Hyperintensionality in Relevant Logics

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Abstract. In this article, we present a definition of a hyperintensionality appropriate to relevant logics. We then show that relevant logics are hyperintensional in this sense, drawing consequences for other non-classical logics, including HYPE and some substructural logics. We further prove results concerning extensionality in relevant logics. We close by discussing related concepts for classifying formula contexts and potential applications of these results.

1 Introduction

Hyperintensionality, being able to distinguish necessarily equivalent formulas, has become an important topic in philosophical logic.[1] The growing importance of hyperintensionality for philosophical concepts has been highlighted by Nolan [24], calling it the "hyperintensional revolution." One can, of course, extend classical logic with hyperintensional operators,[2] but one might wonder whether other logics could offer something distinctive with respect to hyperintensional operators. Recently, Leitgeb [19] defended the non-classical logic HYPE as exhibiting a distinctive combination of simplicity and strength. Among its claimed features is providing a kind of hyperintensionality, a claim disputed by Odintsov and Wansing [25]. We will offer some support to Leitgeb's claim, proceeding via a discussion of relevant logics. Given some of the distinctions that relevant logics draw, such as distinguishing logical truths, it is natural to suspect that relevant logics build in a kind of hyperintensionality. We will argue that this suspicion is borne out by providing some hyperintensional contexts in relevant logics. In so doing, we will draw out some consequences for HYPE and other substructural logics.

In the remainder of this section, we will supply some brief background on relevant logics, in particular the logic R. Then, we will precisely define some concepts

1 See Berto and Nolan [4].

2 Some of the standard examples of hyperintensional operators added to classical logic, often though not always modeled using impossible worlds, include belief operators, knowledge operators, and conditional operators. See Wansing [40], Alechina and Logan [1], and Berto et al. [5], among others, for recent examples, and see Berto and Jago [6, ch. 7] for an overview of the work on epistemic logics. For a general approach to hyperintensional operators, see Sedlár [31].

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to classify formula contexts in Sect. 2, notably extensionality and hyperintensionality. In Sect. 3, we will present our main results concerning hyperintensional contexts in relevant logics, drawing out a consequence for HYPE. Finally, in Sect. 4, we will look at two further definitions for classifying formula contexts and discuss some upshots of our results.

Relevant logics are a family of non-classical logics with a distinctive conditional, or implication, connective.[3] One of the important ways in which the relevant conditional is distinctive can be found in Belnap's variable sharing criterion: If $A \to B$ is valid, then A and B share a propositional variable. The variable sharing criterion is typically taken as a necessary condition on being a relevant logic. We will focus on the standard logical vocabulary of $\{\to, \wedge, \vee, \neg\}$, considering the addition of a modal operator $\Box$, below. The biconditional, $A \leftrightarrow B$, will be defined as $(A \to B) \wedge (B \to A)$. To contrast the relevant conditional and biconditional with the classical material ones, we will use $\supset$ and $\equiv$ for the latter connectives, defining $A \supset B$ as $\neg A \vee B$ and $A \equiv B$ as $(A \supset B) \wedge (B \supset A)$. In the context of relevant logics, and generally any non-classical logic, $A \supset B$ and $A \equiv B$ will be defined as in classical logic.

While there are many relevant logics, our focus will be on the logic R. R is a relatively strong logic. We will present the axioms and rules for it, where $\Rightarrow$ is used to demarcate premises from conclusion in the rules.

(1) $A \to A$
(2) $(A \wedge B) \to A$, $(A \wedge B) \to A$
(3) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(4) $A \to (A \vee B)$, $B \to (A \vee B)$
(5) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(6) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(7) $\neg\neg A \to A$
(8) $(A \to \neg B) \to (B \to \neg A)$
(9) $(A \to B) \to ((B \to C) \to (A \to C))$
(10) $A \to ((A \to B) \to B)$
(11) $(A \to (A \to B)) \to (A \to B)$
(12) $A, A \to B \Rightarrow B$
(13) $A, B \Rightarrow A \wedge B$

The logic R is the least set of formulas containing all the axioms and closed under the rules. Other relevant logics can be obtained by variation of axioms (8)–(11), dropping those axioms or possibly adding others, and by addition of other rules. The focus will be on R, although we will briefly consider some weaker relevant logics towards the end of Sect. 3. Let us now turn to some concepts for classifying formula contexts.

2 Classifying Contexts

Let us begin with some definitions. Following Williamson [41], define a formula context as a pair (C, p), of a formula and an atom. Given a context (C, p), the formula $C(A)$ is what results by replacing every occurrence of p in C with the formula A.

[3] See Dunn and Restall [10], Bimbó [7], or Mares [20] for overviews of the area. See Anderson and Belnap [2] and Routley et al [27] for broader discussions.

Definition 1 (Extensionality). *A formula context* (C, p) *is* extensional *iff for all formulas* A *and* B*,*

– $\models (A \equiv B) \supset (C(A) \equiv C(B))$

This is a fine definition of extensionality for classical logic and its extensions. It is not, however, appropriate for all non-classical logics. The reason is that in many non-classical logics, including relevant logics, the interest is on the primitive conditional connective, and the associated biconditional, rather than the material conditional of the logic, and the associated material biconditional.[4] Therefore, we will replace the definition of extensional context with one that uses the appropriate conditional and biconditional of the logic.

Definition 2 (Extensionality in L). *A formula context* (C, p) *is* extensional *in the logic* L *iff for all formulas* A *and* B*,*

– $\models_{\mathsf{L}} (A \leftrightarrow B) \rightarrow (C(A) \leftrightarrow C(B))$,

where $\models_{\mathsf{L}}$ *is the consequence relation of* L*.*

This is a natural adaptation of Williamson's definition to a non-classical context. For a more general study of extensionality and related concepts, we would need to make the relativity to the chosen conditional and biconditional explicit, so that the two options above would be $(\supset, \equiv)$-extensionality and $(\rightarrow, \leftrightarrow)$-extensionality, respectively. There are alternative definitions of extensionality using different combinations of $\rightarrow$, $\supset$, $\leftrightarrow$, and $\equiv$, but we won't explore those further here.[5] Our interest is not on extensional contexts per se, although we will return to them at the end of the next section. Our interest is, rather, in their use in the definition of non-hyperintensional contexts.

Definition 3 (Non-hyperintensionality, hyperintensionality). *A formula context* (C, p) *is* non-hyperintensional *in* L *iff for all formulas* A *and* B*,*

– $\models_{\mathsf{L}} \Box(A \leftrightarrow B) \rightarrow \Box(C(A) \leftrightarrow C(B))$.

A formula context is hyperintensional *in* L *iff it is not non-hyperintensional.*

A logic L *is hyperintensional iff there is a formula context* (C, p) *that is hyperintensional in* L*.*

Unpacking the definitions, a formula context (C, p) is *hyperintensional* iff there are formulas A and B such that $\not\models_{\mathsf{L}} \Box(A \leftrightarrow B) \rightarrow \Box(C(A) \leftrightarrow C(B))$. An immediate consequence of the definitions is the following proposition.

Proposition 4. *Let* M *be a sublogic of* L*. If* L *is hyperintensional, then so is* M*.*

4 In the context of relevant logics, many of the contraction-free logics lack any theorems not containing '$\rightarrow$', for which see Slaney [32]; so $(\supset, \equiv)$-extensionality will be a less useful concept there. Yet, it still seems sensible to say that those logics have some extensional contexts made up only of the vocabulary $\{\wedge, \vee, \neg\}$. Thanks to an anonymous referee for raising this point.

5 See Humberstone [15,16] and [17, 455] for more on extensionality of connectives.

Hyperintensionality is preserved downwards to sublogics. This will be important for our main result.

Before we proceed, it is worth noting an important intermediate category of formula contexts that we will not discuss below, namely the *intensional contexts*. These are contexts that are not extensional but are non-hyperintensional. Investigation of intensional contexts will be left for future work.

3 Hyperintensionality

Although there are many relevant logics, we will focus on the logic R, which is the strongest of the standard relevant logics.[6] The definitions of extensional, non-hyperintensional, and hyperintensional contexts should be understood as indexed to R, and its modal extensions, with the displayed conditional and biconditional being those of R. One could obtain versions of the definitions for other logics by changing the index.

Once we have settled the question of the base logic, there is a further question concerning which necessity to use in the statement of non-hyperintensionality. For a general study of hyperintensionality, care needs to be taken regarding what modal axioms, if any, should be required to ensure that the non-hyperintensionality definition yields satisfactory results. Williamson uses the necessity of S5 in stating his definition. The necessity of S5 would be a fine necessity for our purposes, but we can obtain stronger results with a different necessity.[7] A logic being hyperintensional is a matter of the invalidity of an instance of the non-hyperintensionality scheme, and, since invalidity is preserved from stronger logics down to weaker logics, using stronger modal principles will give stronger results concerning hyperintensionality. To motivate the appropriate modal principles, we will take a detour through logical necessity.

Anderson and Belnap showed how to define logical necessity in their logic E, a close relative of R, obtained by changing axiom (10) to its rule form, $A \Rightarrow (A \to B) \to B$, and adding a reductio axiom, $(A \to \neg A) \to \neg A$. Anderson and Belnap define $\Box A$ as $(A \to A) \to A$.[8] This can be understood as saying that logic implies A, which is a fair definition of logical necessity. In the context of E, $\Box$, so defined, has an S4-ish logic, and in the context of weaker relevant logics, it obeys weaker principles. In the context of R, however, the defined connective $\Box$ is trivial in the sense that $A \leftrightarrow \Box A$ is a logical truth. Taking this biconditional as a logic's modal axioms gives the modal logic known as TRIV. We will call the extension of R with the TRIV biconditional the logic R.TRIV. While the necessity of R.TRIV is not plausible as a kind of logical necessity, it is useful for the sort of

[6] See Mares [23] for defense of R.

[7] The concept of S5 necessity exhibits some subtleties in the context of relevant logics, for which see Standefer [36].

[8] One can obtain an alternative definition by using the Ackermann truth constant, t, which is glossed as the conjunction of all logical truths. Using the Ackermann constant, $\Box A$ can be defined as $t \to A$. The equivalence of the two definitions is demonstrated by Mares and Standefer [21], among others.

negative results we are after, so we will use it as the necessity in the definitions of non-hyperintensionality and hyperintensionality.

To obtain our main result, namely that many plausible modal extensions of R are hyperintensional, we first prove a lemma using matrix methods. A matrix has a set V of semantic values, with a subset of designated values $D \subseteq V$, and operations on V for interpreting each connective of the language. A valuation v is a function from atoms to V that is extending to the whole language using the operations of the matrix. A valuation v on a matrix is a counterexample to a formula A iff $v(A) \notin D$.

Lemma 5. *The formula* $(p \leftrightarrow q) \to ((p \wedge r) \leftrightarrow (q \wedge r))$ *is not a theorem of* R.

Proof. We will use a three-valued matrix. For the set of values, V, we take $\{0, \frac{1}{2}, 1\}$, with $D = \{\frac{1}{2}, 1\}$. The value of complex formulas is computed using the following tables.

$\to$	0	$\frac{1}{2}$	1	$\neg$
0	1	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	0

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

A valuation v is a countermodel for a formula A iff $v(A) = 0$, which is to say that $v(A)$ is not designated.

Every axiom of R is designated on every valuation and the rules preserve designation.[9] By an inductive argument, this implies that every theorem of R receives a designated value. To show that a formula is not a theorem of R, it suffices to provide a valuation that assigns it 0. In the case of interest, $v(p) = 1$, $v(q) = 1$, and $v(r) = \frac{1}{2}$ will work.[10] This valuation gives $v(p \leftrightarrow q) = 1$, while $v((p \wedge r) \leftrightarrow (q \wedge r)) = \frac{1}{2}$. As $1 \to \frac{1}{2} = 0$,

$$v((p \leftrightarrow q) \to ((p \wedge r) \leftrightarrow (q \wedge r))) = 0,$$

as desired.

The formula scheme $(A \leftrightarrow B) \to ((A \wedge C) \leftrightarrow (B \wedge C))$ is not a theorem of R.[11] With this result in hand, we can turn to our main result.

Theorem 6. *The logic* R.TRIV *is hyperintensional.*

Proof. To show that R.TRIV is hyperintensional, we need a formula context which is hyperintensional. Take the formula context $(s \wedge r, s)$. The formula

$$\Box(p \leftrightarrow q) \to \Box((p \wedge r) \leftrightarrow (q \wedge r))$$

9 This was shown by Robert Meyer. See Anderson and Belnap [2, 470].

10 This countermodel was found using John Slaney's program MaGIC. See https://users.cecs.anu.edu.au/~jks/magic.html.

11 Axioms of this form were studied by Routley et al [27, 345] and by Urbas and Sylvan [38]. Thanks to Andrew Tedder for drawing my attention to these citations.

is not valid in R.TRIV. This is because we can use the fact that $A \leftrightarrow \Box A$ to focus on the equivalent

$$(p \leftrightarrow q) \to ((p \land r) \leftrightarrow (q \land r)),$$

which was shown not to be a theorem of R in Lemma 5.

Thus, we have demonstrated that R.TRIV is hyperintensional. It is worth noting that, for similar reasons, $(p \lor r, p)$ is a hyperintensional context as well. As an immediate corollary, we have the following result.

Corollary 7. *Let* L *be any sublogic of* R.TRIV*. Then* L *is also hyperintensional.*

The sublogics of R.TRIV include all the well-known relevant logics, such as T, E, and B, as well as (multiplicative, additive) linear logic, and further it includes many of their extensions with well-known modal principles. We can extend a base logic L with a non-trivial, primitive necessity operator, $\boxdot$, rather than a defined one. However, as long as L is a sublogic of R, we can, in many cases of interest, embed the result into R.TRIV using the embedding $\tau(\boxdot A) = \Box\tau(A)$, i.e. $\tau(\boxdot A) = (\tau(A) \to \tau(A)) \to \tau(A)$, provided the modal principles for $\boxdot$ are among those of TRIV. For such logics, the countermodel above will suffice to demonstrate hyperintensionality, setting $v(\boxdot A) = v(A)$.

There are modal logics that are not sublogics of TRIV, although the majority of the philosophically significant ones are sublogics of TRIV. Perhaps the most prominent modal logics that are not sublogics of TRIV are provability logics, logics that include the axiom $\Box(\Box A \to A) \to \Box A$.[12] These have not been studied much in the context of relevant logics, although Mares [22] is an exception, studying a provability logic extension of R. Although the above countermodel does not work for Mares's provability logic, the same invalid formula demonstrates that the logic is hyperintensional. For other modal logics that are not sublogics of R.TRIV, it is left open whether they are hyperintensional or not.

As noted above, in relevant logics, one can define a logical necessity operator: $\Box A$ is $(A \to A) \to A$. For the logic R, this necessity obeys the TRIV principles, although for weaker base logics, the defined necessity is more like a familiar kind of necessity. Using this definition, we can view relevant logics as modal logics and use the defined necessity in the definition of hyperintensionality. In this sense, R and its sublogics are hyperintensional.

We will observe one additional corollary of Lemma 5.

Corollary 8. *There are contexts that fail to be extensional in* R.

Proof. As the lemma shows, $(s \land r, s)$ fails to be extensional in R.

For similar reasons, $(s \lor r, s)$ also fails to be extensional in R. While it is perhaps not surprising that R, and all of its sublogics, contain non-extensional contexts, it is worth noting that the particular non-extensional contexts provided involve

[12] See Boolos [8] and Verbrugge [39] for more on provability logics.

only conjunction or only disjunction, both typically thought of as extensional.[13] In the context of R, at least, Williamson's definition of extensional context, with $\supset$ and $\equiv$, would say that $(s \wedge r, s)$ is an extensional context, an $(\supset, \equiv)$-extensional context in the nomenclature of the previous section. This is not the case for many of the weaker relevant logics, which is a consequence of the results of Slaney [32].

By contrast, if we consider the set of connectives often described as intensional, or non-extensional, $\{\to, \neg, \circ\}$, where $\circ$ is the fusion connective, we find that they are all extensional.

Proposition 9. *Let (C, p) be a context built from atoms and only the connectives $\to, \neg$, and $\circ$. Then (C, p) is extensional in* R.

Proof. The connective $\circ$ is definable in R as $A \circ B =_{Df} \neg(A \to \neg B)$. The result is then proved by induction on structure of C, which is straightforward using axioms (8) and (11). The inductive hypothesis is that $\models_{\mathsf{R}} (A \leftrightarrow B) \to (D(A) \leftrightarrow D(B))$, for less complex contexts (D, p).

For the conditional case, the context is $(D \to E, p)$. As $(D(A) \to E(A)) \to (D(A) \to E(A))$ is provable by axiom (1), we can prove

$$(A \leftrightarrow B) \to ((A \leftrightarrow B) \to ((D(A) \to E(A)) \to (D(B) \to E(B))))$$

with the two appeals to the inductive hypothesis and some simple transitivity moves available in R. An appeal to axiom (11) then yields half of the desired result. The other half is obtained similarly.

For the negation cases, we use (8) and the desired result follows immediately.

For logics that lack axioms (8), (10), or (11), the proposition may fail. In weaker logics, some contexts built from the connectives $\{\to, \neg, \circ\}$ can fail to be extensional. As we will see shortly, all the standard relevant logics include the rule form of axiom (9) used in the proof. Let us look at some examples of failures of extensionality in logics lacking axioms (8), (10), or (11). The logic RW is obtained from the axiomatization of R by dropping axiom (11).

Proposition 10. *In* RW*, the context $(r \to r, r)$ is not extensional.*

Proof. We leave it to the reader to find a countermodel using MaGIC.

The logic T is obtained from the axiomatization of R by removing (10) and adding $(A \to \neg A) \to \neg A$. In it, fusion is not definable in terms of negation and conditional. Contexts built from fusion fail to be extensional.

Proposition 11. *In* T*, $(p \circ r, p)$ is not extensional.*

Proof. We leave it to the reader to find a countermodel using MaGIC.

Although fusion fails to be extensional, T still enjoys some extensionality similar to that of R.

[13] Cf. Gabbay [13] corollary 21.

Proposition 12. *In* T*, all contexts constructed from the vocabulary* $\{\to, \neg\}$ *are extensional.*

Proof. The negation and conditional cases from the proof from Proposition 9 can be reproduced here, omitting fusion.

It is worth looking at an example of a failure of extensionality for contexts built from negation that can be found in the logic B. The logic B is the weakest relevant logic that is standardly discussed, and it is obtained from R by dropping axioms (8)–(11) and adding the following rules.

- $A \to \neg B \Rightarrow B \to \neg A$
- $A \to B \Rightarrow (C \to A) \to (C \to B)$
- $A \to B \Rightarrow (B \to C) \to (A \to C)$

Some formulas in the basic vocabulary fail to be extensional in B, beyond the examples provided above.

Lemma 13. *In* B*, the formula context* $(\neg p, p)$ *is not extensional.*

Proof. In B,

$$(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$$

is invalid. We can adapt the matrix from the proof of Lemma 5 to show this. We change the set of designated values to $\{1\}$, replace the conditional table with

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

and all valuations on the resulting matrix assign all the theorems of B designated values.[14] The valuation v where $v(p) = 1$ and $v(q) = \frac{1}{2}$ is a counterexample to the target formula.

To obtain HYPE, or at least its logical truths, from R, we add $A \to (B \to A)$ and trade axiom (8) for its rule form, $A \to \neg B \Rightarrow B \to \neg A$. It follows that we can obtain HYPE by adding some axioms to B. B shares with HYPE the feature of having contraposition as a rule but, crucially, *not* as an axiom, which results in the failure of the pertinent instance of the extensionality scheme above. In fact, this example extends to HYPE as well. This provides an example of hyperintensionality in B.TRIV, as the context $(\neg p, p)$ is also hyperintensional in B.TRIV, and so in all sublogics. A similar point holds for HYPE, and in fact, the same matrix demonstrates the failure of extensionality. Thus, HYPE exhibits hyperintensionality in the same sense as relevant logics. With these results in hand, let us turn to some further concepts for classifying formula contexts and some discussion.

[14] This countermodel was found using John Slaney's program MaGIC.

4 Discussion

Odintsov and Wansing [25] adopt an alternative notion of hyperintensionality, using *self-extensionality*,[15] also known as *congruentiality*,[16] which they argue is closer to the suggestions of Cresswell [9]. Adapting their definition to the present setting, a formula context (C, p) is *congruential* (in L) iff for all formulas A and B,

– if $\models_{\mathsf{L}} A \leftrightarrow B$, then $\models_{\mathsf{L}} C(A) \leftrightarrow C(B)$.

A logic is *congruential* iff all formula contexts are congruential in that logic. Relevant logics and their usual modal extensions are congruential, although there are modal extensions which are not congruential.[17]

It is worth distinguishing congruentiality and hyperintensionality for two reasons. First, it is natural to maintain the distinction between (i) claims that are necessarily equivalent but not logically equivalent and (ii) claims that are both necessarily and logically equivalent. One might think that certain truths of mathematics or metaphysics are necessarily, but not logically, equivalent. Second, hyperintensionality builds in a modal element that is absent in congruentiality in the sense that the former, but not the latter requires a modal operator be used in its definition. Third, and relatedly, congruentiality can be given an alternative definition that does not involve object language biconditionals, instead using mutual entailments, but hyperintensionality cannot be given such definition. Both hyperintensionality and congruentiality are important and interesting classifications of formula contexts, so it is worth distinguishing them.

Let us consider one further concept that could be considered for hyperintensionality in the present context. Although the discussion so far has proceeded at the level of *logics*, independent of any models, one could introduce models for relevant logics, such as the ternary relational models,[18] enabling us to talk about the sets of worlds where formulas hold, using $[A]_M$ for the set of worlds where A holds in a model M. We could introduce a singulary modality, the universal modality $\mathbb{U}$, such that $\mathbb{U}A$ holds at a world iff $[A]_M$ is the set of all worlds in the model M. One could then say a formula context (C, p) is $\mathbb{U}$-*hyperintensional* in L iff for some formulas A and B,

– $\not\models_{\mathsf{L}} \mathbb{U}(A \leftrightarrow B) \rightarrow \mathbb{U}(C(A) \leftrightarrow C(B))$.

We'll say a logic is $\mathbb{U}$-*hyperintensional* iff it has a formula context (C, p) that is $\mathbb{U}$-hyperintensional. The logic R is not $\mathbb{U}$-hyperintensional. Since every sublogic of $\mathsf{R.TRIV}$ is hyperintensional, this tells us that the modal principles of $\mathbb{U}$ are not contained in TRIV, which puts it well outside the usual modal logics. Proponents of relevant logics, however, have a reason not to accept $\mathbb{U}$, as it leads to violations

[15] See Wójcicki [42, 342], who uses the term 'selfextensional', Font [12, ch. 7], Avron [3], for example. Thanks to Rohan French and Andrew Tedder for references.

[16] See Humberstone [18, 19], among others.

[17] See Savić and Studer [28] and Standefer [34] for examples.

[18] See Restall [26, ch. 11] for a good introduction to ternary relational models.

of Belnap's variable sharing criterion.[19] Proponents of relevant logics have reason not to accept that connective and to reject this sense of hyperintensionality. It is not the salient sense of hyperintensionality for the proponent of relevant logics.

It is worth pointing out that relevant logics have a feature that is, in some ways, similar in spirit to hyperintensionality. Classical logic is *monothetic* in the sense that for any two logical truths A and B, $A \leftrightarrow B$ is a logical truth.[20] From the point of view of classical logic, there is only a single logical truth. HYPE is also monothetic, replacing the classical biconditional with the biconditional of HYPE, and similarly for intuitionistic logic. Relevant logics are *polythetic* meaning that there are non-equivalent logical truths, that is, there are logical truths A and B such that $A \leftrightarrow B$ is not a logical truth.[21] In relevant logics, one can draw distinctions between logical truths, much as (classical) hyperintensionality allows one to draw distinctions among necessary truths. One can use logical truths, such as $p \to p$ and $q \to q$, to show that the formula context $(s \wedge r, s)$ is hyperintensional. By contrast, any logic that contains the weakening axiom, $A \to (B \to A)$, and where the conditional obeys *modus ponens* will be monothetic.

The results of this paper show that almost all the common modal extensions of relevant logics have hyperintensional contexts. This result extends to HYPE, although the range of such contexts appears more limited there than in R. As one weakens the logic, the range of hyperintensional contexts grows, a feature that extends to HYPE and other substructural logics as well. Hyperintensionality is of interest in a wide range of philosophical applications of logic, such logics of belief and epistemic logics. There is further work to do to see the extent to which the sorts of hyperintensionality identified here has natural application to, say, logics of belief or epistemic logics. A promising direction for future work is to precisely characterize the range of hyperintensional contexts in the different relevant and substructural logics. This will be useful in better understanding the ways in which logical omniscience can fail in non-classical settings.[22] One can, of course, appeal to various modeling techniques used to obtain hyperintensional contexts over classical logic to obtain such contexts in relevant logics. These modeling techniques will likely interact with the natural hyperintensionality of relevant logics in surprising ways.

To summarize, relevant logics are hyperintensional when considering many natural kinds of necessity, in at least one important sense. One can extend relevant logics with a singulary modal operator for universal necessity, $\mathbb{U}$, to obtain another sense of hyperintensionality. Relevant logics fail to be hyperintensional in that sense, although the relevant logician has antecedent reason not to accept

[19] See Standefer [35,36] for discussion.

[20] See Humberstone [17, 231].

[21] This point was also made by Standefer [33], albeit in discussion of justification logics.

[22] See, for example, Sedlár [29,30], Standefer, Shear, and French [37], and Ferenz [11], among others, for some discussion of logical omniscience in non-classical settings. For a contrasting recent discussion of omniscience in the setting of classical logic, see Hawke, Özgün, and Berto [14].

that modality and not to be interested in that sense of hyperintensionality. Relevant logics and their modal extensions are, generally, congruential, so they are not hyperintensional in the sense preferred by Odintsov and Wansing. Nonetheless, we do agree with Odintsov and Wansing's closing suggestion to study non-self-extensional, or non-congruential, operators, as non-classical logics likely have much to contribute in those areas. Despite being congruent, relevant logics are polythetic, which allows them to draw distinctions in ways reminiscent of hyperintensionality.

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