

# A Study of the Radiation-Reaction on a Point Charge that Moves Along a Constant Applied Electric Field in an Electromagnetic Bopp–Landé–Thomas–Podolsky Vacuum



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**Abstract** The relativistic problem of motion of a classical electrical point charge that has been placed between the plates of a charged capacitor and then released from rest is well-posed in Bopp–Landé–Thomas–Podolsky (BLTP) electrodynamics. That theory introduces a single new parameter, Bopp’s  $\varkappa$ , a reciprocal length. The present article concerns the small- $\varkappa$  regime. Radiation-reaction effects on the motion are shown to appear at order  $\varkappa^3$ . It is found that in the initial phase the motion is accurately accounted for by test particle theory, with the inertia determined by the bare mass of the particle. Subsequently, radiation-reaction effects cause substantial deviations from the test particle motion.

## 1 Introduction

In this tribute to Detlef Dürr we focus on a lesser known scientific passion of Detlef, i.e., “lesser known in comparison to Bohmian Mechanics,” and this is the classical electromagnetic radiation-reaction problem. Here is how Detlef once characterized the situation:

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IN REMEMBRANCE OF  
**Detlef Dürr**

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When people realized that there is a problem, quantum physics was invented. Then everyone began to work on quantum physics and eventually the problem was forgotten. But it still exists. (Detlef Dürr, private communication to the senior author, sometime in the mid 1980s.)

The problem, in a nutshell, is this: The symbolic system of equations of Lorentz electrodynamics with point charges is notoriously ill-defined. The energy and momentum densities of the electromagnetic Maxwell–Lorentz fields of a point charge source are not integrable over any neighborhood of the point charge, and also the Lorentz formula for the electromagnetic “self”-force on such a point charge source is ill-defined. More recently [DeHa2016] it was noted that also the Lorentz formula for the electromagnetic force of one point charge source onto another becomes ill-defined after relatively short times.

Undeterred by infinities [PPV2011], physicists have tried to extract the force of radiation-reaction on an accelerated point charge by analyzing the power emitted by it towards  $|s| = \infty$ , per the retarded Liénard–Wiechert fields. Von Laue [Lau1909] obtained the expression  $\frac{2e^2}{3c^3} P_{\mathbf{u}(\tau)}^\perp \cdot \frac{d^2}{d\tau^2} \mathbf{u}(\tau)$  for the Minkowski force four-vector due to the radiation (cf. [Jac1975]); here,  $\tau$  denotes proper time and  $P_{\mathbf{u}(\tau)}^\perp$  the four-projection onto the subspace that is four-orthogonal to the four-velocity  $\mathbf{u}(\tau)$  ( $= \frac{d}{d\tau} \mathbf{q}(\tau)$ ). Its third derivative of the particle’s spacetime location  $\mathbf{q}(\tau)$  has been the cause of consternation. In particular, it vanishes during intervals of constant four-acceleration when the Larmor formula predicts radiation. Moreover, when non-zero, it changes the initial-value problem from second to third order, and almost all solutions then display unphysical behavior. It has been argued that a way out of this “third-order” dilemma is offered by the fact that the radiation-reaction on the particle should cause only a small correction term to the test particle type force. The co-variant version of the equations of test particle motion reads  $\frac{d^2}{d\tau^2} \mathbf{q}(\tau) = \frac{e}{mc} \mathbf{F}(\mathbf{q}(\tau)) \cdot \mathbf{u}(\tau)$ , where  $\mathbf{F}(\mathbf{s})$  is the Faraday tensor of the applied fields at a spacetime point  $\mathbf{s}$ . Hence the expression  $\frac{d^2}{d\tau^2} \mathbf{u}(\tau)$  in von Laue’s four vector should be interpreted as stand-in for the first-order proper time derivative of  $\frac{e}{mc} \mathbf{F}(\mathbf{q}(\tau)) \cdot \mathbf{u}(\tau)$ , which will not have derivatives of  $\mathbf{q}(\tau)$  higher than second order. The so radiation-reaction-corrected equation of motion, sometimes called the Eliezer–Ford–O’Connell (EFC) equation of motion, reads

$$\frac{d^2}{d\tau^2} \mathbf{q}(\tau) = \frac{e}{mc} \mathbf{F}(\mathbf{q}(\tau)) \cdot \mathbf{u}(\tau) + \frac{2}{3} \frac{e^3}{mc^4} P_{\mathbf{u}(\tau)}^\perp \cdot \frac{d}{d\tau} (\mathbf{F}(\mathbf{q}(\tau)) \cdot \mathbf{u}(\tau)). \quad (1.1)$$

Landau and Lifshitz approximated (1.1) further by replacing all first-order proper time derivatives of  $\mathbf{u}(\tau)$  obtained from  $\frac{d}{d\tau} (\mathbf{F}(\mathbf{q}(\tau)) \cdot \mathbf{u}(\tau))$  by  $\frac{e}{mc} \mathbf{F}(\mathbf{q}(\tau)) \cdot \mathbf{u}(\tau)$ . This approximation to (1.1) is known as the Landau–Lifshitz (LL) equation of motion.

Equation (1.1), and also its Landau–Lifshitz approximation, enjoy some practical successes. Interestingly, this practical success story has a serious blemish. Namely, for a point charge that moves along a constant applied electrostatic field the LL equation of motion simply reproduces the test particle motion, for its radiation-reaction force term vanishes in this situation; cf. [PMD2006]. One may hope to

obtain the radiation-reaction effects on a point charge that moves along a constant electric field by pushing the expansion of the EFC equation further until a non-vanishing radiation-reaction force term is obtained. However, all higher-order terms obtained from such an expansion vanish also.

As an aside, we mention that the LL equation has been derived rigorously as an effective equation of motion in a spacetime adiabatic limit, not for a point charge but for an extended charge distribution's geometric center, and with  $m$  standing for  $m_b + m_f$ , where  $m_b$  is a bare mass and  $m_f$  a field energy contribution; see [Spo2004]. Thus, if one works with extended charge distributions, as in the Abraham–Lorentz-type classical electron theory [Lor1904], and endows the particle with a non-vanishing  $m_b$ , then one may realistically hope to obtain higher-order radiation-reaction corrections that do not vanish for motion along a constant electric field. The LL equation of motion was also obtained in an asymptotic expansion about a “vanishing particle limit,” which captures the motion of a particle with extended charge and mass distributions in the limit of vanishing size, mass, and charge, yet with nonzero charge-to-mass ratio, see [GHW2009]. However, higher-order terms in such expansions will eventually depend on largely arbitrary assumptions about the structure of the extended distributions. Moreover, the formulation of a properly Lorentz co-variant model with extended charged particles [ApKi2001] involves a non-trivial foliation of Minkowski spacetime and poses conceptual challenges for the initial value problem.

In this paper we are interested in the classical theory of motion for a true point charge that interacts with the electromagnetic fields it generates. In [Kie2019] the first well-posedness result of the joint initial value problem for the evolution of the electromagnetic fields and the relativistic motion of  $N$  point charges was announced, not for the ill-defined Lorentz electrodynamics with point charges, but for an electrodynamic model that goes back to work by Bopp [Bop1940, Bop1943], Landé–Thomas [Lan1941, LaTh1941], and Podolsky [Pod1942] (BLTP). The BLTP model replaces Maxwell's law of the electromagnetic vacuum (viz.  $\mathcal{H} = \mathcal{B}$  and  $\mathcal{E} = \mathcal{D}$ ) with a linear differential relation. A well-defined equation of motion was supplied in [Kie2019]. The proof of well-posedness of the joint Cauchy problem will be published in [KTZ2023]. Also the scattering problem for a single particle that encounters a localized potential is well-posed [Hetal2021]. These authors showed that in this problem the “self”-force formula of [Kie2019] can be converted into a formal Lorentz-type expression that involves integration over the whole past of the particle motion, first proposed in [LaTh1941] and further studied in [Zay2014, GPT2015].

In the following we demonstrate that BLTP electrodynamics captures the radiation-reaction on a point charge that is released from rest in a constant applied electric field. We understand our work as part of a proof-of-concept. In principle our approach can handle also more realistic models than BLTP electrodynamics, see [Kie2019].

## 2 BLTP Electrodynamics with a Single Point Charge

The electromagnetic vacuum in BLTP electrodynamics is defined by the two equations

$$\mathcal{H}(t, s) = (1 + \varkappa^{-2} \square) \mathcal{B}(t, s), \quad (2.1)$$

$$\mathcal{D}(t, s) = (1 + \varkappa^{-2} \square) \mathcal{E}(t, s); \quad (2.2)$$

in (2.1) and (2.2), the parameter  $\varkappa^{-1}$  is the ‘‘Bopp length’’ [Bop1940, Bop1943], and  $\square \equiv c^{-2} \partial_t^2 - \Delta$  is the d’Alembertian, with  $c$  the vacuum speed of light. The evaluations  $\mathcal{H}(t, s)$ ,  $\mathcal{B}(t, s)$ ,  $\mathcal{E}(t, s)$ , and  $\mathcal{D}(t, s)$  of the fields at the space point  $s \in \mathbb{R}^3$  and instant of time  $t \in \mathbb{R}$  are defined in any convenient flat foliation of Minkowski spacetime into space & time. These fields satisfy the familiar system of pre-metric Maxwell field equations, which consist of two evolution equations

$$\frac{\partial}{\partial t} \mathcal{B}(t, s) = -c \nabla \times \mathcal{E}(t, s), \quad (2.3)$$

$$\frac{\partial}{\partial t} \mathcal{D}(t, s) = +c \nabla \times \mathcal{H}(t, s) - 4\pi e \delta_{q(t)}(s) \mathbf{v}(t), \quad (2.4)$$

and two constraint equations

$$\nabla \cdot \mathcal{B}(t, s) = 0, \quad (2.5)$$

$$\nabla \cdot \mathcal{D}(t, s) = 4\pi e \delta_{q(t)}(s). \quad (2.6)$$

Here,  $e (> 0)$  is the elementary electric charge,  $\mathbf{q}(t) \in \mathbb{R}^3$  its position and  $\mathbf{v}(t) \in \mathbb{R}^3$  its velocity at time  $t$ .

The particle’s velocity is defined as usual to be the time-derivative of its position vector,

$$\frac{d}{dt} \mathbf{q}(t) =: \mathbf{v}(t). \quad (2.7)$$

In the relativistic generalization of Newton’s point mechanics by Einstein, Lorentz, and Poincaré, the velocity  $\mathbf{v}(t)$ , in turn, changes with time according to

$$\frac{d}{dt} \frac{\mathbf{v}(t)}{\sqrt{1 - \frac{1}{c^2} |\mathbf{v}(t)|^2}} = \frac{1}{m_b} \mathbf{f}(t); \quad (2.8)$$

here,  $m_b \neq 0$  is the *bare inertial rest mass* of the particle, and  $\mathbf{f}(t)$  is the total electromagnetic force acting on it. Following Poincaré (cf. [Mil1998]) we define it as (cf. [Kie2019])

$$\mathbf{f}(t) := e \mathcal{E}^{\text{hom}} - \frac{d}{dt} \int_{\mathbb{R}^3} \Pi^{\text{field}}(t, s) d^3 s, \quad (2.9)$$

where  $\mathcal{E}^{\text{hom}}$  is a constant applied electric field (an idealization of the field between the plates of a capacitor), and  $\mathbf{\Pi}^{\text{field}}(t, s)$  is the momentum vector-density of the Maxwell-BLTP fields

$$4\pi c \mathbf{\Pi}^{\text{field}} = \mathcal{D} \times \mathcal{B} + \mathcal{E} \times \mathcal{H} - \mathcal{E} \times \mathcal{B} - \varkappa^{-2} (\nabla \cdot \mathcal{E}) \left( \nabla \times \mathcal{B} - \frac{1}{c} \frac{\partial}{\partial t} \mathcal{E} \right). \quad (2.10)$$

### 3 The Initial Data

As announced in [Kie2019] and shown in [KTZ2023], BLTP electrodynamics is well-posed as a joint initial value problem for the fields and the point charge, requiring initial data  $\mathcal{B}(0, s)$ ,  $\mathcal{D}(0, s)$ ,  $\mathcal{E}(0, s)$ ,  $(\frac{\partial}{\partial t} \mathcal{E})(0, s)$  for the fields, and  $\mathbf{q}(0)$  and  $\mathbf{v}(0)$  for the particle. The data for  $\mathcal{B}$  and  $\mathcal{D}$  are constrained by the divergence equations, and  $\mathbf{v}(0)$  by  $|\mathbf{v}(0)| < c$ .

In the ensuing sections we discuss these BLTP-dynamical equations for a single point charge moving along the constant applied electric field  $\mathcal{E}^{\text{hom}}$ , starting from rest, with the initial fields the sum of the external field and the electrostatic field of the point charge. Thus, for the particle we have

$$\mathbf{q}(0) = \mathbf{0} \quad \text{and} \quad \mathbf{v}(0) = \mathbf{0}. \quad (3.1)$$

For the fields we have

$$\mathcal{D}(0, s) \equiv \mathcal{E}^{\text{hom}} + e \frac{s}{|s|^3} \quad (3.2)$$

and

$$\mathcal{E}(0, s) \equiv \mathcal{E}^{\text{hom}} + e \frac{1 - (1 + \varkappa|s|)e^{-\varkappa|s|}}{|s|^2} \frac{s}{|s|}. \quad (3.3)$$

We also have  $(\partial_t \mathcal{E})(0, s) \equiv \mathbf{0}$ , as well as  $\mathcal{B}(0, s) \equiv \mathbf{0}$ .

### 4 The Solution of the Field Equations

For the initial data of our problem the electromagnetic fields outside the forward light cone of the initial location of the particle remain precisely the electrostatic fields, i.e., the magnetic field  $\mathcal{H}$  and the magnetic induction field  $\mathcal{B}$  vanish, while the electric displacement field  $\mathcal{D}(t, s)$  is given by (3.2) and the electric field  $\mathcal{E}(t, s)$  is given by (3.3), for all  $t \geq 0$ .

Inside the forward light cone of the initial particle location, but away from the particle position at  $t$ , the fields  $\mathcal{D}(t, s)$  and  $\mathcal{H}(t, s)$  are for all  $t \geq 0$  given by  $\mathcal{D} = \mathcal{E}^{\text{hom}} + \mathcal{D}_{\text{LW}}^{\text{ret}}$  &  $\mathcal{H} = \mathcal{H}_{\text{LW}}^{\text{ret}}$ , with (the acceleration vector of the point charge is highlighted in red)

$$\mathcal{D}_{\text{LW}}^{\text{ret}}(t, \mathbf{s}) = e \frac{c^2 - |\mathbf{v}|^2}{|\mathbf{s} - \mathbf{q}|^2} \frac{c\mathbf{n}(\mathbf{q}, \mathbf{s}) - \mathbf{v}}{(c - \mathbf{n}(\mathbf{q}, \mathbf{s}) \cdot \mathbf{v})^3} \Big|_{\text{ret}} + e \frac{\mathbf{n}(\mathbf{q}, \mathbf{s}) \times [(c\mathbf{n}(\mathbf{q}, \mathbf{s}) - \mathbf{v}) \times \mathbf{a}]}{|\mathbf{s} - \mathbf{q}|(c - \mathbf{n}(\mathbf{q}, \mathbf{s}) \cdot \mathbf{v})^3} \Big|_{\text{ret}}, \tag{4.1}$$

$$\mathcal{H}_{\text{LW}}^{\text{ret}}(t, \mathbf{s}) = \mathbf{n}(\mathbf{q}, \mathbf{s}) \Big|_{\text{ret}} \times \mathcal{D}_{\text{LW}}^{\text{ret}}(t, \mathbf{s}) \tag{4.2}$$

the retarded Liénard–Wiechert fields. Here,  $\mathbf{n}(\mathbf{q}, \mathbf{s}) = \frac{\mathbf{s} - \mathbf{q}}{|\mathbf{s} - \mathbf{q}|}$  is a *normalized* vector from  $\mathbf{q}$  to  $\mathbf{s}$ , the notation “ $\Big|_{\text{ret}}$ ” means that  $(\mathbf{q}, \mathbf{v}, \mathbf{a}) = (\mathbf{q}, \mathbf{v}, \mathbf{a})(t^{\text{ret}})$  to the left of “ $\Big|_{\text{ret}}$ ,” with  $t^{\text{ret}}(t, \mathbf{s})$  being defined implicitly by  $c(t - t^{\text{ret}}) = |\mathbf{s} - \mathbf{q}(t^{\text{ret}})|$ ; inside the initial forward light cone,  $0 < t^{\text{ret}} < t$ . The terms  $\propto \mathbf{a}$  in (4.1) and (4.2) account for the radiation.

Note that the electromagnetic Liénard–Wiechert fields  $\mathcal{H}_{\text{LW}}^{\text{ret}}$  and  $\mathcal{D}_{\text{LW}}^{\text{ret}}$  exhibit both a  $\propto 1/r^2$  singularity and a  $\propto 1/r$  singularity, where  $r$  denotes  $|\mathbf{s} - \mathbf{q}(t)|$ ; they each have a directional singularity at the location of the point charge source, too.

Similarly, inside — and on — the forward light cone of the initial particle location, but away from the particle position at  $t$ , the MBLTP field solutions  $\mathcal{B}(t, \mathbf{s})$  and  $\mathcal{E}(t, \mathbf{s})$  for  $t \geq 0$  are given by  $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$  and  $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$ , with  $\mathcal{B}_0 \equiv \mathbf{0}$  and  $\mathcal{E}_0 \equiv \mathcal{E}^{\text{hom}}$ , and

$$\begin{aligned} \mathcal{E}_1(t, \mathbf{s}) = & e\chi^2 \left( \frac{1 - (1 + \chi|\mathbf{s}|)e^{-\chi|\mathbf{s}|}}{\chi^2|\mathbf{s}|^2} - \frac{1}{2} \right) \frac{\mathbf{s}}{|\mathbf{s}|} + e\chi^2 \int_0^{ct-|\mathbf{s}|} \frac{J_2(\chi\sqrt{c^2(t-t')^2 - |\mathbf{s}|^2})}{c^2(t-t')^2 - |\mathbf{s}|^2} \mathbf{s} d(ct') + \\ & e\chi^2 \frac{1}{2} \frac{\mathbf{n}(\mathbf{q}, \mathbf{s}) - \mathbf{v}/c}{1 - \mathbf{n}(\mathbf{q}, \mathbf{s}) \cdot \mathbf{v}/c} \Big|_{\text{ret}} - \tag{4.3} \\ & e\chi^2 \int_0^{t^{\text{ret}}(t, \mathbf{s})} \frac{J_2(\chi\sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2})}{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2} (\mathbf{s} - \mathbf{q}(t') - \mathbf{v}(t')(t - t')) c dt', \end{aligned}$$

$$\begin{aligned} \mathcal{B}_1(t, \mathbf{s}) = & e\chi^2 \frac{1}{2} \frac{\mathbf{v} \times \mathbf{n}(\mathbf{q}, \mathbf{s})/c}{1 - \mathbf{n}(\mathbf{q}, \mathbf{s}) \cdot \mathbf{v}/c} \Big|_{\text{ret}} - \tag{4.4} \\ & e\chi^2 \int_0^{t^{\text{ret}}(t, \mathbf{s})} \frac{J_2(\chi\sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2})}{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2} \mathbf{v}(t') \times (\mathbf{s} - \mathbf{q}(t')) dt'. \end{aligned}$$

The fields  $\mathcal{B}(t, \mathbf{s})$  and  $\mathcal{E}(t, \mathbf{s})$  are globally bounded in  $\mathbf{s}$  for each  $t$ , and away from the point charge they are Lipschitz-continuous in  $\mathbf{s}$ , including across the initial forward light cone.

Similarly,

$$\begin{aligned} \nabla \cdot \mathcal{E}(t, \mathbf{s}) = & e\chi^2 \frac{e^{-\chi|\mathbf{s}|} - 1}{|\mathbf{s}|} + e\chi^3 \int_0^{ct-|\mathbf{s}|} \frac{J_1(\chi\sqrt{c^2(t-t')^2 - |\mathbf{s}|^2})}{\sqrt{c^2(t-t')^2 - |\mathbf{s}|^2}} d(ct') + \\ & e\chi^2 \frac{1}{(1 - \mathbf{n}(\mathbf{q}, \mathbf{s}) \cdot \mathbf{v}/c)} \frac{1}{|\mathbf{s} - \mathbf{q}|} \Big|_{\text{ret}} - \tag{4.5} \\ & e\chi^3 \int_0^{t^{\text{ret}}(t, \mathbf{s})} \frac{J_1(\chi\sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2})}{\sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2}} c dt', \end{aligned}$$

and

$$\begin{aligned}
(\nabla \times \mathcal{B} - \frac{1}{c} \frac{\partial}{\partial t} \mathcal{E})(t, s) &= e\alpha^2 \frac{1}{(1 - \mathbf{n}(\mathbf{q}, s) \cdot \mathbf{v}/c)} \frac{1}{|s - \mathbf{q}|} \frac{\mathbf{v}}{c} \Big|_{\text{ret}} - \\
e\alpha^3 \int_0^{t^{\text{ret}}(t, s)} &\frac{J_1(\alpha \sqrt{c^2(t-t')^2 - |s - \mathbf{q}(t')|^2})}{\sqrt{c^2(t-t')^2 - |s - \mathbf{q}(t')|^2}} \mathbf{v}(t') dt'.
\end{aligned} \tag{4.6}$$

## 5 Evaluation of the Radiation-Reaction Force

With the help of these solution formulas, the electromagnetic force of the MBLTP field on its point charge source can be computed as follows. Since each electromagnetic field component is the sum of a vacuum field and a sourced field, the bilinear  $\mathbf{\Pi}^{\text{field}}$  decomposes into a sum of three types of terms: the vacuum-vacuum terms, the source-source terms, and the mixed vacuum-source terms. In our problem the vacuum field is  $\mathcal{E}^{\text{hom}}$ ; it does not contribute to  $\mathbf{\Pi}^{\text{field}}$ , but appears separately at rhs(2.9). As explained in [Kie2019], this term is not put in by hand but is a contribution to the momentum balance due to a surface integral at “ $|s| = \infty$ .” Hence the only contribution to rhs(2.9) from  $\mathbf{\Pi}^{\text{field}}$  is the source-source contribution, a “self”-field force in BLTP electrodynamics. Thus, (2.9) is given by

$$\mathbf{f}(t) = e\mathcal{E}^{\text{hom}} + \mathbf{f}^{\text{self}}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t), \tag{5.1}$$

where  $e\mathcal{E}^{\text{hom}}$  is the Lorentz force evaluated with the vacuum field (i.e., a “test particle contribution” to the total force), and (after taking advantage of hyperbolicity; cf. [Kie2019])

$$\mathbf{f}^{\text{self}}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t) \equiv - \frac{d}{dt} \int_{B_{ct}(\mathbf{q}_0)} \left( \mathbf{\Pi}_{\text{source}}^{\text{field}}(t, s) - \mathbf{\Pi}_{\text{source}}^{\text{field}}(0, s - \mathbf{q}_0 - \mathbf{v}_0 t) \right) d^3s \tag{5.2}$$

$$= - \frac{d}{dt} \int_{B_{ct}(\mathbf{q}_0)} \mathbf{\Pi}_{\text{source}}^{\text{field}}(t, s) d^3s, \tag{5.3}$$

with  $\mathbf{\Pi}_{\text{source}}^{\text{field}}$  given by (2.10) with  $(\mathcal{B}, \mathcal{D} - \mathcal{E}^{\text{hom}}, \mathcal{E} - \mathcal{E}^{\text{hom}}, \mathcal{H})$  in place of  $(\mathcal{B}, \mathcal{D}, \mathcal{E}, \mathcal{H})$ . To go from (5.2) to (5.3) we made use of the initial data  $\mathbf{q}_0 = \mathbf{0}$  and  $\mathbf{v}_0 = \mathbf{0}$ , and  $\mathbf{\Pi}_{\text{source}}^{\text{field}}(0, s) \equiv \mathbf{0}$ .

The “self”-field force can be evaluated using retarded spherical coordinates  $(r, \vartheta, \varphi)$  to carry out the  $d^3s$  integrations over the ball  $B_{ct}(\mathbf{q}_0)$ , after which one can differentiate w.r.t.  $t$ . For this very special problem of straight line motion of a charge starting from rest at the origin, this yields

$$\begin{aligned}
 f^{\text{self}}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t) = & \frac{e^2}{4\pi} \left[ -\mathbf{Z}_\xi^{[2]}(t, t) \right. \\
 & - \sum_{0 \leq k \leq 1} c^{2-k} (2-k) \int_0^t \mathbf{Z}_\xi^{[k]}(t, t^r) (t-t^r)^{1-k} dt^r \\
 & \left. - \sum_{0 \leq k \leq 2} c^{2-k} \int_0^t \frac{\partial}{\partial t^r} \mathbf{Z}_\xi^{[k]}(t, t^r) (t-t^r)^{2-k} dt^r \right].
 \end{aligned} \tag{5.4}$$

Here,  $\xi(t) \equiv (\mathbf{q}, \mathbf{v}, \mathbf{a})(t)$  and  $\mathbf{Z}_\xi^{[2]}(t, t) := \lim_{t^r \rightarrow t} \mathbf{Z}_\xi^{[k]}(t, t^r)$ , where

$$\mathbf{Z}_\xi^{[k]}(t, t^r) = \int_0^{2\pi} \int_0^\pi \left(1 - \frac{1}{c} v(t^r) \cos \vartheta\right) \pi_\xi^{[k]}(t, \mathbf{q}(t^r) + c(t-t^r)\mathbf{n}) \sin \vartheta d\vartheta d\varphi, \tag{5.5}$$

with  $v(t)$  defined by  $v(t)|\mathcal{E}^{\text{hom}}| \equiv \mathbf{v}(t) \cdot \mathcal{E}^{\text{hom}}$ , and with  $\mathbf{n} = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$  a normal vector to the retarded sphere of radius  $r = c(t-t^r)$ , where we measure  $\vartheta$  from the  $\mathcal{E}^{\text{hom}}$  direction and  $\varphi$  from an arbitrary axis  $\perp \mathcal{E}^{\text{hom}}$ .

Moreover, the  $\pi_\xi^{[k]}(t, s)$  with  $k \in \{0, 1, 2\}$  and  $s \neq \mathbf{q}$  are defined as follows. We set

$$\mathbf{K}_\xi(t', t, s) := \frac{J_1(\varkappa \sqrt{c^2(t-t')^2 - |s-\mathbf{q}(t')|^2})}{\sqrt{c^2(t-t')^2 - |s-\mathbf{q}(t')|^2}}, \tag{5.6}$$

$$\mathbf{K}_\xi(t', t, s) := \frac{J_2(\varkappa \sqrt{c^2(t-t')^2 - |s-\mathbf{q}(t')|^2})}{c^2(t-t')^2 - |s-\mathbf{q}(t')|^2} (s - \mathbf{q}(t') - \mathbf{v}(t')(t-t')), \tag{5.7}$$

and note that

$$\int_0^{t_\xi^{\text{ret}}(t,s)} \mathbf{K}_{\xi^\circ}(t', t, s) c dt' = \int_0^{ct-|s|} \frac{J_2(\varkappa \sqrt{c^2(t-t')^2 - |s|^2})}{c^2(t-t')^2 - |s|^2} s d(ct'), \tag{5.8}$$

$$\int_0^{t_\xi^{\text{ret}}(t,s)} \mathbf{K}_\xi(t', t, s) c dt' = \int_0^{ct-|s|} \frac{J_1(\varkappa \sqrt{c^2(t-t')^2 - |s|^2})}{\sqrt{c^2(t-t')^2 - |s|^2}} d(ct'). \tag{5.9}$$

We will use  $|_{\text{ret}}$  to mean that  $\mathbf{q}(\tilde{t}), \mathbf{v}(\tilde{t}), \mathbf{a}(\tilde{t})$  are evaluated at  $\tilde{t} = t_\xi^{\text{ret}}(t, s)$ , not  $t_\xi^{\text{ret}\circ}(t, s)$ . Then

$$\begin{aligned}
 \pi_\xi^{[0]}(t, s) = & -\varkappa^4 \frac{1}{4} \left[ \frac{(\mathbf{n}(\mathbf{q}, s) - \frac{1}{c} \mathbf{v}) \times (\frac{1}{c} \mathbf{v} \times \mathbf{n}(\mathbf{q}, s))}{(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^2} \right]_{\text{ret}} \\
 & + \varkappa^4 \frac{1}{2} \left[ \frac{\mathbf{n}(\mathbf{q}, s) - \frac{1}{c} \mathbf{v}}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \times \int_0^{t_\xi^{\text{ret}}(t,s)} \mathbf{v}(t') \times \mathbf{K}_\xi(t', t, s) dt' \\
 & - \varkappa^4 \frac{1}{2} \left[ \frac{\frac{1}{c} \mathbf{v} \times \mathbf{n}(\mathbf{q}, s)}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \times \int_0^{t_\xi^{\text{ret}}(t,s)} c \mathbf{K}_\xi(t', t, s) dt'
 \end{aligned} \tag{5.10}$$



$$\begin{aligned}
 & + \varkappa^4 \frac{1}{2} \left[ \frac{\frac{1}{c} \mathbf{v} \times \mathbf{n}(\mathbf{q}, s)}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \times \left( \frac{1 - (1 + \varkappa |\mathbf{s}|) e^{-\varkappa |\mathbf{s}|}}{\varkappa^2 |\mathbf{s}|^2} - \frac{1}{2} \right) \frac{\mathbf{s}}{|\mathbf{s}|} \\
 & + \varkappa^4 \frac{1}{2} \left[ \frac{\frac{1}{c} \mathbf{v} \times \mathbf{n}(\mathbf{q}, s)}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \times \int_0^{t_{\xi^{\circ}}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi^{\circ}}(t', t, s) dt' \\
 & - \varkappa^4 \int_0^{t_{\xi}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi}(t', t, s) dt' \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt' \\
 & + \varkappa^4 \left( \frac{1 - (1 + \varkappa |\mathbf{s}|) e^{-\varkappa |\mathbf{s}|}}{\varkappa^2 |\mathbf{s}|^2} - \frac{1}{2} \right) \frac{\mathbf{s}}{|\mathbf{s}|} \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt' \\
 & + \varkappa^4 \int_0^{t_{\xi^{\circ}}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi^{\circ}}(t', t, s) dt' \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt' \\
 & - \varkappa^4 c \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{K}_{\xi}(t', t, s) dt' \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{K}_{\xi}(t', t, s) \mathbf{v}(t') dt', \\
 & - \varkappa^3 \frac{1 - e^{-\varkappa |\mathbf{s}|}}{|\mathbf{s}|} \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{K}_{\xi}(t', t, s) \mathbf{v}(t') dt', \\
 & + \varkappa^4 \int_0^{t_{\xi^{\circ}}^{\text{ret}}(t, s)} \mathbf{K}_{\xi^{\circ}}(t', t, s) c dt' \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{K}_{\xi}(t', t, s) \mathbf{v}(t') dt',
 \end{aligned}$$

and

$$\begin{aligned}
 \pi_{\xi}^{[1]}(t, s) = & - \varkappa^2 \left[ \mathbf{n}(\mathbf{q}, s) \frac{(\mathbf{n}(\mathbf{q}, s) \times [\mathbf{n}(\mathbf{q}, s) \times \mathbf{a}]) \cdot \frac{1}{c} \mathbf{v}}{c^2 (1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^4} + \mathbf{n}(\mathbf{q}, s) \times \frac{\mathbf{n}(\mathbf{q}, s) \times \mathbf{a}}{2c^2 (1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \quad (5.11) \\
 & - \varkappa^2 \left[ \mathbf{n}(\mathbf{q}, s) \times \frac{\mathbf{n}(\mathbf{q}, s) \times \mathbf{a}}{c^2 (1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt' \\
 & + \varkappa^2 \left[ \mathbf{n}(\mathbf{q}, s) \times \left[ \mathbf{n}(\mathbf{q}, s) \times \frac{\mathbf{n}(\mathbf{q}, s) \times \mathbf{a}}{c^2 (1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right] \right]_{\text{ret}} \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi}(t', t, s) dt' \\
 & - \varkappa^2 \left[ \mathbf{n}(\mathbf{q}, s) \times \left[ \mathbf{n}(\mathbf{q}, s) \times \frac{\mathbf{n}(\mathbf{q}, s) \times \mathbf{a}}{c^2 (1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right] \right]_{\text{ret}} \times \left( \frac{1 - (1 + \varkappa |\mathbf{s}|) e^{-\varkappa |\mathbf{s}|}}{\varkappa^2 |\mathbf{s}|^2} - \frac{1}{2} \right) \frac{\mathbf{s}}{|\mathbf{s}|} \\
 & - \varkappa^2 \left[ \mathbf{n}(\mathbf{q}, s) \times \left[ \mathbf{n}(\mathbf{q}, s) \times \frac{\mathbf{n}(\mathbf{q}, s) \times \mathbf{a}}{c^2 (1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right] \right]_{\text{ret}} \times \int_0^{t_{\xi^{\circ}}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi^{\circ}}(t', t, s) dt' \\
 & + \varkappa^3 \left[ \frac{1}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{K}_{\xi}(t', t, s) [\mathbf{v}(t_{\xi}^{\text{ret}}(t, s)) + \mathbf{v}(t')] dt', \\
 & + \varkappa^2 \left[ \frac{1}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \frac{1 - e^{-\varkappa |\mathbf{s}|}}{|\mathbf{s}|} \frac{1}{c} \mathbf{v}(t_{\xi}^{\text{ret}}(t, s)) \\
 & - \varkappa^3 \left[ \frac{1}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \int_0^{t_{\xi^{\circ}}^{\text{ret}}(t, s)} \mathbf{K}_{\xi^{\circ}}(t', t, s) c dt' \frac{1}{c} \mathbf{v}(t_{\xi}^{\text{ret}}(t, s)),
 \end{aligned}$$

and

$$\begin{aligned}
\pi_{\xi}^{[2]}(t, s) = & -\varkappa^2 \left[ \frac{1}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(q, s)\right)^2} \frac{1}{c} \mathbf{v} - \left[1 - \frac{1}{c^2} |\mathbf{v}|^2\right] \frac{(\mathbf{n}(q, s) - \frac{1}{c} \mathbf{v}) \times (\frac{1}{c} \mathbf{v} \times \mathbf{n}(q, s))}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(q, s)\right)^4} \right]_{\text{ret}} \\
& + \varkappa^2 \left[ \left[1 - \frac{1}{c^2} |\mathbf{v}|^2\right] \frac{\frac{1}{c} \mathbf{v} \times \mathbf{n}(q, s)}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(q, s)\right)^3} \right]_{\text{ret}} \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi}(t', t, s) dt' \\
& - \varkappa^2 \left[ \left[1 - \frac{1}{c^2} |\mathbf{v}|^2\right] \frac{\frac{1}{c} \mathbf{v} \times \mathbf{n}(q, s)}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(q, s)\right)^3} \right]_{\text{ret}} \times \left( \frac{1 - (1 + \varkappa |s|) e^{-\varkappa |s|}}{\varkappa^2 |s|^2} - \frac{1}{2} \right) \frac{s}{|s|} \\
& - \varkappa^2 \left[ \left[1 - \frac{1}{c^2} |\mathbf{v}|^2\right] \frac{\frac{1}{c} \mathbf{v} \times \mathbf{n}(q, s)}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(q, s)\right)^3} \right]_{\text{ret}} \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi^{\circ}}(t', t, s) dt' \\
& - \varkappa^2 \left[ \left[1 - \frac{1}{c^2} |\mathbf{v}|^2\right] \frac{\mathbf{n}(q, s) - \frac{1}{c} \mathbf{v}}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(q, s)\right)^3} \right]_{\text{ret}} \times \int_0^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt'.
\end{aligned} \tag{5.12}$$

Although this is an intimidating list of integrals, we can already extract an important conclusion: The equation of motion for our point charge does not feature time-derivatives of the particle position  $\mathbf{q}(t)$  higher than second order. This result holds also for BLTP electrodynamics in general [Kie2019]. Hence, BLTP electrodynamics with point charges does not suffer from the  $\ddot{\mathbf{q}}(t)$  problem.

## 5.1 The Small- $\varkappa$ Regime

To make further progress in the evaluation of the integrals we will concentrate our efforts on an asymptotic analysis of the small- $\varkappa$  regime. We make a formal power series expansion about  $\varkappa = 0$  given by  $f^{\text{self}}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t) = \sum_{n=0}^{\infty} \mathbf{F}_0^{(n)}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t)$ , where  $\mathbf{F}_0^{(n)} \propto \varkappa^n$ ; the subscript  $_0$  at  $\mathbf{F}_0^{(n)}$  indicates that we are expanding about  $\varkappa = 0$ . It is manifest that the terms  $O(\varkappa^0)$  and  $O(\varkappa^1)$  vanish identically, so we need to discuss terms  $O(\varkappa^n)$  for  $n \geq 2$ . Several of the spherical integrations can be carried out explicitly in terms of well-known functions. In particular, the contributions  $\mathbf{F}_0^{(2)}$  and  $\mathbf{F}_0^{(3)}$  can be computed explicitly.

### 5.1.1 Radiation-Reaction at $O(\varkappa^2)$

To arrive at the  $O(\varkappa^2)$  contribution we divide the expressions for  $\pi_{\xi}^{[k]}$  by  $\varkappa^2$  and take  $\varkappa \rightarrow 0$ . The only two terms that survive in the limit are those in the first line of rhs(5.11) and rhs(5.12), respectively (indicated below by a superscript  $\cdot^1$ ; later, also superscripts  $\cdot^3, \cdot^4, \cdot^7$  will appear). Carrying out the pertinent integrations in (5.5) one notes that the result only depends on  $t^r$ , not on  $t$ , so that the third line of rhs(5.4) vanishes at  $O(\varkappa^2)$ . Thus,

$$\frac{4\pi}{e^2} \mathbf{F}_0^{(2)}(t) = -\mathbf{Z}_\xi^{[2],1}(t, t) - c \int_0^t \mathbf{Z}_\xi^{[1],1}(t, t') dt' \tag{5.13}$$

Explicitly, (5.13) reads

$$\begin{aligned} \mathbf{F}_0^{(2)}(t) = & -\frac{1}{2} e^2 \varkappa^2 \frac{\mathbf{v}(t)}{v(t)} \left[ 2 \frac{c}{v(t)} - \frac{c^2}{v(t)^2} \ln \frac{1 + \frac{1}{c} v(t)}{1 - \frac{1}{c} v(t)} \right] \\ & + e^2 \varkappa^2 \int_0^t \frac{c^3}{v(t')^3} \left[ \frac{1}{c} v(t') \frac{2 - \frac{v(t')^2}{c^2}}{1 - \frac{v(t')^2}{c^2}} - \ln \frac{1 + \frac{1}{c} v(t')}{1 - \frac{1}{c} v(t')} \right] \frac{1}{c} \mathbf{a}(t') dt' \end{aligned} \tag{5.14}$$

The term in the first line of rhs(5.14) is the contribution from the first line of rhs(5.12), the term in the second line of rhs(5.14) is the contribution from the first line of rhs(5.11). Since for straight-line motion  $\mathbf{v}(t)$  and  $\mathbf{a}(t)$  are collinear, and  $\mathbf{a}(t) = \dot{\mathbf{v}}(t)$ , one can carry out the time integration in the second line of rhs(5.14) in terms of elementary functions of  $v/c$ , and a few algebraic manipulations then give (see the erratum in [Kie2019])

$$\mathbf{F}_0^{(2)}(t) = \mathbf{0}. \tag{5.15}$$

In this problem of straight line motion in a constant external electric field, with the particle starting from rest, the BLTP radiation-reaction force vanishes *exactly* at  $O(\varkappa^2)$ .

### 5.1.2 Radiation-Reaction at $O(\varkappa^3)$

We next evaluate the  $O(\varkappa^3)$  contribution to the radiation-reaction force for small  $\varkappa$ . To arrive at the  $O(\varkappa^3)$  contribution, subtract the  $O(\varkappa^2)$  terms from the expressions for  $\pi_\xi^{[k]}$ , divide the result by  $\varkappa^3$  and take the limit  $\varkappa \rightarrow 0$ . This yields the contributions from three  $\pi_\xi^{[k]}$  terms  $\propto \varkappa^3$ , namely  $\pi_\xi^{[1],4}$ ,  $\pi_\xi^{[1],7}$ , and  $\pi_\xi^{[2],3}$ . They contribute the following force  $\propto \varkappa^3$ ,

$$\begin{aligned} \mathbf{F}_0^{(3)}(t) = & -e^2 \varkappa^3 \mathbf{q}(t) \\ & - e^2 \varkappa^3 \frac{4}{3} \int_0^t \frac{c^2}{v(t')^2} \left\{ 1 + \frac{1}{2} \frac{c}{v(t')} \ln \frac{1 - \frac{1}{c} v(t')}{1 + \frac{1}{c} v(t')} \right\} (t - t') \mathbf{a}(t') dt' \\ & - e^2 \varkappa^3 \frac{2}{3} \int_0^t \left( 1 - \frac{v(t')^2}{c^2} \right) \frac{c}{v(t')} \left\{ 1 + \frac{1}{2} \frac{c}{v(t')} \ln \frac{1 - \frac{1}{c} v(t')}{1 + \frac{1}{c} v(t')} \right\} c dt' \frac{\mathcal{E}_{\text{hom}}}{|\mathcal{E}_{\text{hom}}|} \end{aligned} \tag{5.16}$$

Integration by parts yields for the integral in the second line of rhs(5.16)

$$\int_0^t \frac{c^2}{v(t^r)^2} \left\{ 1 + \frac{1}{2} \frac{c}{v(t^r)} \ln \frac{1 - \frac{1}{c} v(t^r)}{1 + \frac{1}{c} v(t^r)} \right\} (t - t^r) \mathbf{a}(t^r) dt^r = \tag{5.17}$$

$$\int_0^t c \int_0^{v(t^r)/c} \frac{1}{x^2} \left\{ 1 + \frac{1}{2} \frac{1}{x} \ln \frac{1 - x}{1 + x} \right\} dx dt^r = \tag{5.18}$$

$$- \frac{1}{2} \int_0^t \frac{c}{v(t^r)} \left[ 1 + \left( 1 - \frac{v(t^r)^2}{c^2} \right) \frac{1}{2} \frac{c}{v(t^r)} \ln \frac{1 - \frac{1}{c} v(t^r)}{1 + \frac{1}{c} v(t^r)} \right] c dt^r$$

Comparison with the third line of rhs(5.16) reveals cancellations, and we end up with

$$\mathbf{F}_0^{(3)}(t) = -\frac{1}{3} e^2 \varkappa^3 \mathbf{q}(t). \tag{5.19}$$

This is a very surprising result: the  $O(\varkappa^3)$  term of the radiation-reaction force in our initial value problem is a harmonic oscillator force! This result relies on the particular setup of the initial data and the geometry of the problem, but not more.

## 6 The Volterra Equation for the Acceleration

The equation of motion can be recast as a Volterra integral equation for the acceleration,

$$\mathbf{a}(t) = W[\mathbf{v}] \cdot (e\mathcal{E}^{\text{hom}} + \mathbf{f}^{\text{self}}[\mathbf{q}, \mathbf{v}; \mathbf{a}]) (t). \tag{6.1}$$

Here,

$$W[\mathbf{v}] := \frac{1}{m_b} \sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}} [\mathbf{I}_{3 \times 3} - \frac{1}{c^2} \mathbf{v} \otimes \mathbf{v}], \tag{6.2}$$

which for motion along  $\mathcal{E}^{\text{hom}}$  is the same as

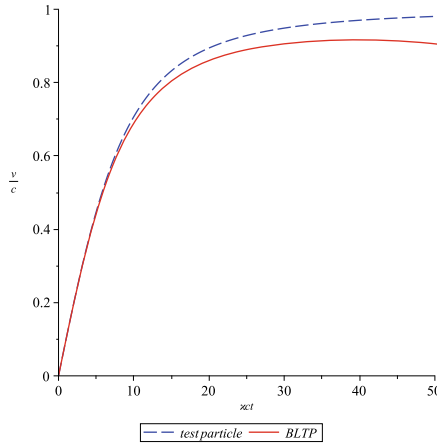
$$W[\mathbf{v}] := \frac{1}{m_b} \left( 1 - \frac{|\mathbf{v}|^2}{c^2} \right)^{3/2} \mathbf{I}_{3 \times 3}. \tag{6.3}$$

In [KTZ2023] we show that the Volterra equation can be uniquely solved to yield  $\mathbf{a}$  as a nonlinear expression in  $\mathbf{q}$  and  $\mathbf{v}$ , posing a second-order initial value problem for  $\mathbf{q}(t)$ .

### 6.1 The Volterra Equation to $O(\varkappa^3)$ (Small $\varkappa$ )

With the radiation-reaction force evaluated to  $O(\varkappa^3)$  we obtain the equation of motion

$$\mathbf{a}(t) = \frac{1}{m_b} \left( 1 - \frac{1}{c^2} |\mathbf{v}(t)|^2 \right)^{3/2} (e\mathcal{E}^{\text{hom}} - \frac{1}{3} e^2 \varkappa^3 \mathbf{q}(t)). \tag{6.4}$$



**Fig. 1** The velocity of a point charge, starting from rest in a constant applied electrostatic field  $\mathcal{E}^{\text{hom}} = 10e\kappa^2$ , versus time, as per test particle theory (dashed curve), resp. BLTP electrostatics with radiation-reaction included to  $O(\kappa^3)$  (continuous curve), when  $\kappa e^2/m_b c^2 = 0.01$ . The period of the velocity of the BLTP motion is  $\kappa cT = 160$ . The test particle’s velocity asymptotes to  $c$

Equation (6.4) is equivalent to the problem of special-relativistic test particle motion in a harmonic oscillator potential, featuring time-periodic solutions with conserved particle energy

$$U = \sqrt{m_b^2 c^4 + |\mathbf{p}|^2 c^2} - e\mathcal{E}^{\text{hom}} \cdot \mathbf{q} + \frac{1}{6} e^2 \kappa^3 |\mathbf{q}|^2. \tag{6.5}$$

It is not clear to us whether this means that the validity of the  $O(\kappa^3)$  approximation is restricted to short times  $\kappa ct \ll 1$  (see Fig. 1, which looks reasonable) or whether such periodic motion over longer times is a genuine feature of BLTP electrostatics as long as  $\kappa e^2/m_b c^2 \ll 1$ . In the latter case BLTP electrostatics would presumably be eliminated for good from the list of contenders for a realistic classical electrostatics.

## 7 Summary and Outlook

We have shown that BLTP electrostatics, as defined in [Kie2019], accounts for radiation-reaction effects on the point charge motion along a constant applied electric field, thereby passing a litmus test that other models (in particular, the Landau–Lifshitz and Eliezer–Ford–O’Connell equations of motion) fail. Our results are based on a small- $\kappa$  expansion of the BLTP force expression. This is acceptable for our “proof-of-concept” demonstration. However, BLTP electrostatics is physically viable at most in the large  $\kappa$  regime [CKP2019]—if at all. An assessment of the large- $\kappa$  regime we leave to some future work.

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