Optimal Portfolio Selection Using a Robust-Bayesian Model

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Abstract. In this paper we implement a portfolio optimization model that integrates the robust portfolio optimization approach and the Bayesian approach with the purpose of modeling the uncertainty of the estimated parameters in the expected returns and in the covariance matrix. The proposed model is implemented using the Wishart and Gamma distribution functions to model the uncertainty with ellipsoidal or quadratic type sets. To do that, we choose a portfolio made up of the shares of the USA DJI index. The results confirm the advantages of the robust approach compared to the traditional mean-variance model, both in performance evaluation and in its diversification.

Keywords: Optimal Portfolio · Robust Optimization · Bayesian Model · Uncertainty Sets

1 Introduction

The mean-variance (MV) model developed by Markowitz [1, 2] represents the most important theoretical development of the modern portfolio theory (MPT) by providing the general formulation for constructing optimal investment portfolios. However, the MV model presents important limitations [3–6], since the MV model generates low diversified portfolios and achieve low performance compared to benchmarks or compared to other more robust portfolio formulations. These limitations are due to the exclusive dependence of the MV model on historical data, which are used to obtain the parameters of expected returns and the covariance matrix, which causes a high sensitivity of the model to these parameters. These limitations have been partially overcome through the approach of robust optimization (RO) of portfolios introduced by [7, 8]. The implementation of this RO approach is based on the formulation of a maximum-minimum problem, known as the worst-case scenario. In this approach, the optimal weights of each asset are obtained when the expected returns take the worst value within the specific uncertainty set, like [9, 10] argued. Thus, by considering parameter uncertainty, RO provides a more consistent solution compared to the MV model. Furthermore, as stated by [11], the RO approach represents a significant advancement of MPT in better adapting to the dynamics of the financial market.

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Since its adoption, RO has offered an intuitive solution supported by convex optimization techniques such as quadratic programming (QP) and second-order cone programming (SOCP), considering the specification of the uncertainty sets and problem constraints [10, 11]. [8] introduced a significant number of RO formulations using convex programming. Later, [12] presented robust formulations for the MV model and portfolio Value-at-Risk (VaR) using ellipsoidal uncertainty sets, while [13] considered interval uncertainty sets for expected returns and the covariance matrix. Based on these works, there has been significant growth in RO to enhance the solutions of the MV model [14] or its extensions using risk measures such as VaR or CVaR [15, 16], as well as in performance measures [17, 18]. Thanks to advances in mathematical programming and the use of computational tools, the RO approach has been progressed, as stated by [10, 11].

On the other hand, RO can be improved from a Bayesian framework, as suggested by [19], who found that expert knowledge can be introduced into the optimization process by incorporating the investor's subjective expectations. This approach is known as Bayesian robust optimization. To do that, [19] incorporated ellipsoidal uncertainty sets from a posterior distribution function resulting from implementing a Bayesian process within the reformulation of the optimization problem to obtain the robust counterpart.

Building on these developments, this study adopts the RO and Bayesian approaches for constructing an optimal portfolio to overcome the limitations identified in the MV approach. For this purpose, the model proposed by [19] is adopted and integrated with the RO approach to achieve the construction of the robust-Bayesian portfolio (RBP). In that sense, we use the Gamma distribution function to improve the portfolio's sensitivity and diversification. The proposed model allows for achieving an RBP for a set of assets in the US market that improves the results of both the MV and RO portfolios.

2 Portfolio Theory

2.1 Mean-Variance (MV) Model

Markowitz [1, 2] developed an optimal solution for the selection of a portfolio of risky assets. The Markowitz formulation considered as inputs of the model the expected return of assets (μ_i) and covariances (σ_{ij}), under the assumption that the returns follow a normal distribution. Thus, a portfolio of n risky assets with expected return $E(R_P) = \mathbf{w}'\boldsymbol{\mu} = \mu_p$ and variance $\sigma_P^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$, where $\boldsymbol{\mu} \in \mathbb{R}^{n\times 1}$ is the vector of expected returns, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is the covariance matrix, and $\mathbf{w} \in \mathbb{R}^{n\times 1}$ is the vector of weights. The optimization problem is a quadratic programming (QP) problem and is solved by minimizing σ_P^2 for a given level of expected return (μ_{p0}) given by:

$$\min_{\{\mathbf{w}\}}\left\{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right\} \text{ s.t. } \boldsymbol{\mu}'\mathbf{w} = \mu_{p0} \text{ and } \mathbf{w}'\mathbf{1} = 1 \tag{1}$$

where, $\mathbf{1} \in \mathbb{R}^{n\times 1}$ is a vector of ones. Additionally, Eq. 1 can also incorporate the restriction on negative or short weights, that is, it is solved for $\mathbf{w} \geq 0$, however, the solution is no longer analytical. Despite the developments that the MV model represents, it has important limitations. For example, using only historical data for the estimation of $\boldsymbol{\mu}$ or $\boldsymbol{\Sigma}$ does not adequately incorporate future uncertainty, resulting in very sensitive or noisy solutions.

2.2 Portfolio Robust optimization

The RO is an optimization approach for portfolio under uncertainty introduced by [7] and [8] and represents an intuitive and efficient way to handle uncertainty for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ through uncertainty sets [10, 11]. Like the Bayesian approach, RO assumes that μ and $\boldsymbol{\Sigma}$ are random variables; however, RO offers an intuitive solution that can be implemented through convex programming, such as quadratic programming (QP) or second-order cone programming (SOCP). Following [20], RO involves formulating a maximum-minimum or minimum-maximum problem and solving it for the entire uncertainty set, even if μ takes its worst possible value. Since the robust counterpart requires reformulating the original optimization problem considering the uncertainty set, the most used types of uncertainty sets are interval sets and ellipsoidal or quadratic sets. The following steps are carried out to implement this: i) defining the uncertainty set for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$; ii) reformulating the original optimization problem to include the uncertainty set; and iii) solving the reformulated problem to obtain the robust optimal portfolio.

By considering the uncertainty set, RO approach provides a more robust and stable solution compared to the traditional mean-variance (MV) approach. It allows for incorporating parameter uncertainty and mitigating the adverse effects of estimation errors. Furthermore, RO can be implemented using various optimization techniques, making it versatile and practical for portfolio optimization under uncertainty. This procedure allows solving the problem that determines the worst possible realization of the parameters before solving the original portfolio selection problem. [12] developed the first comprehensive formulations of this method based on the MV model. However, as suggested [20], the formulation of the robust model depends on the objective function that describes the problem and the specified uncertainty set. For example, if we considered the quadratic utility function of the MV model and the interval-type uncertainty $\mathcal{U}_{\mu}$, the original optimization problem is reformulated in a max-min type as follows:

$$\max_{\{\mathbf{w}\}}\{\min_{\{\boldsymbol{\mu}||\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}_{|}\leq\delta\}}(\mathbf{w}'\boldsymbol{\mu})-\lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\}\text{ s.t. }\quad \mathbf{w}'\mathbf{1}=1 \tag{2}$$

Therefore, the robust version of the problem is given by:

$$\max_{\{\mathbf{w}\}}\left\{\mathbf{w}'\boldsymbol{\mu}-\lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}-\delta|\mathbf{w}|\right\}\quad\text{s.t.}\quad \mathbf{w}'\mathbf{1}=1 \tag{3}$$

Now, if the ellipsoidal uncertainty set is implemented in the same objective function $\mathcal{U}_{\mu}=\left\{\boldsymbol{\mu}|\left(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}\right)'\boldsymbol{\Sigma}_{\mu}^{-1}\left(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}\right)\leq\delta^{2}\right\}$, we obtain the following max-min problem, and the robust version of the problem are given by:

$$\max_{\{\mathbf{w}\}}\{\min_{\{\boldsymbol{\mu}|(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}})'\boldsymbol{\Sigma}_{\mu}^{-1}(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}})\leq\delta^{2}\}}\left(\mathbf{w}'\boldsymbol{\mu}-\lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right)\}\text{s.t. }\mathbf{w}'\mathbf{1}=1 \tag{4}$$

$$\max_{\{\mathbf{w}\}}\left\{\mathbf{w}'\boldsymbol{\mu}-\lambda\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}-\delta\sqrt{\mathbf{w}'\boldsymbol{\Sigma}_{\mu}\mathbf{w}}\right\}\text{s.a. }\mathbf{w}'\mathbf{1}=1 \tag{5}$$

where: $\boldsymbol{\Sigma}_{\mu}$ represents the diagonal matrix of the estimation errors of the covariances or uncertainty matrix. Under the assumption of a multivariate normal distribution $\left(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}\right)'\boldsymbol{\Sigma}_{\mu}^{-\mathbf{1}}\left(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}\right)$ is estimated as a χ^{2} distribution with n degrees of freedom. Given the advantages of the ellipsoidal set over the interval set, as stated by [9, 12, 20, 21], this ellipsoidal set is recommended in the RO. Additionally, [9] found that robust portfolios can perform better than MV portfolios by exhibiting greater stability in their composition over time, which can significantly reduce portfolio rebalancing and thus risk losses. Transaction costs. These results highlight the advantages of the RO approach over the MV model. [11] confirmed the previous results and find that robust portfolios are superior in the design of risk management strategies. Furthermore, [20–22] also found that robust portfolios present superior results to MV portfolios in terms of performance and greater stability over time.

2.3 Bayesian Formulation of the Robust Portfolio

Meucci [19] demonstrated that the Bayesian approach can be integrated into the RO by introducing the prior probability distribution of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, which generates an RBP. To do that, the author used ellipsoidal sets to obtain the robust counterpart by using a conjugate distribution function within the reformulation of the optimization problem. In that sense, RBP combines the investor's subjective beliefs and generates an optimal solution for the uncertainty set used. Following the same assumptions from the MV and RO models, [19] incorporates the investor's prior beliefs using a Wishart inverse-normal distribution: $\boldsymbol{\mu} \sim \mathrm{N}(v_0, \Sigma/T_0)$ and $\boldsymbol{\Sigma}^{-1} \sim \mathrm{W}\left(v_0, \Sigma_0^{-1}/v_0\right)$, where v_0 and T_0 are the hyper parameters. Furthermore, by using the ellipsoidal set of the posterior marginal distribution of $\boldsymbol{\mu}$, [19] get the estimator of the expected value of $\hat{\mu}_{\mathrm{ce}} = \mu_1$ and the estimator of the scattering matrix $\mathbf{S}_{\boldsymbol{\mu}}$. The uncertainty of $\boldsymbol{\Sigma}$, like the previous ellipsoidal set, is also described by the estimator of the expected value of $\boldsymbol{\mu}$ and the scattering matrix estimator as follows:

$$\hat{\boldsymbol{\Sigma}}_{\mathbf{ce}} = \frac{v_1}{v_1 + N + 1}\Sigma_1 \tag{6}$$

$$\mathbf{S}_{\boldsymbol{\Sigma}} = \frac{2\mathrm{v}_1^2}{(\mathrm{v}_1 + \mathrm{N} + 1)^3}\left(\mathrm{D}_{\mathrm{N}}'(\boldsymbol{\Sigma}_{\mathbf{1}}^{-1} \otimes \boldsymbol{\Sigma}_{\mathbf{1}}^{-1})\mathrm{D}_{\mathrm{N}}\right)^{-1} \tag{7}$$

where, $\otimes$ is the Kronecker product. Finally, the robust counterpart is developed. The optimal solution for the established parameters and sets of the PRB is given by:

$$\max_{\{\mathbf{w}'\boldsymbol{\Sigma}_{\mathbf{1}}\mathbf{w} \le \gamma_{\Sigma}^{(i)}\}} \left\{\mathbf{w}'\boldsymbol{\mu}_{\mathbf{1}} - \gamma_{\mu}\sqrt{\mathbf{w}'\boldsymbol{\Sigma}_{\mathbf{1}}\mathbf{w}}\right\} \text{ s.t. } \mathbf{w}'\mathbf{1} = 1 \tag{8}$$

where:

$$\gamma_{\mu} \equiv \sqrt{\frac{\mathrm{q}_{\mu}^2}{T_1}\frac{v_1}{v_1 - 2}} \quad \text{and} \quad \gamma_{\Sigma}^{(i)} \equiv \frac{v^{(i)}}{\frac{v_1}{v_1+N+1} + \sqrt{\frac{2v_1^2 q_{\Sigma}^2}{(v_1+N+1)^3}}} \tag{9}$$

The above formulation is obtained for an inverse Wishart distribution. However, other distribution functions, such as the Gamma, can also be used. As an extension of the previous RBP, we propose a model using the Gamma distribution named as RPBg. This adjustment is made on $\mathbf{\Sigma}$, which is given by:

$$f^{\Gamma}_{\alpha,\beta,\nu,\Sigma}(\Gamma) = \frac{1}{k}|\mathbf{\Sigma}|^{-\alpha}|\mathbf{\Gamma}|^{\alpha-\frac{1}{2}(\rho+1)}\frac{1}{\beta^{\rho\alpha}}e^{\frac{-1}{\beta}tr(\mathbf{\Sigma}^{-1}\Gamma)} \tag{10}$$

where: $\mathbf{\Sigma}^{-1} \sim \mathbf{\Gamma}(\alpha, \beta, (\nu_1, \mathbf{\Sigma_1})^{-1})$ and α and β are the shape and scale parameters, respectively. In that sense, we extended the Meucci's model by using the ellipsoidal uncertainty set. With this adjustment, we found:

$$\hat{\mathbf{\Sigma}}_{\mathbf{ce}} = \left(\frac{\alpha}{\beta(2\alpha+\rho+1)}\right)\Sigma_1 \tag{11}$$

$$\mathbf{S}_{\mathbf{\Sigma}} = \frac{2\alpha^2}{\beta^2(2\alpha+\rho+1)^3}(D'_N\left[\mathbf{\Sigma_1^{-1}} \otimes \mathbf{\Sigma_1^{-1}}\right]D_N)^{-1} \tag{12}$$

By incorporating these adjusted into the optimization model, the PRBg is obtained as:

$$\max_{\{\mathbf{w}'\mathbf{\Sigma_1}\mathbf{w}\le\gamma_{\Sigma}^{(i)}\}}\left\{\mathbf{w}'\boldsymbol{\mu}_{\mathbf{1}} - \gamma_{\mu}\sqrt{\mathbf{w}'\mathbf{\Sigma_1}\mathbf{w}}\right\} \quad \text{s.t.} \quad \mathbf{w}'\mathbf{1} = 1 \tag{13}$$

where:

$$\gamma_{\mu} \equiv \sqrt{\frac{\alpha q^2_{\mu}}{\beta T_1(2\alpha-\rho)}} \quad \text{and} \quad \gamma_{\Sigma}^{(i)} \equiv \frac{v^{(i)}}{\frac{\alpha}{\beta(2\alpha+\rho+1)} + \sqrt{\frac{2\alpha^2 q^2_{\Sigma}}{\beta^2(2\alpha+\rho+1)^3}}} \tag{14}$$

It should be noted that, if the RBPg model uses the parameters $\beta = 2$ and $\alpha = \upsilon/2$, the same results from the RBP model.

3 Numerical Implementation and Results

3.1 Data

The proposed model is implemented for the USA stock market and is taken as the Dow Jones Industrial Average (DJIA) index, which is made up of the 30 most important and representative industrial companies in the USA. In addition, the in-sample analysis period covers from January 2011 to December 2020 and the out-of-sample period covers from January 2021 to December 2022. The data sample is developed by taking the monthly adjusted closing prices of the assets and the index.

3.2 Comparison of Results

We made a comparison of the models and the proposed approach. To do that, we compared the Markowitz MV portfolio and its robust counterparts: the robust portfolio (RP), the RBP and the RBPg taking long positions ($\mathbf{w} \geq 0$) in all of cases; both in the optimal solution and in the performance evaluation in-sample and out-of-sample. Additionally, both the RBP and RBPg portfolios are implemented following the recommendations of Meucci's approach. The results obtained are presented below. Results show significant differences in portfolio composition of the traditional MV model and the robust portfolios. In addition, a notable improvement in the performance results from the RB and RBP portfolios for the out-of-sample period is identified as listed in Table 1b, as well as an improvement in all the optimal portfolios related to the benchmark (DJI), as listed in Table 1 (Fig. 1).

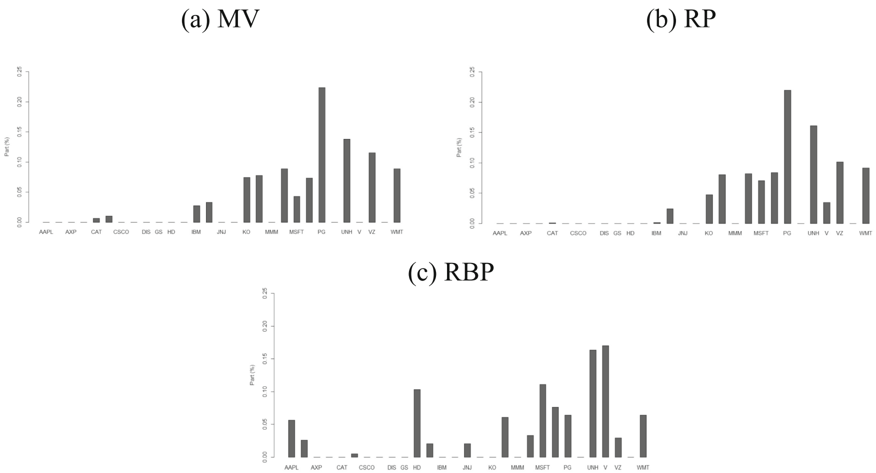

Fig. 1. Optimal weights of portfolios

This performance is measured through risk-adjusted returns using the Sharpe coefficient.

Table 1. Results of the optimal portfolios

	In-sample	Out-sample						
	MV	*RP*	*RBP*	*DJI*	*MV*	*RP*	*RBP*	*DJI*
Return:	0.0115	0.0128	0.0169	0.0081	0.0044	0.0047	0.0051	0.0033
Risk:	0.0280	0.0281	0.0326	0.0393	0.0442	0.0452	0.0502	0.0522
Sharpe Coef.:	0.41	0.4538	0.5177	0.2058	0.0995	0.1047	0.1009	0.0636

Figure 2 shows the historical behavior of the cumulative returns for the optimal portfolios and the benchmark in the in-sample (2a) and out-of-sample (2b) periods. In both cases, a better performance of the RBPg portfolio is identified. In particular, the RBPg portfolio for the out-of-sample period is highlighted. For most of this period, this portfolio achieves a higher cumulative return and presents lower decreases than the others.

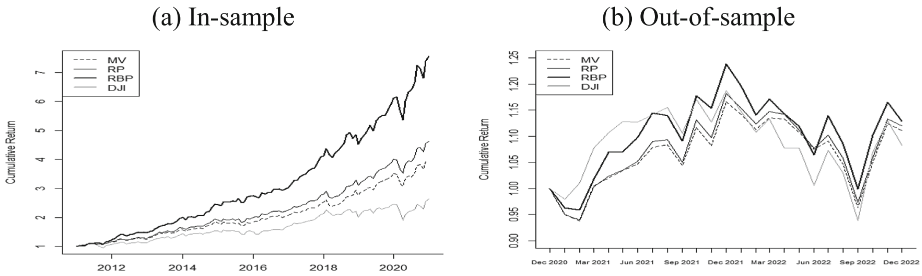

Fig. 2. Cumulative return of portfolios

Results confirm the advantages of robust portfolio models compared to the MV model. Additionally, we implemented a monthly rebalancing exercise of optimal portfolios for the in-sample analysis period. In this exercise, a rolling period of five years is considered for the calculation of the optimal weights, which gives a result of 60 updates for the entire period. Figure 3 illustrates the rebalancing or updating process in the composition of each of the three portfolios (MV, RP, and RBP).

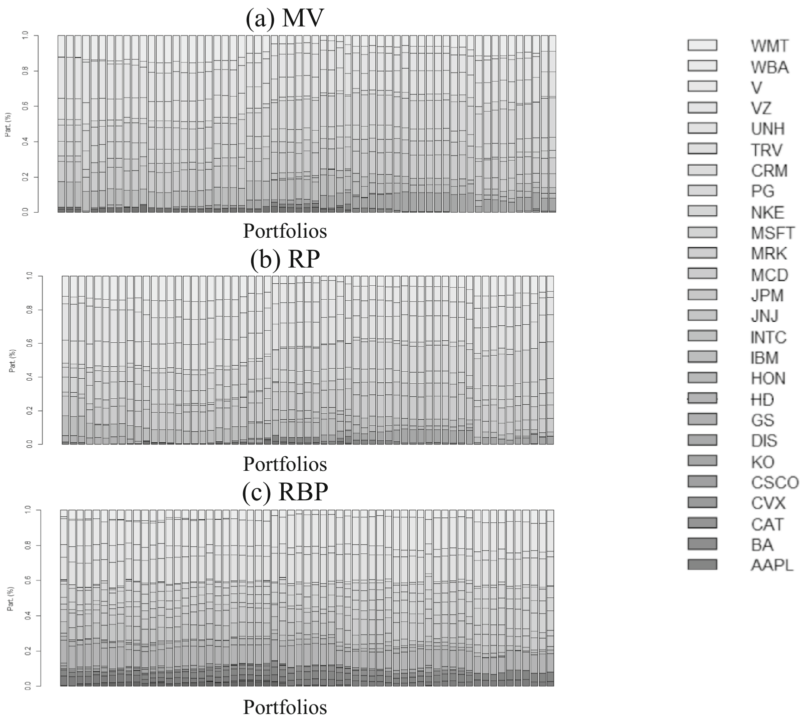

Fig. 3. Portfolio rebalancing

Figure 3a shows the frequent changes that the MV portfolio presents, which confirms the sensitivity problem that this model has compared to the estimated parameters as pointed out above. In addition, the exclusion of a large part of the assets that make up the portfolio is identified. But although the RP portfolio also has a high sensitivity, this is significantly reduced in the RBP portfolio. Figure 3c indicates a small adjustment of the weights of all the assets for the entire period. This consistency of the robust models can also be confirmed by using a concentration indicator, which is useful to identify the high deviation that portfolios can present. For this, we use the Herfindahl–Hirschman index (HHI) measured as the sum of the squares of weights of each n assets that make up the portfolio:

$$HHI = \sum\nolimits_{i=1}^{n} w_i^2 \tag{15}$$

The HHI index is calculated for all portfolios of the previous rebalancing exercise. Figure 4 shows the HHI index for the 60 portfolios. In this case, the HHI indicator is calculated for the RBP portfolio between 700 and 1050, while the indicator for the MV and RP portfolios is higher than these levels. Apart from portfolios 29–30, the RBP portfolio presents lower concentration levels or greater diversification. This is consistent with the better performance of the RBP portfolio, since, in a bearish and high-volatility period such as the one identified for the year 2022, the robust model developed from the worst-case scenario approach minimizes the potential losses that the portfolio may face. However, the variability experienced by the HHI indicator is high, which can lead to a rebalancing of portfolios, although less frequently than the RP and MV models.

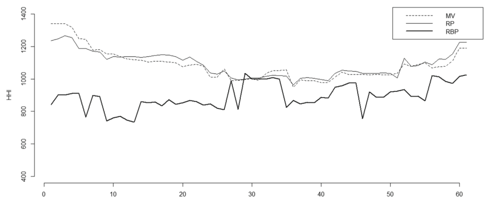

Fig. 4. HHI Index of the portfolios

On the other hand, the RBPg model is implemented following the proposed adjustment by using the gamma distribution. Figure 5 shows the results from the RBPg portfolio, and it is compared with the RBP portfolio. Figure 5a and b confirm a further diversification of the RBPg portfolio. In the first case, this is due to an increase in the assets that are part of the optimal portfolio. In the second case, a lower variability of the HHI index is observed. These results confirm the improvements of the RBP model when using gamma distribution compared to the model proposed by Meucci (2011).

(a) weights of RBP and RBPg portfolios (b) HHI index of RBP and RBPg portfolios

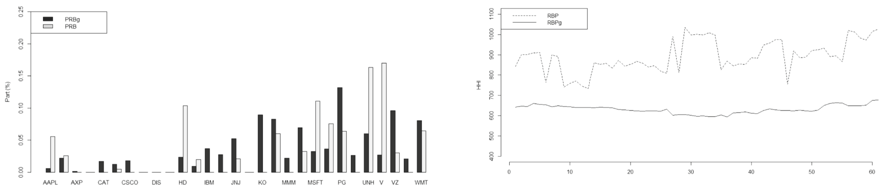

Fig. 5. RBPg portfolio results

4 Conclusions

In this work, a model integrating the RPO and Bayesian approaches was implemented for the development of an optimal portfolio to overcome the limitations of the MV model. RBP was implemented following the original proposal of Meucci (2011). However, an extension was conducted by replacing the Wishart distribution with the gamma distribution to represent the previous and subsequent distribution of the robust counterparts of the portfolio.

This new approach made it possible to build a RP for the US market based on the DJI index, whose results overcome the sensitivity and diversification problems. This means that the robust Bayesian model created a highly consistent portfolio that minimizes rebalancing over the period assessed and achieves better levels of diversification and better performance. Therefore, the proposed approach offers important advantages over Markowitz's MV model by overcoming its main limitations such as the high sensitivity of the portfolio to the estimated parameters and its poor performance outside of the sample. The advantage of the proposed model is that it can be easily replicated in different markets and asset classes. For future work, we recommend adopting alternative approaches to incorporate risk aversion into the model, as well as evaluating the model sensitivity to the distribution parameters. For this, Bayesian models based on the Monte Carlo simulation technique can be implemented. We also recommend reviewing the consistency of the model during recession and crisis periods.

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