Tooth Profile Calculation of a Cylindrical Gear Pair to Achieve a Non-constant Ratio

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Abstract. Mechanical vibration and acoustic noise of gear mechanism due to fabrication and elastic deformation is prevented by purposely given fluctuation of meshing contact cycle. The authors came up with the idea that the pinion and variable rack that achieve non-constant transmission ratio is applied to cylindrical gear pair, and have derived equations of a spur variable gear called "cylindrical variable ratio gear" and showed numerical example previously. In this paper, equations for helical gear pair is newly derived, and consider the constraint to achieve a continuum and periodic fluctuation of transmission. Next, the authors considered evaluation conditions of contact ratio and tip thickness, and show the numerical example that meets those conditions. After that, the numerical example is validated by experiment measuring angular displacement. The result showed that the gear pair fabricated for the experiment achieved a target angular displacement, and was demonstrated the feasibility of proposed calculation method.

Keywords: Cylindrical gear pair · Tooth profile generation · Non-constant ratio · Noise and vibration

1 Introduction

Transmission error of gear mechanism in electric vehicles due to fabrication and elastic deformation causes mechanical vibration and acoustic noise [1]. One of the most popular methods to reduce such vibration and noise is tooth profile modification [2]. On the other hand, some researchers focus on the gear pairs that change cycle of meshing contact to prevent the vibration. Karpov et al. [3] investigated the prevention of resonance oscillations by using non-circular gears. Neubauer et al. [4] came up with the idea of uneven inequidistant gears that have uneven teeth positions and uneven teeth thickness to randomize the periodic excitation. The latter does not cause the fluctuation of transmission ratio in theory, but it cannot be applied to arbitrary gear ratio and both require modification of both of input and output gears.

Then, the authors came up with an idea that helical pinion and rack gears that achieve non-constant transmission ratio [5] can be applied to a cylindrical

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gear pair. The gear pair that consists of a typical involute gear and a typical gear called "cylindrical variable ratio gear" achieves a target non-constant transmission ratio while rotating. Previously, the authors have proposed the calculation method of the spur gear pair and illustrated a numerical example [6]. In this paper, the authors will derive a calculation scheme of the helical gear pair's tooth profiles at first. After that, some evaluation indices to identify usable solution regions between various combination of design parameter values, and numerical example based on the proposed method will be shown. Finally, the transmission ratio of numerical example is experimentally validated by measuring the input/output relationship of angular displacement.

2 Theory of the Cylindrical Variable Ratio Gear Pair

2.1 Definition of the Cylindrical Variable Ratio Gear Pair

First, let us consider the mechanism of gear meshing. In general, a gear pair must contact smoothly without interference or divergence. It can be expressed in the following three conditions [7].

1. Both tooth surfaces of the gear pair must be differentiable curves.
2. Both tooth surfaces of the gear pair must tangent at contact point, i.e., they have a common normal.
3. On the contact point, the velocity components of both gears in the direction of the common normal must be equal.

From the left side of Fig. 1, these conditions can be written as

$$v = \overline{\mathrm{O_1P}}\omega_1 \cos\alpha' = \overline{\mathrm{O_2P}}\omega_2 \cos\alpha', \tag{1}$$

where v and α' denote a velocity component of both gears in the direction of the common normal on the contact point and contact pressure angle, respectively. The definition of transmission ratio is written as $i_{21} = \omega_2/\omega_1$, so the relationship between the transmission ratio and the instantaneous pitch point is written as

$$i_{21} = \frac{\overline{\mathrm{O_1P}}}{\overline{\mathrm{O_2P}}}. \tag{2}$$

Then, the contact points whose normal line passes through the instantaneous pitch point are determined by the transmission ratio. From the above-explained fundamentals, it can be said that once a specification of an involute gear on the driving side to describe its tooth shape and a target transmission ratio as a function of the involute gear's rotation angle are given, tooth surface of the conjugate on the driven side can be calculated as a set of the instantaneous contact points. Then, the authors define such gear as a cylindrical variable ratio gear which meshes an involute gear and achieves fluctuating transmission ratio. The cylindrical variable ratio gear pair has the following characteristics.

1. One of the gear pair is an involute gear, and the other one, a cylindrical variable ratio gear, is a non-involute gear.
2. Both of them have tip and root circles. Basically, the tip and root circles of the cylindrical variable ratio gear are determined accordingly to those of the involute gear.

The calculation process will be explained in the following.

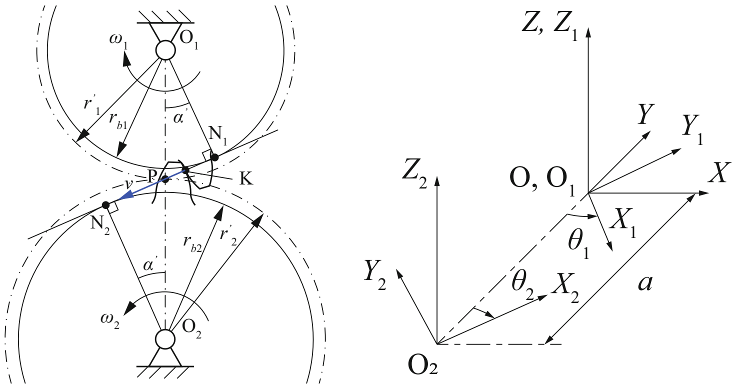


Fig. 1. Left figure shows the meshing condition of a gear pair. Upside gear is an involute gear, and downside gear is conjugate to the involute gear. v is the velocity component of both gears in the direction of the common normal on the contact point. Right figure shows the coordinate frame. $\mathrm{O} - XYZ$ is static. $\mathrm{O}_1 - X_1Y_1Z_1$ and $\mathrm{O}_2 - X_2Y_2Z_2$ rotate with involute gear and cylindrical variable ratio gear, respectively.

2.2 Derivation of the Cylindrical Variable Ratio Helical Gear's Tooth Surface

As a preparation to determine theoretical equations, three-dimensional Cartesian coordinate frames shown in the right side of Fig. 1 are determined. $\mathrm{O} - XYZ$ is considered as a static coordinate frame, and other two coordinate frames are fixed to the involute gear and the cylindrical variable ratio gear which are rotating. For each of those moving coordinates, $\mathrm{O}_1 - X_1Y_1Z_1$ and $\mathrm{O}_2 - X_2Y_2Z_2$, its origin, $\mathrm{O}_{1\mathrm{or}2}$, and $Z_{1\mathrm{or}2}$ axis determine the rotation center and axis of rotation parallel to the Z axis, respectively. The identify number, 1 and 2, are assigned to the involute gear and the cylindrical variable ratio gear, respectively. The origin of the involute gear, O_1, is coincide with O, and O_2 is located at $\vec{a} = (0, -a, 0)^{\mathsf{T}}$, then a represents the distance of rotation centers. The angular displacements of those gears, $\theta_{1\mathrm{or}2}$ are determined by the angle between $X_{1\mathrm{or}2}$ axis and Y axis, as shown in the left side of Fig. 1. It should be noted that initial direction of X_1 and X_2 axes and angular displacement are inverted as shown in the left side of Fig. 1, according to the input-output relationship of the gear transmission.

Position vector $\vec{r}_1$ that represents the teeth surface of the involute helical gear on the moving coordinate $\mathrm{O}_1 - X_1Y_1Z_1$ is written as

$$\vec{r}_1 = \left(r_{1x}\ r_{1x}\ r_{1z}\right)^{\mathsf{T}}, \tag{3}$$

$$\text{where} \begin{cases} r_{1x} = r_{b1}\cos\left(\sigma_1 + u_1 + \phi_1\right) + r_{b1}u_1\sin\left(\sigma_1 + u_1 + \phi_1\right) \\ r_{1y} = r_{b1}\sin\left(\sigma_1 + u_1 + \phi_1\right) - r_{b1}u_1\cos\left(\sigma_1 + u_1 + \phi_1\right) \\ r_{1z} = l \end{cases}.$$

Here, the radius of a base cylinder, $r_{b1} = \left(m_{t1}z_1\cos\alpha_{t1}\right)/2$, is calculated by the transverse module, m_{t1}, number of teeth, z_1, and transverse pressure angle, α_{t1}. Under the consideration of a transverse plane like left side of Fig. 2, σ_1 is the angle between the direction of X_1 axis and the generation point of the involute curve, A_1, and u_1 is the angle between generation point of the involute curve and the tangent point of the base cylinder, N_1, and the normal of tooth surface. ϕ_1 is the angle calculated as $\phi_1 = \left(l\tan\beta\right)/r_1$ by the helix angle, β, and the radius of pitch circle, $r_1 = m_{t1}z_1/2$. Next, let us consider the equation (3) in the static coordinate $\mathrm{O} - XYZ$. The equation is written as

$$\vec{r} = \boldsymbol{M}_{Z_1}^{\theta_1-\pi/2}\vec{r}_1, \tag{4}$$

where $\boldsymbol{M}_{Z_1}^{\theta_1-\pi/2}$ is the rotation matrix that means $\theta_1 - \pi/2$ radian rotation around Z_1 axis.

Considering the static coordinate $\mathrm{O} - XYZ$, the angle of tangent point of the base cylinder and the normal of meshing tooth surface is the instantaneous pressure angle, α'. Therefore, it can be written as

$$\sigma_1 + u_1 + \theta_1 = \alpha'. \tag{5}$$

The equation means that the gear pair contact on the instantaneous meshing line fulfills the instantaneous transmission ratio. Then, u_1 is calculated by the angular displacement of the involute gear, so the contact point of the gear pair can be calculated only by θ_1. Finally, considering the contact point in the moving coordinate frame $\mathrm{O}_2 - X_2Y_2Z_2$, position vector $\vec{r}_2$ that represents the teeth surface of cylindrical variable ratio gear is calculated as

$$\vec{r}_2 = \boldsymbol{M}_{Z_2}^{\theta_2-\pi/2}\left(\boldsymbol{M}_{Z_1}^{\theta_1-\pi/2}\vec{r}_1 - \vec{a}\right), \tag{6}$$

where the output rotation angle, $\theta_2 = \int i_{21}\theta_1$, is determined by the transmission ratio, i_{21} and rotation angle, θ_1, so the teeth surface of a cylindrical variable ratio gear is calculated as Eqs. (5) and (6) by θ_1.

2.3 Constraint to Achieve Periodic Fluctuation of Transmission

To make an endless and continuum rotation of the gear pair happen, a cylindrical variable ratio gear must satisfy the constraint on the transmission ratio. From

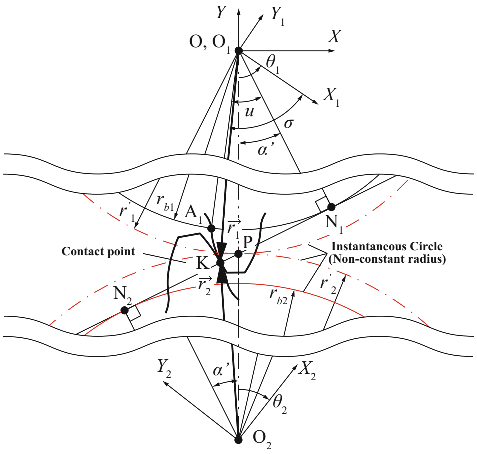


Fig. 2. The meshing condition of a cylindrical variable ratio gear.

the definition of transmission ratio, the relationship of input-output angular displacement is written as

$$\int i_{21} d\theta_1 = \int d\theta_2. \tag{7}$$

Then, let us assume that a cylindrical variable ratio gear can accomplish a full-turn rotation. At that time, the involute gear's corresponding rotation angle can be calculated by a pitch angle, $2\pi/z_1$, times the teeth number of the cylindrical variable ratio gear, z_2. Therefore, Eq. (7) can be rewritten as

$$\int_0^{\frac{2\pi z_2}{z_1}} i_{21} d\theta_1 = 2\pi. \tag{8}$$

Here, transmission ratio can be represented as sum of constant component, $i_{21c} = z_1/z_2$, and variable component, i_{21v}. By neglecting the constant component, Eq. (8) can be rewritten only by using the variable component as

$$\int_0^{\frac{2\pi z_2}{z_1}} i_{21v} d\theta_1 = 0. \tag{9}$$

From the Eq. (9), sum of variable component with respect to the period of the involute gear should be 0. On the other hand, a cylindrical variable ratio gear does not have the constraint of constant part because the teeth of involute gear is uniform, so the gear ratio can be determined by arbitrary integer pair.

3 Numerical Example

3.1 Evaluation Conditions of Cylindrical Variable Ratio Gear

To obtain practical gear tooth profiles, the authors propose the following evaluation conditions of cylindrical variable ratio gear.

a. Contact ratio should be greater than 1.
b. Tip thickness should be greater than 0.

Considering the meshing contact of a cylindrical variable ratio gear pair as that of an involute gear pair at the moment, each parameters can be calculated roughly by an instantaneous transmission ratio as

$$\epsilon = \frac{1}{2\pi}\left\{z_1\left(\tan\alpha_{a1} - \tan\alpha'\right) + z_2\left(\tan\alpha_{a2} - \tan\alpha'\right)\right\}, \tag{10}$$

$$\text{where} \cos\alpha_{a1} = \frac{r_{b1}}{r_{a1}}, \cos\alpha_{a2} = \frac{r_{b2}}{r_{a2}}. \tag{11}$$

In the same way, tip thickness can be written as

$$b_a = \frac{\pi}{2z_2} - \left(\mathrm{inv}\alpha_{a2} - \alpha'\right). \tag{12}$$

However, a cylindrical variable ratio gear cannot be evaluated without exceptions by those parameters. Above contact ratio represents the average of number of meshing contact points in instantaneous transmission ratio, so continuous meshing contact may be maintained even if such contact ratio is less than 1 in the gear whose transmission ratio varies depending on the rotation angle. Above tip thickness also does not represent exact one of a cylindrical variable ratio gear, because the tips of the left and right tooth surfaces may not contact at same time (they may not have same transmission ratio). The exact contact ratio of a cylindrical variable ratio gear can be calculated by number of contact points, so it will be discussed in the following subsection.

3.2 Number of Meshing Teeth

Number of meshing teeth which can be counted as the number of teeth acrossing an instantaneous meshing line. Therefore, by investigating weather each of all teeth meshes or not with respect to every input angle, the number of meshing teeth, namely contact points, in every moment can be calculated. The range of the angular displacement is minimum value from $\angle O_2O_1B_2$ or $\angle O_2O_1N_1$ to $\angle O_2O_1B_1$ or $\angle O_2O_1N_2$ as shown in the left side of Fig. 3. $\angle O_2O_1N_1$ and $\angle O_2O_1N_2$ is instantaneous pressure angle, α', and $\angle O_2O_1B_2$ and $\angle O_2O_1B_1$ is calculated by tip circle and meshing line (instantaneous transmission ratio). Since the tip circle of a cylindrical variable ratio gear is constant, the range of the input angle is calculated by the equation of the involute gear's teeth surface and meshing line. First, the point on involute gear's tooth surface in the left side of Fig. 3 is written as

$$\begin{cases} r = r_{b1}\sqrt{(\tan\alpha_k) + 1} \\ \theta = \sigma_1 + \mathrm{inv}\alpha_k + \theta_1 \end{cases}, \tag{13}$$

where α_k is a variable. "inv" is involute function ($\text{inv}\alpha_k = \tan\alpha_k - \alpha_k$). The equation about B_1, B_2, N_1 and N_2 is written as

$$\overline{O_1B_1} = r_{a1}, \tag{14}$$

$$\angle O_2O_1B_1 = \alpha' - \alpha_{a1}, \tag{15}$$

$$\overline{O_1B_2} = \frac{r_{b1}}{\cos\left\{\arctan\frac{(r_{b1}+r_{b2})\tan\alpha' - r_{b2}\tan\alpha_{a2}}{r_{b1}}\right\}}, \tag{16}$$

$$\angle O_2O_1B_2 = \alpha' - \arctan\frac{(r_{b1}+r_{b2})\tan\alpha' - r_{b2}\tan\alpha_{a2}}{r_{b1}}, \tag{17}$$

$$\overline{O_1N_1} = r_{b1}, \tag{18}$$

$$\angle O_2O_1N_1 = \alpha', \tag{19}$$

$$\overline{O_1N_2} = \frac{r_{b1}}{\cos\left\{\frac{\arctan(r_{b1}+r_{b2})\tan\alpha'}{r_{b1}}\right\}}, \tag{20}$$

$$\angle O_2O_1N_2 = \alpha' - \arctan\frac{(r_{b1}+r_{b2})\tan\alpha'}{r_{b1}}. \tag{21}$$

By substituting the above to r and θ of Eq. (13), the angular displacements, θ_1, at both ends of the meshing line can be obtained. The number of meshing teeth can be calculated because each tooth contributes to meshing within the range of each angular displacement. By using values in Table 1, the transmission ratio i_{21} with respect to angular displacement of involute gear can be written as the upper right of Fig. 3, and the graph with the angular displacement on the horizontal axis and the number of meshing teeth on the vertical axis is shown as bottom right of Fig. 3. From the right side of Fig. 3, the number of meshing teeth tends to decrease when the transmission ratio is large and increase when the transmission ratio is small. If the driving torque is constant, the pressure on the teeth surface increases as the transmission ratio decreases, and the number of contact teeth tends to increase, so the characteristic is advantageous for load capability.

3.3 Illustration of Cylindrical Variable Ratio Gear Pair

After considering equations from (9) to (21), the teeth surface of cylindrical variable ratio gear can be calculated by Eqs. (5) and (6). Then, the appearance of cylindrical variable ratio gear pair on $X-Y$ plane by using numerical example in Table 1 is shown in left side of Fig. 4. The teeth thickness of a cylindrical variable ratio gear changes due to variable transmission ratio.

4 Experimental Validation

4.1 Fabrication of a Setup

By using the numerical example in Table 1 discussed in the previous section, a pair of gear specimens are fabricated to perform experimental validation to check

Table 1. List of design variables and values for numerical example. Subscript numbers, 1 and 2, indicate an involute gear and a cylindrical variable ratio gear, respectively. (A cylindrical variable raito gear is the conjugate to involute gear, so its shift coef. Is not defined. However, considering the involute gear which has same design variables, it can be calculated by center distance etc.)

Symbols	Units	Meanig	Numerical examples
m_{n1}, m_{n2}	mm	Normal module	2.5
α_{n1}, α_{n2}	deg	Normal pressure angle	20
z_{t1}, z_2	–	Number of teeth	23, 55
β	deg	Helix angle	30
h_{a1}, h_{a2}	mm	Addendum	$m_{n1,n2}$
h_{f1}, h_{f2}	mm	Dedendum	$1.25m_{n1,n2}$
x_1, x_2	–	Shift coef	0, -0.038
a	mm	Center distance	112.50
i_{21}	–	Transmission ratio	$23/55\,(1 + 0.03\cos\theta_1)$

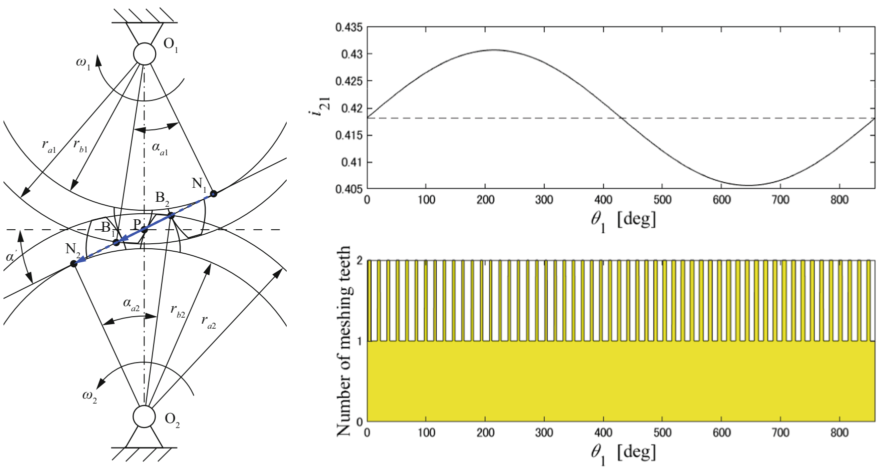


Fig. 3. Left figure shows meshing line of an involute gear pair. Right figure consists of two figures; above shows transmission ratio, and below figure shows the the number of meshing teeth. The number of meshing teeth tends to decrease when the transmission ratio is large and increase when the transmission ratio is small.

weather the target transmission ratio can actually be achieved. Transmission ratio affect angular velocity, so the authors fabricated an experimental equipment to measure the angular displacement while rotating. The right side of Fig. 4 shows the experimental setup. The driven gear which is on the right side in the figure is a cylindrical variable ratio gear, and the other is an involute gear. An encoder is attached to the axis which rotates with the cylindrical variable ratio

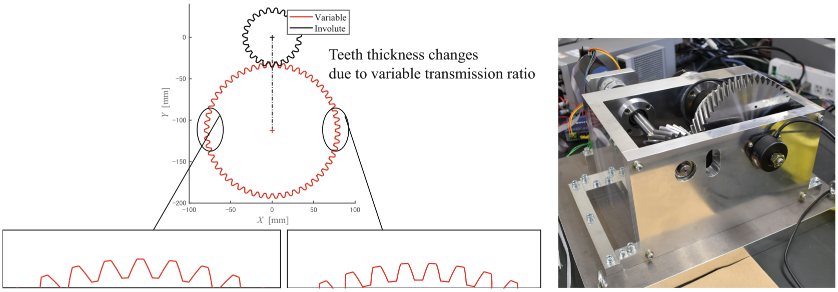


Fig. 4. Left figure shows a cylindrical variable ratio gear pair of numerical example. The above gear is involute gear and the below gear is cylindrical variable ratio gear. Teeth thickness changes due to variable transmission ratio. Right figure shows the experimental setup for a cylindrical variable ratio gear pair. The pinion on the left side is an involute gear, and the gear on the right side is a cylindrical variable ratio gear.

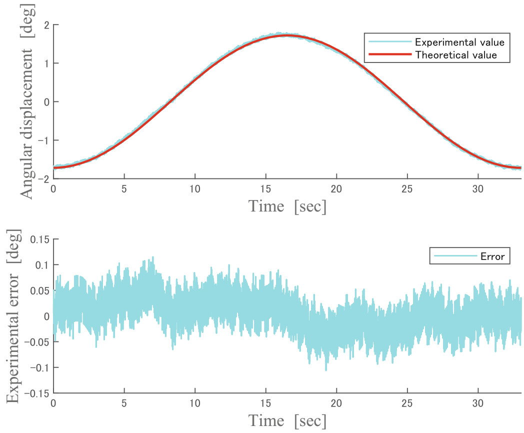
gear, and the angular displacement of that can be measured. The motor rotate $100/23 \simeq 4.34$ rpm constantly, and the encoder measures at 5ms intervals. The time of experiment is $3,901$s $\simeq$ 1h.

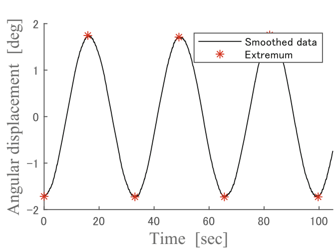
4.2 Experimental Result

Raw data measured by the encoder contains the constant component of the transmission ratio. The waveforms which is removed that are obtained by subtracting the linear approximation from the raw data. Then, upper left of Fig. 5 shows the comparison of the fluctuation of angular displacement between experimental value and theoretical one, and lower left of Fig. 5 shows the experimental error which is calculated by subtracting theoretical value from experimental one. From lower left of Fig. 5, the maximum error is 0.115 deg, and the percentage is 6.7 %. Therefore, considering the experimental values as a sine wave, the amplitude and period of the fluctuation is calculated by extremum of experimental data smoothed by moving average as the right of Fig. 5. From the figure, each experimental error of them is calculated as 0.81 % and 0.22 %, respectively. From the result, cylindrical variable ratio gear manufactured for experiment almost achieved the fluctuation of transmission ratio same as theory.

5 Conclusion

In this paper, the cylindrical variable ratio gear's equation of helical gear pair which achieves the fluctuation of transmission ratio was derived, and the authors considered evaluation conditions and numerical example. Therefore, the numerical example was fabricated for experiment, and measured the fluctuation of the angular displacement. Each experimental errors of them were small, 0.81 % and 0.22 %, respectively, and the result demonstrated the numerical example was validated.





	Theory	Experiment
Amplitude [deg]	1.719	1.733
Period [sec]	33.000	33.074

Fig. 5. Left figures show the result of experimental validation. Upper left shows the comparison of the fluctuation of angular displacement between experimental value which removed the constant part of transmission ratio and theoretical one. Lower left shows the error which is calculated by subtracting theoretical value from experimental one. Right figure shows the experimental data smoothed by moving average and its extremum, and comparison of mean experimental value and theoretical one.

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