Decision with Belief Functions and Generalized Independence: Two Impossibility Theorems

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Abstract. Belief functions constitute a particular class of lower probability measures which is expressive enough to allow the representation of both ignorance and probabilistic information. Nevertheless, the decision models based on belief functions proposed in the literature are limited when considered in a dynamical context: either they drop the principle of dynamical consistency, or they limit the combination of lotteries, or relax the requirement for a transitive and complete comparison. The present work formally shows that these requirements are indeed incompatible as soon as a form of compensation is looked for. We then show that these requirement can be met in non compensative frameworks by exhibiting a dynamically consistent rule based on first order dominance.

1 Introduction

Belief functions [2,23] constitute a particular class of lower probability measures which is expressive enough to allow the representation of both ambiguity and risk (as an example, it perfectly captures the information on which Ellsberg's paradox [8] is built). That is why many decision models based on belief functions have been proposed, e.g. Jaffray's linear utility [16,17], Choquet integrals [1,11,13], Smet's pignistic approach [24,25], Denoeux and Shenoy's interval-valued utility theory [4,5], among others (for more details, the reader can refer to the excellent survey proposed by [3]). Nevertheless, these approaches are often limited when considered in a dynamical context: either they drop the principle of dynamical consistency (this is the case for the Choquet utility), or they limit the combination of lotteries to be purely probabilistic (as in Jaffray's approach) and/or the class of simple lotteries (as in [6,10]), or drop the requirement for a transitive and complete comparison of the decisions (as in [5]).

The present work proposes two impossibility theorems that highlight the incompatibility of the axioms of lottery reduction, completeness and transitivity of the ranking, and independence (and thus, dynamical consistency) when a form of compensation is looked for. We then relax compensation and show that these axioms can be compatible by exhibiting a complete and transitive decision rule which basically relies on first order dominance

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The next section introduces the background on evidential lotteries; the impossibility theorems are presented in Sect. 3. Section 4 finally presents the non-compensatory decision rule.

2 Background and Notations

In this section we present the notations and background on which the further development rely.

Let $\mathcal{X}$ the set of all the possible consequences of the available decisions. $\mathcal{X}$ is assumed to be finite A *mass function* is a mapping f from $2^{\mathcal{X}}$ to $[0, 1]$ such that $\sum_{A \subseteq \mathcal{X}} f(A) = 1$ and $f(\emptyset) = 0$
The mass function f induces the following *belief* and *plausibility* functions:

$$Bel(A) = \sum_{B \subseteq A} f(B) \quad Pl(A) = \sum_{B \cap A \neq \emptyset} f(B)$$

A set $A \subseteq \mathcal{X}$ such that $f(A) > 0$ is called a *focal element* of f. Let $Support(f) = \{A, f(A) > 0\}$ be the set of focal elements of f. If all focal sets are singletons, then $Bel = Pl$ and it is a probability measure.

In a static, one-step probabilistic decision problem, a possible decision is a probability distributions on a set $\mathcal{X}$ of outcomes - a simple "lottery" [26]. The definition naturally extends to the theory of evidence:

Definition 1 (Simple Evidential Lottery).
A simple evidential lottery *is a mass function on* $\mathcal{X}$*. In particular:*

- *A* simple Bayesian (or "linear") lottery *is a mass functions on* $\mathcal{X}$ *the focal elements of which are singletons;*
- *A* set lottery *is a simple lottery with a single focal element* $A \subseteq \mathcal{X}$*,* $A \neq \emptyset$*;*
- *A* constant lottery *provides some consequence* $x \in \mathcal{X}$ *for sure: it contains only one focal element,* $\{x\}$*.*

$\mathcal{M}$ will denote the set of simple evidential lotteries and $\mathcal{P}$ the set of simple Bayesian lotteries. For the sake of readability and by abuse of notation A shall denote the set lottery on A, $\{x\}$ shall denote the constant lottery on x and $\mathcal{X}$ shall denote both the set of consequences and the set of constant lotteries.

Example 1 (Ellsberg's paradox [8]*).* Consider an urn containing 90 balls: 30 balls are red, while the remaining 60 balls are either black or yellow in unknown proportions. Hence a mass distribution of $\{Red, Yellow, Black\}$: $f(\{Red\}) = \frac{1}{3}$, $f(\{Yellow, Black\}) = \frac{2}{3}$ Four possible gambles are presented to the agent:

- A: the agent gets 100 if red is drawn, 0 otherwise;
- B: the agent gets 100 if black is drawn, 0 otherwise;
- C: the agent gets 100 if red or yellow is drawn, 0 otherwise;
- D: the agent gets 100 if black or yellow is drawn, 0 otherwise.

A majority of the agents prefers A to B and D to C [8]. But whatever the utility function considered, there exist no probability distribution such that (i) the expected utility of A is greater than the one of B and (ii) the expected utility of D is greater than the one of C. Ellsberg's example can on the contrary be handled in the framework of belief functions - the four gambles corresponding to the following simple lotteries (notice that f_A and f_D are Bayesian):

$$\begin{array}{llll} f_A(\{100\}) &= \frac{1}{3},\ f_A(\{0\}) = \frac{2}{3}; & f_C(\{0,100\}) = \frac{2}{3},\ f_C(\{100\}) = \frac{1}{3}; \\ f_B(\{100,0\}) = \frac{2}{3},\ f_B(\{0\}) = \frac{1}{3}; & f_D(\{0\}) &= \frac{1}{3},\ f_D(\{100\}) = \frac{2}{3}. \end{array}$$

A compound lottery is a bpa on (simple or compound) lotteries.

Definition 2 (Compound Evidential Lottery).
A compound lottery is a mass function on a set of simple or compound lotteries F. *We shall use the notation* $f = \alpha_1 \cdot F_1 + \ \ldots \ + \alpha_k \cdot F_k$ *where* $F_i \subseteq F$ *and* f *and* $\alpha_i = f(F_i)$.

Example 2 (An Ellsberg's based compound lottery [5]*).* Consider two urns: U1 contains 90 balls, 30 of which are black, and 60 are red or yellow. U2 is identical to Ellsberg's urn. In the first stage you are allowed to draw one ball B1 from U1:

- If B1 is black or red, you are allowed to draw one ball from U2 at the second stage, and you get 100 if it is red, and 0 otherwise (lottery A of Example 1);
- If B1 is yellow, you are allowed to draw one ball from U2, and you get 100 if it is black, and 0 otherwise (lottery B of Example 1).

This is captured by the compound lottery $h(\{f_A\}) = \frac{1}{3}$ and $h(\{f_A, f_B\}) = \frac{2}{3}$ (Fig. 1).

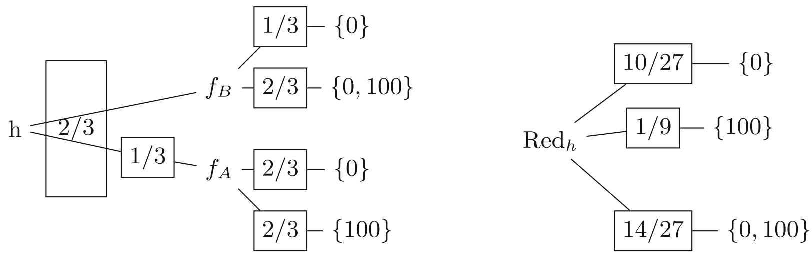

Fig. 1. The compound lottery of Example 2 and its reduction

The reduction of a compound lottery is a simple lottery considered as equivalent to the compound one. When evidential lotteries are dealt with, the operation of reduction is based in Dempster's rule on combination[1]. It is defined by:

[1] A good reference for evidential lottery reduction is Denoeux and Shenoy's work [4]. Jaffrays' work [17] can also be cited but is limited to probabilistic mixtures of bpas.

Definition 3. *Red_l is the simple lottery defined by:*
$\forall A \subseteq \mathcal{X}, \quad Red_l(A) = \sum_{H, l(H)>0} f(H) \cdot m_H(A)$,

$$\text{where} \qquad m_H(A) = \sum_{\substack{(B_1, \ \ldots \ , B_{|H|}),\\ s.\ t. \ \ A = B_1 \cup \ \ldots \ \cup B_{|H|}\\ and\ \forall h_i \in H, B_i \in Support(H_i)}} \prod_{B_i, i=1,|H|} h_i(B_i)$$

The reduction of a simple lottery is the lottery itself. In the following, we shall in particular consider compound lotteries mixing two lotteries only:

Definition 4. *A binary compound lottery on f and g is a compound lottery whose only possible focal elements are $\{f\}$, $\{g\}$ and $\{f, g\}$. Such a lottery is denoted $\alpha \cdot f + \beta \cdot g + \gamma \cdot fg$ (with $\alpha + \beta + \gamma = 1$)*

It is easy to see that the reduction of the binary lottery $l = \alpha \cdot f + \beta \cdot g + \gamma \cdot fg$ is the lottery Red_l defined by

$$\forall A \subseteq \mathcal{X}, Red_l(A) = \alpha \cdot Red_f(A) + \beta \cdot Red_g(A) + \gamma \sum_{B \cup C = A} (Red_f(B) \cdot Red_g(C))$$

As a matter of fact, suppose that f and g are simple; the compound lottery l involves three focal elements:

- $\{f\}$, of probability α provides a series of sets A: here, each A is obtained with a probability $\alpha \cdot f(A)$
- $\{g\}$, of probability β provides a series of sets A: each A is obtained with a probability $\beta \cdot g(A)$
- $\{f, g\}$ of probability γ: the disjunction of f and g is considered: each time f provides a set B with probability $f(B)$ and g provides a set C with probability $g(C)$, the disjunction provides a set $A = B \cup C$, with probability $\gamma \cdot f(B) \cdot g(C)$

Because a set A can be obtained in several ways, the mass of probability of A is the sum of the probabilities of getting this set in the different branches.

It can be shown that the reduction of any compound lottery is equivalent to a binary compound lottery. For the sake of simplicity and without loss of generality, the next sections consider binary lotteries only.

A decision rule amounts to a preference relation $\succsim$ on the lotteries. The rule of expected utility for instance makes use of a utility function u on $\mathcal{X}$: for any two distributions $f, g \in \mathcal{P}$, $f \succsim_{EU} g$ iff the expected value of the utility according to f is at least equal to the expected value of the utility according to g.

Considering credal lotteries (i. e. sets of probability distributions) [12] have proposed a rule based on the lower expectation of the utility. When applied on the set $\mathcal{P}(f) = \{p, \forall A, Bel(A) \leq P(A) \leq Pl(A)\}$, the lower expectation of the utility is equal to the Choquet integral based on the Bel measure [11,13,22]. So, for any simple evidential lottery f, one shall maximize

$$Ch(f) = \Sigma_A f(A) \cdot min_{x \in A} u(x)$$

Example 3. Choquet integrals capture many situations which cannot be captured by expected utility, and in particular the Ellsberg paradox. Setting $u(x) = x$ is easy to check that:

$$Ch(f_A) = \frac{1}{3} * 100 = 10/3$$
$$Ch(f_B) = \frac{2}{3} * min(0, 100) = 0$$
$$Ch(f_C) = \frac{1}{3} * 100 + \frac{2}{3} * min(0, 100) = 100/3$$
$$Ch(f_D) = +\frac{2}{3} * 100 = 200/3$$

So, $Ch(f_A) > Ch(f_B)$ and $Ch(f_D) > Ch(f_C)$, which captures Ellsberg' Example Jaffray's approach leads to the same values and the same preference order when letting $\alpha_{B_*(a),B^*(a)} = 1$ whatever a (i. e. following the most cautious approach).

Lottery reduction allows to consider the rule in a dynamical context: to compare compound lotteries, compare their reductions. Nevertheless it may happen that the Choquet value of the reduction of $l = \alpha \cdot f + \beta \cdot h + \gamma f \cdot h$ outperforms the one of $l' = \alpha \cdot g + \beta \cdot h + \gamma \cdot g \cdot h$ while $Ch(g) > Ch(f)$ (see Example 4). In decision trees this means that if an agent prefers l to l' ex-ante, then when reaching f he can be tempted to exchange it for g: the Choquet integral is not dynamically consistent. This can lead to *Dutch Books* [19] or a negative value of information [27] - this also makes the use of dynamic programming algorithms difficult.

Example 4. Let $x, y, z \in \mathcal{X}$ with $u(x) \succ u(y) \succ u(z)$. Choose p such that $1 > p \cdot u(x) + (1-p) \cdot u(z) > u(y)$. Finally, let $f = p \cdot x + (1-p) \cdot z$ and $g = y$. It holds that $Ch(f) = p \cdot u(x) + (1-p) \cdot u(z) > Ch(g) = u(y)$.

Consider $h = 1 \cdot f \cdot y$ and $h' = 1 \cdot g \cdot y$. h' always provides consequence y, so $Ch(h') = u(y)$. h has two focal elements, $m(\{x, y\}) = p$ and $m(\{z, y\}) = 1 - p$, so $Ch(h) = p \cdot u(y) + (1-p) \cdot u(z)$. So $Ch(h) < Ch(h')$.

Jaffray [16] circumvents the difficulty by considering linear compound lotteries only, i. e. compound lotteries of the form $\lambda \cdot f + (1 - \lambda) \cdot g$ (this kind of linear modelhas been more recently studied by [14,21]) Giang [10] also restricts the framework to recover dynamical consistency, assuming that all the simple lotteries are consonant. In the same line of thought, Dubois et Al. [6] limit lotteries to hybrid probability-possibility functions. We shall also cite other approaches, like Smets's pignistic utility [25]. Nevertheless, each approaches either restrict the field of application (like Jaffray's), or the requirement of a complete and transitive order, or is not dynamically consistent.

3 Toward Impossibility Theorems

The decision rules on evidential lotteries are often limited when considered in a dynamical context: either they drop the principle of dynamical consistency (e. g. the Choquet integral), or they restrict themselves to particular classes of lotteries (as in Jaffray's approach), or they drop the requirement for a transitive an complete comparison of the decisions (as in [5]). All circumvent an implicit difficulty, that the current work aims at highlighting, under the form of impossibility theorems. To this extend, let us first present the main axioms.

Consider a relation $\succsim$ on the set of simple and compound evidential lotteries that can be built on a set of consequences $\mathcal{X}$ - $\succsim$ may be e.g. the preference relations built by the Choquet rule or by Jaffray's linear utility. Let $\succ$ denote the asymmetric part of $\succsim$ and $\sim$ its symmetric part. A first requirement is that the preference can compare any act to any other in a transitive way:

Axiom 1 (Completeness and transitivity (A1)). *$\succsim$ is complete and transitive*

I. e. , $\sim$ is transitive and defines equivalence classes, totally ordered by $\succ$. Jaffray's linear utility and the Choquet integral do satisfy axiom A1, while the rule defined by Denoeux and Shenoy [5] defines a transitive but incomplete preference (it may happen that neither $f \succsim g$ nor $g \succsim f$)

Independence. The crucial axiom when comparing lotteries is the axiom of independence, which ensures the dynamical consistency of the decision rule. This axiom has been proposed by Von Neumann and Morgenstern in the context of probabilistic lotteries: for any $0 < \alpha \leq 1$, $f \succsim g \iff \alpha \cdot f + (1-\alpha) \cdot h \succsim \alpha \cdot g + (1-\alpha)h$.

Axiom 2 (Independence). *For any probabilistic lotteries $f, g, h \in \mathcal{P}$,*

– *if* $0 < \alpha \leq 1$ *then* $f \succ g \Rightarrow \alpha \cdot f + (1-\alpha) \cdot h > \alpha \cdot g + (1-\alpha)h$

We extends this axiom to evidential lotteries in two steps:

Axiom 3 (Generalized Weak Independence (wGI)). *For any evidential lotteries f, g, h, any $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$: $f \succsim g \Rightarrow \alpha \cdot f + \beta \cdot h + \gamma \cdot fh \succsim \alpha \cdot g + \beta \cdot h + \gamma \cdot gh$*

A direct but important consequence of wGI is that, when two lotteries are indifferent to each other, the one can be replaced by the other in any composition.

Proposition 1 (Substitution). *If wGI holds, then for any evidential lotteries f and g such that $f \sim g$, and any h, $h \sim h_{f \leftarrow g}$*
where $h_{f \leftarrow g}$ is the compound lottery in which f is replaced by g, i. e. $h_{f \leftarrow g}(A) = h(A)$ if $g \notin A$ $h_{f \leftarrow g}(A) = h(A) + h(A \cup \{f\})$ if $g \in A$

The Von Neumann's and Morgenstern's independence requirement moreover requires that if $\alpha > 0$ then $f \succ g \Rightarrow \alpha \cdot f + \beta \cdot h + \gamma \cdot fh \succ \alpha \cdot g + \beta \cdot h + \gamma \cdot gh$ - one recognizes here the principle of independence proposed by Jensen [18] and used by Jaffray [16] in his axiomatization of the linear utility. We shall use the following generalization of Von Neumann and Morgernstern's axiom:

Axiom 4 (Generalized Independence (GI)). *For any evidential lotteries f, g, h, any $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$:*

(i) $f \succsim g \Rightarrow \alpha \cdot f + \beta \cdot h + \gamma \cdot fh \succsim \alpha \cdot g + \beta \cdot h + \gamma \cdot gh$ *(wGI)*
(ii) If $\alpha > 0$ *then* $f \succ g \Rightarrow \alpha \cdot f + \beta \cdot h + \gamma \cdot fh \succ \alpha \cdot g + \beta \cdot h + \gamma \cdot gh$

Generalized Independence is fundamental in the context of special decision since it guaranty the dynamic consistency and dynamic programming [20].

Lottery Reduction. A fundamental notion for the comparison of compound lotteries is the equivalence of a compound lottery and its reduction; comparing compound lotteries then amounts at comparing their reductions. By construction, all the rules presented in the previous Section do satisfy lottery reduction

Axiom 5 (Lottery Reduction (LR)). *For any evidential compound lottery* $l \sim Red_l$

As soon as GI holds, the substitution property applies, and the axiom of lottery reduction implies that any compound lottery can be replaced by its reduction: for any two lotteries f, h, $h \sim h_{f \leftarrow Red_f}$.

The previous axioms obviously imply a form of monotony w. r. t. ambiguity: an ambiguity between a decision and another cannot be better than getting the best one for sure, or worst that getting the worst one for sure.

Proposition 2 (Monotony w. r. t. ambiguity). *If LR, GI and A1 holds, then,*

(i) for any f, g *such that* $f \succsim g$, $f \succsim 1 \cdot fg \succsim g$
(ii) for any $x_1, x_2 \in \mathcal{X}$ *such that* $\{x_1\} \succsim \{x_2\}$, $\{x_1\} \succsim \{x_1, x_2\} \succsim \{x_2\}$.

Proof. By GI, $f \succsim g$ implies $1 \cdot fg \succsim 1 \cdot gg$. By LR, $1 \cdot gg \sim g$ - so by transitivity $1 \cdot fg \succsim g$. Similarly $f \succsim g$ implies by GI $1 \cdot ff \succsim 1 \cdot fg$ and LR and A1 then imply $f \succsim 1 \cdot fg$. Item (ii) is a particular case item (i), setting $f = \{x_1\}, g = \{x_2\}$

Certainty Equivalence. The notion of certainty equivalence is often used in decision theory, in particular for the elicitation of the utility functions and uncertainty measures. The certainty equivalent of a decision is a constant act that identifies the certain amount that generates indifference to a given decision. When considering ambiguous acts, and typically set-decisions, we shall require that any such act admits a certainty equivalent.

Axiom 6 (Restricted certainty equivalent (RCE)).
For any non empty subset A *of* $\mathcal{X}$, $\exists x \in \mathcal{X}$ *such that* $A \sim x$

Compensation. The GI axiom ensures dynamical consistency when comparing lotteries. It is not obeyed by the Choquet integral (see Example 4). Nevertheless the Choquet integral ensures a form of compensation under uncertainty, classically captured by the so-called "continuity" axiom [18].

Axiom 7 (Continuity). *For any evidential lotteries* f, g *and* h *such that* $f \succ g \succ h$ *there exists* $\lambda, \theta \in (0, 1)$ *such that* $\lambda f + (1 - \lambda)h \succ g \succ \theta f + (1 - \theta)h$

This axiom, proposed by Jensen [18] in its axiomatization of expected utility and taken up by Jaffray [16] in his axiomatization of a linear utility theory of belief functions, claims that a bad lottery h can always be compensated by a good lottery f ($\lambda f + (1 - \lambda)h \succ g$) and a good lottery f can be deteriorated by a bad one ($g \succ \theta f + (1-\theta)h$). This axiom states in particular that there exists neither a probability distribution that is infinitely desirable nor a probability distribution that is undesirable.

The continuity axiom expresses a form of compensation under probabilistic uncertainty (under risk). We shall finally consider compensation under ambiguity: an ambiguity between two possible outcomes is always better than getting the worst of them for sure, but worse than getting the best of them for sure.

Axiom 8 (Compensation under ambiguity (C)).
If $\{x_1\} \succ \{x_2\}$ *then* $\{x_1\} \succ \{x_1, x_2\} \succ \{x_2\}$

This axiom strengthens monotony w. r. t. ambiguity, which is a natural consequence of A1, GI and LR.

Most rules encountered in the literature generally the axiom of continuity (e. g. Jaffray's and the Choquet integral; this axiom is one on the stone edges of Jaffray's characterization), but are not fully compensatory with respect to ambiguity and either relax the axiom of independence, or the composition of lotteries. This suggests that there may be some range of incompatibility between theses properties as soon as ambiguous compound lotteries are allowed.

The main results of this paper is that as soon as a transitive rule based on lottery reduction applies on evidential lotteries without any restriction, it cannot satisfies both Generalized Weak Independence Axiom and the Compensation Axiom and/or Continuity and allow more than two distinct consequences

Theorem 1. *If there exists two distinct consequences in $\mathcal{X}$ which are not equivalent for $\succsim$, then A1, wGI, LR, RCE and C are inconsistent.*

Proof. Suppose that $\exists x_1, x_3 \in \mathcal{X}$ such that $\{x_1\} \succ \{x_3\}$. Then by C, $\{x_1\} \succ \{x_1, x_3\} \succ \{x_3\}$. By CE, there exists a x_2 such that $\{x_1, x_3\} \sim \{x_2\}$. So, $\exists x_1, x_2, x_3 \in \mathcal{X}$ such that $x_1 \succ x_2 \succ x_3$

- Suppose first $\{x_2\} \sim \{x_1, x_3\}$. By C, $\{x_1\} \succ \{x_2\}$ implies $\{x_1\} \succ \{x_1, x_2\} \succ \{x_2\}$. Moreover $\{x_2\} \sim \{x_1, x_3\}$ implies $1 \cdot \{x_1\} \cdot \{x_2\} \sim 1 \cdot \{x_1\} \cdot \{x_1 x3\}$ by wGI. By LR, $1 \cdot x_1, \{x_1 x3\} \sim \{x_1, x_3\}$ and $1 \cdot \{x_1\} \cdot \{x_2\} \sim \{x_1, x_2\}$. By substitution (which derives from wGI and A1) $\{x_1, x2\} \sim \{x_1, x_3\}$.
  From $\{x_2\} \sim \{x_1, x_3\}$ and A1, we also get $\{x_1, x_2\} \sim \{x_2\}$ which contradicts $\{x_1\} \succ \{x_1, x_2\} \succ \{x_2\}$.
- Suppose now that $\{x_2\} \succ \{x_1, x_3\}$. By wGI, we get $1 \cdot \{x_2\}.\{x_2, x_3\} \succsim 1 \cdot \{x_1, x_3\}, \{x_2, x_3\}$. By LR, $1 \cdot \{x_2\}.\ \{x_2, x_3\} \sim \{x_2, x_3\}$ and $1 \cdot \{x_1, x_3\} \cdot \{x_2, x_3\} \sim \{x_1, x_2, x_1\}$. By substitution $\{x_2, x_3\} \succsim \{x_1, x_2, x_3\}$.
  But by C $\{x_2\} \succ \{x_2, x_3\}$ and thus, because $\{x_1\} \succ \{x_2\}$ we get $\{x_1\} \succ \{x_2, x_3\}$ by A1; by C again, $\{x_1\} \succ \{x_2, x_3\}$ implies $1 \cdot \{x_1\} \cdot \{x_2, x_3\} \succ \{x_2, x_3\}$ - LR and substitution, then imply $\{x_1, x_2, x_3\} \succ \{x_2, x_3\}$
- Let us finally suppose that $\{x_1, x_3\} \succ \{x_2\}$. On the one hand by wGI we get $1 \cdot \{x_1, x_2\} \cdot \{x_1, x_3\} \succsim 1 \cdot \{x_2\} \cdot \{x_1, x_2\}$; by LR, $1 \cdot \{x_1, x_2\} \cdot \{x_1, x_3\} \sim \{x_1, x_2, x_3\}$ and $1 \cdot x_2 \cdot \{x_1, x_2\} \sim \{x_1, x_2\}$. So, by substitution $\{x_1, x_2, x_3\} \succsim \{x_1, x_2\}$
  By C we have $\{x_1, x_2\} \succ \{x_2\}$ and $\{x_2\} \succ \{x_3\}$, so by A1, $\{x_1, \{x_2\}\} \succ \{x_3\}$.

By CE, there exists a x such that $x \sim \{x_1, x_2\}$, so by A1 $x \succ \{x_3\}$. By C, this impies $x \succ \{x, x_2\}$. By substitution, we get $\{x\} \succ 1 \cdot \{x_1, x_2\}\{x_3\}$. By LR, $1 \cdot \{x_1, x_2\}\{x_3\} \sim \{x_1, x_2, x_3\}$. Thus by A1, $\{x\} \succ \{x_1, x_2, x_3\}$. Since $\{x\} \sim \{x_1, x_2\}$, by A1, we get $\{x_1, x_2\} \succ \{x_1, x_2, x_3\}$, which contradicts $\{x_1, x_2, x_3\} \succsim \{x_1, x_2\}$

So neither $x_2 \succsim \{x_1 x_3\}$ nor $\{x_1, x_3\} \succsim \{x_2\}$ which contradicts axiom A1.

Theorem 2. *If there exists a group of at least four consequences in $\mathcal{X}$ which are pairwise distinct for $\succsim$, then A1, wGI, LR and C are inconsistent.*

Proof (Theorem 2). Let $\{x_1\} \succ \{x_2\} \succ \{x_3\} \succ \{x_4\}$. Let us observe that, thanks to the continuity axiom $\lambda\{x_1\}+(1-\lambda)\{x_3\} \succ \{x_2\} \succ \theta\{x_1\}+(1-\theta)\{x_3\}$. Then:

- By LR $1 \cdot \lambda\{x_1\} + (1-\lambda)\{x_3\} \cdot \{x_1, x_3\} \sim \{x_1, x_3\} \sim 1 \cdot \theta\{x_1\} + (1-\theta)\{x_3\} \cdot \{x_1, x_3\}$ and $1 \cdot \{x_2\} \cdot \{x_1, x_3\} \sim \{x_1, x_2, x_3\}$. By wGI and A1 we have $\{x_1, x_3\} \sim \{x_1, x_2, x_3\}$
- By LR $1 \cdot \lambda\{x_1\} + (1-\lambda)\{x_3\} \cdot \{x_2, x_3\} \sim \lambda\{x_1, x_2, x_3\} + (1-\lambda)\{x_2, x_3\}$, $1 \cdot \theta\{x_1\} + (1-\theta)\{x_3\} \cdot \{x_2, x_3\} \sim \theta\{x_1, x_2, x_3\} + (1-\theta)\{x_2, x_3\}$ and $1 \cdot x_2 \cdot \{x_2, x_3\} \sim \{x_2, x_3\}$. By substitution $\{x_1, x_3\} \sim \{x_1, x_2, x_3\}$. wGI and A1then imply $\lambda\{x_1, x_3\} + (1-\lambda)\{x_2, x_3\} \succsim \{x_2, x_3\} \succsim \theta\{x_1, x_3\} + (1-\theta)\{x_2, x_3\}$. Since $\lambda, \theta \in (0, 1)$ we get $\{x_1, x_3\} \succsim \{x_2, x_3\}$ and $\{x_2, x_3\} \succsim \{x_1, x_3\}$. Hence by A1 we have $\{x_2, x_3\} \sim \{x_1, x_3\} \sim \{x_1, x_2, x_3\}$
- By LR $1 \cdot \lambda\{x_1\} + (1-\lambda)x_3 \cdot \{x_1, x_2\} \sim \lambda\{x_1, x_2\}\{+(1-\lambda)\{x_1, x_2, x_3\}$, $1 \cdot \theta x_1 + (1-\theta)x_3 \cdot \{x_1, x_3\} \sim \theta\{x_1, x_3\} + (1-\theta)\{x_1, x_2, x_3\}$ and $1 \cdot x_2 \cdot \{x_1, x_2\} \sim \{x_2, x_3\}$. By substitution $\{x_2, x_3\} \sim \{x_1, x_2, x_3\}$, wGI and A1 we have $\lambda\{x_1, x_2\} + (1-\lambda)\{x_2, x_3\} \succsim \{x_1, x_2\} \succsim \theta\{x_1, x_2\} + (1-\theta)\{x_2, x_3\}$. Since $\lambda, \theta \in (0, 1)$ we get $\{x_2, x_3\} \succsim \{x_1, x_2\}$ and $\{x_1, x_2\} \succsim \{x_2, x_3\}$. Hence by A1 we have $\{x_1, x_2\} \sim \{x_2, x_3\} \sim \{x_1, x_3\} \sim \{x_1, x_2, x_3\}$
- By Proposition 2, $\{x_1\} \succ \{x_2\} \succ \{x_3\}$ implies $\{x_1, x_2\} \succsim \{x_2\}$ and $\{x_2\} \succsim \{x_2, x_3\}$; but the previous point has shown that $\{x_1, x_2\} \sim \{x_2, x_3\}$ - hence, by A1, $\{x_2\} \succsim \{x_2, x_3\}$. Using A1 we have $\{x_1\} \succ \{x_2\} \sim \{x_1, x_2\} \sim \{x_2, x_3\} \sim \{x_1, x_3\} \sim \{x_1, x_2, x_3\}$

We thus get $\{x_2\} \sim \{x_2, x_3\}$. Applying the same reasoning to x_2, x_3 and x_4 we obtain that $\{x_2\} \succ \{x_3\} \sim \{x_2, x_3\} \sim \{x_3, x_4\} \sim \{x_2, x_4\} \sim \{x_2, x_3, x_4\}$ - hence $\{x_3\} \sim \{x_2, x_3\}$

From $\{x_2\} \sim \{x_2, x_3\}$ and$\{x_3\} \sim \{x_2, x_3\}$, we get by A1 $\{x_2\} \sim \{x_3\}$ which contradicts $\{x_2\} \succ \{x_3\}$

The decision rules proposed in the literature escape the impossibility theorems in some ways: either by restricting the composition of lotteries (like Jaffray's approach), by relaxing the axiom of independence (like the pignistic and Choquet approaches), or the by relaxing axiom of completeness as done by Denoeux and Shenoy. The above impossibility theorems justify these approaches: it shows for instance that relaxing completeness is a way to keep the other axioms - and especially continuity.

4 A Dominance-based Rule for the Comparison of Evidential Lotteries

The question is then to determine whether there is a way to satisfy the principle of independence without dropping Axiom A1 or lottery reduction nor restricting either type of lottery considered nor their composition of lotteries to the linear case. The answer is actually yes - in this section, we provide a cautious rule which applies on evidential lotteries without any restriction, and satisfies the above principles.

Consider a set of consequences $\mathcal{X}$ equipped with a complete and transitive preference relation $\geq$ and let $f \geq x$ be the event "f provides a consequence as least as good as x". For a simple lottery, $Bel(f \geq x) = \Sigma_{A, \forall y \in A, u(y) \geq u(x)}\ m(A)$ measures to what extent the DM is certain to reach as least a consequence as good as x; for a compound, let $Bel(f \geq x) = Bel(Red_f \geq x)$. The cumulative belief vector for f is thus $\boldsymbol{f} = (Bel(f \geq x))_{x \in \mathcal{X}})$

The rule is based on the lexicographic comparison of the cumulative belief vectors. Recall that for any two real vectors of same length $a \succsim_{lexi} b$ iff there exists a i^* such that for any $i < i^*$, $a_i = b_i$ and $a_{i^*} > b_{i^*}$

Definition 5 (Lexi-Bel dominance rule).

$$f \succsim_{lexi_{Bel}} g \text{ iff } (Bel(f \geq x))_{x \in \mathcal{X}}) \succsim_{lexi} (Bel(g \geq x))_{x \in \mathcal{X}})$$

i. e. $f \succsim_{lexi_{Bel}} g$ iff $\exists x^* \in \mathcal{X}$ s. t. $\forall y < x^*, Bel(f \geq y) = Bel(g \geq y)$ and $Bel(f \geq x^*) > Bel(g \geq x^*)$

Example 5. Let us consider again the lotteries at work in Ellsberg's paradox. For each of the four lotteries, the probability of getting at least the worst consequence is obviously equal to 1. Moreover: $Bel(A \geq 100) = \frac{1}{3}$, $Bel(B \geq 100) = 0$, $Bel(C \geq 100) = \frac{1}{3}$, $Bel(D \geq 100) = \frac{2}{3}$. The cumulative vectors are $\boldsymbol{A} = (1, \frac{1}{3})$, $\boldsymbol{B} = (1, 0)$, $\boldsymbol{C} = (1, \frac{1}{3})$, $\boldsymbol{D} = (1, \frac{2}{3})$. Thus $A \succ_{lexi_{Bel}} B$ and $D \succ_{lexi_{Bel}} C$

Proposition 3. $\succsim_{lexi_{Bel}}$ *is complete, transitive and satisfies axioms LR, RCE and GI.*

Proof.

- A1 is obeyed since the lexi comparison of vectors is complete and transitive.
- Lottery reduction is satisfied by construction.
- Consider any $A \subseteq \mathcal{X}$ and the consequence $\underline{a} = minA$ It holds that $Bel(A \geq x) = 1$ if $x \geq \underline{a}$ and $Bel(A \geq x) = 0$ if $x > \underline{a}$. In the same way, $Bel(\underline{a} \geq x) = 1$ if $x \geq \underline{a}$ and $Bel(\underline{a} \geq x) = 0$ if $x > \underline{a}$. So, the set decision A and the constant decision $\{(\underline{a})\}$ have the same cumulative vector. Hence they are equivalent for the lexi bel decision rule. Restricted CE thus holds.
- As to GI, it holds that for any h, f

$Bel(Reduction(\alpha f + \beta h + \gamma f h) \geq x)$
$(a) \quad = \alpha \sum_{B \subseteq [x,+\infty[} f(B) + \beta \sum_{B \subseteq [x,+\infty[} h(B)$
$\quad + \gamma \sum_{B \subseteq [x,+\infty[} \sum_{C \cup D = B} f(C)h(D)$
$(b) \quad = \alpha Bel(f \geq x) + \beta Bel(h \geq x) + \gamma \cdot (Bel(f \cup h \geq x))$
$(c) \quad = \alpha Bel(f \geq x) + \beta Bel(h \geq x) + \gamma \cdot Bel(f \geq x) \cdot Bel(h \geq x)$
Step (a) follows from definition of the reduction and the definition of the Bel measure. Step (b) follows from the definition of the union of two mass functions ($(f \cup h)(B) = \sum_{C \cup D = B} f(C)h(D)$). Then step (c) is based on Prop. 2 in [7].

Suppose that $f \sim_{lexi_{Bel}} g$, i. e. that $Bel(f \geq x) = Bel(g \geq x)\ \forall x$; Then for each x, $\alpha \cdot Bel(f \geq x) + \beta \cdot Bel(h \geq x) + \gamma Bel(f \geq x) \cdot Bel(h \geq x) = \alpha \cdot Bel(g \geq x) + \beta \cdot Bel(h \geq x) + \gamma Bel(g \geq x) \cdot Bel(h \geq x)$, i. e. the cumulative vectors of the two compound lotteries are equal - so $\alpha f + \beta h + \gamma f h \sim_{lexi_{Bel}} \alpha g + \beta h + \gamma g h$.

Suppose that $f \succ_{lexi_{Bel}} g$, i. e. that there exists a x^* such that $\forall x < x^*, Bel(f \geq x) = Bel(g \geq x)$ and $Bel(f \geq x^*) > Bel(g \geq x^*)$ For each $x < x*$, $Bel(\alpha f + \beta h + \gamma f h \geq x) = Bel(\alpha g + \beta h + \gamma g h \geq x)$ as previously. Moreover, $Bel(f \geq x*) > Bel(g \geq x*)$. So, if $\alpha > 0$ or $Bel(h \geq x*) > 0$ the value of $Bel(\alpha f + \beta h + \gamma f h \geq x^*)$ is strictly greater than the value of $Bel(\alpha g + \beta h + \gamma g h \geq x^*)$. The first compound lottery is strictly preferred to the second one. When $\alpha = 0$ and $Bel(h \geq x*) = 0$ $Bel(\alpha f + \beta h + \gamma f h \geq x) = 0$ and $Bel(\alpha g + \beta h + \gamma f g \geq x) = 0$ for each $x \geq x^*$, i. e. the two compound lotteries are equivalent for $\sim_{lexi_{Bel}}$
So, if $f \succsim_{lexi_{Bel}} g$ the first compound lottery is weakly preferred to the second one and $\alpha > 0$, $f \succ_{lexi_{Bel}} g$ leads to a strict preference: GI is obeyed.

So, the lexi-Bel rule do satisfy the axioms looked for - this proves their compatibility. This rule is nevertheless very pessimistic. Beyond its characterization, further work include the proposition of more optimistic rules, e.g. using the principles that we have developed in the possibilistic context [9].

5 Conclusion

This paper shows that the axioms of lottery reduction, completeness and transitivity of the ranking, and independence (and thus, dynamical consistency) can be compatible when considering evidential lotteries in their full generality, but this supposes to reject the compensation principles, and in particular the continuity axiom and the principle of compensation under ambiguity. The impossibility theorems presented in this paper provide a first step toward the handling of independence in decision with belief functions. The next step is to consider more general lotteries, based on families of probabilities ("credal lotteries"). It is worth noticing that, unless in particular cases, the rules proposed for imprecise probabilities in their full generality either relax completeness or are dynamically inconsistent: Troffaes and Huntley [15] for instance relax completeness. Miranda and Zaffalon [28] preserve it for particular lower probabilities. Further work includes the investigation of an impossibility theorem in the credal context.

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