.1 An Intuitive Approach

In a first step, the articles are sorted according to their value, starting with the one with the highest value, then the one with the second highest value (etc.), resulting in Table 7.

Table 7 Items Sorted by Value

Item	7	3	10	6	9	1	4	2	5	8
Weigth in $g(w_i)$	700	655	680	520	210	200	180	135	80	130
Value in \$ (v_i)	25,900	15,065	14,960	14,040	7,560	7,000	4,320	3,780	3,200	2,600

Source: Own representation according to Honerlage et al. (2017) [21].

In the second step, the items are then gradually packed into the knapsack. Another item is added until the maximum weight of the knapsack is reached. In this case, items 7 and 3 would be packed and the knapsack would be filled to a weight of 1,355g. Since item 10 would exceed the maximum load capacity of the knapsack with 680g, this is skipped and the following item is added. This is item 6 with a weight of 520g, so by adding this item the total weight amounts to 1,875g. This means that the backpack still has a capacity of 125g. The table shows that only one item can be added to the knapsack, and that is the one weighing 80g. With a weight of 1,955g, the maximum carrying the weight of the knapsack is almost reached and no more units can be added in this case. If now the terms yes and no are replaced with a mathematical notation, the following representation results:

{1,1,0,1,0,0,0,0,1,0}

Through this approach, Jack achieves a total profit of 58,205 Euros, which means he has reached his goal of a minimum profit of 40,000 Euros. This is graphically illustrated in Table 8.

Table 8 Possible Approach of Knapsack Packing

Item	7	3	10	6	9	1	4	2	5	8
Weigth in $g\,(w_i)$	700	655	680	520	210	200	180	135	80	130
Value in \$ (v_i)	25, 900	15, 065	14, 960	14, 040	7, 560	7, 000	4, 320	3, 780	3, 200	2, 600
Cum. Weight in g	700	1, 355	–	1, 875	–	–	–	–	1, 955	–
Cum. Value in \$	25, 900	40, 965	–	55, 005	–	–	–	–	58, 205	–

Source: Own representation according to Honerlage et al. (2017) [21].

In the following, however, two questions should be asked:

1. Is there any possibility to further optimize the result?
2. Where is the result maximum and how far is the result calculated by the heuristic method from this? [20]

This will be explained in the following chapter.

2.3.2 Optimization of the Approximation Method

In the previous chapter, the first approach was explained in order to achieve a high profit. This chapter now is concerned with optimizing this approach and then comparing it with the analytically calculated optimum. Instead of focusing only on the profit that an item makes, the following calculation is about determining the value of the item in relation to its weight. The so-called profit density is calculated as follows:

$$\text{profit density} = \frac{\text{value } (v_i)}{\text{weigth } (w_i)} \tag{5}$$

In the following, the calculation of the profit density results in values that are shown in Table 9. These values are sorted from the highest to the lowest result.

Table 9 Data Sorted by Profit Density

Item	5	7	9	1	2	6	4	3	10	8
Weigth in $g(w_i)$	80	700	210	200	135	520	180	655	680	130
Value in Euro (v_i)	3, 200	25, 900	7, 560	7, 000	3, 780	14, 040	4, 320	15, 065	14, 960	2, 600
Profit density	**40**	**37**	**36**	**35**	**28**	**27**	**24**	**23**	**22**	**20**

Source: Own representation according to Honerlage et al. (2017) [21].

The sorting by profit density shown in the table is defined as part of the Greedy Algorithms [22]. Behind the Greedy Algorithm is a procedure that limits itself to sorting a basic set in descending order by its values and then adding a part of the total set to a knapsack until a certain limit is reached [23].

This step is now applied to the example described above. Jack sorts the items according to their profit density, then adds one item after another to the knapsack. He checks the cumulative weight in each step. As long as the limit of the knapsack has not been reached, he adds another item. It is still necessary to check whether the value restriction is fulfilled. This step is shown in Table 10:

Table 10 Weight Restriction in the Profit Density Order

Item	5	7	9	1	2	6	4	3	10	8
Weigth in $g(w_i)$	80	700	210	200	135	520	180	655	680	130
Value in Euro (v_i)	3, 200	25, 900	7, 560	7, 000	3, 780	14, 040	4, 320	15, 065	14, 960	2, 600
Profit density	**40**	**37**	**36**	**35**	**28**	**27**	**24**	**23**	**22**	**20**
Cum. Weight in g	80	780	990	1, 190	1, 325	1, 845	–	–	–	1, 955
Cum. Value in Euro	3, 200	29, 100	36, 660	43, 660	47, 440	61, 480	–	–	–	64, 080

Source: Own representation according to Honerlage et al. (2017) [21].

Jack can now add items 5, 7, 9, 1, 2, and 6 without exceeding the capacity limit of the knapsack. The next ones, items 4, 3, and 10 would be too heavy and have to be skipped. Only the item with the lowest profit density but a value of an additional 2,600 Euros can be added to the knapsack. In a mathematical notation, it would be written as follows:

{1,1,1,1,1,1,0,0,0,1}

With a total weight of 1,975g, Jack makes a profit of 64,080 Euros. Optimization of the result has taken place in this case, as Jack was able to increase his profit by about 6,000 Euros.

Here, too, it is advisable to check whether the result can be further improved. For this purpose, the model applied at the beginning is checked again and modified. Instead of skipping the item at the point where the weight restriction is reached the last possible item that could have been added to the knapsack is swapped with the succeeding one. Therefore, the steps of the preceding example have to be repeated. First, calculate the profit density, then sort the items by that, and afterward take the

items with the highest profit density. Then items have to be changed in exactly one position [16]. In this case items 3 and 6 would be swapped (see Table 11).

Table 11 Modified Weight Restriction in the Profit Density Order

Item	5	7	9	1	2	3	4	6	10	8
Weigth in $g(w_i)$	80	700	210	200	135	655	180	520	680	130
Value in Euro (v_i)	3, 200	25, 900	7, 560	7, 000	3, 780	15, 065	4, 320	14, 040	14, 960	2, 600
Profit density	**40**	**37**	**36**	**35**	**28**	**23**	**24**	**27**	**22**	**20**
Cum. Weight in g	80	780	990	1, 190	1, 325	1, 980	–	–	–	–
Cum. Value in Euro	3, 200	29, 100	36, 660	43, 660	47, 440	62, 505	–	–	–	–

Source: Own representation according to Honerlage et al. (2017) [21].

Due to the higher weight of item 3, it is impossible to add an additional item to the knapsack. The heuristic research comes to an end here as soon as it does not achieve an improvement with the first corrective procedure. As this is the case here because the cumulative profit is smaller than in the previous calculation, no further correction is made. Here the mathematical notation is:

{1,1,1,1,1,1,0,0,0,0}

In order to be able to evaluate such a heuristic procedure, a comparison with an analytical determination of the profit maximum is required. In order to be able to judge whether the heuristic method is sufficiently suitable to be used, conditions must be defined. Such a condition could be that the result of the heuristic procedure reaches at least 80% of the optimum of an analytical determination. For knapsack problems, profit is usually the decisive factor for the evaluation.

One way to identify the optimum of all possibilities is an enumeration. For this purpose, every possible result is considered and the gain of the individual possibilities is compared under the condition of the weight restriction. In the case of a bank robbery, this would mean that 1,024 possibilities would have to be calculated. Consequently, this method identifies the optimal solution, but due to its exhaustive search for larger quantities which is really time-consuming and thus a corresponding inefficiency, it should not be used [20].

Another possibility is the method of dynamic optimization. Dynamic optimization (DO) offers solutions for decision problems where a sequence of interdependent decisions has to be made in order to achieve an optimum for an overall problem [4]. Due to the scope of the dynamic optimization procedure, this is explained in more detail in another chapter.

Looking at the example, if the current time value of the objects in the knapsack is to be maximized, an exact determination of the optimum (dynamic optimization, enumeration, etc) would make sense. However, this can only be achieved under the condition of a high expenditure of time and/or programming. The application of a heuristic method, on the other hand, is a comparatively easy procedure. It is possible to identify an approximately good solution with little effort.

3 Conclusion

A heuristic is a rule of thought that can be used to reduce the effort required to solve complex problems. In the first instance, they provide a solution-optimized approach. Due to the lower complexity, the solution only offers a local optimum and not a global optimum, as analytical methods do. In Operations Research, heuristics solve optimization problems such as order, allocation, grouping, and selection problems. In addition, many business areas of companies consciously and unconsciously use heuristic methods. They save time and include both quantitative and qualitative features.

Even if various heuristic procedures exist, they are all based on the same principles. The opening procedures are used to determine the first valid solution. To solve the Traveling-Salesman Problem, the neighbor and insertion heuristics have to be mentioned in the context of opening procedures. If an existing solution should be optimized by a heuristic procedure, it is called an improvement procedure. The procedures presented in this essay to solve the TSP are the r-opt method and Simulated Annealing. In addition, the knapsack problem deals also with the procedure for improvement in an opening procedure. By adding or removing items from the backpack, the trial and error principle is used to identify the best possible solution. Dynamic warehousing has to be categorized as incomplete exact procedures. The search has been stopped when the quality of the result was enough.

A heuristic method should be used if a satisfactory but possibly not optimal solution quality is accepted and if the effort to solve a problem should be kept low so that the applicability is high compared to analytical methods. Even if they do not provide optimal results, it often makes sense to use such a procedure. Heuristics are used for decision-making when it is important to find quick and economically acceptable solutions. In the age of digitization and the advance of computing capacities on "supercomputers", heuristic methods are progressively losing importance. Through the advance of artificial intelligence and the inclusion of measurable and non-measurable decision parameters, it is increasingly possible to achieve near-optimal to optimal results. Nevertheless, the decision for or against a heuristic method should be made depending on the time required, the given information, and the precision of the results.

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Part II
Simulation

Simulation Processes in Business and Economics: Fundamentals of the Monte Carlo Simulation

Alexander Wachholz & Richard Malzew

Abstract Simulations, especially Monte Carlo simulations, are a common tool in operations research. They enable the study of complex scenarios, evaluation of cost-effectiveness, risk mitigation, and efficient testing of improbable situations. Monte Carlo simulations facilitate decision-making by creating realistic experimental models for various operations research purposes.

1 Fundamentals of Simulation Processes

1.1 Definition and Purpose

Fig. 1 Purpose of Simulations. Source: Own representation according to Lipinski et al. (2017) [2].

The purpose of simulations is to gain knowledge that can be transferred to real processes [1]. 1 It is irrelevant here whether, within the framework of the simulation, the

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systems are real or imaginary. The result of the simulation describes an experimental model with realistic dynamic processes and serves as a basis for decision-making. As can be seen in Figure 1, there are different purposes for simulations. They can be used, for example, to assess cost-effectiveness, and reduce security risks and complexity. Simulations have the advantage that situations can be tested which are considered impossible in practice. In many cases, simulations help to save time because they are faster in execution than real-time tests. Simulations can be helpful for a baseline systematic presentation. Simulations describe in the broadest sense the preparation, execution, and evaluation of simulations with a simulation model [2].

Figure 2 shows a classification model for simulations. In this model, simulations can be divided into the situation, complication, question and answer (SCQA) model - also known as the minto-pyramid principle. With regard to the first dimension of SCQA, the situation contains the description of the status quo, followed by a problem in the situation, which is further explained within the com-plication. Additionally, the complication generates the tension in the respective situation. The third dimension, the question, will emerge within the complication and lead to possible answers. There are two different ways to check the answers resulting from the previous steps. Firstly, the answers can be tested in real experi-ments. Secondly - for the reasons mentioned above - they can also be tested with a simulation model. After the test results have been substantiated, an adequate decision can be made which leads to a solution [3].

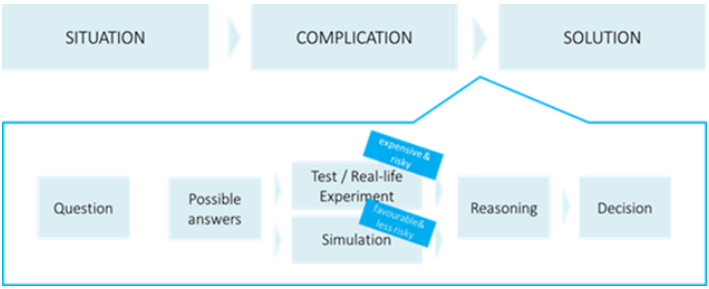

Fig. 2 Classification in the SCQA Model. Source: Own representation according to Minto (2002) [3].

1.2 Workload

A further advantage of simulations is that they can be adjusted in time, which has a temporal advantage in the case of very slow or very fast systems - e.g., slow motion of evolutionary processes or similar. Simulations are also useful if the complexity of the real system does not allow for observations or if the economic profitability is not

given. An important area of application is also the training of personnel for complex systems such as pilot training in a flight simulator [2].

1.3 Fields of Application

Three main areas in which simulations are used can be distinguished as scientific cases, technical cases, and economic cases. This work focuses on economic issues, but the main areas of application are meteorology, medicine, mobility, traffic planning, and physics. The focus in these specific areas depends on various variables. On the one hand, corporate planning is geared toward the economic context, and simulations are primarily used for decision-making [4]. On the other hand, the focus in medical education is on the scientific context. In these cases, it is used primarily for research purposes [5].

1.4 Types of Simulations

Basically, two types of simulations can be distinguished: the deterministic simulation and the stochastic simulation. In a deterministic simulation, the input and output variables are clearly defined [6]. This means that for each input parameter, an output parameter exists in a 1:1 relationship. It is assumed that all components and the interrelationships are fully predictable. However, it should be noted that in most cases it is not realistic to set clearly defined input and output parameters in a complex system [7]. In contrast to deterministic simulation, stochastic simulation works with random variables. This results in a dependence on random variables. A stochastic simulation performs many iterations based on random variables and produces a different output in each iteration in a one-to-n (1:n) relationship [6]. Depending on the random variables, the input parameters are not reproducible. A variant of a stochastic simulation is the Monte Carlo simulation, which generates random numbers to simulate a random process [6].

2 Monte Carlo Simulations

2.1 Introduction

The Monte Carlo method is an example of one of the stochastic simulations described in the first chapter. Sobol describes it as a numerical technique for solving mathematical problems by modeling random numbers [8]. It uses mathematical principles such as probability theory, the law of large numbers, the central limit theorem, and the sampling method with a large number of samples. One of the most common Monte

Carlo simulations is the approximation of the expected value via the Monte Carlo evaluator in mathematical models with undefined or not clearly defined integrals [9].

A Monte Carlo simulation is a computer-based method for assessing future events and supports decision-makers in evaluating various options [10].

The advantage of the Monte Carlo method is that it allows the modeling of uncertain situations, plays it a hundred or a thousand times on a computer, and estimates the probabilities of the occurrence of a certain situation (most Monte Carlo simulations use at least 1,000 runs) [10].

The method also illustrates the consequences of moderate decisions. The procedure simulates random numbers for each factor involved. The combination of different random numbers for the numerous factors involved yields the result of the scenario. This allows mathematical problems to be solved - in contrast to deterministic methods - with random numbers for certain parameters [11].

The first known Monte Carlo simulation was used by the scientist Georges Louis Leclerc de Buffon in the 18th century. He wanted to calculate the probability that a needle thrown by him would end up in a defined pattern. So, he threw the needle several times, then noted and analyzed the results. This test is known as a precursor to the Monte Carlo simulations [12].

The most famous founding father of the Monte Carlo simulation is Stanislaw Ulam, who was a Polish-American scientist. In 1946 he applied the method for the first time together with John von Neumann, a Hungarian-American computer scientist. They used Monte Carlo simulations during the Second World War to calculate neutron diffusion. Later, these simulations contributed to the development of the atomic bomb. The name "Monte Carlo Method" was chosen for reasons of secrecy and is based on the names of the famous casino in the Monte Carlo district of Monaco [13].

As shown in Figure 3, the process of using a Monte Carlo simulation is divided into five steps: developing the model logic, defining the input distribution, determining the number of iterations, running the simulation, and analyzing the results.

The process starts with the first step of developing the model logic, which means that the user determines which parameters influence the other variables (usually based on mathematical formulas or models). Examples of input variables are sales or variable costs, and those for output parameters are risk factors or present values. Furthermore, the user of the Monte Carlo method is able to determine the strength of the influence. At least one parameter is required, but usually, at least two or more parameters are included in the simulation.

The second step is to define the input distribution. This means defining which possible results in the parameters can assume and which probable distribution results from the combination of the parameters. Normally, a moderate value in the middle of the scale is much more likely than an extreme value.

In the next step, the user sets the number of iterations that the computer program should execute or repeat steps one and two. Since the simulation is based on the law of large numbers and the central limit theorem, it is very important that the number of iterations is high enough to obtain reliable data. Various experts recommend

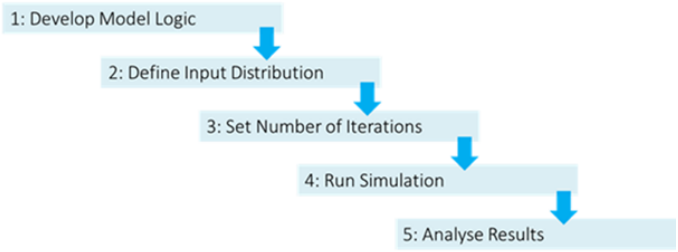

Fig. 3 Five Steps of a Monte Carlo Simulation. Source: Own representation according to Gleißner (2017) [14].

performing the simulation at least 1,000 times, whereas Haufe (Ed.) recommends repeating the Monte Carlo simulation at least 10,000 times to obtain reliable data.

It depends, above all, on the model observed and the uncertainties contained therein. The more uncertainties there are, the more iterations are required to obtain reliable results. Overall, a higher number of iterations usually leads to a higher quality of the Monte Carlo method.

The fourth step is to run the simulation. The computer program determines random numbers for the parameters from the first step using the distribution from the second step and converts them into a result. This is repeated according to the number of iterations specified in step three. At the end of this step, there are many hundreds or thousands of results for the same scenario.

These results are analyzed in step five. From the distribution of the results, the user gets the probability of a certain event, e.g., for the probability that the net present value of an investment will be positive [14]. Supported by the computer program, it is also possible to assign the results to different scenarios. In the above example, you can determine the probability that the net present value of an investment will be [14].

The following case study is a simple example demonstrating how a Monte Car-lo simulation with the Microsoft Office Excel Add-In @Risk from Palisade works.

2.2 Case Study - Project Management

The case study supports the planning process of a project where specific tasks must be done, and the end date of the project has to be determined. In this example, specific tasks must be done in this project. Some tasks can only be started when another task has been completed. The specific duration of a task is in most cases not known, so a distribution of the expected duration of every task can be defined.

With the help of Risk, the project is simulated and returns a distribution of the simulated end dates. This result could be used by a project manager for the determination of a project end date considering the probability of the project duration.

The significance of the result highly depends on the accuracy of the estimated task durations. Weekends and other influences have been excluded in this case study.

Step 1: Develop Model Logic

The start day is the only mandatory known input in this example. The other input values are the project task durations. A task can only start the day after the previous task is completed. If no necessary previous task exists, the task can be started at the beginning of the project. The formula for the end date of every single task (i) is computed by:

$$\text{EndDate}_i = (\text{IF}(\text{Prev}_i ! = NULL, \text{StartDate}, \text{EndDate}_{\text{Prev}} + 1)) + \text{Duration}_i \quad (1)$$

The simulation result corresponds to the end date of the last task of the project, so the formula for the project end date is:

$$\text{ProjectEnd} = \max(\text{EndDate}) \quad (2)$$

The project consists of five tasks shown in Figure 4. Some tasks are dependent on another task. The start day of the project is set for Jan. 6th, 2020.

Known Inputs	
Start Date	06.01.2020

Uncertain Inputs

Task Name	Start	Previous	Duration	End
Task 1	06.01.2020		10	16.01.2020
Task 2	17.01.2020	Task 1	12	29.01.2020
Task 3	06.01.2020		9,666667	15.01.2020
Task 4	30.01.2020	Task 2	5	04.02.2020
Task 5	05.02.2020	Task 4	3	08.02.2020

Output	
Project end date	08.02.2020

Fig. 4 Case Study - Model Logic. Source: Own representation.

Step 2: Define Input distribution

The uncertain parameters are the estimated durations of the defined tasks. Considering that the project has different types of tasks, every task needs its own parameter distribution for the duration. For this example, some available distributions in @Risk are used for the different parameters, displayed in Figure 5.

Task	Distribution type	Parameters	@Risk Command
Task 1	Normal	Mean = 10; StdDev = 2	RiskNormal(10;2)
Task 2	Triangle	Min = 10; Most Propably = 11; Max = 15	RiskTriang(10;11;15)
Task 3	Triangle	Min = 8; Most Propably = 9; Max = 12	RiskTriang(8;9;12)
Task 4	Uniform	Min = 4; Max = 6	RiskUniform(4;6)
Task 5	Fix value	Value = 3	3

Fig. 5 Case Study - Distributions for the Task Durations. Source: Own representation.

For task 5, no distribution is used and the duration is defined as a fixed value, so this task will take 3 days in any case. Task 4 will take a random time between 4 and 5 days.

Step 3 and 4: Set the Number of Iterations / Run Simulation

The number of iterations is set to 10,000, so the project will be simulated 10,000 times (see Figure 6). Then the simulation can be executed via the "Start Simulation" button.

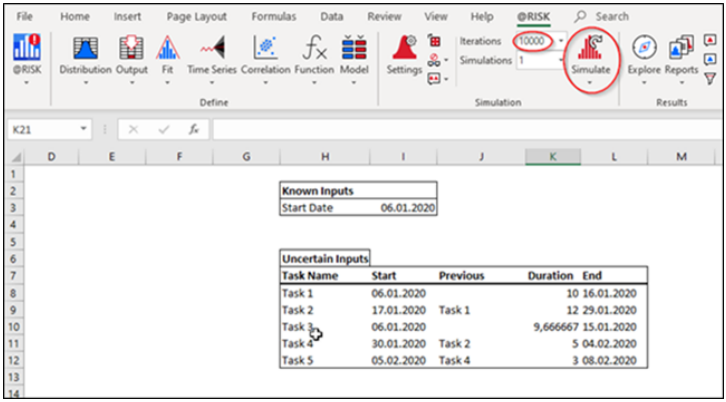

Fig. 6 Case Study – Iterations. Source: Own representation by using @Risk (Excel Add-In) (2020) [15].

Step 5: Analyse Results

As shown in Figure 7, the approximate end date of the project is the 8th of February, based on the density of the simulated project ends.

Table 1 shows the main statistic values of the simulation. In about 5% of the executed simulations, the tasks have been ready after the 11th of February, and another 5% of the simulations determined a project ended before the 4th of February. All simulation results are between Jan. 31st and Feb. 16th, 2020.

With this result, the project management has decision support for estimating or determining the project end, which was generated by the Monte Carlo simulation.

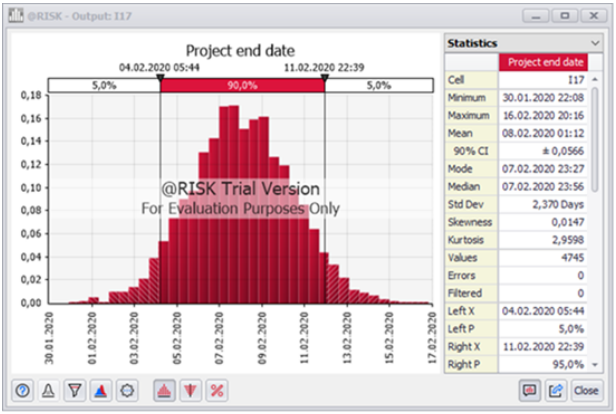

Fig. 7 Case Study - Determined Project End Date. Source: Own representation by using @Risk (Excel Add-In) (2020) [15].

Table 1 Case Study – Conclusion

STATISTIC	VALUE	INTERPRETATION
Minimum	30.01.2020	The day of minimum duration of the project.
Maximum	16.02.2020	The day of maximum duration of the project.
Mean	08.02.2020	The calculated mean value of all simulations (with the highest probability).
5^{th} Percentile	04.02.2020	5% probability that the project lasts at least this day.
95^{th} Percentile	11.02.2020	95% probability that the project lasts longer than this day.

Source: Own representation by using @Risk (Excel Add-In) (2020) [15].

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Markov Chain Monte Carlo Methods

Thomas Neifer

Abstract The Markov Chain Monte Carlo (MCMC) methods based on the Bayes theorem are used when an a posteriori distribution does not have a tractable form and is therefore not fully known or directly usable (e.g., for maximum a posteriori parameter estimation). MCMC methods overcome intractability by drawing parameter values from known distributions and correlating these drawings until they approximately match the target distribution. MCMC methods represent a powerful class of algorithms for processing data and knowledge, which is why they are also called a "quantum leap in statistics".

1 Introduction

Especially in the case of multivariate distributions with many variables, it proves to be complex in practice and therefore not pragmatic to solve them analytically. Markov Chain Monte Carlo Simulations (in the following MCMC) are used to sample a probability distribution and can iteratively approximate multivariate distributions with sufficient computing power.

Since the computing capacity has grown continuously over the last years, new possibilities to approximate even the most complex distributions with sufficient accuracy are available. MCMC is also used to estimate the distribution of parameters in a series of observations. Practitioners have so far dealt little with this technique and have mostly (wrongly) chosen other approaches, although MCMC can achieve very good results. This is mainly due to the fact that very few applications are discussed in the literature. Most of them are coin tosses or general - often discussed and not very practical - use cases in the context of statistical probability distributions. This article gives an overview of the different Markov Chain Monte Carlo techniques and practically applies the Metropolis algorithm within Simulated Annealing to solve the Traveling Salesman Problem in Python.

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2 Theoretical Foundations

2.1 Bayes Theorem

Inferential statistics allow decisions to be made for a given data set based on the knowledge of the parameters of statistical models. Classical approaches here only consider the data for estimating the model parameters. In contrast to classical methods, Bayesian statistics include not only the data but also the previous information (a priori) about the underlying problem in the parameter estimation. Furthermore, Bayesian statistics do not distinguish between parameters and observations in a model. They are all considered random variables [1].

Let Y be a random variable with associated probability distribution $P(Y = y)$ and let $P(Y = y|\theta)$ be the conditional probability distribution, which depends on the unknown parameter θ. The Bayesian approach assumes a distribution for both the data and the parameter, including a probability distribution P(θ) in the analysis, which has an a priori (before the data or "new information") and an a posteriori form. The assumed distribution for the data and the a priori distribution for the parameter are combined by the Bayes theorem, resulting in the a posteriori distribution of the parameter:

$$P(\theta|y) = \frac{P(y|\theta)P(\theta)}{P(y)} \tag{1}$$

$P(y)$ can be expressed as:

$$P(y) = \int P(y|\theta)P(\theta)\delta\theta \tag{2}$$

A priori probability:

The a priori probability describes a probability value which can be determined a priori based on previous knowledge. If no further previous knowledge is available, the principle of indifference is suitable. This principle is based on the Laplace probability and defines the a priori probability as $p = 1/n$ by assuming a discrete equal distribution [2]. For a priori distributions this means that this corresponds to the prior knowledge of the distribution for an unknown parameter θ of the total population before drawing a sample. This parameter is then to be estimated by observations y of the random variable Y based on the sample data [3].

A posteriori probability:

θ is an unknown parameter of a probability distribution, which is to be estimated by observations y of a random variable Y. Furthermore, if an a priori distribution for θ before the sample, as well as the conditional distribution of the sample data $P(y|\theta = \theta_0)$, exist, then the a posteriori distribution for θ of the total population can be determined by means of these data using Bayes' theorem [3].

Likelihood:

The Likelihood $P(y|\theta = \theta_0)$, which is also called inverse probability, corresponds to the distribution of the sample data under the condition of prior knowledge of the model parameter θ.

Bayes Theorem:

The theorem of Bayes proves that for two events, A and B, the conditional probability for A under the condition of the occurrence of B ($P(A|B)$) is calculated by the probability for B under the condition of the occurrence of A ($P(B|A)$) [4]. For a finite number of events $A_i, i = 1, \ldots, N$ the theorem of Bayes can be applied to calculate the a posteriori probability $P(A_i|B)$ as:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{N} P(B|A_j)P(A_j)} \tag{3}$$

2.2 Multivariate Distributions

We speak of a multivariate distribution of a random sample if it consists of several dimensions. In the case of exactly two dimensions, we also speak of a bivariate distribution. Multi-dimensional samples can also be understood as objects of investigation with several properties, in which the common distribution of the properties is the key question. Exemplary questions are [5]:

- What is the probability of finding an apartment that is larger than 100 m² (X) and has less than 3 rooms (Y)?
- What is the probability to get a clubs card and an 8?
- What is the probability of meeting a person with black hair and blue eyes?

The above examples are bivariate distributions with the dimensions X and Y. If the distribution of one dimension is considered separately from the others, e.g. only X, the result is a univariate distribution, the so-called marginal distribution. The conditional distribution on the other hand is the distribution of one dimension under the condition of the other dimensions. A conditional distribution describes, for example, the distribution of living space if an apartment has 4 rooms (see Figure 1).

2.3 Markov Chain

A Markov chain is a stochastic process and indicates the probability of a transition from one state to another (transition probability). It is also possible with a certain probability that no transition of state takes place and the chain remains in its current state [7].

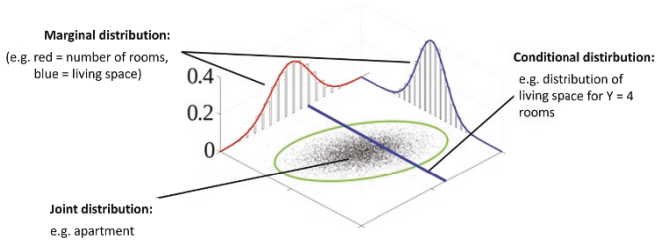

Fig. 1 Example of a Bivariate Gaussian Distribution. Source: Own representation according to IkamusumeFan (n.y.) [6].

In contrast to methods such as time series analysis, Markov Chains assumes that a prognosis is already possible with a partial view of the entire process history and that not all realizations have to be considered. The Markov property describes that Markov Chains "forget" their past after a certain period [8].

The order r of a Markov Chain determines how many (past) realizations are considered for the determination of the next state. Thus, the first order ($r = 1$) of a Markov chain considers only the current, but not further past realizations.

If X_n denotes the current state of a stochastic (i.e. random) process, which can take N different realizations from the state space S, the Markov property can be defined as [8]:

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0), i, j \in N_0 \tag{4}$$

Thus, the future state $X_{n+1} = j$ is conditionally dependent on r preceding states depending on the order of the Markov chain. Thus, a conditional probability exists. In the case of the first order, the conditional probability that the process X_{n+1} will be realized (i.e., that the state j will occur) exclusively considers the condition of the state i or the realization of the process X at the current time n.

2.4 Monte Carlo Simulation

Some problems (such as those in stock exchanges, in the field of comic radiation, or in nuclear physics) are too complex to be solved analytically with exact formulas [9]. Indeed, in 1946 Stan Ulam first tried to solve the probability of a game success of a Patience (Solitaire) deck combinatorically. Since this proved to be too complicated, he instead chose a more practical way in which he randomly selected different solitaire decks as starting points and counted their success [10]. In this way, he was able to approximate the probability of success over several random experiments.

In statistics, Monte Carlo simulations include those methods that repeat random experiments very often to approximate arbitrary distributions. For this purpose,

random numbers $x_1, x_2, \ldots, x_n$ are generated n times, each of which lies between 0 and 1. If n is chosen to be correspondingly large, this results in an equal distribution over $x_1, x_2, \ldots, x_n$. Subsequently, these random numbers are transformed into a target distribution via further steps [5].

Monte Carlo simulations can be used in many ways. A simple example is the approximation of π using the basic idea that the circular area A of the unit circle with radius $r = 1$ results from $A = \pi * r^2 = \pi * 1^2 = \pi$ [11]. The unit circle is divided into four equal squares $(\pi/4)$ with the edge length $r = 1$. Now, N times random points $p(x, y)$ are generated, where x and y are independent of each other and each of them lies in the value range $[0; 1]$. If n is the number of all points to which $x^2 + y^2 \leq 1$ applies, then N is the result of a sufficiently large N:

$$\pi \approx 4 * \frac{n}{N} \tag{5}$$

The condition $x^2 + y^2 \leq 1$ ensures, based on the Pythagorean theorem, that each point counted in n (or its coordinates) is smaller than or equal to the circle radius $r = 1$. In other words: The condition applies only to points that lie within the circle (or a quarter of the circle). The more points (capital N) are randomly generated, the more points lie within the circular area. This ensures that the ratio n/N between points inside and points outside the circular area is always approximated more exactly. The more exact the ratio is, the more accurate π is approximated.

3 Markov Chain Monte Carlo Simulation

If individual iterations of a Monte Carlo simulation are linked together so that the next random draw depends on (exclusively) the current random draw, a Markov process or Markov chain is created. The combination of both methods is called Markov-Chain Monte Carlo simulation in the literature. Many algorithms follow the MCMC scheme so that this procedure can be called a separate class [12].

As a simulation method in MCMC procedures, the parameter values of θ are drawn from approximate distributions and these drawings are corrected in the direction of the target distribution (a posteriori distribution, $p(\theta|y)$) [3]. Thereby a Markov chain is generated, which has $(\theta|y)$ as stationary distribution (i.e. as an eigenvector with an eigenvalue of 1, which remains unchanged during state transitions). The relation to the Bayesian statistics is shown here in the approximation of the a posteriori target distribution.

Each Markov chain should be aperiodic, irreducible and have a stationary distribution. At step t, the transition process should now converge to this stationary distribution as $t \to \infty$ [13]. To achieve convergence with the target distribution, the concepts of aperiodicity, irreducibility, and reversibility should be mentioned here. Irreducibility means that the Markov chain has a positive probability to reach other states. Aperiodicity demands that the chain should not be "trapped" in cycles. With respect to reversibility, the following should apply [10]:

$$p(\theta^t)p(\theta^{t-1}|\theta^t) = p(\theta^{t-1})p(\theta^t|\theta^{t-1}) \tag{6}$$

The a posteriori (multivariate) distribution of parameters often has no tractable form and is therefore neither completely known nor directly usable. MCMC simulation methods are used here, which overcome the intractability by drawing the values of θ from approximated known distributions from data (y) and correcting these drawings until they approximate the target distribution (posterior, $P(\theta|y)$).

MCMC methods perform a sequential simulation in which the distribution of each simulated value depends on the previous one and the drawings, therefore, form a Markov Chain. The Markov property ensures that the approximate distributions improve at each step of the simulation by convergence to the target value [3], [13].

The stationary distribution of the Markov process corresponds to the target distribution $P(\theta|y)$. Therefore, the goal is to run the simulation until the current distribution comes very close to the stationary distribution [3].

For the approximation of the distribution different algorithms are used, which will be presented in the following.

3.1 Metropolis(-Hastings) Approximation Algorithm

For the approximation of distributions and the determination of MAP parameter estimates, the Metropolis- [14], its later generalization Metropolis-Hastings- [15], and the Gibbs sampling algorithm [16] can be differentiated. The relationship between the algorithms can be seen in Fig. 2.

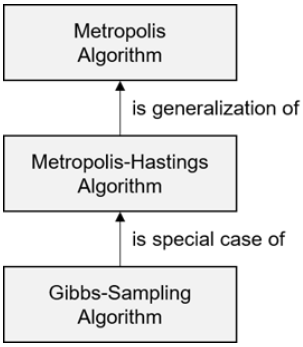

Fig. 2 Differentiation of Metropolis-, Metropolis-Hastings- and Gibbs Sampling Algorithm. Source: Own representation.

In very simplified terms, all three methods approximate distributions (or more generally speaking integrals) via MCMC simulations by very often taking samples from an initial distribution (in the Metropolis algorithms this is the a posteriori distribution, in Gibbs sampling the likelihood distribution). While the Metropolis

algorithm from the 1950s requires a symmetrical distribution for this, Hastings generalized the algorithm so that arbitrary output distributions can be used. In both Metropolis algorithms, the selection of the next step in the Markov chain is stochastic. In Gibbs sampling, on the other hand, this is done deterministically.

Metropolis-Hastings is a family of MCMC simulation methods that are used in Bayesian modeling to sample the a posteriori distribution. It is based on the Metropolis algorithm, which is an adaptation of a random walk (i.e. random step sequence, random movement) with an acceptance or rejection rule, which thus approaches a specific target distribution. A proposed distribution $J_t(\theta^*|\theta_{t-1})$ is used, from which samples are taken. The proposed distribution depends in each case on the values of the last iteration and must be symmetrical, i.e. $J_t(\theta_a|\theta_b) = J_t(\theta_b|\theta_a)$ for all θ_a and fulfill θ_b and t. If the ratio of the density of the current and the previous value is greater than the sample from the uniform equal distribution, θ^* is accepted as the new value θ^t [3]. The Metropolis algorithm can be defined as follows:

Metropolis Algorithm:

1. Initialize $t = 1$ and define start values $\theta_1^{(0)}, \theta_2^{(0)}, ..., \theta_d^{(0)}$ of vector $\theta = (\theta_0, \theta_1, ..., \theta_d)$;
2. Sample θ^* from proposed distribution $q(\theta^{t-1})$;
   a. Calculate:

$$\alpha(\theta^{(t-1)}, \theta^*) = \min\{1, \frac{p(\theta^*|y)}{p(\theta^{t-1}|y)}\},$$

   b. Calculate:

$$u \sim U[0,1]. \text{ If } u < \alpha(\theta^{(t-1)}, \theta^*),$$
$$\text{then } \theta^{(t)} = \theta^*, \text{ else: } \theta^{(t)} = \theta^{(t-1)};$$

3. Set $t = t + 1$ and repeat step 2 until the desired a posteriori distribution was drawn.

As a generalization of the Metropolis algorithm, the difference with the Metropolis-Hastings algorithm is that the proposed distribution $J(\theta^*|\theta)$ does not have to be symmetrical. With a stationary distribution $p(\theta|y)$ and a proposed distribution $q(\theta^*|\theta)$, the Metropolis-Hastings algorithm involves extracting a value θ^* at a given θ from the proposed distribution $q(\theta^*|\theta)$. The Markov chain finally approaches θ^* with an acceptance probability

$$\alpha(\theta^{(t-1)}, \theta^*) = \min\{1, \frac{p(\theta^*|y)q(\theta^{(t-1)})}{p(\theta^{(t-1)}|y)q(\theta^*)}\}, \tag{7}$$

otherwise, it remains with θ.

The lack of symmetry constraint optimizes the speed of the random walk process. Convergence to the target distribution is achieved in the same way as with the Metropolis algorithm [3]. The Metropolis-Hastings-Algorithm can be defined as follows:

Metropolis-Hastings Algorithm:

1. Initialize $t = 1$ and define start values $\theta_1^{(0)}, \theta_2^{(0)}, ..., \theta_d^{(0)}$ of vector $\theta = (\theta_0, \theta_1, ..., \theta_d)$;
2. Sample θ^* from proposed distribution $q(\theta^{t-1})$;
   a. Calculate:

$$\alpha(\theta^{(t-1)}, \theta^*) = \min\{1, \frac{p(\theta^*|y)q(\theta^{(t-1)})}{p(\theta^{t-1}|y)q(\theta^*)}\},$$

   b. Calculate:

$$u \sim U[0,1]. \text{ If } u < \alpha(\theta^{(t-1)}, \theta^*),$$
$$\text{then } \theta^{(t)} = \theta^*, \text{ else: } \theta^{(t)} = \theta^{(t-1)};$$

3. Set $t = t + 1$ and repeat step 2 until the desired a posteriori distribution was drawn.

3.2 Gibbs Sampling Algorithm

In its basic version, Gibbs sampling represents a special case of the Metropolis-Hastings algorithm. However, extended versions exist in the meantime, which is regarded as separate framework concepts for sampling from a finite set of variables.

If the common distribution of several variables is unknown or difficult to sample, but the conditional distributions of each variable are known or easy to sample, Gibbs-Sampling is used. Here, for each variable from its respective distribution successively an instance is produced, which depends conditionally on the current values of the other variables. The sequence of sampling represents a Markov chain whose stationary distribution corresponds to the common distribution sought [3].

The Gibbs sampler iteratively runs through the subvectors $\theta = (\theta_1, \ldots, \theta_d)$ of a parameter vector θ and draws each subset depending on the value of all others. This is therefore an iteration with d steps. A sequence of the d subvectors of θ is selected and θ_j^t is taken again from the conditional distribution, which contains all other components of θ:

$$p(\theta_j|\theta_{-j}^{t-1}, y) \tag{8}$$

Here, θ_j^{t-1} represents all subvectors of θ except θ_j. Each subvector θ_j is thereby updated subject to the newest values of the other subvectors of θ, i.e. the values of iteration t for the already updated subvectors and the values of iteration $t-1$ for the others [3]. The Gibbs sampler takes the proposal for this from a conditional distribution of the desired equilibrium distribution, which is always accepted [13].

The idea behind this is to sample successively from each conditional distribution of θ_j to obtain samples from the common a posteriori distribution. The Gibbs sampling algorithm can be defined as follows:

Gibbs Sampling Algorithm:

1. Initialize $t = 1$ and define start values $\theta_0^{(0)}, \theta_1^{(0)}, ..., \theta_{j-1}^{(0)}$ of vector $\theta = (\theta_0, \theta_2, ..., \theta_j - 1)^T$;
2. Draw samples:

$$\begin{aligned}
\theta_0^t &\sim p(\theta_0|\theta_1^{(t-1)}, \theta_2^{(t-1)}, ..., \theta_d - 1^{(t-1)}, y) \\
\theta_1^t &\sim p(\theta_1|\theta_1^{(t)}, \theta_2^{(t-1)}, ..., \theta_d - 1^{(t-1)}, y) \\
&\vdots \\
\theta_d - 1^t &\sim p(\theta_j - 1|\theta_1^{(t)}, \theta_2^{(t)}, ..., \theta_d - j - 2^{(t)}, y) \qquad (9)
\end{aligned}$$

3. Set $t = t + 1$ and repeat step 2 until the desired sample for every θ_j was observed.

4 Solving Traveling Salesman Problem using Metropolis Algorithm in Python

The Traveling Salesman Problem (TSP) is an NP-hard problem of combinatorial optimization. It describes the situation of an action traveler who wants to visit N places exactly once and finally return to his hometown [17]. The shortest route to travel to the places is searched for. In principle, the solution sounds obvious and simple: For N cities, which represent the nodes, all possible paths and the distance of each path must be determined. Then every possible sequence (route) is determined and the smallest possible one is selected. However, the number of possible sequences for N cities increases with their faculty $(N-1)!/2$ and for five cities exist accordingly: $(5-1)!/2 = 5!/2 = 60$ possible routes. For ten cities these would already be $(10-1)!/2 = 181,440$ possible routes [18].

If the brute force method[1] underlying the above procedure is applied here, even a high-performance computer would quickly need several years to determine the optimal route as N increases [20]. This is due to the fact that this is a super polynomial (to a maximum exponential) increasing runtime. Traditional linear optimization problems are limited to a polynomial of the size of the input [21].

Here, heuristic methods are used, which approximate an optimum. In addition to e.g. Nearest Neighbor algorithms, there is a possibility to solve the TSP in Simulated Annealing, which is based on the above-mentioned metropolis algorithm [22].

Simulated Annealing was developed in analogy to the creation of solids by cooling molten metals. If the cooling is sufficiently slow, an equilibrium of the metal structures is achieved with minimal energy input. If a poor structure is reached during the process, the additional energy supplied allows the structure to break away from the previous structure to the desired equilibrium [23], [24].

In terms of the TSP, a starting solution, i.e. a random route, is assumed, which is optimized by an iterative process through random combinations of locations in connection with a probabilistic acceptance criterion (in this case a shorter total distance). If the new route is shorter, it is considered as a new solution candidate, in the opposite case it is not considered and a new iteration starts [24].

To solve the TSP by means of Simulated Annealing an algorithm in Python was developed, which will be presented in the following. The required libraries include numpy, matplotlib.pyplot, copy, and mpmath.

Import Libraries

```
import numpy as np
import matplotlib.pyplot as plt
import copy
import mpmath as mp
```

For the specification of the locations, a dictionary was created for each location, which contains the name of the location as well as the corresponding latitude and longitude data. In this example, a traveling salesman is supposed to visit ten places in Germany: Berlin, Munich, Hamburg, Cologne, Bremen, Stuttgart, Bonn, Nuremberg, Dresden, and Dortmund.

Define Cities

```
#define city as city = {lat1: , lon1: , lat2: , lon2: ]:
city1 = {"name": "berlin", "lat": 52.520008, "lon": 13.404954}
city2 = {"name": "munich", "lat": 48.135124, "lon": 11.581981}
city3 = {"name": "hamburg", "lat": 53.551086, "lon": 9.993682}
```

[1] A brute force algorithm is based on the idea that it identifies an optimal case by testing all possible cases [19].

```
city4 = {"name": "cologne", "lat": 50.938361, "lon": 6.959974}
city5 ={"name": "bremen", "lat": 53.079296, "lon": 8.801694}
city6 = {"name": "stuttgart", "lat": 48.7784485, "lon": 9.1800132}
city7 = {"name": "bonn", "lat": 50.73585, "lon": 7.10066}
city8 = {"name": "nuremberg", "lat": 49.45387, "lon": 11.07730}
city9 = {"name": "dresden", "lat": 51.050407, "lon": 13.737262}
city10 = {"name": "dortmund", "lat": 51.51423, "lon": 7.46528}
```

These are then added to a list for better iteration over the single dictionaries and the number of cities is defined as parameter N.

Append Dictionaries to List and Define Number of Cities

```
# append dictionaries to list:
cities = [globals()['city%d'%i] for i in range(1,N+1)]

# define number of cities:
N = len(cities)
```

To measure the distances between the locations, the haversine formula is used, which measures the distances through [25]–[28]:

$$d = 2r \sin^{-1}(\sqrt{sin^2(\frac{\phi_2 - \phi_1}{2}) + \cos(\phi_1)cos(\phi_2)\sin^2(\frac{\varrho_2 - \varrho_1}{2})}), \qquad (10)$$

where: $r =$ earth radius, $\phi =$ latitude, $\varrho =$ longitude.
This was stored as a function to call it for each iteration from the list of locations.

Calculate Haversine Distance

```
def haversine_distance(lat1, lon1, lat2, lon2):
   r = 6371
   phi_1 = np.radians(lat1)
   phi_2 = np.radians(lat2)
   delta_phi = np.radians(lat2 - lat1)
   delta_rho = np.radians(lon2 - lon1)
   s = np.sin(delta_phi / 2)**2 + np.cos(phi_1) * np.cos(phi_2)
       * np.sin(delta_lambda / 2)**2
   d = r * (2 * np.arctan2(np.sqrt(s), np.sqrt(1 - s)))
   return np.round(d, 2)
```

Next, a list of place names is generated, which are read from the dictionaries above. Then the distances between the places are calculated using the 'haversine_distance' function defined above and stored in a new dictionary.

Calculate Distances

```
# get city names:
city_names = []
for i in cities:
    city_names.append(i.get("name"))

# get distances between cities
distance = {}
for i in range(0, len(cities)):
    for j in range(0, len(cities)):
        if i == j:
            continue
        else:
            distance["{}_{}".format(cities[i].get("name"),
            cities[j].get("name"))] = haversine_distance(
            cities[i].get("lat"), cities[i].get("lon"),
            cities[j].get("lat"), cities[j].get("lon"))
```

An exemplary extract from the dictionary "distance" shows the distances between the cities (see Table 1).

Table 1 Distances between Cities

Route	km	Route	km	Route	km
Berlin-Munich	504.41	Berlin-Hamburg	255.25	Berlin-Cologne	477.24
Berlin-Bremen	315.60	Berlin-Stuttgart	511.49	Berlin-Bonn	478.03
Berlin-Nuremberg	377.82	Berlin-Dresden	165.0	Berlin-Dortmund	421.43
...	...	...	...	Dortmund-Berlin	421.43
Dortmund-Munich	477.74	Dortmund-Hamburg	283.77	Dortmund-Cologne	73.06
Dortmund-Bremen	196.32	Dortmund-Stuttgart	327.8	Dortmund-Bonn	90.21

Source: Own representation.

Now the obtained distances are compared in a matrix (see Table 2).

Create Distance Matrix

```
distance_array = []
for i in range(0, len(city_names)):
    distance_list = []
    for j in range(0, len(city_names)):

      #print("{}_{}".format(city_names[i], city_names[j]))
       distance_list.append(distance.get(
       "{}_{}".format(city_names[i], city_names[j])))
    distance_array.append(distance_list)

distance_matrix = np.array(distance_array)
distance_matrix = distance_matrix.astype(float)
distance_matrix = np.nan_to_num(distance_matrix)
```

Table 2 Distance Matrix

	B	**M**	**H**	**C**	**BR**	**S**	**BN**	**N**	**DD**	**DO**
B	0	504.4	255.3	477.2	315.6	511.5	478.0	377.8	165.0	421.43
M	504.4	0	612.4	456.4	583.6	191.0	434.2	151.2	359.4	477.7
H	255.3	612.4	0	356.4	95.0	533.7	370.0	461.7	376.9	283.8
C	477.2	456.4	356.4	0	269.4	288.1	24.6	336.3	474.3	73.1
BR	315.6	583.6	95.0	269.4	0	479.0	285.5	433.1	405.7	196.3
S	511.5	191.0	533.7	288.1	479.0	0	264.0	157.2	412.5	327.8
BN	478.0	434.2	370.0	24.6	285.5	264.0	0	317.4	466.6	90.2
N	377.8	151.2	461.7	336.3	433.1	157.2	317.4	0	259.4	439.1
DD	165.0	359.4	376.9	474.3	405.7	412.5	466.6	259.4	0	439.1
DO	421.4	477.7	283.8	73.1	196.3	327.8	90.2	343.1	439.1	0

B = Berlin, M = Munich, H = Hamburg, C = Cologne, BR = Bremen, S = Stuttgart, BN = Bonn, N = Nuremberg, DD = Dresden, DO = Dortmund

Source: Own representation.

Now a function is needed to calculate the total distance for a given route. This function takes as input the distance matrix and an initial (random) route sequence "seq" and is defined by:

Calculate Total Distance

```
# Calculate total distance for a given sequence:
def calc_distance(distance_matrix, seq):
    td = 0
    for i in range(len(seq)):
        td = td + distance_matrix[seq[i % N], seq[(i+1) % N]]
    return td
```

Before we use Simulated Annealing to estimate the best possible route, we need to define a rejection criterion, the number of iterations, and the initial route sequence.

Create Rejection Criterion

```
# free parameters relating to reject probability when given sequence
# is not shorter than the actual shortest sequence:
A = float(pow(1, -10))

# number of iterations:
ITERATIONS = 100

# initial random sequence:
seq = np.arange(N)

# print initial route sequence length:
print("Initial distance: " + str(calc_distance(distance_matrix, seq))
+ " km.")
```

In the example, the initial distance is 3,992.8 km. Next, a list "all_distances" is created, which is to contain the determined distances within the iterations. Afterward, the algorithm iterates according to the given number and defines the parameters a and b as combinations of the locations. With the parameters p and u it checks the abort criterion. If p is greater than u, then the current route is the shortest route, if not, then it remains the previous route. Afterward, the shortest distance is displayed.

Metropolis Algorithm

```
all_distances = []
for i in range(ITERATIONS):
    for a in range(0, N):
        for b in range(a, N):
            tdt = calc_distance(distance_matrix, seq)
            all_distances.append(tdt)
            seq_tmp = copy.copy(seq)
            seq_tmp[[a, (b) % N]] = seq_tmp[[(b) % N, a]]
            deltad = calc_distance(distance_matrix, seq_tmp) - dt
            p = min(1, mp.exp(-1 * deltad / A))
            u = np.random.rand()
            if u < p:
                seq = seq_tmp

print("The shortest distance is: " + str(calc_distance(
distance_matrix, seq)) + " km.")
```

The shortest distance is 1,674.8 km for the ten cities in the example. The route is displayed with the following code:

Display Shortest Route

```
route = []

for i in seq:
    route.append(cities[i].get("name"))

print("The best route is: " + ' -> '.join(str(x) for x in route))
```

In this case, the best possible route is bremen -> hamburg -> berlin -> dresden -> nuremberg -> munich -> stuttgart -> bonn -> cologne -> dortmund.

Finally, Figure 3 below shows the improvement of the routes over the different iterations. Here we see that after about 80 iterations the algorithm has determined the shortest distance.

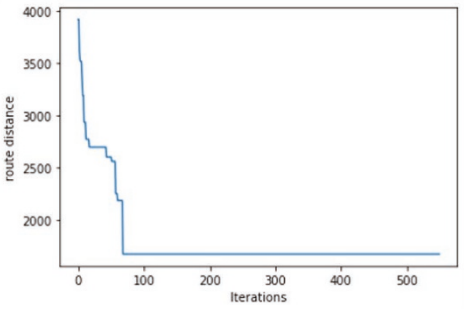

Fig. 3 Simulated Annealing Iterations and Route Distance. Source: Own representation.

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Part III
Nonlinear Optimization

Nonlinear Optimization: The Nelder-Mead Simplex Search Procedure

Franz W. Peren*

Abstract This work explores the field of nonlinear optimization, focusing on the complexity of solving real-world problems with nonlinear functions, continuous variables, and various constraints. It addresses the iterative nature of these optimization problems and their convergence to locally optimized solutions.

1 Introduction

Optimization focuses on the maximization or minimization of real problems, quantified by a mathematical function. Nonlinear optimization, also known as nonlinear programming, differs from linear programs regarding various developments and properties.

Linear programs are defined by the following characteristics according to Cottle and Thapa [1]:

1. The objective function is linear.
2. All constraints are linear and involve at least one equation or inequality.
3. They have many different variables.
4. They can be solved by finite algorithms.

Nonlinear programs are generally defined by the following characteristics [1]:

1. They include at least one nonlinear function.
2. One or more continuous variables exist.
3. They contain inequality constraints, equality constraints or no constraints.

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4. Their included functions have properties, which may include continuity, differentiability, and/or convexity/concavity.
5. Occasionally they have complicated optimality criteria.
6. They can be solved by convergent (but not usually finite) solution algorithms with associated rates of convergence.

Accordingly, a nonlinear optimization system must have continuous variables and at least one nonlinear function. The system may include inequality constraints, equality constraints or no constraints. A nonlinear optimization problem is strongly influenced by the properties of the objective functions and the properties of the admissible domain [2].

Nonlinear optimization is also characterized by its iterative nature, which is determined by sequences of problem solutions and objective function values. Ideally, the sequences converge to feasible solutions that provide (locally) optimized objective function values [1].

2 Basic Properties of Nonlinear Optimization

Optimization in this context means maximizing or minimizing an objective function by the comparison of usually different action alternatives regarding the accomplishment of a target criterion and the selecting of the best of various given alternatives [3]. The objective function might possibly be a quality or expense factor of a technical facility or e.g., a process that is affected by different parameters. In general, not every adjustment of the relevant parameters will be realizable but will rather be parameters of a feasible array (feasible points) [4]. If the feasible region is not limited, there is an **unconstrained optimization problem**. Unconstrained optimization problems have no restrictions so that the admissible area is the complete space of real numbers [5].

In the case of a **constrained optimization problem**, the valid space of the optimum is restricted to a smaller admissible range of values. Those constraints may be as follows:

$$h_j = 0 \quad \text{with} \quad j = 1, ..., n_h \quad \rightarrow \quad \text{equality constraints}$$

$$g_j \leq 0 \quad \text{with} \quad j = 1, ..., n_g \quad \rightarrow \quad \text{inequality constraints}$$

Equality constraints restrict the function value range by the given equations. Those restrictions may have the form of constant, linear, or nonlinear equations [5].

A constrained optimization problem can be summarized mathematically as follows [6]:

(a)	minimize/maximize	$f(x)$	objective function
(b)	subject to	$x \in X$	X = the amount of all (possible) x-values

(c) unconstrainted $X = \mathbb{R}^n$
constrainted $X \neq \mathbb{R}^n$

Example: constrainted optimization problem

$f = f(x) \rightarrow \min!$ with $x \in \mathbb{R}$

subject to $X = \{x \in \mathbb{R} | x \geq 1\}$

$f(x)$ is restricted by the constraint $x \geq 1$

The objective function possesses local and global minimums as well as maximums. The identification of global extrema can be quite intricate due to the existence of a huge number of local extrema, and every local minimum or maximum might be a global extremum. Therefore, the determination of local extrema is often primarily addressed [7]. Global extrema can also be located at the edges of the domain.

Extrema, respectively optimal points x^{opt} of the objective function $f = f(x)$, are defined as follows [4]:
x^{opt} is an extremum (optimal point) of the function $f = f(x)$, if

(a) $x^{\text{opt}} \in X$ x^{opt} is a feasible point of X
(b) X is the set of all feasible points
(c) $f(x^{\text{opt}}) \leq f(x)$ for every $x \in X$
(d) $f(x^{\text{opt}}) \geq f(x)$ for every $x \in X$

3 Nonlinear Optimization Methods

Based on *Meywerk*, a distinction should be made between so-called *search strategies* and *gradient strategies* (Figure 1) [5]. For search strategies, *deterministic* and *stochastic* search strategies are distinguished. For gradient strategies, a distinction should be made between *quasi-Newton methods* and *Gauss-Newton methods*.

3.1 Search Strategies

Search strategies are based on searching procedures for solving nonlinear optimization problems. Search strategies can be divided into deterministic or stochastic methods. Search strategies are often heuristically-based. The requirements for identifying the objective function are lower for search strategies than for gradient strategies. In

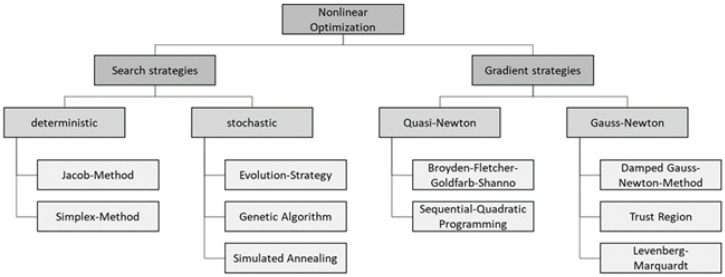

Fig. 1 Optimization Methods of Nonlinear Optimization according to Meywerk (2007) [5].

search strategies, as the name suggests, the objective function is often only approximatively reflected. In search strategy procedures the objective function is narrowed down by repeated iterations according to the selected algorithm [5].

3.2 Deterministic Search Strategies

In deterministic methods all future events are uniquely defined by preconditions. The target results of the next step are uniquely defined by the previous step. The individual steps are reproducible and always provide the same solution for the same initial value [8]. Deterministic search strategies mainly include complete enumerations or heuristic search techniques.

A popular deterministic search strategy is, for example, the simplex method, which is also called the *Nelder-Mead simplex search procedure* or polyhedron method, according to *John Nelder* and *Roger Mead*.[2] This simplex method has no parallel to the simplex method of linear optimization.

The method was introduced in 1965 and serves as a deterministic method for generating the solution of nonlinear functions with several parameters. It is an algorithm for multidimensional unconstrained optimizations without derivatives. The method is simple and robust, which compares points in a specific space. These points are also called simplexes. A simplex is an object with n+1 vectors, which are formed in an n-dimensional space.

The contour line diagram of Figure 2 shows a minimum. The target is to localize this minimum, which could be done by an examination of coordinate by coordinate. Since the function is determined by two parameters, x, and y, we have to rasterize in the x-direction and y-direction.

This optimization process works for higher dimensional problems as well.

[2] John Nelder (1924 – 2010) and Roger Mead (1938 – 2015) were British statisticians.

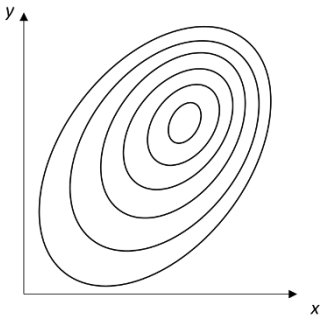

Fig. 2 Scheme of a Two-Dimensional Optimization Problem according to Wiley-VCH Verlag GmbH & Co. KGaA (Ed.) [9].

3.3 The Nelder-Mead Simplex Search Procedure

The aim of this method is to contract the simplex, which, for example, consists of $\mathbb{R}^2$ in the form of a triangle, which should be as small as possible because the minimum of the multidimensional function is located inside the triangle.[3]

The Nelder-Mead simplex search procedure is one of the most popular algorithms for multidimensional unconstrained optimization without derivatives. The basic algorithm is very robust and easy to use.

This method is used widely to solve parameter estimation and statistical problems where the function values are uncertain. It can also be used by solving problems based on discontinuous functions.

The Nelder-Mead simplex search procedure is designed to solve unconstrained optimization problems of minimizing a given nonlinear function $f = f(x_0, \ldots, x_n)$ with $x_i \in \mathbb{R}$ and $i = 1, \ldots, n$ for which the local extremes are to be determined.

The Nelder-Mead search algorithm is simplex-based. A simplex S in $\mathbb{R}^n$ is defined as the convex entity of $n + 1$ vertices with $x_0, \ldots, x_n \in \mathbb{R}^n$. For example, a simplex in $\mathbb{R}^2$ is a triangle (Figure 3):

[3] The following part of the article is largely based on Singer S.& Nelder J. (2009) [10].

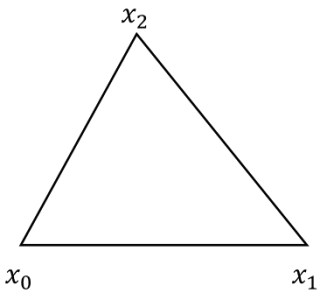

Fig. 3 Simplex in in $\mathbb{R}^2$ (Triangle) according to Singer, S.; Nelder, J. (2009) [10].

A simplex in $\mathbb{R}^3$ is a tetrahedron (Figure 4):

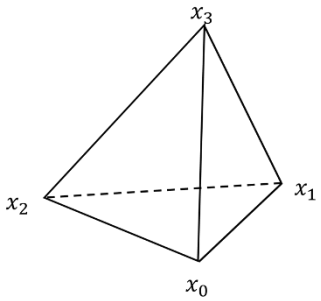

Fig. 4 Simplex in $\mathbb{R}^3$ (Tetrahedron) according to Singer, S.; Nelder, J. (2009) [10].

A simplex-based deterministic search method begins with a set of $n + 1$ points with $x_0, \ldots, x_n \in \mathbb{R}^n$ that are considered the vertices of a working simplex S.

The starting point can be freely selected, which is best done by covering the point where the minimum is expected. Starting with that working simplex S, the function values of the corners (the vertices) of the simplex S have to be compared with each other. If we are searching for the minimum, the corner (the vertex) with the highest function value has to be improved. This algorithm continues as long as the minimum is (approximately) localized.

The method performs a sequence of transformations of the working simplex S, aimed at decreasing the function values at its corners, the vertices of S. This process is terminated when the working simplex S becomes sufficiently small or when the function values improve marginally.

The Nelder-Mead simplex search procedure typically requires only one function evaluation at each step, while other search methods use multiple function evaluations simultaneously. The general algorithm of the Nelder-Mead simplex search procedure works as follows [10]:

- The initial working simplex S must be constructed.
- The following steps must be repeated until the termination test is satisfied and the minimum is located:
  - the termination test requirements must be calculated;
  - if the termination test is not satisfied, the working simplex S must be transformed by four possible options: reflection, expansion, contraction, and compression [11].
- The best vertex of the current simplex S and the associated function value must be returned.

The initial simplex S is usually constructed by generating $n + 1$ vertices $x_0, \ldots, x_n \in \mathbb{R}^n$ around a given input point $x_{\text{input}} \in \mathbb{R}^n$. In practice, the most frequent choice is $x_{\text{input}} = x_0$ to allow proper restarts of the algorithm. Then the remaining n vertices are generated to obtain one of two standard shapes of the simplex S:

- S is right-angled at x_0, based on coordinate axes:
  $x_j = x_0 + h_j e_j \quad \text{with} \quad j = 1, \ldots, n$
  where h_j is a stepsize of one unit in the direction of the unit vector e_j in $\mathbb{R}^n$.
- S in a regular simplex, where all edges have the same length.

One iteration of the Nelder-Mead simplex search procedure consists of the following three steps [11]:

1. **Ordering:** Determination of the indices h, s, l as follows:
   a. Index h of the worst vertex in the current working simplex S
      $f_h = \max_j f_j$
   b. Index s of the second worst vertex in the current working simplex S
      $f_s = \max_{j \neq h} f_j$
   c. Index l of the best vertex in the current working simplex S
      $f_l = \min_{j \neq h} f_j$

   Sometimes the vertices of S are ordered with respect to the function values:
   $f_0 \leq f_1 \leq \ldots \leq f_{n-1} \leq f_n \quad \text{with} \quad h = n, s = n - 1, \text{ and } l = 0.$

2. **Centroid:** Calculation of the centroid c of the best side. The best side is the opposite of the worst vertex x_h

$$c = \frac{1}{n} \sum_{j=1}^{n} x_j \quad \text{with} \quad j \neq h$$

3. **Transformation:** Computing the new working simplex from the current one. First, try to replace only the worst vertex x_h with a better point by using reflection, expansion, or contraction to the best side. Consider that all test points always lie on the line defined by x_h and c. If this succeeds, the accepted point becomes

the new vertex of the working simplex. If this fails, the simplex is to compress towards the best vertex x_l. In this case, n new vertices are computed.

Simplex transformations of the Nelder-Mead simplex search procedure are controlled by four parameters:

$$\begin{aligned} \alpha &\quad \text{for reflection with} \quad \alpha > 0 \\ \beta &\quad \text{for contraction with} \quad 0 < \beta < 1 \\ \gamma &\quad \text{for expansion with} \quad \gamma > 1 \text{ and } \gamma > \alpha \\ \delta &\quad \text{for compression with} \quad 0 < \delta < 1 \end{aligned}$$

In practice, the most frequent used standard values are $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = 2$, and $\delta = \frac{1}{2}$.

The following algorithm describes the transformation process of the simplex S. The effects of various transformations are shown in the corresponding Figures 5 to 9. The new working simplexes are shown in red [11], [12].

- **Reflection**
  Reflection is the first method that has to be checked. The largest point of the simplex, described by a triangle, is mirrored on the mirror line (Figure 5). The first step for reflection is the calculation of the centroid (mirror center):

$$c = \frac{1}{n} \sum_{j=1}^{n} x_j \quad \text{with} \quad j \neq h$$

The reflection point x_r is calculated as follows:

$$x_r = c + \alpha(c - x_h) \quad \text{and}$$

$$f_r = f(x_r)$$

If $f_l \leq f_r < f_s$, x_r is to be accepted and no further iterations are needed.

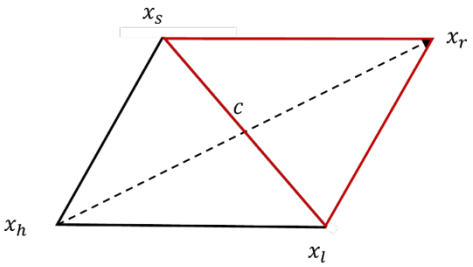

Fig. 5 Reflection of the Largest Point of the Simplex according to Singer, S.; Nelder, J. (2009) [10].

- **Expansion**
  The expansion doubles the path of the reflection (Figure 6).
  If $f_r < f_l$, the expansion point is calculated as follows:

  - $x_e = c + \gamma(x_r - c)$, and
  - $f_e = f(x_e)$

  If $f_e < f_r, x_e$ is to be accepted and no further iterations are needed.

  The simplex is expanded only if $f_e < f_r < f_l$. This kind of "greedy minimization"[4] is mostly used in implementations, but also in theory [14].

  In the original Nelder-Mead publication, x_e is also accepted if $f_e < f_l$, and $f_r < f_l$, regardless of the relationship between f_r and f_e. In the case that $f_r < f_e, x_r$ would be a better point than x_e so that x_e must be still accepted for the new simplex [15].

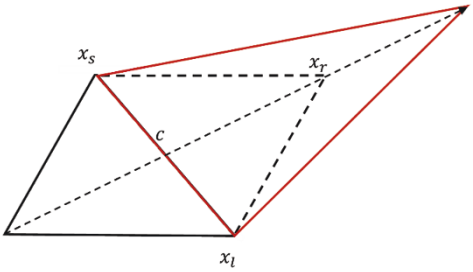

Fig. 6 Expansion of the Simplex according to Singer, S.; Nelder, J. (2009) [10].

- **Contraction**
  If $f_r \geq f_s$, the contraction point x_c must be computed by using the better of the two points x_h and x_r.

  – **Outside** (Figure 7):
    If $f_s \leq f_r < f_h$, the contraction point is calculated as follows:
    · $x_c = c + \beta(x_r - c)$, and
    · $f_c = f(x_c)$
    If $f_c \leq f_r, x_c$ is to be accepted and no further iterations are needed. Otherwise, a compression (shrink transformation) is to be performed.
  – **Inside** (Figure 8):
    If $f_r \geq f_h$, the contraction point is calculated as follows:

[4] A greedy algorithm is an algorithm that always takes the best immediate solution while finding an answer. In many cases, greedy algorithms do not find the optimal solution, but a greedy heuristic is able to find out a locally optimal solution that approximates a globally optimal solution in a reasonable amount of time. Cf. Black, P.E. (2005) [13].

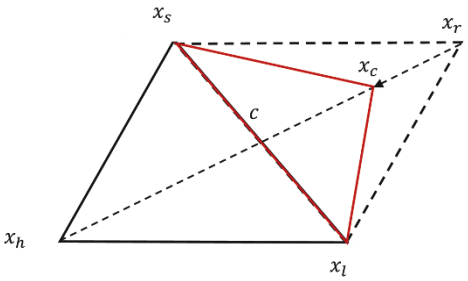

Fig. 7 Outside Contraction of the Simplex according to Singer, S.; Nelder, J. (2009) [10].

- $x_c = c + \beta(x_h - c)$, and
- $f_c = f(x_c)$

If $f_c < f_h$, x_c is to be accepted and no further iterations are needed. Otherwise, a compression (shrink transformation) is to be performed.

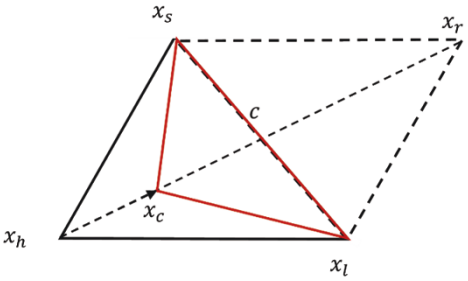

Fig. 8 Inside Contraction of the Simplex according to Singer, S.; Nelder, J. (2009) [10].

- **Compression** This last method is used when the other methods were not applicable. All vectors must be compressed; in practice, the vectors $x_s \rightarrow x_l$ and $x_h \rightarrow x_l$ are mostly halved (Figure 9). The compressed vectors build the new simplex. All n new vertices have been computed:

  – $x_j = x_l + \delta(x_j - x_l)$, and
  – $f_j = f(x_j)$ with $j = 0, \ldots, n$ and $j \neq l$

  The Nelder-Mead simplex algorithm has to be iterated again. The iterations end when the minimum is (approximately) identified or the reason for termination, e.g., a limited number of iterations, has been reached.

The Nelder-Mead simplex search procedure works in many practical problems, like parameter estimation and process control. Often, an accurate solution in form of a full optimization or by using a complete enumeration is not necessary or

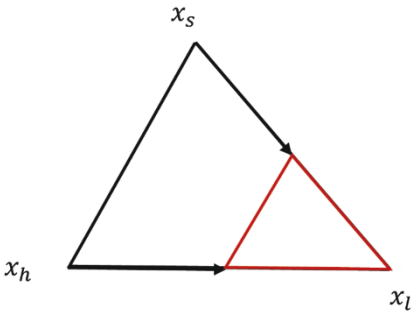

Fig. 9 Compression of the Simplex according to Singer, S.; Nelder, J. (2009) [10].

maybe (temporarily) impossible to compute. The Nelder-Mead algorithm frequently produces useful results. This method typically requires only one function evaluation per iteration, except in compression transformations, which are extremely rare in practice. For practical problems, this heuristic method is often faster than other procedures. Because it is simple to understand and quickly produces quite satisfactory results, the Nelder-Mead simplex search procedure is still popularly used in practice [10].

4 Conclusion

An overview of nonlinear optimization methods, differentiated by so-called *search strategies* and *gradient strategies*, is given in Figure 1. Nonlinear optimization methods optimize (minimize or maximize) a given objective function when the objective function and/or the constraints are nonlinear.

Maximization problems can be converted into minimization problems. Nonlinear optimization problems can be unrestricted or the objective function can involve unknown parameters in which the unknown parameters may be restricted by constraints. Figure 1 shows a wide range of possibilities to solve nonlinear optimization problems.

Search strategies are used when the objective function is not completely known. The objective function is identified by using a search algorithm by repeated iterations. Search strategies can be distinguished between *deterministic* and *stochastic* search methods. *Deterministic search strategies* are used when all possible future events can be uniquely determined by preconditions. The same initial values always lead to the same solution. *Stochastic search strategies* are used for solving problems with inherent random noise. Those stochastic methods narrow down the target range of the objective function by using stochastic processes that perform randomly defined operations. The solution is stochastic and usually changes with the same initial input values.

Gradient methods are based on differentiable functions. To find a local optimum (minimum or maximum), repeated steps are taken in the opposite direction of the gradient of the objective function beginning at a starting point. The steps in the opposite direction of the gradient (or approximate gradient) of the function at the respective current point are repeated because they direct to the steepest descent and lead to a local maximum of that function, where a differentiable function is concave. Gradient methods can be differentiated between the *Gauss-Newton method* and *quasi-Newton methods* (Figure 1).

Gradient descent is a first-order iterative optimization algorithm for finding a local minimum of a differentiable function. The idea is to take repeated steps in the opposite direction of the gradient (or approximate gradient) of the function at the current point because this is the direction of the steepest descent. Conversely, stepping in the direction of the gradient will lead to a local maximum of that function; the procedure is then known as **gradient ascent**.

A *Gauss-Newton algorithm* is used to solve nonlinear minimization problems by using the method of least squares, which is a standard approach in regression analysis to approximate the solution of overdetermined systems (sets of equations in which there are more equations than unknown parameters) by minimizing the sum of the squares of the residuals [16]. The Gauss–Newton algorithm[5] can only be used to minimize a sum of squared function values, but second derivatives are not required [17].

Quasi-Newton methods are similar to the Gauss-Newton method. They can be used as an alternative to the Gauss-Newton algorithm to identify local maxima or minima of nonlinear functions. Their use makes sense if the Hessian matrix[6] is unavailable to compute at every iteration. The Gauss-Newton method requires the Hessian matrix to find extrema or the Jacobian matrix[7] in order to search for zeros.

While a textbook should give a general overview of solving quantitative problems, this article is intended to show only one option of how nonlinear optimization problems can be solved. Here the Nelder-Mead simplex search procedure is selected to give a practical example because it is simple to understand and use, and it represents a typical heuristic method, which is often faster and more robust than alternative (optimization) methods. In practice, it can also be useful to linearize nonlinear problems or developments by using dummy variables (see chapter 6: Regression Analysis Using Dummy Variables).

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[5] Isaac Newton (1642 - 1727) was an English mathematician; Johann Carl Friedrich Gauß (1777 - 1855) was a German mathematician.

[6] *The Hesse matrix or Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function or scalar field. It describes the local curvature of a function with several variables.* It was developed by Ludwig Otto Hesse (1811 - 1874), a German mathematician.

[7] The *Jacobian matrix of a vector-valued function with several variables is the matrix of all its first-order partial derivatives. It generalizes the gradient of a scalar-valued function with several variables. It was developed by Carl Gustav Jacob Jacobi (1804 - 1851)*, a German mathematician.

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Dynamic Programming

Thomas Neifer & Dennis Lawo

Abstract Dynamic Programming, or dynamic optimization, is an optimization approach that simplifies complex problems by breaking them into smaller, interconnected subproblems. This method eliminates redundancy and significantly improves efficiency. DP finds practical applications in various real-world problems within Operations Research, enhancing decision-making processes. Its usefulness is shown by two examples with a practical application in Python.

1 Introduction

Linear programming deals with the optimization of an objective function under certain restrictions, if these are convex (or linear). A common solution method is the simplex method, which is one of the fastest algorithms for solving such an optimization problem. Depending on the complexity, however, the runtime of the algorithm can be exponential [1]. However, it has also been shown that on average a polynomial runtime can be obtained with random input data [2].

In the context of *dynamic programming*, or *dynamic optimization*, an optimization problem is decomposed into smaller subproblems, so that the solution can be reduced from possibly exponential to merely polynomial complexity. Furthermore, dynamic optimization (DO) is not exclusively used to solve convex problems but allows the solution of various structures. For example, a linear problem can be solved by DO by decomposing it into smaller subproblems and following the corresponding DO algorithm [3]. The term dynamic programming was coined by Richard Bellman, who introduced it in the 1940s [4].

If the solution to a decision problem is assumed, in which the decisions are interdependent in time, an optimum can be found for the entire problem by means of the DO. Due to the sequential character of DO, some authors refer to a better name as stepwise or sequential optimization [5].

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The special property of dynamic programming is shown in its ability not to have to calculate things twice [6]. This can be illustrated by the simple example of the Fibonacci series, which is programmed exemplarily in Python. Formally, the discrete Fibonacci sequence $(0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ for the n-th Fibonacci number F_n can be defined as:

$$F_n = F_{n-2} + F_{n-1}$$

with start values:

$$F_0 = 0 \quad \text{and} \quad F_1 = 1$$

If we ignore the DO approach at the outset, a function for the recursive calculation of Fibonacci numbers can be formulated as follows:

Recursive Calculation of Fibonacci Numbers

```
def fibonacci(n):
    if n < 2:
       return n
    else:
       return fibonacci(n-1) + fibonacci(n-2)
```

Both initial values $F_0 = 0$ and $F_1 = 1$ are mapped here by an if condition. As long as n is less than 2, only n is returned. Otherwise, the calculation of $n \geq 2$ takes place over the repeated and additively linked call of the function, in each case for $n - 1$ and $n - 2$. Caused by this recursion without intermediate storage of the result, for example with the calculation of $n = 50$ through 'fibonacci(50)' again the functions 'fibonacci(49)' and 'fibonacci(49)' are executed. Furthermore, this also calls the functions 'fibonacci(48)' and 'fibonacci(47)' again for 'fibonacci(49)'. It quickly becomes apparent that a double and thus redundant call of 'fibonacci(48)' occurs here.

If this is visualized in a tree structure, the redundancy of the recursive approach without memoization becomes clear (see Figure 1).

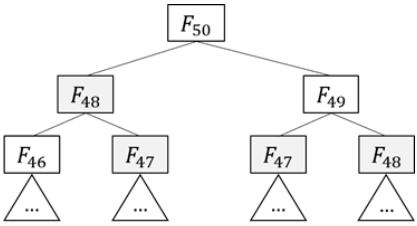

Fig. 1 Recursive Tree Structure. Source: Own representation according to Logofătu (2014) [7].

The connection of the Fibonacci series with the golden ratio (Φ) further reveals the complexity of the approach: it can be shown that the quotient of consecutive numbers of the Fibonacci sequence converges with $n \rightarrow \infty$ to $\Phi = 1,61803..$ [8]. Accordingly, a Fibonacci number F_n, assuming that zero corresponds to the first number in the series, can be calculated as [9]:

$$F_n = \frac{\Phi^n}{\Phi + 2}$$

From this, it can be seen that the n-th Fibonacci number can essentially be described by Φ^n and the running time of the algorithm can accordingly be described by an exponential relationship [8].

Here, DO enables an enormous runtime reduction due to the polynomial relationship. This happens by the simple buffering of the determined Fibonacci value and the with a new calculation preceding the examination of whether the value is not already known. A function written in Python, which corresponds to the DO, could be designed as follows:

DO Approach with Memoization

```
dict = { }
def fibonacci(n):
    if n < 2:
       return n
    elif n not in dict:
       dict[n] = fibonacci(n-1) + fibonacci(n-2)
    return dict[n]
```

This simple memoization allows the runtime to be reduced from several years to a few milliseconds, even for $n > 100$ [7]. For the sake of completeness, the second possible solution to the problem using dynamic programming via a bottom-up approach will also be illustrated below:

Alternative DO Bottom-Up Approach

```
dict = { }
def fibonacci(n):
    dict = {0: 0, 1:1}
    for i in range(2, n + 1):
       dict[i] = dict[i-2] + dict[i-1]
    return dict[i]
```

The bottom-up approach calculates the solution to the problem from the bottom up, contrary to the recursive approach. In addition to the predefined start values for

zero and one, the function is no longer called repeatedly, but the auxiliary object 'dict' is filled from bottom to top. In Operations Research (OR) there are several problems for which DO can be used. Among them are the Knapsack problem, the Traveling Salesman problem, and the order quantity problem. After the theory of DO is explained in the following, it is presented in the context of more complex and application-oriented problems of OR.

2 Theoretical Foundations

As we have already noted, DO is based on the simple premise that an optimization problem can be solved through an iterative decision process by dividing it into several subproblems. The smaller subproblems are now solved first in order to compose larger partial solutions. Only those subproblems are calculated that are actually needed to solve the larger problems. Values that have already been calculated do not have to be calculated again [10]. In this case, such a decision process is distributed over several stages (or periods), where each stage consists of a set of possible states and decisions [5].

Before the specific properties of DO models are discussed, the general form of dynamic optimization problems is described.

2.1 Definition and Properties of DO Models

The DO represents a complex procedure of the OR, which results in particular from the difficulty of the suitable modeling of the optimization problem, the design of the solution procedure as well as from stochastic influences of the problem. Therefore, there are requirements for the mathematical representation of the optimization problem, which is characterized in particular by the type of variables as well as the existing restrictions. For simplification, we will assume a minimization problem in the following, which uses an additive linkage of the stepwise objective functions. However, this can easily be transferred to a multiplicative linkage as well as to a maximization problem [5].

If the model is deterministic and discrete and is to be minimized by the sum of the stage-related objective functions, the mathematical formulation is as follows [11]:

$$F(x_1, X_2, ..., x_n) = \sum_{k=1}^{n} f_k(z_{k-1}, x_k) \rightarrow \min!$$

The following restrictions must be observed [5]:

$$z_k = t_k(z_{k-1}, x_k) \quad \text{for} \quad k = 1, ..., n$$
$$z_k \in Z_k \quad \text{for} \quad k = 1, ..., n$$

$$z_0 = \alpha$$

$$x_k \in X_k(z_{k-1}) \quad \text{for} \quad k = 1, ..., n$$

where: n : Number of stages of a decision process
z_k : State of the system at the end of the stage k
Z_k : Set of all possible states at the end of stage k
$z_0 = a$: Initial state
Z_n : Set of possible or given final states after n stages
x_k : Decision in stage k
$X_k(z_{k-1})$: Set of all decisions that can be selected in stage k starting from state $k-1$
$t_k(z_{k-1}, x_k)$: Transformation function that defines the state z_k of which the system transitions in stage k after the decision x_k is made at the end of stage $k-1$ in state z_{k-1}.
$f_k(z_{k-1}, x_k)$: Stage-related objective function, which describes the influence of the decision x_k made in dependence on the state z_{k-1} on the objective function value. In general, for each DO problem it must hold that f_k depends only on the state z_{k-1} of the preliminary stage and the decision x_k. This corresponds to the Markov property [11], [12].

DO models can be classified according to the following criteria (see Table 1 [5]:

Table 1 DO Model Properties

Time intervals of the periods or stages	Distinction between discrete and continuous models. While discrete models represent changes of state at the point in time or in discrete steps, continuous models allow continuous changes of state.
Disturbance variable	In deterministic models, the disturbance variable b_k (external influences on the model) can only assume exactly one value. In stochastic models, however, b_k itself represents a random variable, which means that it can assume different values with known probabilities.
State and decision variables	State and decision variables can be single-valued or multi-valued in the form of vectors.
Finiteness of state space	Z_k and X_k can be either finitely bounded or infinite with respect to their set.

Source: Own representation according to Domschke et al. (2015) [5].

Figure 2 illustrates the causalities between the above terms once again.

Discrete and deterministic DO models can be clearly represented in a digraph $G = (V, E)$ (see Figure 3) [5]. A digraph, or directed graph, is composed of a finite, nonempty set of nodes V and a set of arrows E. The elements of E are also called directed edges [14]. Here, V represents the union set of all Z_k for $k = 1, \ldots, n$, where

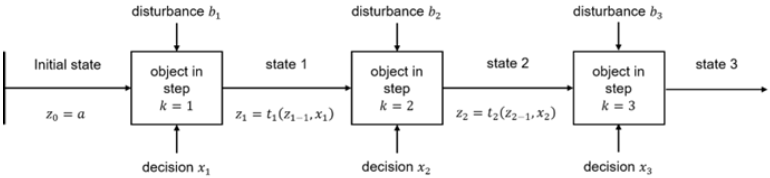

Fig. 2 Graphical Representation of a DO Model. Source: Own representation according to Schwarz (2010) [13].

each node $z_k \in Z_k$ describes an acceptable state. E consists of arrows (z_{k-1}, Z_k) with $z_{k-1} \in Z_{k-1}$ as well as $z_k \in Z_k$ for $k = 1, \ldots, n$. An arrow represents the system transition from a state z_{k-1} to z_k, which can be explained by $z_k = t_k(z_{k-1}, x_k)$. Due to the deterministic model assumption, the explicit specification of b_k is omitted here, since it is a constant.

With each transition from z_{k-1} to z_k, the objective function is influenced, because with each state change period-related costs are associated, the amount of which is determined by the stage-related objective function $f_k(z_{k-1}, x_k)$.

Furthermore, $z_0 = \alpha$ as the initial state and $z_n = 0$ as the final state applies as usual [5].

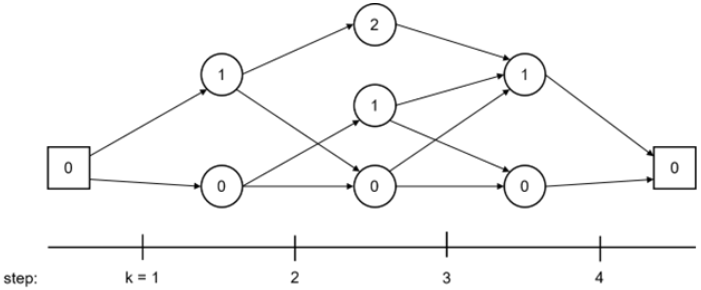

Fig. 3 Digraph of an Exemplary DO Model. Source: Own representation according to Domschke et al. (2015) [5].

2.2 Solution Principle of Dynamic Optimization

Finding an optimal policy for a discrete and deterministic DO model with initial state $z_0 = \alpha$ and a set Z_n of possible final states is called problem $P_0(z_0 = \alpha)$. Similarly, the task of determining an optimal policy for transforming a state $z_{k-1} \in Z_k$ into one

of the possible final states $\in Z_n$ is defined as problem $P_{k-1}(z_{k-1})$. Here, the optimal objective function value of a problem $P_k(z_k)$ is $F_k^*(z_k)$.

Here, a policy denotes a sequence of decisions $(x_j, x_{j+1}, ..., x_k)$ that transforms a system to a state $z_k \in Z_k$ given a state $z_{j-1} \in Z_{j-1}$. Thus, assuming a minimization problem, a sequence of decisions $(x_j^*, x_{j+1}^*, ..., x_k^*)$ is called an optimal policy if it transforms a system from a given state $z_{j-1} \in Z_{j-1}$ to a state $z_k \in Z_k$ while minimizing an objective function [11].

In order to determine an optimal overall policy of a system, the following applies in accordance with the optimality principle going back to Bellman (1957) [15]:

Theorem 0.1 *Let $(x_1^*, ..., x_{k-1}^*, x_k^*, x_n^*)$ be an optimal policy that transforms a system from an initial state $z_0 = \alpha$ to a final state z_n and let z_{k-1}^* be a state that the system reaches at stage $k-1$ then follows:*

$(x_k^*, ..., x_n^*)$ is an optimal (partial) policy which, starting from the state z_{k-1}^* in stage $k-1$, transforms the system to the given or allowed final state z_n (backward recursion) and $(x_1^*, ..., x_{k-1}^*)$ is an optimal (partial) policy that transforms the system from a given initial state $z_0 = \alpha$ to a state z_{k-1}^* in stage $k-1$ (forward recursion) [5].

Accordingly, an optimal policy exists exactly when each sub-policy is also optimal. The DO uses Bellman's optimality principle to derive an optimal overall policy by forward or backward recursion [3].

2.3 Bellman's Functional Equation Method

Bellman's Functional Equation Method describes a methodology for process analysis and optimization, which consists of the phases of decomposition, backward, and forward recursion. In decomposition, the decision process is decomposed into several sub-problems, where only the decision options are considered. By alternating the solution of the sub-problems or their alternative decomposition into further sub-problems, the smallest sub-problems are identified and finally solved. Thus, in the next step, the solutions for the next larger (sub-)problems are created. By means of backward recursion, optimal decisions of all intermediate states of the (sub)problems are now made backward, taking into account the objective function, with the final state as the starting point. Forward recursion, on the other hand, considers the existing initial state as the starting point. The optimal decisions made under the objective function are made on the basis of those decisions made in the previous backward recursion [16].

The goal is to observe a trajectory $(x_0, x_1, ..., x_n)$ which satisfies the optimality principle. For this Bellman defines the following equation:

$$S(x) = \min_{y \in U(x)} g(x, y) + S(y)$$

where the value of the trajectory is defined as the summed value of the different sub-problems. This means that:

$$g(x_i, x_{i+1}, ..., x_j) = \sum_{k=i}^{j-1} g(x_k, x_{k+1})$$

where: x : Finite set of possible system states

$x_i, i = 0, ..., n$: System States subdivided into several successive sub-problems x

$U(x)$: Set of subsequent states for each system state x in the frame of stages 0 to $n - 1$

$(x_i, x_{i+1}, ..., x_j)$: State sequences (trajectories), which allow transitions between the states

$0 \leq g(x_i, x_{i+1}, ..., x_j)$: Non-negative evaluation function for the trajectories and their sections

S : Optimal value that can be assigned to each state, where the target set x_n contains at least one optimal value

$S(x)$: Value of an optimal trajectory. Assumes the value ∞ if no trajectory is present [16].

The Bellman-Ford algorithm thus works simplistically according to the following principle (pseudocode) [17]:

Step 1: Assign x_i to infinity and x_0 to zero for $i \neq 0$
Step 2: for each edge(u, v) do $n - 1$ times:
    $x_i = \min\{x_i, x_u + \text{weight}_{uv}\}$
Step 3: for any edge(u, v):
    if $x_u + \text{weight}_{uv} < x_i$:
        negative-weight cycle

3 Applications

3.1 Basic Example: Finding the Shortest Route

Let us assume that there are seven cities A to G (nodes), which have the following traffic connections (edges) including the specified distances to each other (see Table 2):

If this is visualized by means of a graph, Figure 4 results.

Table 2 Distances Between the Seven Cities

From	To	Distance
A	B	17
A	C	17
A	D	1
A	E	10
A	G	2
B	D	6
B	E	16
C	F	5
C	G	7
D	F	7
F	E	18
F	G	8

Source: Own representation.

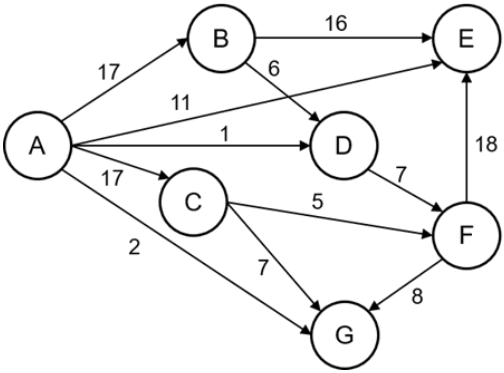

Fig. 4 Graph: Distances Between the Seven Cities. Source: Own representation.

Initiation and Phase 1:

The goal is to find the shortest distance from the starting point A to the other cities. Let us consider the example with A as the starting point. Accordingly, the worst estimate applies for $k = 0$, where A is assigned the value 0 and B to G is assigned the value infinity.

In phase 1, all edges and the respective distances to the respective city are assigned as well as the predecessor, i.e. the city crossed before. For A, the value 0 is still used, since this is the starting point and no distance has to be covered.

The first edge is now A → B, which has a distance of 0 (coming from A) + 17 (to B). Since 17 is less than infinity, the new value is assigned, as well as the information that the predecessor node is A. The next edge, B → E, includes a distance of 16. Thus, starting from the starting point A, E is 17 (to B) + 16 away, making a total of 33. Since 33 is less than infinity, the new value is assigned as well as B as the

predecessor node. The edge B → D is treated analogously: Starting from A, 17 + 6 and consequently 23 are needed to get to D via B. 23 is less than infinity and is therefore assigned as the new value. Since, in addition to A → B → E, there is also a direct connection between A and E, which, at 11, is smaller than 33, the smaller value is used here, starting from A. The same applies to the following edge A → D: Since 1 is smaller than 23, the new value is used.

Edge D → F requires 1 (from A to D) + 7 (from D to F) and therefore 8. Since the edge F → E comprises a total of 26 (8 from A to F + 18 from F to E) and this is more than 11 (A → E), 11 remains.

The remaining edges are treated in the same way, resulting in Table 3.

Table 3 Finding the Shortest Route: Phase 0 and 1

k	A	B	C	D	E	F	G
0	0	∞	∞	∞	∞	∞	∞
1		17 A	17 A	~~23 B~~ 1 A	~~33 B~~ 11 A	8 D	2 A

Source: Own representation.

Phase 2:

Phase 2 now iterates over the edges and nodes again and checks in an analogous way whether an improvement of the previous results occurs. Since this is not the case, the minimum distances of the cities to the starting point A were identified (see Table 4).

Table 4 Finding the Shortest Route: Phase 2

k	A	B	C	D	E	F	G
0	0	∞	∞	∞	∞	∞	∞
1	0	17 A	17 A	1 A	11 A	8 D	2 A
2	0	17 A	17 A	1 A	11 A	8 D	2 A

Source: Own representation.

3.2 Bellman-Ford Algorithm in Python

Next, let's look at how an algorithm can solve the example from chapter 3.1. For this purpose, we define the problem from Figure 4 again as input for the algorithm:

Initiation of Nodes and Edges

```
# Definition of nodes A - G as a list (A = 0, B = 1, etc.):
nodes = [0, 1, 2, 3, 4, 5, 6]

# Definition of edges as dictionary:
edges = {(0,1): 17, (0,2): 17, (0,3): 1, (0, 4): 11, (0, 6): 2,
               (1,3): 6, (1,4): 16,
               (2,5): 5, (2,6): 7,
               (3,5): 7,
               (5,4): 18, (5,6): 8}
```

Then, a function is defined that reflects the Bellman-Ford algorithm:

Definition of the Bellman-Ford function

```
# Definition of the Bellman-Ford function. The required arguments are the
# nodes, the edges with their respective weights as well as the start node.

def bellmanford(nodes, edges, sourceNode = 0):
    # Initiation: start node is assigned 0, the rest infinite
    pathDistances = {v: float('inf') for v in nodes}
    pathDistances[sourceNode] = 0
    # Path definition
    paths = {v: [] for v in nodes}
    paths[sourceNode] = [sourceNode]
     # Bellman-Ford algorithm:
    for _ in range(len(nodes) - 1):
        for (u, v), w_uv in edges.items():
            if pathDistances[u] + w_uv < pathDistances[v]:
                pathDistances[v] = pathDistances[u] + w_uv
                paths[v] = paths[u] + [v]
    return pathLengths, paths
```

The determination and output of the result are as follows:

Apply the Bellman-Ford function

```
shortestDistances, shortestPaths = bellmanford(nodes, edges)

print(shortestDistances)
print(shortestPaths)
```

The output is now analogous to the above result from chapter 3.1:

```
print(shortestDistances)
{0: 0, 1: 17, 2: 17, 3: 1, 4: 11, 5: 8, 6: 2}
print(shortestPaths)
{0: [0], 1: [0, 1], 2: [0, 2], 3: [0, 3], 4: [0, 4], 5: [0, 3, 5], 6: [0, 6]}
```

For example, to get from the starting point 0 (A) to point 5 (F), a distance of 8 (0 + 1 + 7) must be covered. This corresponds to the path [0, 3, 5] and therefore A → D → F.

4 Conclusion

Dynamic programming is an important tool for breaking down complex real-world optimization problems into small, manageable sub-problems. The Bellman-Ford algorithm makes it possible to achieve optimization by an iterative approach. Recursion is of essential importance, but it becomes more and more time and resource-consuming. For this purpose, the memorization of the subproblem solutions serves to counteract this weakness. All in all, dynamic optimization is suitable for the optimization of many different real-world problems and thus represents a powerful tool for operation research.

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Part IV
Project Management

Network Analysis Method

Matthias Krebs*

Abstract This work provides an overview of the methods of the network technique in project management. The network diagram technique has evolved over the years to provide valuable insights into project planning and critical path analysis. This paper describes its practical application using modern project management software and presents an Excel-based approach to creating diagrams. Understanding these methods provides project managers with important tools for effective project planning and execution.

1 Introduction

Network analysis methods[2] are an efficient way to calculate or optimize the activities in a project. The intention of these methods is:

> "Breaking down a complex project's data into its component parts (activities, events, durations, etc.) and plotting them to show their interdependencies and interrelationships" [1].

A fundamental flow chart for the network analysis methods is shown in Figure 1.

The flow charts of the network analysis methods are called "network diagrams" or "precedence diagrams" (PD). Each diagram has a specific start, a defined amount of following nodes, and a clear project finish. Network analysis methods are nothing new; the first illustrations of the network technique trace back to 1849 [2].

In 1956, the "Critical Path Method" was invented by the company "Du Pont de Nemours & Co." A short time later the technique became popular in Europe as well. In 1958, the "Metra Potential Method" was invented by the company "Metra"

* The following presentation of Gantt Charts has been developed at the Bonn-Rhein-Sieg University, Sankt Augustin, Germany, in 2020 under the leadership of Prof. Peren. It is based on an unpublished manuscript from Sebastian Hartmann and Lyn Scholz, which has been developed in the same course years before. Kai Florian Fliescher was also involved in creating this article.

[2] Term is synonymous for "network technique."

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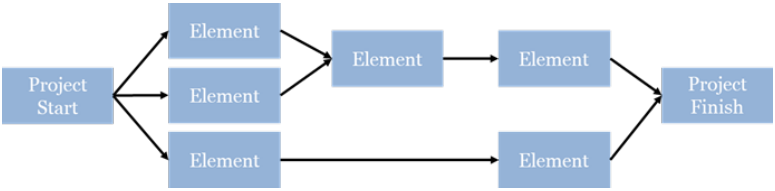

Fig. 1 Fundamental Flow Chart: "Network Diagrams". Source: Own representation.

in France. In the same year, the "Program Evaluation And Review Technique" was developed by the U.S. Navy. In the following years, several network programs were developed and modified in America and Europe. Today the network technique is commonly used to handle complex projects, for example, the building of airports and the purpose of logistics and transportation [3].

The Precedence Diagramming Method (PDM) was developed in the early 1960s by H.B. Zachry in cooperation with IBM. It has largely replaced Arrow on Node diagramming. PDMs show activities as boxes that have properties of the activities they represent assigned to them. This diagramming method is utilized to draw the project schedule network diagrams – for example, the critical path network diagram and the critical chain network diagram.

2 Theory

The network diagrams are theoretically based on graph theory. The network diagrams are also part of the "Project Management Professional" examinations [4].

2.1 The Graph in the Context of Network Analysis Methods

The graph theory is used to show the characteristics of a number of nodes, which are connected by edges [2].

As shown in Figure 2, the graph (network) consists of nodes (circles or rectangles) that are connected by edges. All network analysis methods are based on the mathematic model of a directed, finite and acyclic graph [2]. A graph is directed where the edges start from a specific node and ends on another defined node. The edges are represented by arrows. The attribute finite points out that a specific number of arrows start from each node and it also addresses the finite countable amount of all nodes in the network. Last but not least, acyclic means that every graph cannot start and end at exactly the same point. It is not allowed to connect the end of a graph with the beginning of it.

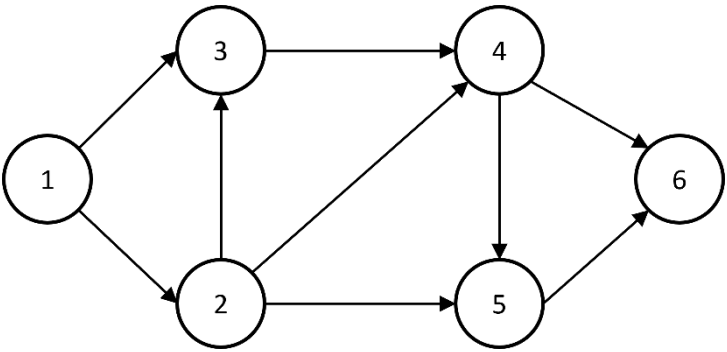

Fig. 2 Directed, Finite, and Acyclic Graph. Source: Own representation according to Berner et al. (2013) [2].

2.2 Benefits of Network Analysis Methods

Network analysis methods have many benefits. The essential benefits of network analysis methods are:

- A clear setup and structure of the project:
  "all project activities are shown in sequence with relevant interrelationships" [5]
- It can easily show the critical path and floats:
  "network diagram of a project shows how activities are interrelated with each other from the beginning of the project till the end" [5]
- It can track the current project progress:
  "it shows the order of activities in a project and total path from the beginning of the project till the end." [5]

As noted, the most well-known types of network analysis methods were developed within three years, from 1956 to 1958. A few years later, in 1970, these methods were standardized by the German organization "Deutsches Institut für Normung e.V." (DIN).

2.3 Different Types of Illustration Facilities

Terms, methods, and illustration techniques are defined in the voluntary standard DIN 69900 [6] and mainly differ in their graphical model and evaluation as shown in Figure 3. DIN 69900 distinguishes three different network analysis methods [2]:

- The "Activity-On-Arrow" network diagram (Critical-Path-Method, CPM)

- The "Activity-On-Node" network diagram (Metra Potential Method, MPM)
- The "Event-On-Node" network diagram (Program Evaluation and Review Technique, PERT)

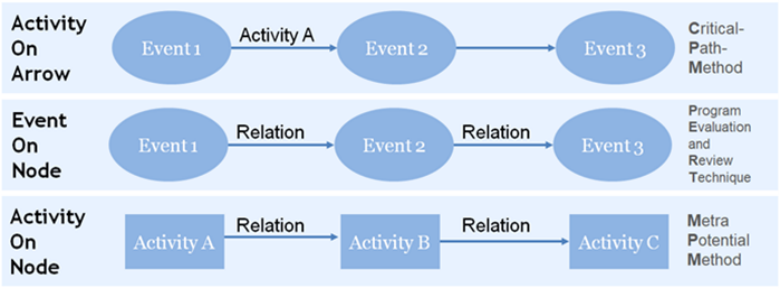

Fig. 3 Types of Network Diagram Illustrations. Source: Own representation based on Berner et al. (2013) [2].

The terms on the left side describe the different types of illustration which are associated with the particular method on the right side and vice versa.[3] The fundamental characteristics of these three types of illustrations are [2], [7]:

- Activity-On-Arrow diagram
  This network diagram is activity-oriented. The connection of the events (the nodes, shown as cycles) is implemented with arrows. The arrows represent the activities. The sequence of the activities in the project is equal to the sequence of the arrows. In order to define the arrangement of relationships in the network diagram, additional dummy paths might be necessary.
- Event-On-Node diagram
  This network diagram is event-oriented. The nodes (cycles), are exactly defined dates. The arrows define the arrangement of relationships between the nodes.
- Activity-On-Node diagram
  This network diagram is activity-oriented. Activities are implemented by nodes (rectangles). The arrangement of relationships between the activities is displayed by arrows.

[3] The illustration of the method is commonly called together with the named methods, but this need not always be like that. The methods are usual implemented with a particular type of illustration.

2.4 The Metra Potential Method (MPM)

As mentioned in Figure 3, this paperwork will address the Metra Potential Method (MPM), illustrated by the Activity-On-Node (AON) network diagram.

2.4.1 The Network Diagram

Generally, a network diagram illustrates the course of a project and consists of activities, arrows, and further parameters [8]. This fundamental structure of the network diagram is demonstrated in Figure 4.

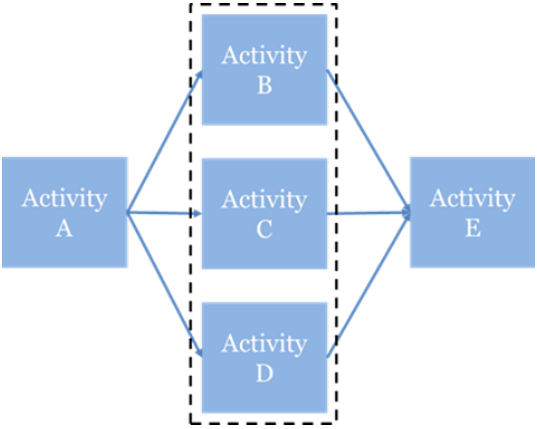

Fig. 4 The Network Diagram 1. Source: Own representation.

The project in this example starts with “Activity A” and ends with “Activity E”. In this network diagram, three activities occur between the initial “Activity A” and final “Activity E”.

2.4.2 Characteristics of the Network Diagram

In order to build a network diagram, some specific rules have to be considered [2], [9].

Networks typically flow from the left to the right side in order to describe the sequence of the project [10]:

- Activities cannot begin until all of its preceding activities are finished;
- Arrows indicate precedence and flow. They can cross over each other;

- Activities must have at least a specific start, end, and duration;
- Identify each activity with a unique identification code. This identification code must be greater than its predecessors;
- Loops are not allowed;
- Conditional statements are not allowed;
- The network has unique start and stop nodes.

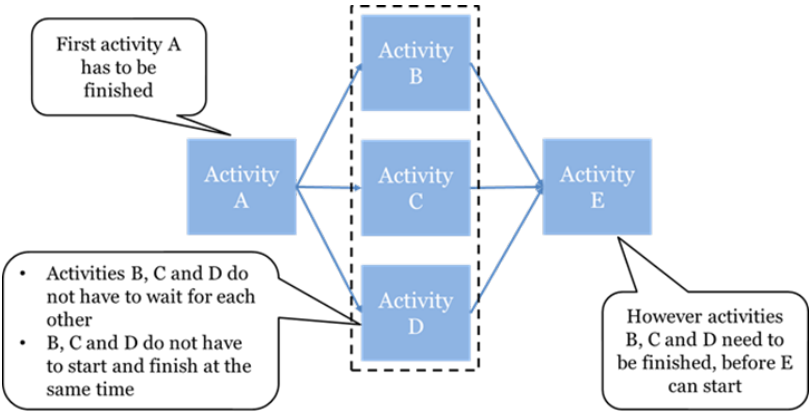

Fig. 5 The Network Diagram 2. Source: Own representation.

First, "Activity A" has to be finished. The succeeding "Activities B, C, and D" are not related to each other and do not have to start and finish at the same time. However, "Activities B, C, and D" need to be finished before "Activity E" can start.

Additionally, this example has a so-called "burst event" and a "merge event."

If more than one activity follows the predecessor, it is called a "burst event." Vice versa, it is called a "merge event" when a single activity follows after more than one predecessor. Figure 6 highlights these events within the sequence.

2.4.3 Characteristics of the Network Diagram

Defined information is recorded for each activity. In detail, the necessary parameters for the Metra Potential Method are:

- i = identification code of the activity;
- O = activity owner;
- TF_i = total float;
- FF_i = free float;
- t_i = duration required to perform activity i;

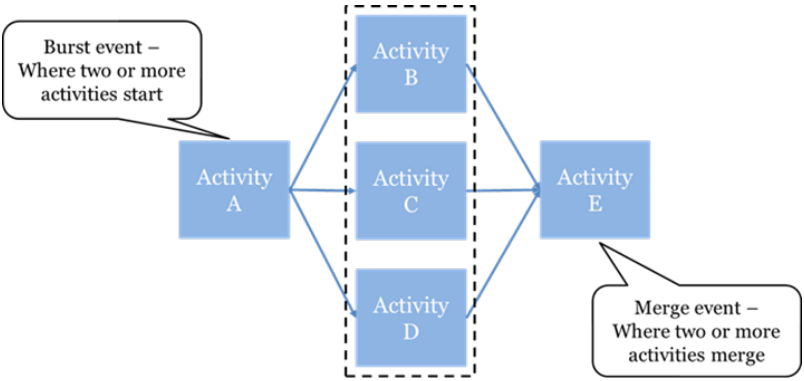

Fig. 6 The Network Diagram 3. Source: Own representation.

- EST_i = earliest possible start for activity i;
- EFT_i = earliest possible finish for activity i;
- LST_i = latest possible start for activity i;
- LFT_i = latest possible finish for activity i.

i	O	t_i
Activity description		
EST_i	TF_i	EFT_i
LST_i	FF_i	LFT_i

i = identification code
O = activity owner
TF_i = total float
FF_i = free float

t_i = duration required to perform activity i
EST_i = earliest possible start for activity i
EFT_i = earliest possible finish for activity i
LST_i = latest possible start for activity i
LFT_i = latest possible finish for activity i

Fig. 7 Information Recorded for each Activity. Source: Own representation based on Rudorfer and Serbezov (2012) [11].

As shown in Figure 7, each activity (or task) has a specific identification code. In addition, a specific owner supervises the execution and takes the responsibility for the task. Each task has a description and duration. EST_i and EFT_i define the earliest possible start and finish for activity i. LST_i and LFT_i specify the latest possible start and finish for activity i. Furthermore, there are two types of floats: the total and the free float. Both give a certain amount of time an activity could be delayed without

disrupting the project. Detailed definitions and calculations can be found in section 3.4.

3 Case Study - Moving

Moving from one place to another must be well planned and takes a lot of time to execute smoothly. Therefore, Mr. and Mrs. Smith decided to use a network in form of a PD to keep their moving process on schedule.

3.1 Step 1: Identify activities, predecessors and determine durations

Mr. and Mrs. Smith developed a list of seven activities required to complete their project "Moving" (see Table 1).

Table 1 Activities of Case Study "Moving"

Activity	Description	Predecessor	Duration (weeks)	Process Owner
A	Plan to move	-	1	(X) Mr
B	Pack moving boxes	A	2	(O) Mrs
C	Arrange or a truck	B	2	(X) Mr
D	Arrange for recolation assistants	B	3	(O) Mrs
E	Apartment relocation	C, D	2	(X) Mr
F	Renovate old apartment	E	1	(X) Mr
G	House-warming party	E	3	(O) Mrs

Source: Own representation.

In the following, the activities are shortened with the individual identification code. Activities A and F take one week to complete. B, C, and E have durations of two weeks, whereas activities D and G take three weeks to be finished. In addition, Mr. and Mrs. Smith determined the dependencies between activities. Activity A must be completed before activity B can begin. Activity C and D have activity B as immediate predecessors. The predecessors C and D must be completed before successor activity E can start. Activities F and G start if activity E is finished. Mrs. Smith takes responsibility for activities B, D and G. Mr. Smith takes responsibility for activities A, C, E, and F.

3.2 Step 2: Illustrate Dependencies in a Network Diagram

Once they have identified the relations between the activities, they can visualize the sequences of the activities in a network diagram as shown in Figure 8.

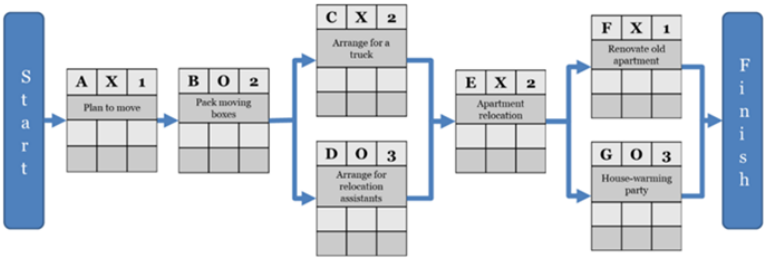

Fig. 8 Network Diagram (Dependencies). Source: Own representation.

3.3 Step 3: Forward and Backward Calculation

The forward calculation is used to calculate the earliest time each activity can start and finish. The earliest finish time (EFT) of the final activity determines the total duration of the project. Starting with the earliest start time (EST) for the initial activity, the EST is "time zero." The EFT of the activity is its EST plus the duration required to complete the activity. The EST of the following activities is equal to the EFT of its predecessor. In case the activity has more than one activity preceding it, the latest EFT of these predecessors will be used [10]. The calculation of the EFT is visualized in Figure 9.

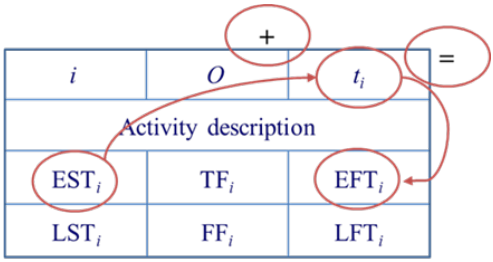

Fig. 9 Calculation of EFT. Source: Own representation.

Using the case study, EST equals the respective predecessor's EFT, and EFT itself for each activity is calculated as followed in Table 2. Both are added to the network diagram shown in Figure 10.

Table 2 Forward Pass Calculation.

Activity	EST (EFT predecessor)	Duration	EFT (EST + duration)	Activity
A	0	1	0 + 1 = 1	A
B	1	2	1 + 2 = 3	B
C	3	2	3 + 2 = 5	C
D	3	3	3 + 3 = 6	D
E	6	2	6 + 2 = 8	E
F	8	1	8 + 1 = 9	F
G	8	3	8 + 3 = 11	G

Source: Own representation.

Since activity E has two predecessors (activities C and D), its EST is 6 because activity D is the one with the latest EFT.

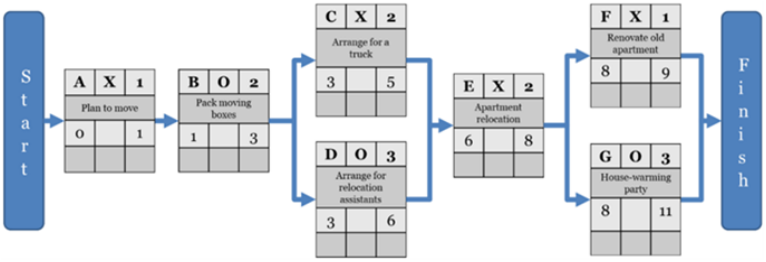

Fig. 10 Network Diagram (Forward Pass). Source: Own representation.

The backward calculation determines the latest times each activity can start and finish without delaying the completion of the project. It determines which activities can be delayed without prolonging the project (these activities have a total float > 0) and which are critical (these activities have a total float = 0). Starting with the latest finish time (LFT) for the final activity, the LFT is equal to its EFT, determined through the forward calculation. The latest start time (LST) of an activity is equal to its LFT minus the time required to perform the activity. The LFT for any other activity is equal to the latest start time (LST) of the activities directly succeeding[4] it. If there is more than one activity succeeding, the earliest LST of all direct successors will be used [12]. The calculation of the LST is visualized in Figure 11.

[4] Successor is that activity that logically comes after another activity in a plan, because a network diagram always flows from left to right.

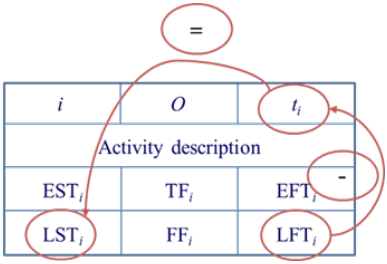

Fig. 11 Calculation of LST.
Source: Own representation.

Based on the data given in the case study, the LST and LFT are determined in Table 3 and also added to the network diagram shown in Figure 12.

Table 3 Backward Pass Calculation.

Activity	LFT (LST succeeding)	Duration	LST (LFT + duration)	Activity
G	11	3	11 – 3 = 8	G
F	11	1	11 – 1 = 10	F
E	8	2	8 – 2 = 6	E
D	6	3	6 – 3 = 3	D
C	6	2	6 – 2 = 4	C
B	3	2	3 – 2 = 1	B
A	1	1	1 – 1 = 0	A

Source: Own representation.

The LFT for the final activities F and G is 11 because 11 weeks are needed to complete the whole project. Since activity E has two succeeding activities, its LFT is 8 because LST of G is earlier than F.

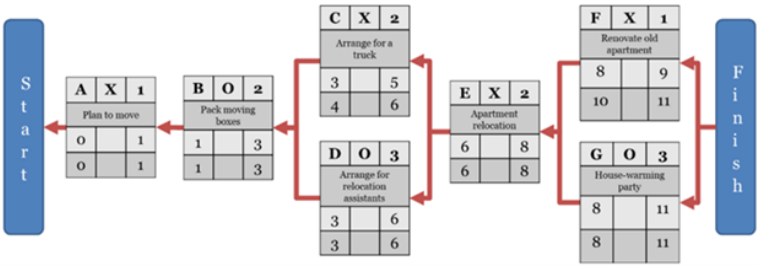

Fig. 12 Network Diagram (Backward Pass). Source: Own representation.

3.4 Step 4: Float Calculation

There are two types of floats: the total float and the free float, which can be determined for each activity.[5] The total float is the amount of time that an activity may be delayed without delaying the completion of the project [13].

The formula: $LFT_i - EFT_i$ or $LST_i - EST_i$

The free float is the amount of time that an activity can be delayed without delaying the EST of the succeeding activity [13].

The formula: $EST_{\text{succeeding}} - EFT_i$

The total and free float calculation is visualized in Figure 13.

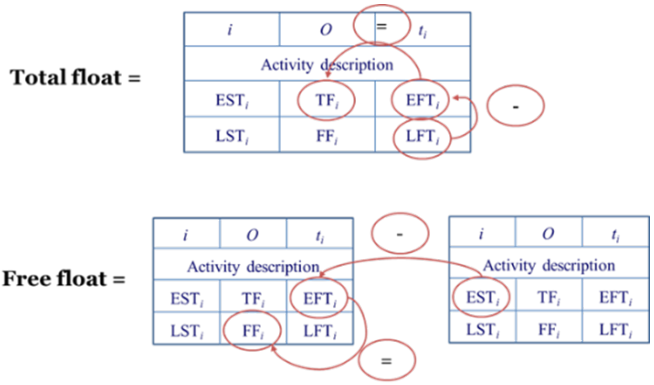

Fig. 13 Total and Free Float Calculation. Source: Own representation.

Using the project "Moving," the floats can be determined as in Table 4 and were put into the network diagram shown in Figure 14.

Table 4 Total and Free Float Calculation.

Activity	Total float (LFT-EFT)	Free float ($EST_{\text{succeeding}}$-EFT)
A	1 – 1 = 0	1 – 1 = 0
B	3 – 3 = 0	3 – 3 = 0
C	6 – 5 = 1	6 – 5 = 1
D	6 – 6 = 0	6 – 6 = 0
E	8 – 8 = 0	8 – 8 = 0
F	11 – 9 = 2	11 – 9 = 2
G	11 – 11 = 0	11 – 11 = 0

Source: Own representation.

[5] Moreover, there are other types of floats. Compare to Berner et al. (2013) [2].

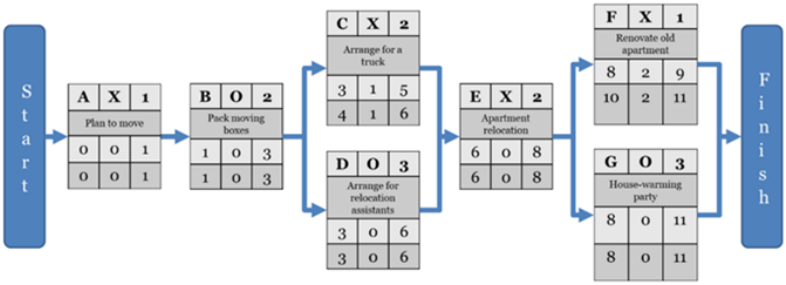

Fig. 14 Network Diagram (Floats). Source: Own representation.

3.5 Step 5: Critical Path

The critical path is the sequence of activities with the longest duration path on the network diagram and determines the duration of the whole project. Since these activities have no total floats, none of these activities can be delayed without affecting the completion of the project. The critical path is an important project management tool allowing project management to focus efforts on the most important tasks [14].

Therefore, the following critical path "A-B-D-E-G" for the project "Moving" is visualized in Figure 15.

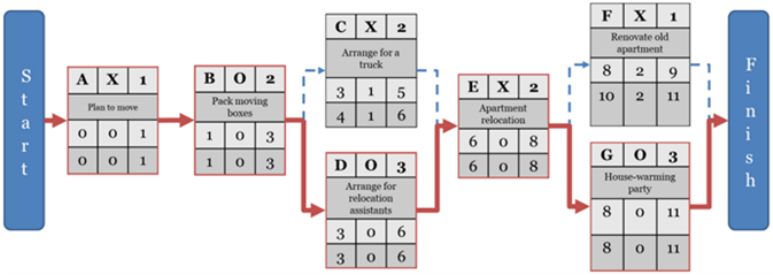

Fig. 15 Critical Path. Source: Own representation.

3.6 Case Study Insights

To sum up, the project "Moving" is completed after 11 weeks. Mr. and Mrs. Smith identify the critical activities A, B, D, E, and G. These activities require special atten-

tion because they cannot delay without extending the project. Non-critical activities are C and F because they have a total float of 1 and 2 weeks respectively.

4 Computer-based Application with OmniPlan3 (macOS

OmniPlan is a planning and project management software from The Omni Group, headquartered in Seattle, Washington. The first version, OmniPlan 1.0, was released in 2006. At present, the latest software version is 3.x and was released in 2015. It is commercial software with a standard and pro version, but also a free trial version for fourteen days.[6] The following paragraphs show how to create the case study "Moving" with the OmniPlan 3 software.

The first step is to create a new project. To start a project in the network view, press the fourth button in the "View switcher" at the top left corner. A new task will open immediately. To set the project information, press the "information" button (i) at the top right. It is possible to enter information about the entire project, such as project name, start and end date, the initial flow direction, and also the task effort (duration). Like the case study, the project name is "Moving," the duration is from 01/08/2018 to 03/23/2018 and is displayed in weeks. Project setup is done as shown in Figure 16.

To create an activity, right-click within the working space and add a new task. The task will depict as a rectangle. After that, it is possible to set the task information with the "information" button. As shown in Figure 16, information like name and effort for the task can be entered. If you enter the effort, the software will automatically calculate the task's start and end date.

After creating all required tasks, it is time to connect them to get a network diagram. As you can see on the left side of Figure 17, the activities "Plan to Move" and "Pack Moving Boxes" are shown as single tasks. On the right side, both activities are connected. To establish a connection between two single tasks, use the drag-and-drop function. The connection is shown as a line between the tasks.

For most projects, it is not sufficient just to connect single tasks. Many projects have burst events where instead of only one task, two or more tasks follow the predecessor. The same applies to merge events that unite several predecessors. Within OmniPlan, it is possible to solve this problem very easily by connecting the burst or merge events via the drag-and-drop function. Figure 18 shows the burst event "Pack Moving Boxes" which is followed by both "Arrange for Relocation Assistants" and "Arrange for a Truck." The merge event "Apartment Relocation" follows "Arrange for Relocation Assistants" and "Arrange for a Truck" and is shown in Figure 19.

After creating all tasks with their connections, the software shows the complete network diagram as in Figure 21. The software is also able to evaluate the critical path and floats. To select the critical path, navigate to the "Milestones" tab within the

[6] Referring to the free trial version of OmniPlan Pro for Mac v3.13 at https://www.omnigroup.com/omniplan.

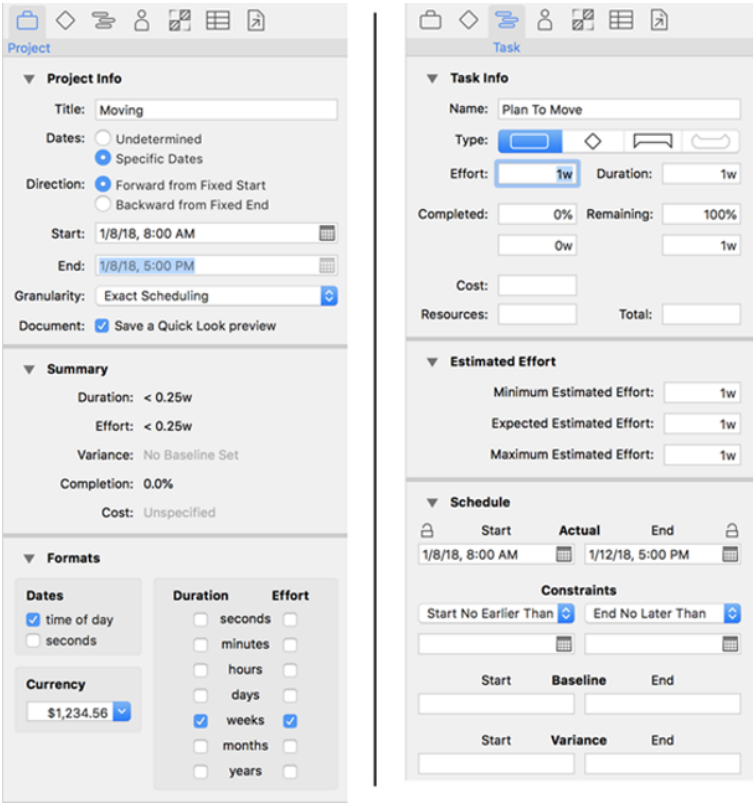

Fig. 16 Project Information (Left) and Task Information (Right). Source: Own representation.

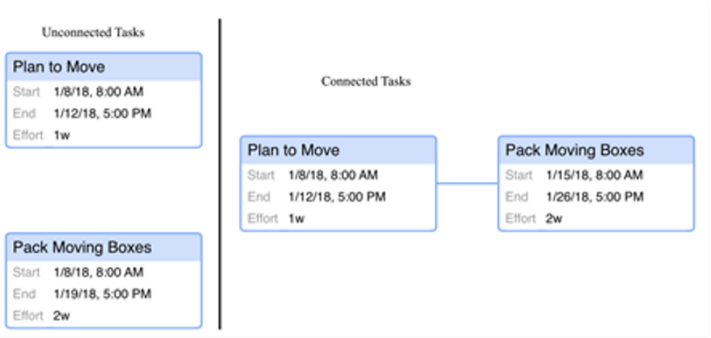

Fig. 17 Connections. Source: Own representation.

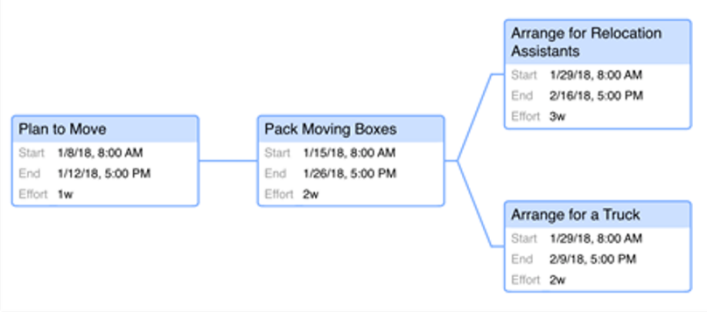

Fig. 18 Burst Event. Source: Own representation.

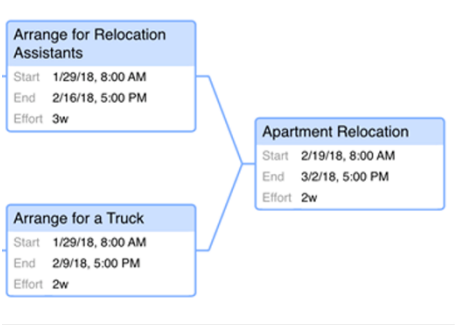

Fig. 19 Merge Event. Source: Own representation.

info menu on the right as shown in Figure 20. The critical path is shown in Figure 21 as a sequence of red rectangles in the network diagram.

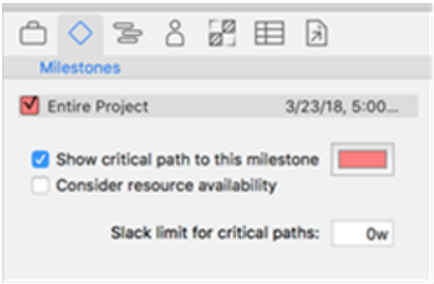

Fig. 20 Show Critical Path. Source: Own representation.

To determine the free and total floats, go to "view" and then "presentation options." There it is possible to choose add "Free Slack" and "Total slack," as OmniPlan

calls them. The floats are then shown within tasks below their effort (duration). Figure 21 finally summarizes the complete case study "Moving" in a network diagram, including the critical path and the free and total floats.

Fig. 21 Case Study "Moving" Network Diagram. Source: Own representation.

5 Using Excel to Create a Precedence Diagram

As shown in section 4, there are special programs available in order to create network planning diagrams. However, creating a precedence diagram and thus calculating the critical paths can also be done by using Microsoft Excel, a widespread office tool, as seen in Figure 22.

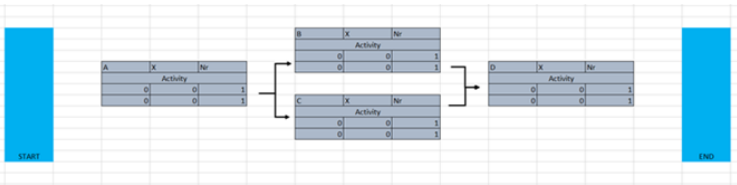

Fig. 22 Precedence Diagram Created in Microsoft Excel. Source: Own representation.

Arrows can be inserted by using the "Insert Object" functionality of Excel as shown in Figure 23. When creating such diagrams, using Excel may not be the best practice, but it requires no additional software except for the most present MS Office applications and can be done easily and quickly. A major advantage is that the calculations of the critical path can also be done swiftly.

For CPM computations, three main rows have to be created: activities list, duration of activities, and possible activity paths. Each activity gets its own column with data from Chapter 3. In the next step, all activities have to be mapped to paths. In our example, we have two paths in total. If the activity is part of the respective path "1"

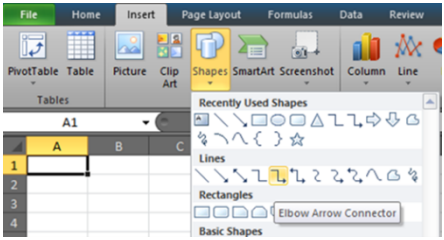

Fig. 23 Insert Arrows Using Standard Functionalities of Excel. Source: Own representation.

is entered, "0" if it is not part of this path. This is a binary flag for the upcoming calculation. Excel sheet as shown in Figure 24 is created.

	A	B	C	D	E	F	G	H
1	**Activities**	A	B	C	D	E	F	G
2	**Name**	Plan to move	Pack moving boxes	Arrange for a truck	Arrange for relocation assistants	Apartment relocation	Renovate old apartment	House-warming party
3	**Duration (weeks)**	1	2	2	3	2	1	3
4	**Paths**							
5	*Path 1*	1	1	1	0	1	1	0
6	*Path 2*	1	1	0	1	1	0	1

Fig. 24 Full Case Study Shown in Excel in Order to Calculate CPM. Source: Own representation.

As we have learned in section 2, the critical path is the path with the longest duration. To figure out the paths' durations, the Excel function "SUMPRODUCT" is used as shown in Figure 25. It combines the arrays "Path 1" and "Path 2" respectively with task durations and thus calculates the total duration.

I5 =SUMPRODUCT($B$3:$H$3;B5:H5)

	A	B	C	D	E	F	G	H	I
1	**Activities**	A	B	C	D	E	F	G	
2	**Name**	Plan to move	Pack moving boxes	Arrange for a truck	Arrange for relocation assistants	Apartment relocation	Renovate old apartment	House-warming party	
3	**Duration (weeks)**	1	2	2	3	2	1	3	
4	**Paths**								**Path duration**
5	*Path 1*	1	1	1	0	1	1	0	**8**
6	*Path 2*	1	1	0	1	1	0	1	**11**

Fig. 25 Caluclating the Total Duration of Each Path by Using Excel's "SUMPRODUCT". Source: Own representation.

6 Conclusion

The first network analysis methods were developed within three years, from 1956 to 1958. They are based on the graph theory. The methods have different graphical models (network diagrams) and differ in their way of evaluation and level of detail for each activity. Essential insights are the different floats, which identify the amount of time that an activity can be delayed without delaying its successors or project completion. Furthermore, the critical path is a significant insight that determines the project's duration and critical activities that require special attention.

One of these methods mentioned above is the Precedence Diagramming Method. The main benefit of PDM is that it shows dependencies between activities, so it can be an important communication tool for stakeholders, as it is a good metric to visualize the context of activities.

These methods optimize the business planning of companies. On the one hand, these methods are useful to control and supervise complex projects. A specific, modular view for each activity inspires all stakeholders of a project to interact with each other. Furthermore, network techniques are time-based and network diagrams clearly illustrate the whole structure of a project. It can focus on the most important key elements.

On the other hand, increasing project details requires that control effort rises as well. It is necessary to keep in mind that the network technique should be used for projects with a huge number of tasks. For less complex projects it might be sufficient to use the more visual and easier-to-create "Gantt diagrams."

Finally, it is a management decision on which technique should be applied to a specific project.

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The Peren-Clement Index

Reiner Clement & Franz W. Peren

Abstract The Peren–Clement index is a method of country-specific risk analysis for direct investments. This analysis tool provides a guideline when deciding which foreign markets offer the possibility of additional business engagement and investment and the extent of an existing engagement or investment can be increased or should be reduced. The Peren-Clement index can be used as an early detection system, which evaluates probabilities and risks of an investment in a certain foreign market, which are determined by the political situation, its social, economical and judicial environment as well as its predictable or anticipated future developments of the analysed country.

1 Introduction

Companies are often faced with questions about the criteria that determine investment decisions in other countries and foreign business locations, or even competing locations as the case may be. Due to the difficulty of deciding between different locations, it is to be expected [1], [2] that these companies will include a variety of criteria in their considerations and subject the prospective locations to comparative scrutiny [3]–[7]. In comparison with a domestic investment decision, effecting a direct investment is regarded as an especially difficult and complex decision, particularly because of uncertain framework conditions.

1.1 Definition of a Foreign Direct Investment

A foreign direct investment (FDI) is distinct from a portfolio investment due to the following characteristics (Fig. 1):

F. W. Peren, T. Neifer (eds.), *Operations Research and Management*,
Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-031-47206-0_13

(1) It involves financial participation by an investor in an enterprise located in another country.
(2) The aim is to exert a sustained influence on the business policy of this enterprise.
(3) In accordance with international standards, a sustained influence is assumed if the respective interest constitutes at least 10% of the share capital of the company in the target country [8].

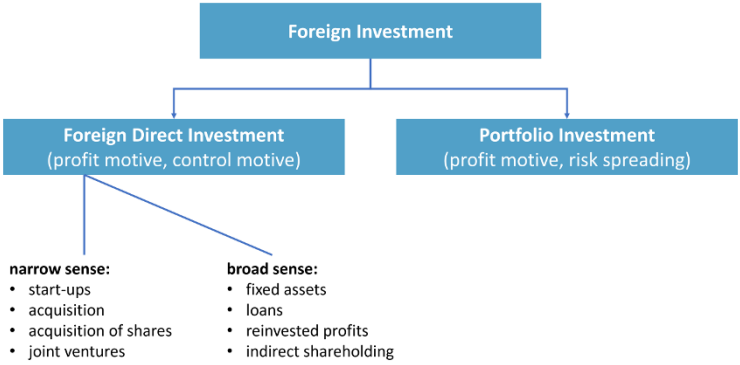

Fig. 1 Forms of Foreign Investment. Source: Own representation according to Holwegler, B.; Trautwein, H.M. (1998) [9].

Foreign direct investments may take different forms. Apart from the acquisition of shares and cooperation between enterprises, such as in the form of joint ventures, there are also complete takeovers of foreign enterprises or newly established subsidiaries (greenfield investments). A wider view also regards equity interests as foreign direct investments. These may take the form of non-cash contributions, loans, or reinvested profits.

1.2 Structural Features

Foreign direct investments have varying structural effects. A distinction must be made between substitutive and additive foreign direct investments in the target country. In the case of substitution, domestic investments are displaced by the influx of foreign capital. Additive foreign direct investments increase the investment volume in the target country. The time aspect is a key consideration for this evaluation. For example, an additive direct investment may indirectly result in displacement of domestic enterprises if the foreign enterprise in the target country ties up scarce

financial resources or labor. Conversely, company takeovers or the acquisition of interests may have an additive effect if the survival of domestic enterprises is secured or the reinvestment of capital increases the domestic capital stock.

In the country of origin, a distinction based on horizontal, vertical, and conglomerate foreign direct investments makes sense. The classification is based on the production process of the investing enterprise. In the case of horizontal foreign direct investments in the target country, the same products are produced as in the country of origin. Here production can take place at the same time, but it can also be outsourced. Vertical foreign direct investments include both upstream and downstream production stages within the output chain. In the case of backward direct investments, raw materials are extracted, or preliminary works are manufactured for use in domestic production abroad. Forward foreign direct investments result in the establishment of downstream production stages abroad. These include sales and marketing organizations for the disposal of domestic goods. Conglomerate foreign direct investments are a type of investment where production at the foreign and domestic sites is unrelated either in vertical or horizontal terms [9].

2 Theory of Direct Investment

Within the scope of traditional theories of international trade – which, above all, are shaped by the *Heckscher-Ohlin theorem* – foreign direct investments and the existence of multinational enterprises only lend themselves to conditional explanation. International foreign direct investments appear as capital flows from rich to poor countries, motivated by lower labor costs in poor countries [10]. Capital will flow from rich to poor countries until the ratio between real capital and labor is the same in all countries and the returns on investment as well as the wage rates in the respective countries have converged. International movements of capital, therefore, equalize differences in the levels of production, given the full competition in the markets for goods and capital. As portfolio investments, they react to differences in returns where an investor does not aim to actively exert an influence on the corporate policy of foreign enterprises. Foreign direct investments, however, can be explained only through the assumption of incomplete markets and by taking transaction costs into consideration.

2.1 Justification for Foreign Direct Investment

Economic theory holds that providers of foreign direct investments are primarily multinational enterprises. These are characterized by the fact that they have production sites in at least two countries [10]. Multinationals are to be distinguished from companies that produce and export goods or services from only one country.

In view of the growing importance of foreign direct investments, methods that provided explanations for the foreign direct investments carried out by multinational enterprises were developed at end of the 1960s. With his OLI theory, *Dunning*, in particular, undertook the attempt for the first time to integrate different approaches and to develop a general explanatory model for foreign direct investments [11]–[13]. The aim of this approach is to explain which forms of market cultivation (i.e., exports, licenses, or foreign direct investments) should be adopted by companies in foreign markets as a function of three company-specific advantage categories.

Accordingly, foreign investments should be made if the following three conditions are all fulfilled (Table 1):

Table 1 Decision Matrix with Different Forms of Foreign Activity

Foreign Activity	Owner-specific competitive advantage	Internalization advantages	Location-specific advantages
Direct investment	yes	yes	yes
Exports	yes	yes	no
Licensing	yes	no	no

Source: Own representation according to Dunning, J. H. (1982) [12].

1. **Ownership-specific advantages:** The potential investor has company-specific advantages over the local competitors in the target country. These include, for example, technological know-how, research and development capacities, patents, management skills, a superior labor organization, exclusive raw materials or sales channels, as well as the exploitation of returns of scale and specialization effects.
2. **Internalization incentive advantages:** The potential investor has internalization advantages. This means that there must be an incentive to exploit existing company-specific competitive advantages within the company's own organization instead of making use of them through the external market. Internalizing advantages consist, for example, of the avoidance of transaction and negotiation costs.
3. **Location-specific advantages:** The host country has location-specific competitive advantages over the country of the potential investor. These include cost advantages, a larger market, or more favorable framework conditions for investments.

An investor will take action using foreign direct investments only if the three advantages specified above coincide. If only ownership-specific competitive advantages and internalization incentive advantages apply, then it pays to serve the foreign market through exports in order to be able to exploit the superior location-specific advantages in the home country. If there is also a lack of internalization incentive advantages due, for example, to inadequate legal framework conditions for the protection of intellectual property, then an investor will contract out production by licensing to foreign enterprises.

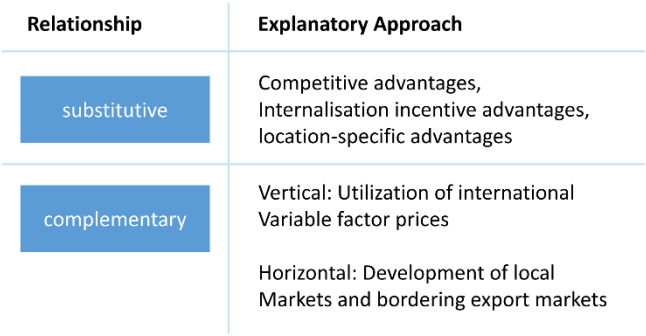

Fig. 2 Relationship Between Foreign Trade and Foreign Direct Investments.

In the systematic approach formulated by *Dunning*, foreign direct investments and foreign trade appear as mutually exclusive alternatives when it comes to supplying a foreign market (i.e. substitutive relationship). We can use this approach to explain direct investment flows between highly and less-developed countries. However, this concept does not give sufficient consideration to foreign direct investment relationships between highly developed national economies.

Complementary relationships between foreign trade and foreign direct investments can be integrated if the method employed by *Dunning* is expanded to include the production of several products and/or multilevel production. The result is vertical or horizontal foreign direct investments (Fig. 2). By combining the conventional trade theory approach with *Dunning's* assumptions, it is possible to explain why certain factor allocations can be carried out preferably within the company rather than through the market. These models fall within the term of *New Trade Theory*, which aims at providing a realistic view of foreign direct investments.

According to *New Trade Theory*, the foreign activities of multinational enterprises are not characterized primarily by the fact that they merely pursue (labor) cost advantages in less developed countries. Instead, their primary strategy consists in developing worldwide output networks in which their technical knowledge can be

more efficiently used than in exclusively national networks [14]. In this case, multinational enterprises have corporate headquarters that offer so-called *headquarter services*, which can be utilized by all of the company's domestic and foreign operational facilities. The results of research and development represent one example. If a new technology is developed for more efficient production, then the costs for the development of this technology are incurred only once, regardless of the number of plants in which it is used. If the respective company expands its production to additional foreign business locations, then it does not have to duplicate the technological development costs, but it is able to use them abroad without additional costs. In the theory of multinational enterprises, international foreign direct investments principally serve to exploit company-specific economies of scale through the establishment of global output networks. However, precautions have to be taken against the outflow of knowledge to competing companies.

Vertical Foreign Direct Investments

In the case of vertical foreign direct investments, it is not how a market is served but rather the decision regarding the spatial degree of concentration of the production processes that stands at the fore. Vertical foreign direct investments arise when the respective enterprise geographically separates individual production stages. This type of foreign direct investment aims for the utilization of different international factor prices. Using different factor costs for capital and labor, *Helpman* and *Krugman* explain why it may make sense to distribute activities to several countries and why this results in multinational enterprises and FDI. Thus, *headquarter services* are established in the parent country, which is rich in capital, while commodity production takes place abroad where labor is more abundantly available. This extension explains bilateral foreign direct investments between industrialized countries endowed with similar advantages. Here, the reference model from foreign trade theory is also the classical *Heckscher-Ohlin model*. Vertical foreign direct investments tend to foster foreign trade. Forward vertical foreign direct investments tend to result in greater exports, while backward vertical foreign direct investments instead tend to boost the imports of the country of origin.

Therefore, when it comes to vertical foreign direct investments, a different level of development between the countries involved, relatively low trade barriers, geographical proximity, and different factor attributes as well as factor price differences can be highlighted as the main factors of influence [15].

Horizontal Foreign Direct Investments

The essential reason for horizontal direct investment is the development of the local market as well as neighboring export markets.[1] The production sites on location at least partly replace the exports of the parent company. In most cases, some *headquarter services* will at least remain in the country of origin and will not be duplicated and/or multiplied.

The following company-specific characteristics should be noted for horizontal foreign direct investments [15]: The scale economies of the individual foreign production sites are relatively small in relation to the economies of scale of domestic

[1] See for example Markusen, J.R. (2000) [16].

production, and the product-specific transport costs must be relatively high as well so that external production is worthwhile in comparison with transport from a single, central production location. The following sectors will be more likely to profit from such a strategy:

- growth industries,
- sectors that require a high degree of proximity to the customer, and
- sectors that do not offer tradable services or only with difficulty.

A further important indicator is the low export share of the respective enterprises. Since production sites are primarily established in order to serve the foreign market in the case of horizontal foreign direct investments, a large share of production remains in the recipient country and/or the sales region.

All in all, it is not always possible to make a clear distinction between horizontal and vertical foreign direct investments in direct investment theory since the foreign production sites purchase several *headquarter services* from the parent company even in cases of horizontal foreign direct investments. Thus, in practice, horizontal foreign direct investments often have vertical features. Hence, the importance of business location factors for horizontal and vertical foreign direct investments can more likely be distinguished in analytical terms (Tab. 2).

Most major factors of influence affect both types of direct investment; however, rarely in the same manner and with the same impact. In practice, the characteristics of horizontal and vertical foreign direct investments often merge.

Table 2 Determinants of Foreign Direct Investment

Factors of influence	Horizontal FDI	Vertical FDI
Market size	large	unclear
Level of development	similar	different
Relative factor endowment	similar	different
Relative factor costs	unclear	different
Trade barriers	high	low
Geographical distance	high	low

Source: Own representation according to Köller, M. (2006) [15].

2.2 Valuation Perspectives

The global growth rate of foreign direct investments has constantly increased since around the mid-eighties. In essence, this represents companies' responses to changes in their competitive environment. The successful efforts to liberalize world trade within the scope of the GATT and subsequent WTO negotiations have particularly contributed to this development. The target countries and the forms of foreign direct investments have changed as well. Greater focus has been placed on developing

countries and emerging economies. *Cooperative organization models* such as *strategic alliances* have grown in importance as they offer companies the possibility of realizing their aims with less capital requirements.

Various perspectives may be taken into consideration in order to evaluate the effects of FDI (Fig. 3). However, the concrete effects essentially depend on the structural characteristics of the direct investment as well as their respective motives:

- The relationship between foreign direct investments and foreign trade at the global level and/or between different groups of countries.
- Effects of FDI in the target country: the effects on growth, employment, structural change, and productivity stand out in particular.
- Effects of FDI in the country of origin: among other things, this includes aspects involving site competition, international competitiveness as well as the repatriation of investments.
- Evaluation of FDI from the investor's standpoint: the motives behind the direct investment, e.g. costs, procurement, or sales as well as their effects on management ratios.

Valuation Perspectives	Questions
global	Development of world trade and foreign direct investments, impacts on country groups
destination country	Substitutive and additive: Consequences for growth, employment and structural change
contry of origin	Horizontal and vertical: Consequences for growth, employment and structural change
investor	Profitability, cost-benefits, risks

Fig. 3 Valuation Perspectives of Foreign Direct Investments.

Global View

In the past two decades, foreign direct investments were a substantial driver of globalization worldwide. The *United Nations Conference on Trade and Development (UNCTAD)*, which has operated as a subsidiary organization of the UN General

Assembly since 1964, is concerned with the development and composition of global FDI flows and holdings. The annually published World Investment Report (WIR) offers the most current statistics and figures with regard to FDI while reporting on trends and developments in the international investment environment. The first part of the report usually provides an overview of current developments and data, while the second part of the report is devoted to a central topic of interest which changes on an annual basis. The reports that have been published since 1991 are downloadable online. Numerous country profiles with regard to foreign direct investment activity are also available at the UNCTAD website. UNCTAD also maintains a statistics portal on foreign direct investments on the Internet.

The *Organization for Economic Co-operation and Development (OECD)* offers a database with statistics on transnational foreign direct investment activity. However, the available statistics only rarely allow a separation between horizontal and vertical foreign direct investment. Often the analysis is necessarily limited to differentiating between the various business locations of the foreign direct investment and representing aggregated flows and/or holdings.

Target Countries

When evaluating the effects of FDI in target countries, it is necessary to distinguish between substitutive and additive foreign direct investments. In contrast to the traditionally positive evaluation provided by policymakers in industrial countries, the overall economic benefit of foreign direct investments was regarded as controversial – particularly in developing countries and emerging markets. There were concerns that national sovereignty would be undermined by powerful multi-national enterprises in particular along with a loss of control of national resources. For this reason, foreign direct investments were often restricted by various mechanisms (e.g. with regard to equity investments or legal forms of cooperation).

In the past two decades, a fundamental change in this attitude could be observed in the majority of developing countries and emerging markets as well. Many of these countries now proactively attempt to attract enterprises that may in the future engage in foreign direct investments. A scramble for foreign direct investments can even be seen in some cases. Behind this change in attitude is the realization that many of the causes of insufficient economic development can be attributed to internal factors in the respective country, rather than a malfunctioning of global economic interdependencies. Thus the (re-)integration of national economies into the world market, driven mainly by foreign direct investments, appears to make sense [17].

Foreign direct investments offer the opportunity for many countries to integrate themselves into the output chains of multinational enterprises, offering comparative or absolute advantages. Thus, foreign direct investments may significantly contribute to an improvement in the allocation of national resources and induce or accelerate structural change in the country concerned. Countries that are integrated into multinational output chains are also able to gradually create additional efficiencies, where applicable, to supplement and/or replace labor-intensive functions with ones that are more technology-intensive and, over the course of time, take on the role of a global supplier of individual components, products or product groups for multinational enterprises that emanates beyond the respective local region. The local and

regional environments of foreign subsidiaries are also able to profit from the presence of foreign investors. Thus, the effects on employment in developing countries and emerging markets through the influx of foreign capital are largely positive. As a rule, domestic labor is employed in the foreign branches, even if a part of the management comes from the country of origin.

Apart from the accumulation of real capital, foreign direct investments at the same time lead to an increase in human capital and knowledge. Thus, they represent by far the greatest opportunity for developing countries and emerging markets to increase their growth potential otherwise under their own steam. Against this backdrop, many countries feel compelled not only to open up their markets to foreign direct investments but also to facilitate the investment decision of foreign investors through incentives [18].

Foreign direct investments (in particular newly established companies) may exacerbate regional differences in the target country. An unequal spatial distribution of investments may accelerate the development of growth or the stagnation and depopulation of areas [19].

Countries of Origin

With respect to countries of origin, the discussion on the possible repercussions of FDI is shaped in particular by the distinction between vertical and horizontal foreign direct investments:

- Market-oriented foreign investments are driven by a desire to develop new markets or to further penetrate the foreign market concerned. In this case, companies create production capacities abroad that do not jeopardize their domestic capacities. In the meantime, the establishment of sales channels abroad can contribute to increased export activities from the country of origin and result in the creation or protection of domestic jobs. The production of goods or the provision of services in the target country can induce investments in the country of origin. This is possible, on the one hand, due to export effects that can lead to an expansion of investments or new investments in the country of origin if the given capacities are inadequate. Investments may, on the other hand, take place if the production of upstream, intermediate, and finished products is developed in the country of origin due to new developments at the foreign business location.
- If the foreign direct investment is motivated by cost optimization, then the labor-intensive parts of the output chain may, among other things, be outsourced if labor costs, for example, are lower in the target countries. However, the relocation of jobs does not necessarily provide proof of negative consequences on employment in the country of origin as a result of foreign direct investments. Firstly, some of the domestic jobs would be lost in any case due to the labor costs advantages of competitors. Secondly, relocation of parts of the output chain in favor of business locations with cheaper labor costs may ultimately result in competitive advantages for the company. Superior jobs are possibly secured in the home country if inferior, low-paid production segments are relocated abroad [20], [21]. However, foreign investments may also replace past exports if, for example, the foreign market is served directly through local manufacturing. In

part, there may also be an increase in (re-)imports in order to serve the home market from abroad. In this case, domestic jobs may also be lost.

Which effects (complementary effect; substitution effect) prevail depends not only on the reason for the activity abroad, but also the industry, the period under review, and many other conditions according to individual cases (e.g. the level of development and economic situation of the countries involved, Tab. 3). In various empirical investigations, it was determined that growth in employment in foreign subsidiaries was higher than in the parent company in the country of origin; however, the growth rate of parent companies turned out to be higher for domestic companies without international subsidiaries [19].

Table 3 Examples of Economic Effects of Foreign Direct Investment in the Country of Origin and the Target Country

	Country of origin	**Target country**
Employment	Complementary effects result in increased demand for labor; substitutive effects result in the reduction of domestic employment	Positive effects due to demand for labor; increase in the quality of human capital
Competitiveness	Increase because of cost optimization; access to know-how; optimization of the output chain	Increase due to modernization and adaption of corporate structures to global standards; technology transfer
Resources	Access to raw materials, technology, human capital	Loss of resources
Spatial and sectoral effect	-	Possible increase in spatial and secotral disparaties
Balance of payment	Return of profits	Foreign currency; export revenues; reinvestment of profits

Source: Own representation according to Austrian Institute for SME Research (2010) [19].

Investors

In contrast to a decision for an export country, the decision to make a direct investment is a long-term constitutive decision. For example, it is not always possible to manufacture at the location with the lowest production costs because this location may change over time. If the business location with the best labor costs were continually sought after, then a company might be forced to constantly relocate. Thus, the choice of a business location always takes into account the uncertainty of future developments. Foreign business transactions are, of course, always subject to risks and uncertainties, above all, when the investor is generally unable or only slightly able to influence prevailing circumstances or is unable to judge the creditworthiness of the local contracting parties. For this reason, the business location factors that are relevant for investment carry different weight depending on the situation affecting the individual decision made by an enterprise. Therefore, in order to also strategically

evaluate and justify the decision-making, an enterprise must consider a variety of factors when selecting the optimal business location.

At the center of the observation below is the uncertainty associated with an FDI in the respective target country from the perspective of the company in a country of origin. However, in order to also create an evaluation framework for small- and medium-sized enterprises, it should not only consider multinational companies. Contrary to the case of multinational companies, the type and scope of foreign business activity among small and medium-sized enterprises have not been investigated sufficiently thus far, although there has been a clear increase in the internationalization of German small- and medium-sized enterprises in recent decades.

3 Direct Investments and Site Selection

Risk indices, among other things, are also available in order to underscore such a decision. The original version of the present Index (Peren-Clement lndex – PCI) was published in 1998 [22]. The Index is included in various reference works [23], [24] and is well-established in the international literature on risk evaluation while also serving as a business tool [24]–[27]. Further development of the Index in the present Version 2.0 has three aims:

1. Dynamic adjustment of the index's criteria and consideration of changes that have occurred in the international environment since its last iteration. These include:

    - the intensification in the globalization of economic relationships that have led, in particular, to the transfer of knowledge and technology to emerging markets and developing countries;
    - the emergence of outsourcing and offshoring in various industries;
    - the increasing openness and transparency of corporate decisions due to globally available information and communication technologies;
    - progress in the reduction of tariffs and other barriers to trade.

2. Critics of risk indices insist that they do not take the specifics of the particular industry or individual company into consideration.[2] This criticism is intended to be addressed through the possibility of a two-dimensional representation that offers a customized assessment.
3. In contrast to other indices, a theoretical substantiation of the investment decision is undertaken within the scope of the PCI. What is essential here is the connection between two crucial schools of thought on strategic planning: the resource- and market-based points of view [29], [30].

    - In the case of the resource-based view, it is not the market structure but rather the uniqueness of the resources at a location that forms the starting point

[2] cf. e.g. Daum, A.; Greife, W. & Przywara, R. (2009) [28].

for competitive advantages. However, these resources only have limited mobility and tradability. They interact with the internal resources of an enterprise (inside-out perspective).

- The market-based view employs an outside-in perspective. Thus, the structure of the industry (production and sales) prescribes how a company is able to position itself in a market.

These two perspectives are not mutually exclusive but rather complement each other in important ways. Ultimately, companies must address both of these perspectives within the scope of an international investment decision and consider them as complementary elements in their strategic planning.

In general, the location quality of a region can be neither directly observed nor directly measured. Thus, suitable indicators (variables) have to be found for an empirical investigation. Within the scope of econometric analysis, both the reason for tapping a new market as well as the exploitation of local production conditions can be proven to be significant motives for foreign direct investments. Furthermore, the political and legal framework conditions that define a location also play a substantial role [31]–[34].

3.1 Framework for Decision-Making

The investment decisions made by companies should follow a systematic and progressive structure. In doing so, a distinction must be made between the macro-environment and the micro-environment of the individual company. At the site-specific micro level, core competencies can be found that have a positive impact on the competitiveness of the individual enterprise. Moreover, concrete production and sales-oriented motives that reflect the supply and demand side of an ob-served market can also be found (Fig. 4).

Other systems add the dimension of performance factors (outputs) to the production factors (inputs) and market factors [35]. In this case, it is assumed that the level of performance of the planned or existing location (performance) cannot be derived directly from the production and market factors. The performance factors include quantitative indicators such as, for example, productivity, process performance, and lead times or qualitative indicators, e.g., innovative capacity or flexibility when it comes to product adaptation. As a rule, this classification works well in practice and is also compatible with the decision-making framework of the PCI. The decision-making framework in the following (PCI 2.0) is divided into 20 criteria. These 20 criteria are assigned a point value of 50 points. Each criterion is then weighted with 0 to 2 points so that the 20 criteria have a maximum possible overall score of 100 points.[3]

[3] The weighting is based on an extensive review of the literature and evaluation of studies.

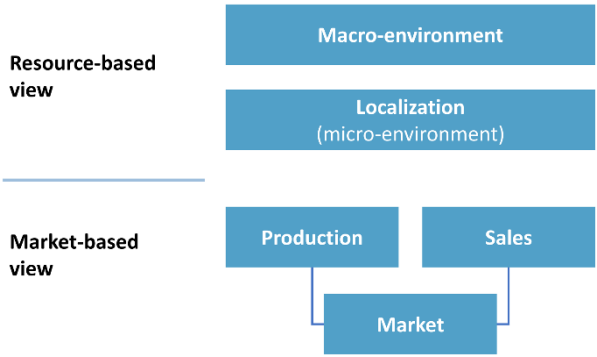

Fig. 4 Framework for Decision-Making with Regard to International Locations.

3.1.1 Macro-environment

This category is largely predetermined for companies, as the acting forces are exogenous (Tab. 4). The enterprise observes or anticipates these factors and responds with strategy adjustments. A series of current international statistics and aggregate assessments in the form of indices are available to enterprises to evaluate the macroeconomic environment. However, the various norms underlying these indices, as well as their different characteristics with regard to the measurement and quantification of the respective risks and opportunities of foreign direct investments, must be taken into consideration.

A few examples:

- The detailed analyses of the *World Economic Forum* are, for example, suitable for evaluating the features of *economic policy, legal certainty,* and *financial solvency* [36]. A high score in the *Global Competitiveness Report* (e.g. Switzerland in 2014, first place with 5.67 points) implies a low risk for foreign direct investments, and a low score (e.g. Algeria in 2014, ranked 100 with 3.79 points) corresponds to accordingly high risk [37].
- The political social stability indicator describes the capabilities of a government to realize its stated program, the unity of the government, its legislative strength, and public support. Thus, a high index value expresses an accordingly high degree of stability and, consequently, is associated with a lower risk for foreign direct investments. The relevant data are made available, for example, within the scope of *International Country Risk Guides (ICRG)* [38].
- The data contained in the *Index of Economic Freedom* from the *Heritage Foundation* [39] can be used to evaluate bureaucratic obstacles. This index measures the

degree of economic freedom based on property rights and the extent of government regulation of the market. Other parameters include government corruption, restrictions on foreign trade, income and corporate taxes as well as the rule of law. The highest possible score is 100. A high score (e.g. Hong Kong in 2014, ranked in first place with 90.1 points) is concomitant with greater freedoms and tends to be associated with a relatively low risk for foreign direct investments. A low score (e.g. Indonesia in 2014, ranked 100th with 58.5 points) indicates a clearly higher risk due to bureaucratic obstacles to foreign direct investments.

Table 4 Macro-Environment

Macro environment	**Points**	**Weighting**	**Total**
Political and social stability		4	
Bureaucratic obstacles		2	
Economic policy		3	
Legal security		3	
Solvency		3	
Total		**15**	

3.1.2 Localization (micro-environment)

In the globalized world, it is often not national economies but rather regions that compete with each other in order to attract productive enterprises. When it comes to this dimension, the factors that are of importance are those that are not offered – or are only offered to a much lesser extent or quality by other business locations – or are not easily imitated (Tab. 5). In theoretical terms, this dimension can be linked to the resource-based view [40].

This is equally true for business locations when it comes to their *human capital, transport connections (accessibility), existing management skills, access to markets,* and the *quality of life* that is offered. All of the factors specified are in turn dependent upon additional factors. For example, the quality of life at a particular business location is usually more significant as a "soft factor" (i.e. the less criminality there is, the better the health system, and the lower the regional price level, the higher the purchasing power of a monetary unit). Other favorable effects on the quality of life are cultural activities, education and training opportunities, environmental quality as well as local recreation and leisure opportunities [1].

As a rule, these factors are not transferable and can only be duplicated by competing business locations within limits. The importance of these relationships is emphasized, among other things, by the factor conditions in the *Porter Diamond model* [41].

Above all, the interactive effects between corporate and location-specific resources are important in this context. On the one hand, locally bound resources have

an impact on the attractiveness of a business location for companies while, on the other hand, the company's investments contribute to the generation and development of know-how at a business site [42]. The existing resources at a particular site location then provide for competitive advantages when complementary site-specific resources are added to corporate skills and resources so that a change in location or an alternative location would result in a competitive disadvantage for the relevant enterprise [40], [43].

Table 5 Localization (Micro-Environment)

Localization	**Points**	**Weighting**	**Total**
Human capital		4	
Transport connections		2	
Managment skills		2	
Access of markets		2	
Quality of life		3	
Total		**13**	

But even if a country provides for an environment that is generally friendly to investments, foreign direct investments will only take place if there are additional concrete reasons for the company. Empirical studies and accompanying literature have distinguished between production- and market-oriented elements. Thus, both sides of the market – supply, and demand – are addressed.

In theoretical terms, both dimensions can be linked to the market-based view decisively shaped by Porter [44]. This view makes analysis of the market an essential task. A company's competitive advantages result in particular from proper selection of a market and/or a target segment of a market as well as superior positioning (over the course of time) within this market (outside-in view).[4]

Production

The prevailing economic and property rights (Tab. 6) are of fundamental importance for making investment decisions in new locations. They determine the manner in which rights of action with regard to economic goods are ascribed to the economic players. These include, for example, the right to exploit the asset value (right of use and decision); the right to modify or change the ownership of a property (right of change), or the right to transfer the asset value either in part or as a whole (transfer right). Furthermore, freedom of contract and the principle of liability should be clarified prior to making a decision [4].

All in all, it may be assumed that *economic* and *property rights* that do not correspond to free market principles substantially obstruct a company's activities and have a negative impact on prospective investment decisions in new locations. In this context, empirical analyses also demonstrate the importance of protective rights involving intellectual property (e.g. patent protection).

[4] cf. for example Sakarya, S.; Eckman, M. & Hyllegard, K.H. (2007) [45]; Weber, J. (2008) [46].

Companies are often established based on *cost-oriented motives*. For example, these include labor cost advantages and tax benefits, as well as purchasing and procurement advantages [47]. However, with regard to labor cost advantages, it is important to note the relation between wages and qualification (unit labor costs). In wealthy countries, low costs alone neither define the competitive advantages of enterprises nor the growth dynamics of countries. At a macroeconomic level, low costs and a high per capita income may even be contradictory over the long term. The stock of human capital thus becomes a kind of procurement advantage in addition to cost orientation.

Companies often require additional *capital* in order to make investments. If an investment cannot be financed through retained earnings and no additional capital is contributed by its current shareholders, then the required funds may be procured either through a loan or acceptance of external equity. Correspondingly, positive effects may be a well-functioning credit market as well as the presence of sufficient investment capital. If foreign direct investments entail the international movement of capital, then the free convertibility of national currencies becomes more important.

Complementary industries that are capable of providing raw materials and manufacturing supplies are likewise relevant for production. The motive for purchase protection and procurement security is essential, particularly for foreign direct investments in countries with abundant resources. Thus, for example, uncertainties in the pricing of primary products can be reduced if company subsidiaries provide the key raw materials.

Investment incentives are also of importance with regard to production-oriented motives. These can be provided by the host country, the home country of the investing company, or multinational entities. According to the *OECD Investment Reform Index*, incentives may take the following three forms [48]:

- Regulatory incentives, e.g., the design of work, environmental and social standards.
- Financial incentives, e.g., in the form of grants for training specialists by the government, through subsidized loans, or by means of guarantees for loans (credits).
- Fiscal and tax incentives, e.g., tax reductions for foreign investors such as corporate income tax reductions or a temporary tax allowance and the establishment of special tax-privileged areas.

Sales

Often business decisions to establish a new business location are based on the desire to develop a promising sales market that has only been insufficiently served thus far or not at all [4], [49]. Of primary importance are the *size and dynamics of the market* (Tab. 7). High *per capita income* serves as an indicator of effective demand and turns the respective geographical location into a possibly interesting sales market for local market development investments [34]. If there are trade barriers between the current business location of a company and the potential sales market under consideration, then this too may require that the company establish its own

Table 6 Production

Production	Points	Weighting	Total
Economic and property rights		2	
Manufacturing costs		2	
Capital procurement		3	
Complementary industries		2	
Investment incentives		2	
Total		**11**	

site in the target country. Trade barriers include protectionist measures that restrict free international competition as a whole. These include both tariff and non-tariff *trade barriers*. In addition, satisfactory distribution structures as well as a sufficient degree of *reliability on the part of local distributors* are substantially important for the successful development of new markets. Existing or potential *network structures* at the business location and/or in the target country must also be given sufficient consideration and included in the evaluation of a possible direct investment.

Table 7 Sales

Sales	Points	Weighting	Total
Size and dynamic of the market		3	
Per capita income		2	
Avoidance of tariff barriers		2	
Reliability of local contracting parties		2	
Distribution structures		2	
Total		**11**	

3.2 Risk Assessment

The design of the PCI is based on a cost-benefit analysis – which is also referred to as a cost-utility analysis, a point-rating system or a scoring model. Thus, the Index belongs to the quantitative, non-monetary methods of analysis in decision theory.

The aim is to analyze alternative courses of action with the purpose of organizing preferences for the decision-maker in a multidimensional scoring system. Results are ordered by specifying the utility values (total values) of the alternatives. As with other comparable indices, e.g. *Business Environment Risk Intelligence Index (BERI)* [50], the assignment of these points is subjective. An alternative is the *Delphi method*, where points are averaged from various experts.

The following points can be awarded based on the degree of fulfillment of the relevant criteria:

0 = not acceptable
0.5 = questionable
1 = acceptable
1.5 = good
2 = very good

A distinction is made between 20 criteria with a total weighting of 50 (see Tab. 4 to Tab. 7). A site can receive a maximum of 100 points when each factor is multiplied by the points identified above. A classification of the foreign risk for each business location is made possible based on its maximum value (Tab. 8).

Table 8 Graduation of Country Risks

Gradation of country risks (maximum 100 points)	**Points**
Hardly recognizable risk	> 80
Low risk	70 – 79
Moderate risk, barriers in daily operations, risk coverage recommended	60 – 69
Relatively high risk, investment environment with serious defects, inevitable risk coverage	50 – 59
Location is not recommended for direct investments	< 50

The use of critical variables is particularly beneficial (knock-out variables). If certain key factors are specified as knock-out variables and a location receive a score of 2 points or less among these key factors, then the direct investment should be rejected altogether. This would also apply even if all other factors received positive valuations and the overall score produced a good result that makes the location appear to be a favorable choice.

Examples:

- A company wants to use human capital abroad. However, the actual use appears to be questionable. In this case, the resulting value is:
  0.5 (*questionable*) x 4 (weighting of the criterion *human capital*)[5] = 2 points.
- A company aims to benefit from the cost advantages in production in particular. In this case the total evaluation for the criterion must be at least acceptable so that the value is above 2:
  1.0 (*acceptable*) x 3 (weighting of the criterion *manufacturing costs*)[6] = 3 points.

[5] cf. Tab. 2.

[6] cf. Tab. 3.

Unlike other indices, the PCI allows for customized and company-specific representation. Thus, the assessment of the investment risk takes place on the basis of a two-dimensional representation. Therein both the national macroeconomic and individual corporate views are combined in a useful manner.

3.3 Case Study

Table 9 Structural Arrangement of PCI

Macro Environment	**Points**	**Weighting**	**Sum**
Political and social stability	...	4	...
Bureaucratic obstacles	...	2	...
Economic policy	...	3	...
Legal security	...	3	...
Solvency	...	3	...
Sum		**15**	**...**
Localization	**Points**	**Weighting**	**Sum**
Human capital	...	4	...
Transport connections	...	2	...
Managment skills	...	2	...
Access of markets	...	2	...
Quality of life	...	3	...
Sum		**13**	**...**
Production	**Points**	**Weighting**	**Sum**
Economic and property rights	...	2	...
Manufacturing costs	...	2	...
Capital procurement	...	3	...
Complementary production sectors	...	2	...
Investment incentives	...	2	...
Sum		**11**	**...**
Sales	**Points**	**Weighting**	**Sum**
Size and dynamic of the market	...	3	...
Per capita income	...	2	...
Avoidance of tariff barriers	...	2	...
Reliability of local contractors	...	2	...
Distribution structures	...	2	...
Sum		**11**	**...**
Total score PCI			**...**

Company A and company B aim to internationalize through foreign direct investment abroad and are considering the two countries/regions X and Y. Both companies employ the PCI (Tab. 9) for measurement as well as a comparative and quantitative evaluation of the respective country risks. The PCI shows a total score of 71 for country X (Tab. 10). This value implies a relatively low investment risk.

Investment Alternative 1: Country X PCI = 71

Table 10 Risk Assessment for Country/Region X

Macro Environment	**Points**	**Weighting**	**Sum**
Political and social stability	1.5	4	6
Bureaucratic obstacles	2	2	4
Economic policy	2	3	6
Legal security	1.5	3	4.5
Solvency	1.5	3	4.5
Sum		**15**	**25**
Localization	**Points**	**Weighting**	**Sum**
Human capital	0.5	4	2
Transport connections	2	2	4
Managment skills	0	2	0
Access of markets	1.5	2	3
Quality of life	2	3	6
Sum		**13**	**15**
Production	**Points**	**Weighting**	**Sum**
Economic and property rights	2	2	4
Manufacturing costs	2	2	4
Capital procurement	2	3	6
Complementary production sectors	2	2	4
Investment incentives	2	2	4
Sum		**11**	**22**
Sales	**Points**	**Weighting**	**Sum**
Size and dynamic of the market	0	3	0
Per capita income	0.5	2	1
Avoidance of tariff barriers	2	2	4
Reliability of local contractors	0.5	2	1
Distribution structures	1.5	2	3
Sum		**11**	**9**
Total score PCI			**71**

The calculation for the alternative country/region Y also results in a total core of 71 points (Tab. 11). Company A and company B now know that both foreign direct investments would only be associated with a relatively low risk and thus both tend toward positive valuation. However, which location would be the more suitable for the individual company cannot be deduced from national macroeconomic risk assessments performed to date.

Investment Alternative 2: Country /Region Y PCI = 71

Table 11 Risk Assessment for Country/Region Y

Macro Environment	**Points**	**Weighting**	**Sum**
Political and social stability	2	4	8
Bureaucratic obstacles	1	2	2
Economic policy	1	3	3
Legal security	2	3	6
Solvency	1.5	3	4.5
Sum		**15**	**23.5**
Localization	**Points**	**Weighting**	**Sum**
Human capital	1.5	4	6
Transport connections	2	2	4
Managment skills	1.5	2	3
Access of markets	2	2	4
Quality of life	1.5	3	4.5
Sum		**13**	**21.5**
Production	**Points**	**Weighting**	**Sum**
Economic and property rights	2	2	4
Manufacturing costs	0.5	2	1
Capital procurement	0	3	0
Complementary production sectors	0	2	0
Investment incentives	0.5	2	1
Sum		**11**	**6**
Sales	**Points**	**Weighting**	**Sum**
Size and dynamic of the market	2	3	6
Per capita income	2	2	4
Avoidance of tariff barriers	1.5	2	3
Reliability of local contractors	1.5	2	3
Distribution structures	2	2	4
Sum		**11**	**20**
Total score PCI			**71**

It is important to compensate for this evaluation deficit that occurs in all of the risk indices published to date by adding a further individual corporate dimension to the evaluation model.

The two companies pursue their respective internationalization with their own company-specific aims (Tab. 12). While it is most important for company A to develop the overseas market and to expand available resources by acquiring additional resources from the business site abroad, company B focuses on the cost benefits that can be generated through production at the foreign location.

Table 12 Company-specific Internationalization Aims

Internationalization aims of Company A:	**Target weighting - in %**
1. Sales in overseas market	50%
2. Expansion of resources through localization	30%
3. Safeguarding of foreign location-strategic importance (macro environment)	15%
4. Generation of cost benefits (production)	5%
	100%
Internationalization aims of Company B:	**Target weighting - in %**
1. Generation of cost benefits (production)	70%
2. Safeguard foreign location / strategic importance (macro environment)	15%
3. Expansion of resources through localization	10%
4. Sales in overseas market	5%
	100%

The companies now combine this second dimension with the weighting of their individual aims and the individual factors of the PCI. The company-specific factors *macro-environment, localization, production,* and *sales* are distributed to 100% (and/or 1.0 target weighting in total). The result is a customized, two-dimensional overall score that now makes a targeted decision possible for the particular company (Fig. 5).

The two-dimensional cumulative scores – measured in *Pecle* – now clearly demonstrate, in contrast to all prevailing risk indices, which site is best suited for a direct investment for which investor at the current point in time. Company A should decide on direct investment in country/region X since this is where it is best able to realize its individual corporate aims at present. For company A the country/region X achieves a total score of 20.275 Pecle, while country/region Y scores with only 13.85 Pecle. The reason for the now visible distinction is that country/region A offers the better starting position with the factors sales and localization, which are the most important factors for company A when it comes to choosing the location ("sales on the foreign market: 50%" and "expand resources through localization: 30%").

Company B intends to internationalize in order to realize cost benefits in production ("generation of cost benefits in production: 70%"). Therefore, company B should decide on country/region Y. With a company-specific overall core of 21.1

Company specific country results: Country X

Company A:

Factor	PCI	Target weights	Sum [Pecle]
Macro-environment	23.5	0.15	3.525
Localization	21.5	0.30	6.450
Production	6	0.05	0.300
Sales	20	0.50	10.000
Total Score PCI	**71**	**1.00**	**20.275**

Company B:

Factor	PCI	Target weights	Sum [Pecle]
Macro-environment	23.5	0.15	3.525
Localization	21.5	0.10	2.150
Production	6	0.70	4.200
Sales	20	0.05	1.000
Total Score PCI	**71**	**1.00**	**10.875**

Company specific country results: Country Y

Company A:

Factor	PCI	Target weights	Sum [Pecle]
Macro-environment	25	0.15	3.750
Localization	15	0.30	4.500
Production	22	0.05	1.100
Sales	9	0.50	4.500
Total Score PCI	**71**	**1.00**	**13.850**

Company B:

Factor	PCI	Target weights	Sum [Pecle]
Macro-environment	25	0.15	3.750
Localization	15	0.10	1.500
Production	22	0.70	15.400
Sales	9	0.05	0.450
Total Score PCI	**71**	**1.00**	**21.100**

Fig. 5 Individual Assessment of Country Risks.

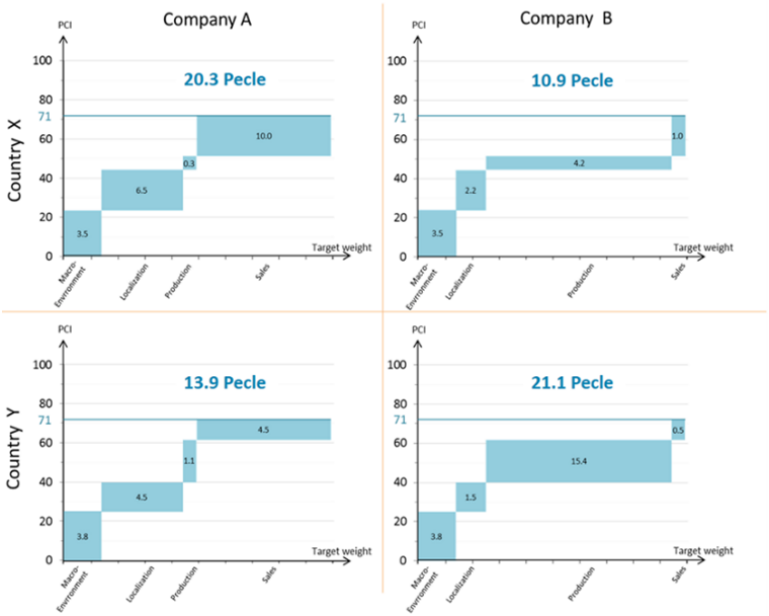

Fig. 6 PCI - Simultaneous Assessment at the Macroeconomic Level and Company-specific Goals with Foreign Direct Investments.

Pecle, country/region Y is substantially better suited for direct investment for company B than country/region X (total score of 10.875 Pecle).

Using the example of country/region X versus country/region Y (Fig. 6) the two-dimensional representation clearly shows the advantage of such a combinatorial view at the macroeconomic level on the one hand and individual corporate objectives on the other hand.

4 Conclusion

The change in global framework conditions by creating an individual corporate-level scoring framework for the Peren-Clement Index (PCI) has resulted in the re-ordering of the macroeconomic factors that have previously been taken into consideration. The various levels of assessment that should be taken into consideration in any investment decision are now combined. Thus, the choice of an international business location often takes place in stages: first, the macro-environment is regarded, and then the micro-environment, which offers core competencies and has a positive impact on a company's competitiveness. In addition, there are concrete production- and sales-oriented motives that reflect the supply and demand side, which also must be incorporated into the investment decision.

Within the context of an international investment decision, the Index presented here connects two main currents of thought in strategic planning that are known as the resource-oriented and market-oriented views:

- In the case of the resource-based view, it is not the market structure but rather the uniqueness of the resources at a location that forms the starting point for competitive advantages. However, these resources only have limited mobility and tradability. They interact with the internal resources of an enterprise (*inside-out perspective*).
- The market-based view employs an *outside-in perspective*. Thus, the structure of the industry (production and sales) prescribes how a company is able to position itself on a market.

These two perspectives are not mutually exclusive but rather complement each other. Ultimately, companies must address both of these perspectives within the scope of an international investment decision and take them into consideration as complementary elements in their strategic planning.

Practical decision-making is facilitated through a two-dimensional correlation between the economically relevant macro level and the individual corporate aims behind an intended direct investment. The use of risk indices in business planning for the evaluation of global investments has thus acquired a fundamental new quality.

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The Peren Theorem

The Mathematical Frame in Which We Live

Franz W. Peren

Abstract The Peren Theorem addresses the critical issue of unsustainable resource consumption by humans and its potential consequences for the planet. This theorem establishes a mathematical relationship that shows that human consumption of natural resources, if it exceeds the Earth's regenerative capacity in the long term, will lead to the complete depletion of the planet's natural environment. This article discusses the urgent need to shift to sustainable consumption patterns and reevaluate the definition of wealth.

1 Synopsis

Humans consume the natural resources of the Earth faster than the Earth is able to regenerate them. Mankind on the whole lives above its means and often at the expense of future generations. Current economic activity, with the aim of maximising monetary profits and generating quantitative growth and prosperity, cannot be continued. The *Peren Theorem* demonstrates that the consumption of natural resources within a closed system, as represented by Earth, is only possible if their consumption is able to naturally regenerate. If this balance is disturbed for too long a period, this will then result in the natural death of the planet. With an increasing global population, the per capita consumption of natural resources of all humans living on or from the Earth must be proportionately reduced.

2 The Current Human Lifestyle Cannot be Continued

Humans consume the natural resources of the Earth faster than the Earth is able to regenerate them. For many years now, human demand for natural resources has exceeded the Earth's capacity to regenerate these resources. According to the *Global*

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Footprint Network [1], the *Earth Overshoot Day* in the year 2017 took place on August 2nd of that year [1]. In the year before, it was on August 13th, 2016.

Mankind on the whole lives above its means and at the expense of future generations. The entire natural resources consumed after *Earth Overshoot Day* can no longer be replenished by the Earth in the same year concerned. If such an imbalance remains over the long term, then the natural resources of the Earth will be consumed up to the natural death of the planet.

The *Living Planet Report 2012* of the *World Wide Fund For Nature (WWF)* [2] shows that roughly up to the year 2030 humans would require two planets in order to cover their need for natural resources if mankind continues to live as it has largely deemed proper thus far. Natural resources are wasted particularly within and for so-called highly developed economies; however, the definition of this concept is misleading because development that results in a human lifestyle that permits the Earth to bear homo sapiens only for approximately seven months per year with itself being consumed for the remaining five months of the same year – both at the expense of other organisms as well as future generations of its own species – can hardly be regarded as "highly developed". Current economic activity with the aim of maximising monetary profits and generating quantitative growth and prosperity for the companies and citizens of participating national economies already takes place today at the expense of those who either do not or are only partially able to benefit from this predatory ecological exploitation, or who consciously choose to not take part.

3 The Peren Theorem

In the year 2012, the author developed the *Peren Theorem* [3-6] in discussion with Wiltrud Terlau, director of the International Centre for Sustainable Development (IZNE)[1], and Reiner Clement, professor of economics at Bonn-Rhein-Sieg University in Sankt Augustin, Germany:

> *"If the users within a closed system employ its natural resources in such measure that its natural regeneration is exceeded over the long term, then the natural environment of this system will be completely exhausted."*

[1] International Centre for Sustainable Development – IZNE (2017): https://www.h-brs.de/en/izne, accessed 9 December 2022.

For a closed system stability is[2]:

$$R_T \leq R_{regen}$$

$$\text{where} \quad R_T = R_H + R_O$$

$$\text{and} \quad R_H = \sum_{I=1}^{N} r_I = r_H N$$

$$\Leftrightarrow r_H = \frac{\sum r_I}{N} = \frac{R_H}{N}$$

with

R_T = consumption of natural resources as a whole

R_{regen} = regeneration of natural resources as a whole

R_H = human consumption of natural resources

R_O = consumption of natural resources not caused by humans

r_I = individual per capita consumption of natural resources by humans

r_H = average per capita consumption of natural resources by humans

N = number of people who live on Earth or access its natural resources

I = human individuals who live on Earth or access its natural resources; $1, \ldots, N$

[2] Stability is to be understood here as emancipatory in its meaning, i.e. if within a well-defined, temporal interval the inequation $R_T \leq R_{regen}$ is temporarily violated, then it is nevertheless valid altogether during this period. The scope and location of such a time period are to be selected in such a way that they contain the respectively current point in time and so that the strategic aim of a stable balance between consumed and regenerated natural resources is achieved not only over the longer term, but also for the benefit of those directly affected within the system taken into consideration.

4 Options for Securing Human Livelihood

In relation to mankind and the closed system of the Earth, this mathematical relationship implies that humans have the following options[3] in order to secure their existence on Earth:

1. Other consumers of natural resources on this planet are reduced; a practice that mankind already pursues. The habitats of animals and plants are diminished by humans with the consequence that plants and animals are decimated.
2. Mankind reduces itself until this theorem inverts into a positive balance, i.e. until terrestrial consumption caused by humans lies below the natural regeneration of the Earth over the long term.
3. Substantial numbers of mankind leave the Earth. Accordingly, these humans do not use any or hardly any terrestrial natural resources.
4. Mankind modifies the scope and quality of its consumption of natural resources so that this permits a regeneration of natural resources to the extent required. This would require substantial abandonment of the luxury that is understood by large parts of mankind today as prosperity. Individuals would then be entitled to far less natural resources on average than currently claimed and consumed on the average per capita.
5. The recourse to natural resources, i.e. the use, resp. the consumption, of water, soil, air, natural energies and/or sources of energy, plants and animals are put at a clearly higher price than the currently irrational case of disparity in relation to the true value of natural resources. Individual mobility would require a different quality and a clearly higher price. The consumption of meat would have to be more expensive and thus reduced. Global output chains would largely have to be shifted to local production because transportation would have to be priced in accordance with the demand for natural resources. Travel (over long-distances) would also have to be made substantially more expensive and limited.
6. Mankind substitutes natural resources in favor of synthetic materials; whereas the ecological requirements for the manufacture, transport, recycling and/or disposal of such plastics would also have to be attributed to human consumption of natural raw materials.
7. A more intensive circular economy, i.e. more efficient recycling of already used natural resources could slow down the process of exhaustion of the natural environment of the Earth. However, if increases in efficiency or technical progress results in rebound effects so that increases in efficiency mean that the consumer uses any savings obtained in order to demand more products or services which again consume (additional) natural resources, then increases in efficiency can also result in a so-called backfire, i.e. to rebound effects of more than one hundred percent.

[3] The following list is by no means exhaustive.

5 Individual Prosperity Effects

The *Peren Theorem* mathematizes and emancipates a life cycle matter of course. Like every mathematical statement, this theorem is also logically true and thus indisputable in rational terms. If mankind within its terrestrial existence should be interested in a natural environment so that it secures a required (minimum) measure of quality of life for humans – that is certainly evaluated differently by each individual – then operational implementation of this theorem as soon as possible is imperative.

Conversely, this theorem also implies that an increasing global population[4] has to be accompanied with a proportionate reduction in the average per capita consumption of natural resources if it is to continue to be true that:

$$R_T = R_H + R_O$$
$$\text{where} \quad R_H = \sum r_I = r_H N.$$

Given p percent increase in the global population and unchanged average per capita consumption, then ceteris paribus, i.e. unchanged consumption of natural resources not caused by humans R_O the entire consumption of natural resources caused by mankind R_H would likewise exhibit proportionate by a factor of $\left(1 + \frac{p}{100}\right)$:

$$R_H \left(1 + \frac{p}{100}\right) = r_H N \left(1 + \frac{p}{100}\right).$$

If human consumption of natural resources is meanwhile to be kept constant even with an increasing global population, then the formal relationship of the *Peren Theorem*[5]

$$R_H \stackrel{!}{=} r_H N \left(1 + \frac{p}{100}\right)$$

determines the following average per capita consumption of natural resources r_H

$$r_H = \frac{R_H}{N} \left(1 + \frac{p}{100}\right)^{-1}$$

where human consumption of natural resources as a whole R_H would remain unchanged in relation to the original state prior to the respectively considered period

[4] A generally comprehensible overview on population development can be found, for example, on Wikimedia Foundation Inc. (Ed.) (2020): https://en.wikipedia.org/wiki/Population_growth, accessed 9 December 2022, and the literature cited therein.

[5] The aim is that human consumption of natural resources altogether R_H remains unchanged despite world population growth. Therefore R_H is to be equated with $r_H N(1 + \frac{p}{100})$, whereby average human per capita consumption of natural resources r_H is ultimately reduced by the growth factor of the world population $(1 + \frac{p}{100})^{-1}$ within the period under consideration.

of increase in the global population.

Concomitant with positive population growth of p percent during a certain period, the average per capita consumption of natural resources r_H would have to be proportionately reduced by the factor

$$\left(1 + \frac{p}{100}\right)^{-1}$$

In particular, the inhabitants of wealthy national economies, above all the industrialized countries, whose individual human consumption of natural resources r_I is clearly above the average per capita consumption worldwide r_H could by no means continue to maintain their prosperity and lifestyle.

If the global population grows meanwhile with unchanged or even increasing (average) prosperity, as understood and lived today, then the consumption of natural resources would additionally accelerate through an (exponentially) increasing global population with simultaneous shortening of the period of total exhaustion of the natural resources of the Earth.

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