Pool Games in Various Information Environments

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Abstract. The emergence of Blockchain digital technology provides one of the most prominent transaction mechanisms in an increasing variety of digital and augmented environments. In the Blockchain habitat, interactions among autonomous agents, called miners, form mining pools that aggregate computational power in order to increase their possible gains. A Pool game models mining pools that compete against each other in order to improve their outcome by strategically committing their miners. Current studies in Pool games make the assumption that pools have complete and correct information about the situation. In this work we drop this assumption, studying Pool games under various information environments such as incomplete information and erroneous information.

Keywords: Pool games · misinformation games · Bayesian games

1 Introduction

Nowadays, the emergence of digital environments, automated procedures, and big data have brought into light many intangible, crowd-sourcing, and sophisticated digital transaction methodologies. One of the emerging attractive solutions that provide security, accessibility, and privacy in big data systems is the Blockchain, see [3]. It was proposed in [12] to serve as the main concept in the digital economy, providing transparent and secure transactions in distributed and decentralized environments (for more details see [5]). Hence, Blockchain technology provides an appealing and applicable methodology for versatile practices from Insurance and Commerce to the Internet of Things and Security.

In practice, a Blockchain is a distributed synchronized secure database containing validated blocks of transactions. A block is validated by special nodes, called miners, via the solution of a computationally demanding problem, called the proof-of-work puzzle. The miners compete against each other and the first one to solve the problem, provides a full proof of work, and announces it. The block is then verified by a predefined agreement protocol called consensus. After

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the new block reaches the consensus, it is added to the distributed database, and the miner that generated the block is rewarded according to the, commonly and a priori known, protocol of the transaction.

In order to increase their outcome, miners form mining pools implemented by a pool manager (see [10]), where all of them provide proof of work concurrently and share their revenues accordingly. In this work, we focus on open pools that allow any miner to join them. The utility of a pool is the total sum of the revenues received by its miners. The information available to a pool includes the set of its miners, a predefined protocol for the reward of newly generated blocks, and the set of adversary pools.

As stated in [4], a miner may attack an adversary pool by providing partial proof of work to its pool manager. The attacking miner shares the revenue obtained in the pool but does not contribute, thus the utility of the attacked pool deteriorates and becomes less attractive to other miners.

Therefore, pools may have incentives to commit miners attack and deliver partial proof of work (called infiltration rate) to opponent pools in order to improve their revenues, see [10]. Hence, the use of game-theoretical tools in the Blockchain environment is a direct way to study, model and analyze these kinds of interactions (see [1,4,11]).

In the literature, pools are aware of the number of miners at their disposal and can estimate accurately the amount of attacking miners. Thus, they have complete and correct information regarding the interaction with other pools. This is a highly unrealistic scenario, as, in many situations, none of these assumptions hold, due to the presence of side information, bounded rationality, computational restrictions, etc. In this paper, we study the Pool game considering various information environments as stated in the following subsection.

1.1 Contributions

A pool manager may experience several scenarios for modeling uncertainty or erroneous beliefs with regards to the mining power (how many miners) or the infiltration rate (how many attackers) of a pool. In order to cope with these issues and provide a more realistic analysis of the Pool game setting, we plug in the model of [4] incomplete and incorrect information, considering two cases.

First, we address the case where the pools have incomplete information regarding the infiltration situation. In particular, the pools are not aware of the size of the incoming attack. In Subsect. 4.1, we model this scenario using the notion of Bayesian games as provided in [9], prove the convergence of revenues and compute the equilibria of the Bayesian Pool game.

Second, in Subsect. 4.2, we consider the case where the pools have incorrect information regarding the specifications of the interaction. More specifically, they think they know the actual mining power of the pool and the accurate number of incoming attacks. We model this scenario using the concept of misinformation games as introduced in [16]. In order to cope with the iterative nature of the Pool game, we apply the *Adaptation Procedure*, which was introduced in [14]. Hence, the misinformed pools have the machinery to re-evaluate their information and adapt their decisions in the next round. We prove the convergence of the revenues and compute the equilibria of the misinformed Pool game.

Another contribution of this work is about the convergence of the revenues of the Pool games for the case of complete-correct information. Namely, in Lemma 1, Theorem 1, and Corollary 1 we prove the convergence of the density revenues for non-constant infiltration rates, as opposed to Lemma 1 in [4].

Summarizing the various information scenarios, in the correct-complete case, [4], the pools know their mining power and estimate correctly the infiltration rates. In the setting of incomplete information, pools know their mining power but assign probabilities in the infiltration rate. In the setting of incorrect information, pools know incorrectly their mining power and the infiltration rates they face. Throughout our analysis, we assume that all of the pools are of equal capabilities and all miners are identical.

2 Related Work

Several studies cope with complete and correct information settings (e.g. [2,4]). The Pool game is presented in detail in [4]. A different approach is introduced in [2], where the authors provide allocation methodologies so that the miners cooperate, and avoid the development of centralized pools. In the same spirit, in [8,17] authors introduced models where the miners can either cooperate or employ a block withholding attack in a pool. In [10] authors study the Pool game model, in the complete-correct information environment, from the perspective of system rewards and punishments and analyze the outcome of the interaction. Further, in [7] authors study models, where miners play a complete-information stochastic game from the perspective of miners. In this study, we focus on the cases of incomplete and incorrect information.

In the context of Bayesian games, authors in [19] consider the case where a user knows the distribution of others' valuations, and focuses on truthful mechanisms. In [18] authors propose a characterization of Blockchain protocols regarding the rational and Byzantine behaviors. Furthermore, in [6] authors present a probabilistic model based on the information propagating over a Blockchain habitat (e.g. a Bitcoin network). They probabilistically identify the users initiating the transactions and do not implement their framework in the case of incomplete information. Authors in [15] plug in Bayesian game theory into Blockchain transactions, and provide an auction model.

In summary, existing works that apply game theory in Blockchain environments with incomplete information, mainly focus on the development of sufficient and robust mechanisms that regulate the situation, rather than on interactions among pools. To the best of our knowledge, there are no works that deal with the situation where pools experience subjective views of information.

3 Preliminaries

We consider a normal-form *game* $G = \langle N, S, U \rangle$ that consists of a set of players N, a set $S = S_1 \times \ldots \times S_{|N|}$ of players' joint decisions, where S_i is player's

i set of pure strategies and a utility matrix $U = (U_1, \ldots, U_{|N|})$, where $U_i \in \mathbb{R}^{|S_1| \times \ldots \times |S_{|N|}|}$ is player's i utility matrix.

A mixed strategy for player i that represents a probabilistic mixture of pure strategies, is a tuple $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{i|S_i|})$ where $\sigma_{ij} \geq 0$ and $\sum_j \sigma_{ij} = 1$. Let Σ_i be the set of all possible mixed strategies of player i. In a game with $|N|$ players, a strategy profile is an $|N|$-tuple $\sigma = (\sigma_1, \ldots, \sigma_{|N|}), \sigma_i \in \Sigma_i$, and σ_{-i} is the strategy profile all but player i. Further, we will use the Forbenius norm $\|\cdot\| : \mathbb{R}^{n \times m} \to \mathbb{R}$ in Sect. 4.

The players' behaviour in a normal-form game is predicted through the *Nash* equilibrium:

Definition 1 (Nash equilibrium [13]). *A strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_{|N|}^*)$ is a Nash equilibrium, if and only if, for any i and for any $\sigma_i \in \Sigma_i$, $f_i(\sigma_i^*, \sigma_{-i}^*) \geq f_i(\sigma_i, \sigma_{-i}^*)$, where f_i is the utility function of player i, defined as $f_i : \Sigma \to \mathbb{R}$, such that:*

$$f_i(\sigma_i, \sigma_{-i}) = \sum_{k \in S_1} \cdots \sum_{j \in S_{|N|}} U_i(k, \ldots, j) \cdot \sigma_{1,k} \cdot \ldots \cdot \sigma_{|N|,j}, \tag{1}$$

We denote a Nash equilibrium by ne and the set of nes in G by $NE(G)$.

3.1 Incomplete Information

In classical game theory, incomplete information is addressed by Bayesian games. A Bayesian normal-form game is defined as a tuple $BG = \langle N, S, \Theta, p, U \rangle$, where N, S are defined as previously. $\Theta = \Theta_1 \times \ldots \times \Theta_{|N|}$ is the set of joint types of players, p is a common prior distribution over the types, and U is the set of utility functions, $U = (U_1, \ldots, U_N)$, whereas $U_i : (S \times \Theta)_i \to \mathbb{R}$.

A player's type is private information and is used to make decisions and update her beliefs about the likelihood of opponents' types (using the conditional probability $p(\theta_{-i}|\theta_i)$, where $\theta_i \in \Theta_i$). In this setting, a pure strategy is given by a mapping from the type space to the strategy space, $s_i : \Theta_i \to S_i$. In other words, s_i maps every information type $\theta_i \in \Theta_i$ that player i has to the pure strategy that she could play in that type. As in the case of correct-complete information, a mixed strategy σ_i is a probabilistic mixture of pure strategies.

Each player calculates her expected utility given that she knows her own type but not the types of the opponents (*ex-interim* concept)[1] by the following formula,

$$\mathbb{E}[U_i(\sigma, \theta_i)] = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \mathbb{E}[U_i(\sigma, (\theta_i, \theta_{-i}))], \tag{2}$$

where θ_i is the type of player i.

Player i's best response curve to strategy profile σ_{-i} is given by $BR_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}[U(\sigma_i, \sigma_{-i})]$.

Definition 2 (Bayes-Nash equilibrium, [9]). *A* Bayes-Nash equilibrium *is a mixed-strategy profile σ such that $\sigma_i \in BR_i(\sigma_{-i}), \forall i \in N$.*

[1] There are also the concepts of *ex-post*, and *ex-ante* utilities, see [9].

Repeated Bayesian Games Consider a finite discrete time horizon T, with $T > 0$. In our study we will analyze two cases: i) undiscounted utilities, and ii) discounted utilities. In the first case, the utilities in BG are evaluated as limits of arithmetic averages. In the latter case, we allow every player to evaluate her utility sequence with a discount factor $\delta \in (0, 1)$. The utility formulas for both cases are provided by Table 1. Further, the players' types are intrinsic characteristics and are fixed throughout the interaction.

Table 1. Utility formulas for the repeated Bayesian game.

case	formula	
Undiscounted	$\frac{1}{T}\sum_{t\in[T]} U_i(\sigma^t, \theta_i)$	$i \in [\lvert N\rvert]$, $T > 0$
Discounted	$(1-\delta)\sum_{t\in[T]} \delta^t U_i(\sigma^t, \theta_i)$	

3.2 Incorrect Information

Misinformation games were introduced in [16] and defined as a tuple $mG = \langle G^0, G^1, \ldots, G^{|N|}\rangle$, where all G^i are normal-form games and G^0 contains $|N|$ players. Further, G^0 is called the *actual game* and represents the game that is actually being played, whereas G^i ($i \in N$) represents the game that player i thinks that is being played (called the *game of player* i). Moreover, no assumptions are made as to the relation among G^0 and G^is, thereby allowing all types of misinformation.

The outcome of a misinformation game is dictated by the equilibrium strategy profiles that each player picks in her view.

Definition 3 (Natural misinformed equilibrium). *A strategy profile* $\sigma^* = (\sigma_1^{*,1}, \ldots, \sigma_{|N|}^{*,|N|})$ *is a natural misinformed equilibrium, if and only if, for any* i *and for any* $\hat{\sigma}_i \in \Sigma_i^i$, $f_i^i(\sigma_i^{*,i}, \sigma_{-i}^{*,i}) \geq f_i^i(\hat{\sigma}_i, \sigma_{-i}^{*,i})$, *where* f^i *is the utility function of player* i *in the game* G^i *and defined as:* $f_i^i : \Sigma^i \to \mathbb{R}$, *such that:*

$$f_i^i(\sigma_i^i, \sigma_{-i}^i) = \sum\nolimits_{k\in S_1^i} \cdots \sum\nolimits_{j\in S_{|N|}^i} U_i^i(k, \ldots, j) \cdot \sigma_{1,k}^i \cdot \ldots \cdot \sigma_{|N|,j}^i, \tag{3}$$

Further, we denote by $NME(mG)$ the set of *nme*s in mG. Observe that players obtain the utilities provided by G^0, and not the utilities they realise in G^is.

Evidently, the misinformed players may come across an outcome different than the one they expect in their own game. In the case where the interaction is iterative, the misinformed players update their information according to choices they made, the choices their opponents have made, and the corresponding information that the environment provides to them. This process is formalized by the *Adaptation Procedure* that was introduced in [14]. More specifically, the Adaptation Procedure occurs in discrete time steps $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It starts at $t = 0$ with the misinformation game mG. Then, at each time step $t \geq 0$, the players choose a Nash strategy profile in their game, and new *nme*s are formed. As the outcome and the respective utilities are announced, the players re-adjust their choices and update their utilities according to the information they have received. Formally, the Adaptation Procedure is provided by the following definition.

Definition 4 (Definition 4.4 in [14]). *For a set M of misinformation games, we set $\mathcal{AD}(M) = \{mG_{\boldsymbol{u}} \mid mG \in M, \boldsymbol{u} \in \chi(\sigma), \sigma \in NME(mG)\}$. We define as Adaptation Procedure of a set of misinformation games M to be the iterative process such that:*

$$\begin{cases} \mathcal{A}D^{(0)}(M) = M \\ \mathcal{A}D^{(t+1)}(M) = \mathcal{A}D^{(t)}(\mathcal{A}D(M)) \end{cases}$$

for $t \in \mathbb{N}_0$.

where $\chi(\sigma)$ is the set of indices associated with the strategies in the support of the strategy profile σ and $mG_{\boldsymbol{u}}$ is the updated game. Namely, $\boldsymbol{u}$ provides the position in the subjective utility matrices where the update takes place, and is determined by the strategic choices of the players.

The Adaptation Procedure terminates (see Definition 4.5 in [14]), if there exists a time point $t \in \mathbb{N}_0$ such that any further iterations do not provide new information to the players, that is $\mathcal{AD}^{(t+1)}(M) = \mathcal{AD}^{(t)}(M)$. Interestingly, the *nme* where the players do not obtain new information is stable; this is called stable misinformed equilibrium *sme*. We denote by $SME(mG)$ the set of *sme*s in mG.

4 The Pool Game

We consider the case where a set of N pools, with a total of m miners, compete with each other in order to maximize their outcome. This situation is introduced in [4] as the Pool game. In particular, the pools try to maximize their revenue density by optimizing their infiltration rates to the adversaries. As stated in [4], the revenue density of pool i is the ratio between the average revenue that miner v earns and the average revenue it would have earned as a solo miner. We denote the revenue density of pool i at time step t by $r_i(t)$.

In this study, the interaction is evolved in discrete time steps, and the total number of miners that each pool has in its disposal remains constant throughout the game. Moreover, each pool can compare the rates of partial and full proofs of work it receives from its miners, in order to find the rate of infiltrators attacking it, see [4]. Also, it can compute the revenue rates of each of the other pools. Initially, we restate the basic concepts of Pool games in the case where each pool has correct and complete information about the infiltration rates and the density revenue.

Let $m_i(t)$ be the total number of miners in the disposal of pool i, whereas $m_{ii}(t)$ is the number of miners pool i assigns to mine honestly in pool i, and

$m_{ij}(t)$ is the number of miners used by pool i to infiltrate pool j at time step t (infiltration rate). Thus, in general, it holds $m_i(t) \geq \sum_j m_{ij}(t)$. Clearly, in each time step pool i mines with power $m_{ii}(t)$, and shares its reward among $m_i(t)+\sum_{j\in[|N|]\setminus\{i\}} m_{ji}(t)$ members. For our analysis we use the following vector that measures the direct mining revenue density,

$$\mathbf{u}(t) = \left(\frac{m_1(t) - \sum_{j\in[|N|]\setminus\{1\}} m_{1j}(t)}{m_1(t) + \sum_{j\in[|N|]\setminus\{1\}} m_{j1}(t)}, \dots, \frac{m_N(t) - \sum_{j\in[|N|]\setminus\{N\}} m_{Nj}(t)}{m_N(t) + \sum_{j\in[|N|]\setminus\{N\}} m_{jN}(t)}\right)^T$$

Further, in time step t pool i gains revenue $m_{ij}(t)r_j(t-1)$ through infiltrating pool j with $m_{ij}(t)$ miners, and distributes it among members $m_i(t) + \sum_{j\in[|N|]} m_{ji}(t)$. We construct the $|N| \times |N|$ infiltration matrix as follows

$$\mathbf{IR}(t) = \begin{pmatrix} \frac{m_{11}(t)}{m_1(t)+\sum_j m_{j1}(t)} & \cdots & \frac{m_{1\,|N|}(t)}{m_1(t)+\sum_j m_{j1}(t)} \\ \vdots & \ddots & \vdots \\ \frac{m_{|N|1}(t)}{m_{|N|}(t)+\sum_j m_{j|N|}(t)} & \cdots & \frac{m_{|N||N|}(t)}{m_{|N|}(t)+\sum_j m_{j|N|}(t)} \end{pmatrix} \tag{4}$$

Plugin together the $\mathbf{u}(t)$ and $\mathbf{IR}(t)$ we end up with the density revenue vector at time step t,

$$\mathbf{r}(t) = \mathbf{u}(t) + \mathbf{IR}(t) \cdot \mathbf{r}(t-1) \text{ with } \mathbf{r}(0) = \mathbf{u}(0) \tag{5}$$

Moreover, the direct mining rate, $R_i(t)$, of pool i at time step t, is the number of its miners, $m_i(t)$, minus the miners it uses for infiltration, $\sum_{j\in[|N|]\setminus\{i\}} m_{ij}(t)$, and is divided by the total mining rate in the system, namely the number of all miners apart from the perpetrators. So, we have the following formula

$$R_i(t) = \frac{m_i(t) - \sum_{j\in[|N|]\setminus\{i\}} m_{ij}(t)}{m - \sum_{j\in[|N|]\setminus\{i\}} \sum_{k\in[|N|]\setminus\{j\}} m_{jk}(t)} \tag{6}$$

Hence, for the revenue density of pool i we have

$$r_i(t) = \frac{R_i(t) + \sum_{j\in[|N|]\setminus\{i\}} m_{ij}(t) r_j(t)}{m_i(t) + \sum_{j\in[|N|]\setminus\{i\}} m_{ji}(t)} \tag{7}$$

with that we define the revenue density vector $\mathbf{r}(t) = (r_1(t), \dots, r_n(t))^T$. In case where $|n| = 2$, the infiltration rates are $m_{12}(t)$ and $m_{21}(t)$, and the formula (7) takes the form

$$\begin{aligned} r_1(m_{12}(t), m_{21}(t)) &= \frac{m_{22}(t)R_1(t) + m_{12}(t)(R_1(t) + R_2(t))}{m_{11}(t)m_{22}(t) + m_{11}(t)m_{12}(t) + m_{22}(t)m_{21}(t)}, \\ r_2(m_{12}(t), m_{21}(t)) &= \frac{m_{11}(t)R_2(t) + m_{21}(t)(R_1(t) + R_2(t))}{m_{11}(t)m_{22}(t) + m_{11}(t)m_{12}(t) + m_{22}(t)m_{21}(t)} \end{aligned} \tag{8}$$

with $m_{11}(t), m_{22}(t) > 0$ and $m_1(t) + m_2(t) \leq m$. Further, each pool controls only its own infiltration rate. In each round of the Pool game, each pool will optimize its infiltration rate of the other. Clearly, an equilibrium exists where neither Pool_1 nor Pool_2 can improve its revenue by changing its infiltration rate.

As stated in [4], the values of $m_{12}(t), m_{21}(t)$ at the equilibrium can be computed by solving the following system of first-order ordinary differential equations

$$\begin{cases} \dfrac{\partial r_1(m_{12}(t), m_{21}(t))}{\partial m_{12}(t)} = 0 \\ \dfrac{\partial r_2(m_{12}(t), m_{21}(t))}{\partial m_{21}(t)} = 0 \end{cases} \tag{9}$$

In the rest of the analysis we assume that pool i has two pure strategies; attack or to non-attack the adversary. The density revenue for the pure strategy attack is r_i, and for the pure strategy non-attack is $\tilde{r}_i$. Hence the pure strategy profiles are (attack, attack), (attack, non-attack), (non-attack, attack), and (non-attack, non-attack). Moreover, from [4] we have the following ordering for the density revenues of the pools

$$\begin{aligned} &\text{For Pool}_1: \begin{cases} (attack, non-attack) > (non-attack, non-attack) \\ (attack, attack) > (non-attack, attack) \end{cases} \\ &\text{For Pool}_2: \begin{cases} (non-attack, attack) > (non-attack, non-attack) \\ (attack, attack) > (attack, non-attack) \end{cases} \end{aligned} \tag{10}$$

With this at hand we can produce the payoff matrix as provided in Table 2.

Table 2. Pool game with two pools, Fig. 9 in [4].

Pool_1 \ Pool_2	attack	non-attack
attack	(r_1, r_2)	$(r_1, \tilde{r}_2)$
non-attack	$(\tilde{r}_1, r_2)$	$(\tilde{r}_1, \tilde{r}_2)$

Observe that the game provided by Table 2 is a Prisoner's Dilemma, meaning that $(attack, attack)$ is a dominating pure strategy profile.

Moreover, in [4] it is proved that the pool revenues converge, in the case where infiltration rates are constant. In the following we prove that the convergence of density revenues holds for cases where the infiltration rates are not constant.

Lemma 1. *Consider a Pool game with $|N|$ pools, and $m_i(t)$, $m_{ij}(t)$ non-zero polynomials of equal degree $d \in \mathbb{N}$ with non-negative coefficients $\forall i, j \in [|N|]$ and $\forall t \in \mathbb{N}$. Then the pool density revenues converge.*

Proof. Let $m_i(t) = \sum_{k\in[d]} \alpha_{i,k} t^k$ and $m_{ij}(t) = \sum_{k\in[d]} \beta_{ij,k} t^k$, with $\alpha_{i,k}, \beta_{ij,k} \geq 0$ $\forall i, j, k$. Observe that the elements of the $\mathbf{IR}(t)$ are

$$(\mathbf{IR}(t))_{ij} = \frac{m_{ii}(t)}{m_i(t) + \sum_j m_{ji}(t)} = \frac{\sum_{k\in[d]} \beta_{ii,k} t^k}{\sum_{k\in[d]} \alpha_{i,k} t^k + \sum_j \sum_{k\in[d]} \beta_{ji,k} t^k}$$

Hence taking the limit we have,

$$\lim_{t\to+\infty} (\mathbf{IR}(t))_{ij} = \lim_{t\to+\infty} \frac{\sum_{k\in[d]} \beta_{ii,k} t^{k-d}}{\sum_{k\in[d]} \alpha_{i,k} t^{k-d} + \sum_j \sum_{k\in[d]} \beta_{ji,k} t^{k-d}} = \frac{\beta_{ii,d}}{\alpha_{i,d} + \sum_j \beta_{ji,d}}$$

In the $(\mathbf{IR}(t))_{ij}$ the denominator can not be equal to 0, as $m_i(t)$, $m_{ij}(t)$ are non-zero polynomials. Hence, on the limit the $\mathbf{IR}(t)$ has constant elements, then using Lemma 1 in [4] we conclude. □

In the rest of the analysis, we consider $\mathbf{m}_i(t)$ as non-negative and continuous functions in time $\forall i$, and we have the following result

Theorem 1. *Consider a Pool game with $|N|$ pools, with bounded $\mathbf{u}(t)$, and $\mathbf{IR}(t)$ such that $\|\mathbf{IR}(t)\| \leq 1$ $\forall t \in \mathbf{N}$. Then the pool density revenues converges to $\mathbf{u}(t)$.*

Proof. From the Eq. (5) we have

$$\mathbf{r}(t) - \mathbf{u}(t) = \sum_{k=1}^{t} \left(\prod_{d=1}^{k} \mathbf{IR}(t-d) \right) \mathbf{u}(t-k) \tag{11}$$

Now, fix t and consider the sequences $\{\alpha_k\}_{k\in[t]} := \mathbf{m}(t-k)$ and $\{\beta_k\}_{k\in[t]} := \prod_{d=1}^{k} \mathbf{IR}(t-d)$. Observe that since $\|\mathbf{IR}(t)\| \leq 1$ we have $\| \prod_{d=0}^{k-1} \mathbf{IR}(t-d)\| \leq \prod_{d=0}^{k-1} \|\mathbf{IR}(t-d)\| \to 0$, as $t, k \to \infty$ thus, $\beta_k \to 0$[2]. Further, from the assumptions the α_k is a bounded sequence, so $\sum_k \alpha_k \beta_k \to 0$[3]. But $\sum_k \alpha_k \beta_k \to 0$ is the right-hand side of (11). Thus, we conclude. □

Theorem 1 provides a pointwise convergence of $\mathbf{r}(t)$ on $\mathbf{u}(t)$. Also, using Theorem 1 we can deduce the following remark.

Corollary 1. *Consider a Pool game with $|N|$ pools, with a convergent $\mathbf{u}(t)$, and $\mathbf{IR}(t)$ such that $\|\mathbf{IR}(t)\| \leq 1$ $\forall t \in \mathbf{N}$. Then the pool density revenues converges.*

Until now we present the case where every pool at any time step knows the revenue density of all other pools $r_j(t-1)$ and its total infiltration rate $\sum_{j=1}^{p} m_{ji}(t)$. In the following two Subsections we will drop these assumptions and we mitigate the cases where the pools have i) some distribution over the revenue densities, and ii) incorrect revenue densities. Interestingly, as the Pool game is an iterative Prisoner's Dilemma, then it has a dominant equilibrium strategy profile. Thus, in the case where a mediator provides side information to the pools, this will not affect their choices. In a nutshell, a correlation device can not alter the outcome of the Pool game in Table 2 (Fig. 1).

[2] It holds, $\lim_{n\to\infty} A^n = 0$ iff the spectral radius of the square matrix A is less than 1, which holds as the spectral radius of a matrix is less or equal than the matrix norm.
[3] Intuitively, we use the fact that if α_k is bounded and if $\beta_k \to 0$, then $\sum_k \alpha_k \beta_k \to 0$.

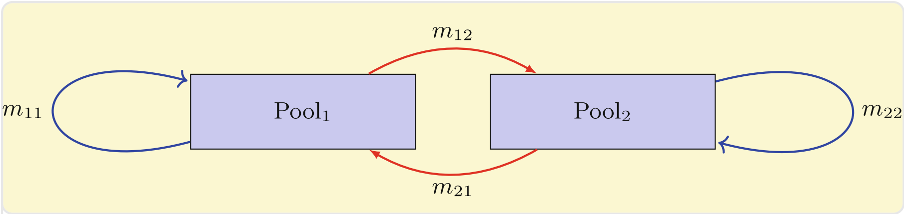


Fig. 1. Pool game with $N = \{\text{Pool}_1, \text{Pool}_2\}$.

4.1 Incomplete Information

From [4] we know that a pool can estimate the rates with which it is attacked. Now, assume that the estimation has a level of uncertainty. E.g., at time t, Pool_2 estimates with probability p_1 that Pool_1 attacks her with the correct infiltration rate $m_{12}(t)$ and with probability p_2 that Pool_1 attacks her with infiltration rate $\hat{m}_{12}(t)$, with $p_1 + p_2 = 1$. At the same time, the Pool_1 does not experience any uncertainty in her estimations, and believes that Pool_2 attacks her with the correct infiltration rate $m_{21}(t)$. Hence, we have a Bayesian game, where Pool_1 has one type $\Theta_{\text{Pool}_1} = \{\theta_{\text{Pool}_1,1}\}$ and Pool_2 has two types $\Theta_{\text{Pool}_2} = \{\theta_{\text{Pool}_2,1}, \theta_{\text{Pool}_2,2}\}$. The density revenues $r_1, r_2, \tilde{r}_1, \tilde{r}_2, r'_1, r'_2, \hat{r}_1$, and $\hat{r}_2$ are compute via the formulas (6)–(8).

Table 3. Information types in Bayesian Pool game with two pools.

Pool_1 \ Pool_2	attack	non-attack
attack	(r_1, r_2)	$(r_1, \tilde{r}_2)$
non-attack	$(\tilde{r}_1, r_2)$	$(\tilde{r}_1, \tilde{r}_2)$

(a) Types: $\theta_{\text{Pool}_1,1}, \theta_{\text{Pool}_2,1}$

Pool_1 \ Pool_2	attack	non-attack
attack	(r_1, r'_2)	$(r_1, \hat{r}_2)$
non-attack	$(\tilde{r}_1, r'_2)$	$(\tilde{r}_1, \hat{r}_2)$

(b) Types: $\theta_{\text{Pool}_1,1}, \theta_{\text{Pool}_2,2}$

While the utility functions of the pools are given in Table 4.

Table 4. Utility functions u_{Pool_1} and u_{Pool_2} for the Bayesian Pool game from Table 3.

r	c	Θ_{Pool_1}	Θ_{Pool_2}	u_{Pool_1}	u_{Pool_2}
attack	attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,1}$	r_1	r_2
attack	attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,2}$	r'_1	r'_2
attack	non-attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,1}$	$\tilde{r}_1$	r_2
attack	non-attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,2}$	r'_1	$\hat{r}_2$
non-attack	attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,1}$	$\tilde{r}_1$	r_2
non-attack	attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,2}$	$\hat{r}_1$	r'_2
non-attack	non-attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,1}$	$\tilde{r}_1$	$\tilde{r}_2$
non-attack	non-attack	$\theta_{\text{Pool}_1,1}$	$\theta_{\text{Pool}_2,2}$	$\hat{r}_1$	$\hat{r}_2$

Using the methodology provided in Subsect. 3.1 we construct the induced utility matrix provided in Table 5.

Table 5. Induced utility matrix.

	$s_1 s_1$	$s_1 s_2$	$s_2 s_1$	$s_2 s_2$
s_1	$(r_1, p_1 r_2 + p_2 r'_2)$	$(r_1, p_1 r_2 + p_2 \hat{r}_2)$	$(r_1, p_1 \tilde{r}_2 + p_2 r_2)$	$(r_1, p_1 \tilde{r}_2 + p_2 \hat{r}_2)$
s_2	$(\tilde{r}_1, p_1 r_2 + p_2 r'_2)$	$(\tilde{r}_1, p_1 r_2 + p_2 \hat{r}_2)$	$(\tilde{r}_1, p_1 \tilde{r}_2 + p_2 r_2)$	$(\tilde{r}_1, p_1 \tilde{r}_2 + p_2 \hat{r}_2)$

Next, we can compute the Bayes-Nash equilibria. Then given the finite horizon of the repeated procedure we can derive the final utility for each one of the pools.

Corollary 2. *Consider the Bayesian Pool game BG. If all the utility matrices in the Information types in the BG have constant infiltration rates, then the pool revenues converge.*

For notational convention, let $\mathbf{IR}^i(t)$ be the infiltration matrix, and $\mathbf{r}^i(t)$ be the revenues density, in the ith Information type. Then, the Lemma 1 and Theorem 1 can be transfused in the case of Bayesian Pool games. Namely,

Lemma 2. *Consider a Bayesian Pool game BG with $|N|$ pools. If for all Information types $i \in \Theta$ in the BG, $m^i_j(t)$, $m^i_{jk}(t)$ are non-zero polynomials of equal degree $d \in \mathbb{N}$ with non-negative coefficients $\forall i, j \in [|N|]$ and $\forall t \in \mathbb{N}$, then the pool density revenues converge.*

Proof. From Lemma 1 in each $\mathbf{r}^i(t)$ and the distribution p over the Θs we have that $\lim_{t \to \infty} \mathbf{r}(t) = \lim_{t \to +\infty} \sum_i p_i \mathbf{r}^i(t)$ that converges. □

Corollary 3. *Consider a Bayesian Pool game BG with $|N|$ pools. If for all Information types $i \in \Theta$ in the BG, $\mathbf{u}^i(t)$ converge, and $\mathbf{IR}^i(t)$ are such that $\|\mathbf{IR}^i(t)\| \leq 1$ $\forall t \in \mathbb{N}$, then the pool revenues converge.*

4.2 Incorrect Information

In the previous Subsection we presented the case where the pools experience uncertainty over the density revenues. Now, assume that the pools have incorrect information regarding the mining power and the density revenues. E.g., at time t, Pool$_1$ knows the Pool game in Table 6b and the Pool$_2$ knows the Pool game in Table 6c, whereas the actual situation captured by Table 6a. This is a case of incorrect information and is described by the misinformed Pool game mG with density revenues matrices as provided in Table 6.

Table 6. Misinformed Pool game.

	s_1	s_2
s_1	(r_1, r_2)	$(r_1, \tilde{r}_2)$
s_2	$(\tilde{r}_1, r_2)$	$(\tilde{r}_1, \tilde{r}_2)$

(a) Actual Game

	s_1	s_2
s_1	$(\dot{r}_1, \dot{r}_2)$	$(\dot{r}_1, \hat{r}_2)$
s_2	$(\hat{r}_1, \dot{r}_2)$	$(\tilde{r}_1, \hat{r}_2)$

(b) Pool_1 game

	s_1	s_2
s_1	$(\bar{r}_1, \bar{r}_2)$	$(\bar{r}_1, \hat{r}'_2)$
s_2	$(\hat{r}'_1, \bar{r}_2)$	$(\tilde{r}_1, \hat{r}'_2)$

(c) Pool_2 game

From the analysis of Sect. 4 at each time step each pool will solve independently the system (9). Namely,

$$\text{Pool}_1 : \begin{cases} \dfrac{\partial \dot{r}_1(m^1_{12}(t), m^1_{21}(t))}{\partial m^1_{12}(t)} = 0 \\ \dfrac{\partial \dot{r}_2(m^1_{12}(t), m^1_{21}(t))}{\partial m^1_{21}(t)} = 0 \end{cases}, \quad \text{Pool}_2 : \begin{cases} \dfrac{\partial \hat{r}'_1(m^2_{12}(t), m^2_{21}(t))}{\partial m^2_{12}(t)} = 0 \\ \dfrac{\partial \hat{r}'_2(m^2_{12}(t), m^2_{21}(t))}{\partial m^2_{21}(t)} = 0 \end{cases} \tag{12}$$

Then the agglomeration of the solution $m^1_{12}(t)$, from the left system, and $m^2_{21}(t)$ from the right system, will provide the *nme*. Given the *nme*, the Adaptation Procedure will evaluate the information of the pools and then the procedure will proceed to the next time step. Thus, the matrices as given in Table 7.

Table 7. Misinformed Pool game after the first step of the Adaptation Procedure.

	s_1	s_2
s_1	(r_1, r_2)	$(r_1, \tilde{r}_2)$
s_2	$(\tilde{r}_1, r_2)$	$(\tilde{r}_1, \tilde{r}_2)$

(a) Actual Game

	s_1	s_2
s_1	(r_1, r_2)	$(\dot{r}_1, \tilde{r}_2)$
s_2	$(\hat{r}_1, \dot{r}_2)$	$(\tilde{r}_1, \hat{r}_2)$

(b) Pool_1 game

	s_1	s_2
s_1	(r_1, r_2)	$(\bar{r}_1, \hat{r}'_2)$
s_2	$(\hat{r}'_1, \bar{r}_2)$	$(\tilde{r}_1, \hat{r}'_2)$

(c) Pool_2 game

Since all the games in mG are Prisoner's Dilemmas, the Adaptation procedure will update the $(attack - attack)$ joint decision according the utilities of the actual game, and will provide $\mathcal{AD}^{(1)}(M)$, that is the misinformed Pool game for the $t = 1$. Observe, that the ordering between r_1 and $\hat{r}_1$, $\hat{r}'_1$, and r_2 and $\tilde{r}_2$, $\hat{r}'_2$ affects the progress of the Adaptation Procedure. Namely,

Corollary 4. *Given the misinformation game in Table 6, if* $r_1 > \max\{\hat{r}_1, \hat{r}'_1\}$ *and* $r_2 > \max\{\hat{r}_2, \hat{r}'_2\}$ *then* $\mathcal{AD}^{(1)}(M) = \mathcal{AD}^{(0)}(M)$, *and the Adaptation Procedure terminates in one step.*

In case the Corollary 4 holds, the misinformed Pool game has a unique *sme*, that is $(attack, attack)$. On the other hand,

Lemma 3. *Given the misinformation game in Table 6, if* $r_1 < \max\{\hat{r}_1, \hat{r}'_1\}$ *or* $r_2 < \max\{\hat{r}_2, \hat{r}'_2\}$ *then the Adaptation Procedure terminates at most in* $|S|$ *steps.*

Proof. From Proposition 4.11 in [14] we have that the Adaptation Procedure in the misinformed Pool game Table 6 is finite. It is easy to see that at most the Adaptation Procedure will update the total number of the joint pure strategies of the misinformed game, that is $|S|$. □

In case where the Adaptation Procedure updates all the joint pure strategies of the subjective Pool games, then we end up with a unique *sme*, that is $(attack, attack)$. In any intermediate situation where the Adaptation Procedure terminates in time steps either $t = 2$ or $t = 3$, we need more information in order to conclude about the *sme*s.

Next, we have the following results regarding the convergence of the density revenues. We start with the case where the infiltration rates are constant in all games in the misinformation game.

Lemma 4. *Consider the finite misinformation Pool game mG with constant infiltration rates for all G^is and G^0, then the pool density revenues converge.*

Proof. From the Corollary 4 and Lemma 3 the Adaptation Procedure terminates. Then, from Lemma 1 in [4] we conclude. □

Abusing notation, we denote as $\mathbf{u}^0$, $\mathbf{u}^i$ the direct mining revenue densities, and m^0_{ij}, m^i_{ij} are the infiltration rates in the actual game G^0 and in the G^i respectively.

Lemma 5. *Consider the finite misinformation Pool game mG, then if $m^i_j(t)$, $m^i_{jk}(t)$ are non-zero polynomials of equal degree $d \in \mathbb{N}$ with non-negative coefficients $\forall i, j \in [|N|]$ and $\forall t \in \mathbb{N}$, then the pool density revenues converge.* □

Proof. Using Lemma 1 for each G^i we have that each $\mathbf{r}^i(t)$ converges. Further, from Lemma 3 the Adaptation Procedure for mG terminates in finite time, thus the revenue densities converge for the mG.

Lemma 6. *Consider the finite misinformation Pool game mG, $\mathbf{u}^i(t)$ are bounded, and $\mathbf{IR}^i(t)$ are such that $\|\mathbf{IR}^i(t)\| \leq 1$ $\forall t \in \mathbb{N}$, then the pool revenues converge.*

Proof. Using Theorem 1 for each G^i we have that each $\mathbf{r}^i(t)$ converges. Further, from Lemma 3 the Adaptation Procedure for mG terminates in finite time, thus the revenue densities converge for the mG. □

Interestingly, we can attain convergence of the density revenues of the mG in the case where the subjective games G^i have general infiltration rates. This is provided by te following result.

Corollary 5. *Consider the finite misinformation Pool game mG, such that $\mathbf{u}^0(t) > \mathbf{u}^i(t)$, $\forall i \in [|N|]$ and the Adaptation Procedure terminates after $|S|$ steps. If one of the following holds*

- *the infiltration rates of the G^0 are constant*
- *the $m^0_i(t)$, and $m^0_{ij}(t)$ are non-zero polynomials with non-negative coefficients of equal degree $\forall i \in [|N|]$*
- *$\forall t$ the $\mathbf{u}^0(t)$ converges and $\|\mathbf{IR}^0(t)\| \leq 1$*

Then the density revenues for the mG converge.

In the case where the assumptions of the Corollary 5 hold then $SME(mG) = \{\sigma |\ \sigma := (\sigma_1, \ldots, \sigma_{|N|}),\ \sigma_i \in ne_j$ for some $ne_j \in NE(G^0)\}$. In other words, the *sme*'s of the mG are all the combinations of the Nash equilibria strategy profiles of the pools in G^0.

5 Numerical Experiments

The theoretical results of Sect. 4 cope with the Pool game in various information environments. Here, we empirically demonstrate the evolution of the density revenues of the Pool game, as provided by the equation $\mathbf{r}^*(t) = \mathbf{u}^*(t) + \mathbf{IR}^*(t) \cdot \mathbf{r}^*(t-1)$ analyzed in Sect. 4[4], considering the cases of i) complete and correct information, ii) incomplete information, and iii) incorrect information. In what follows $T = 1000$, the number of pools is $|N| = 2$, where each one starts with 100 miners. As shown in Table 8 the number of miners does not remain constant. Further, we take $\mathbf{r}(0) = (m^*_{12}(0), m^*_{21}(0))^T$ for all cases.

To demonstrate our numerical results we use the functions provided in Table 8. More specifically, we use linear functions (second column) to study the case where pools attract miners that increase proportionally in time. Second, we pick cubic functions (third column) as they are "relatively simple" polynomial functions that experience critical points. Apparently, the properties of the functions affect the behavior of the pools.

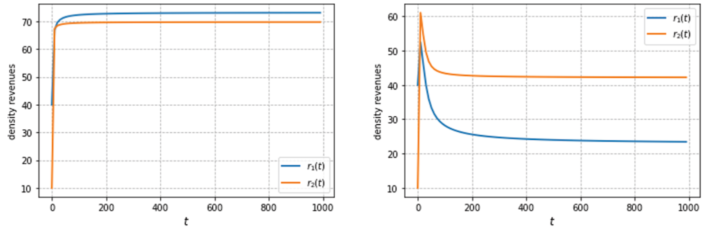
Table 8. Polynomial functions for $m(t)$s.

	Figs. 2a, 3a, 3b, 4a, 4b	Figs. 2b, 3c, 3d, 4c, 4d
$m_1(t)$	$200t + 100$	$6t^3 + 10t^2 + 4t + 100$
$m_{11}(t)$	$110t + (100 - m_{12}(0))$	$5t^3 + 7t^2 + t + (100 - m_{12}(0))$
$m_{12}(t)$	$40t + m_{12}(0)$	$4t^3 + 2t^2 + 2t + m_{12}(0)$
$m_2(t)$	$156t + 100$	$8t^3 + 9t^2 + 4t + 100$
$m_{21}(t)$	$20t + m_{21}(0)$	$3t^3 + 3t^2 + 2t + m_{21}(0)$
$m_{22}(t)$	$91t + (100 - m_{21}(0))$	$4t^3 + 5t^2 + t + (100 - m_{21}(0))$

Complete - Correct Information. In this case, the pools have complete and correct information regarding the Pool game. In Figs. 2a, 2b we compute the density revenues with initial values $m_{11}(0) = 60$, $m_{12}(0) = 40$, $m_{21}(0) = 10$, and $m_{22}(0) = 90$. Further, the density revenue functions for the case where the $m(t)$s are provided by Table 8.

Incomplete Information. For the case of incomplete information we provide experiments both for the undiscounted and the discounted cases, as they were presented in Subsect. 3.1, whereas we compute the density rates, using the settings in Table 9. In Figs. 3 are shown both the undiscounted (Figs. 3a, 3c) and the discounted cases (Figs. 3b, 3d). Further, in Figs. 3a, 3b, 3c, and 3d the $m(t)$s are polynomial functions and are provided by Table 8. Clearly, the numerical results are inline with Lemma 2, and Corollaries 2, and 3.

[4] The asterisk refers to the different information environments.

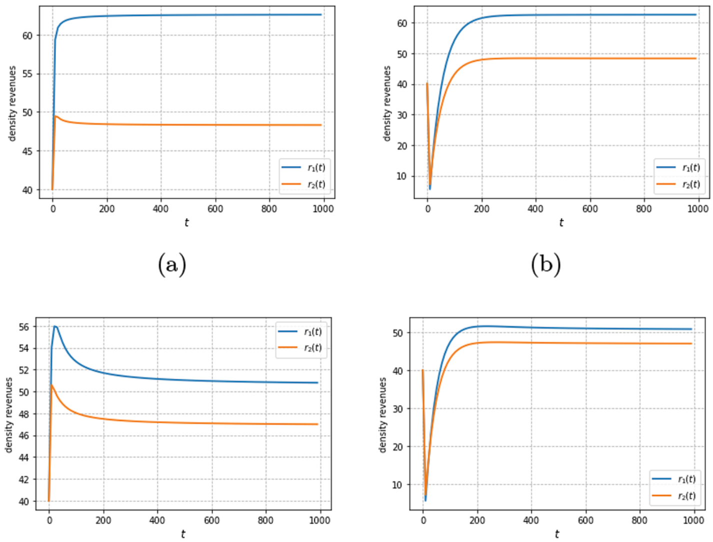


(a) (b)

Fig. 2. Realisations for the density revenues for the complete-correct information environment.

Table 9. Initial infiltration rates, distribution over Information sets, and δ.

Case	$m_{11}(0)$	$m_{12}(0)$	$m_{21}(0)$	$m_{22}(0)$	p	δ
Undiscounted	90	10	30	70	.4	-
	60	40	40	60	.6	
Discounted	90	10	30	70	.4	.8
	60	40	40	60	.6	



(a) (b) (c) (d)

Fig. 3. Realisations for the density revenues regarding incomplete information environment, see Table 9.

Incorrect Information. The pools have subjective views regarding the Pool game. This case is analysed using misinformation games, as presented in Subsect. 3.2 for the values of Table 10. The asterisk in Table 10 simply implies that these values are according to the game in the second column.

In Figs. 4a–4d we compute the density revenues as provided by the equations (12), using polynomial infiltration rates, see Figs. 4a, 4b, 4c, and 4d. Since all the actual and the subjective Pool games are in the class of Prisoners' Dilemma the misinformed Pool game has a unique *nme*. So, the Adaptation procedure will terminate in one step. Observe that eventually, the density revenues converge to the values close to the density revenues in the case of complete-correct information. This happens because $m(t)$s are increasing functions and masking the effect of the update. Thus, the structure of $m(t)$s can tune the effect of misinformation. Finally, the numerical results are in line with Lemmas 4, 5, and 6.

Table 10. Initial infiltration rates for a misinformed Pool game.

Game	$m_{11}(0)$	$m_{12}(0)$	$m_{21}(0)$	$m_{22}(0)$
Actual	60	40	20	80
G^{Pool_1}	70	30	20	80
G^{Pool_2}	90	10	40	60

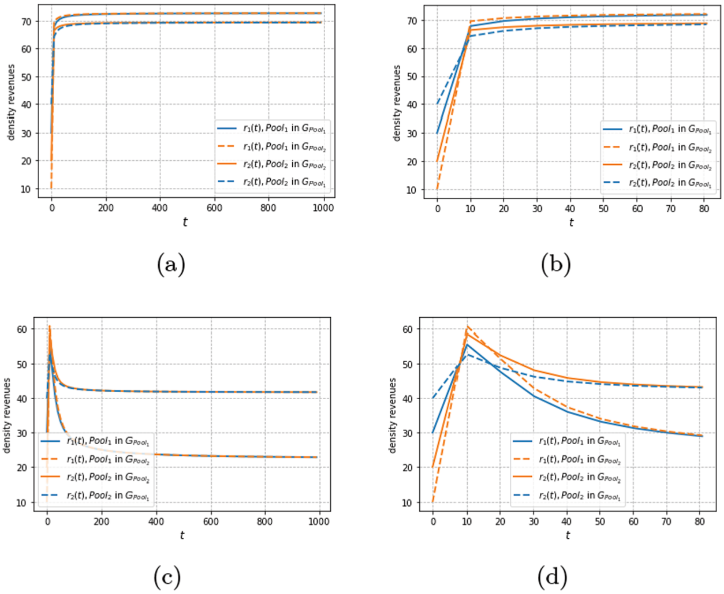


Fig. 4. Realisations for the density revenues regarding the settings that presented in Table 10, for the incorrect information environment.

As a general remark, $m(t)$s influence the density revenues. In cases where $\mathbf{r}(t)$ attain a critical point, the pools may have incentives to stop/continue the interaction. For example, in Fig. 2b the pools attain the maximum density revenues early in time. On the other hand, in Fig. 3b the pools attain the minimum density revenues early and then they recover. As a result, the pools can exploit the properties of $m(t)$s for their benefit.

6 Conclusions

In this paper, we transfuse and study the Pool game model, which was introduced in [4], under different information environments. In particular, we consider the cases where the pools i) experience uncertainty, and ii) have erroneous information regarding the interaction. We provide theoretical results regarding the convergence of the density revenues in all cases, and we generalize the convergence results in [4]. In parallel, we demonstrate experimentally the theoretical results in all the aforementioned information environments.

Our analysis provides several insights regarding the behavior of the pools. First, we show experimentally, for all information environments, that the behavior of the pools is affected severely by the formulas of infiltration rates. In that direction, if the formulas are linear then we can expect the Pool game to converge quicker compared to the case of cubic formulas. Second in case of incorrect information the pools are not necessarily to understand the actual interaction in order to converge.

As the Blockchain framework is becoming more and more involved in versatile and demanding activities is of paramount importance to study and analyze it in more realistic environments. With this work, we make a first step in this direction. To that end, some future directions are to study protocols other than withholding attacks, to develop a mechanism that regulates the efficiency of a Pool game, and to measure the inefficiency of the Pool game due to uncertainty, and misinformation.

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References

1. Altman, E., et al.: Blockchain competition between miners: a game theoretic perspective. In: Frontiers in Blockchain (2020)
2. Chen, Z., Sun, X., Shan, X., Zhang, J.: Decentralized mining pool games in blockchain. In: 2020 IEEE International Conference on Knowledge Graph (ICKG), pp. 426–432 (2020)
3. Deepa, N., et al.: A survey on blockchain for big data: approaches, opportunities, and future directions. Future Gener. Comput. Syst. **131**, 209–226 (2020)

4. Eyal, I.: The miner's dilemma. In: 2015 IEEE Symposium on Security and Privacy, pp. 89–103 (2014)
5. Huang, H., Kong, W., Zhou, S., Zheng, Z., Guo, S.: A survey of state-of-the-art on blockchains: Theories, modelings, and tools. arXiv:2007.03520 (2020)
6. Juhász, P., Stéger, J., Kondor, D., Vattay, G.: A Bayesian approach to identify bitcoin users. PLoS ONE. **13**, e0207000 (2016)
7. Kiayias, A., Koutsoupias, E., Kyropoulou, M., Tselekounis, Y.: Blockchain mining games. In: Proceedings of the 2016 ACM Conference on Economics and Computation, EC 2016, pp. 365–382 (2016)
8. Lewenberg, Y., Bachrach, Y., Sompolinsky, Y., Zohar, A., Rosenschein, J.S.: Bitcoin mining pools: a cooperative game theoretic analysis. In: Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2015, pp. 919–927 (2015)
9. Leyton-Brown, K., Shoham, Y.: Essentials of Game Theory: A Concise Multidisciplinary Introduction. Synthesis Lectures on Artificial Intelligence and Machine Learning, pp. 1–88. Springer Nature, Switzerland (2008). https://doi.org/10.1007/978-3-031-01545-8
10. Li, W., Cao, M., Wang, Y., Tang, C., Lin, F.: Mining pool game model and Nash equilibrium analysis for pow-based blockchain networks. IEEE Access **8**, 101049–101060 (2020)
11. Liu, Z., et al.: A survey on applications of game theory in blockchain. arXiv:1902.10865 (2019)
12. Nakamoto, S.: Bitcoin: A peer-to-peer electronic cash system (2008)
13. Nash, J.F.: Non-cooperative games. Ann. Math. 286–295 (1951)
14. Papamichail, M., Varsos, C., Flouris, G.: Implementing the adaptation procedure in misinformation games. In: Proceedings of the 12th Hellenic Conference on Artificial Intelligence (2022)
15. Shi, Z., Zhou, H., de Laat, C.T.A.M., Zhao, Z.G.: A Bayesian game-enhanced auction model for federated cloud services using blockchain. Future Gener. Comput, Syst. **136**, 49-66 (2022)
16. Varsos, C., Flouris, G., Bitsaki, M., Fasoulakis, M.: A study of misinformation games. In: Pacific Rim International Conference on Artificial Intelligence (2021)
17. Wu, D.A., dong Liu, X., Yan, X., Peng, R., Li, G.: Equilibrium analysis of bitcoin block withholding attack: a generalized model. Reliab. Eng. Syst. Saf. **185**, 318–328 (2019)
18. Zappalà, P., Belotti, M., Potop-Butucaru, M.G., Secci, S.: Brief announcement: game theoretical framework for analyzing blockchains robustness. In: International Symposium on Distributed Computing (2020)
19. Zhao, Z., Chen, X., Zhou, Y.: Bayesian-Nash-incentive-compatible mechanism for blockchain transaction fee allocation. arXiv:2209.13099 (2022)