Chapter 2
Multi-Dimensional Variational Problems

This short chapter establishes the bases of the multi-dimensional case, starting from the techniques which were described in the previous chapter for the 1D case and highlighting the necessary modifications. After discussing the existence of the minimizers and their properties in terms of a multi-dimensional Euler–Lagrange equation, which typically takes the form of an elliptic PDE, a section is devoted to the study of harmonic functions, which are the solutions to the simplest example of a variational problem, that of minimizing the Dirichlet energy $u \mapsto \int |\nabla u|^2$.

As a choice, throughout the chapter, the multidimensional case will only be treated when the target space is, instead, one-dimensional. Many notions can be easily extended to vector-valued maps without particular difficulties.

2.1 Existence in Sobolev Spaces

In this section we will discuss the classes of problems for which we are easily able to prove the existence of a solution in the Sobolev space $W^{1,p}(\Omega)$. In the sequel Ω will be a bounded (and, when needed, smooth) open set in $\mathbb{R}^d$. We will consider optimization problems in $W^{1,p}(\Omega)$ for $p > 1$.

Box 2.1 Memo—*Sobolev Spaces in Higher Dimensions*
Given an open domain $\Omega \subset \mathbb{R}^d$ and an exponent $p \in [1, +\infty]$ we define the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) \ : \ \forall i \exists g_i \in L^p(\Omega) \text{ s.t.} \int u \partial_{x_i} \varphi = - \int g_i \varphi \text{ for all } \varphi \in C_c^\infty(\Omega)\}.$$

(continued)

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Box 2.1 (continued)
The functions g_i, if they exist, are unique, and together they provide a vector which will be denoted by ∇u since it plays the role of the derivative in the integration by parts. The space $W^{1,p}$ is endowed with the norm $||u||_{W^{1,p}} := ||u||_{L^p} + ||\nabla u||_{L^p}$. With this norm $W^{1,p}$ is a Banach space, separable if $p < +\infty$, and reflexive if $p \in (1, \infty)$.

The functions in $W^{1,p}(\Omega)$ can also be characterized as those functions $u \in L^p(\Omega)$ such that their translations u_h (defined via $u_h(x) := u(x+h)$) satisfy $||u - h_h||_{L^p(\Omega')} \leq C|h|$ for every subdomain Ω' compactly contained in Ω and every h such that $|h| < d(\Omega' \partial\Omega)$. The optimal constant C in this inequality equals $||\nabla u||_{L^p}$.

Sobolev Injections If $p < d$ all functions $u \in W^{1,p}$ are indeed more summable than just L^p and they actually belong to L^{p^*}, where $p^* := \frac{pd}{d-p} > p$. The injection of $W^{1,p}$ into L^q is compact (i.e. it sends bounded sets into precompact sets) for any $q < p^*$ (including $q = p$), while the injection into L^{p^*} is continuous. If $p = d$ then $W^{1,p}$ injects compactly in all spaces L^q with $q < +\infty$, but does not inject into L^∞. If $p > d$ then all functions in $W^{1,p}$ admit a continuous representative, which is Hölder continuous of exponent $\alpha = 1 - \frac{d}{p} > 0$, and the injection from $W^{1,p}$ into $C^0(\Omega)$ is compact if Ω is bounded.

The space $W_0^{1,p}(\Omega)$ can be defined as the closure in $W^{k,p}(\Omega)$ of $C_c^\infty(\Omega)$ and the Poincaré inequality $||\varphi||_{L^p} \leq C(\Omega)||\nabla\varphi||L^p(\Omega)$, which is valid for all $\varphi \in C_c^\infty(\Omega)$ if Ω is bounded (this is not a sharp condition, for instance bounded in one direction—a notion to be made precise—would be enough), allows us to use as a norm on $W_0^{1,p}(\Omega)$ the quantity $||u||_{W_0^{1,p}} := ||\nabla u||_{L^p}$.

If $p = 2$ the space $W^{1,p}$ can be given a Hilbert structure exactly as in 1D and is denoted by H^1. Higher-order Sobolev spaces $W^{k,p}$ and H^k can also be defined exactly as in dimension 1 (and $W_0^{k,p}$ is again defined as the closure of $C_c^\infty(\Omega)$ for the $W^{k,p}$ norm).

We can also define negative-order Sobolev spaces: the space $W^{-k,p'}$ is defined as the dual of the space $W_0^{k,p}$.

Theorem 2.1 *Given a function $g \in W^{1,p}(\Omega)$ and a measurable function $F : \Omega \times \mathbb{R} \to \mathbb{R}_+$ such that $F(x, \cdot)$ is lower semicontinuous for a.e. x, let us consider the problem*

$$\min \left\{ J(u) := \int_\Omega \left(F(x, u(x)) + |\nabla u(x)|^p \right) \mathrm{d}x : u \in W^{1,p}(\Omega),\ u - g \in W_0^{1,p}(\Omega) \right\}. \tag{2.1}$$

This minimization problem admits a solution.

Proof Take a minimizing sequence u_n such that $J(u_n) \to \inf J$. From the bound on $J(u_n)$ we obtain an upper bound for $||\nabla u_n||_{L^p}$. Applying the Poincaré inequality for $W_0^{1,p}$ functions to $u_n - g$ we obtain

$$\begin{aligned} ||u_n||_{L^p} &\leq ||u_n - g||_{L^p} + ||g||_{L^p} \\ &\leq C||\nabla(u_n - g)||_{L^p} + ||g||_{L^p} \\ &\leq C||\nabla u_n||_{L^p} + C||\nabla g||_{L^p} + ||g||_{L^p}, \end{aligned}$$

which allows us to bound $||u_n||_{L^p}$.

Hence, $(u_n)_n$ is a bounded sequence in $W^{1,p}$ and we can extract a subsequence which weakly converges in $W^{1,p}$ to a function u (see Boxes 2.2 and 2.3). This convergence implies the strong convergence $u_n \to u$ in $L^p(\Omega)$ and, up to a subsequence, a.e. convergence $u_n(x) \to u(x)$. Moreover, we also have $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^d)$ by applying the continuous linear mapping $W^{1,p}(\Omega) \ni v \mapsto \nabla v \in L^p$.

The space $W_0^{1,p}(\Omega)$ is a closed subspace (by definition, as it is defined as the closure of C_c^∞ in $W^{1,p}$) of $W^{1,p}(\Omega)$ and, since it is a vector space and hence a convex set, it is also weakly closed (see Proposition 3.7). Then, from $u_n \rightharpoonup u$ we deduce $u_n - g \rightharpoonup u - g$ and $u - g \in W_0^{1,p}(\Omega)$. Hence, the limit u is also admissible in the optimization problem.

We now need to show that J is l.s.c. for the weak convergence in $W^{1,p}(\Omega)$. The semicontinuity of the L^p norm for the weak convergence easily provides

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \leq \liminf_n \int_\Omega |\nabla u_n|^p \,\mathrm{d}x.$$

The a.e. pointwise convergence together with the lower semicontinuity of F and the use of Fatou's lemma provides

$$\int_\Omega F(x, u(x)) \,\mathrm{d}x \leq \liminf_n \int_\Omega F(x, u_n(x)) \,\mathrm{d}x.$$

This shows that u is a minimizer. □

Box 2.2 Memo—*Weak Convergence and Compactness*

On a normed vector space $\mathcal{X}$ we say that a sequence x_n weakly converges to x if for every $\xi \in \mathcal{X}'$ we have $\langle \xi, x_n \rangle \to \langle \xi, x \rangle$, and we write $x_n \rightharpoonup x$. Of course, if $x_n \to x$ (in the sense of $||x_n - x|| \to 0$), then we have $x_n \rightharpoonup x$. On the dual space $\mathcal{X}'$ we say that a sequence ξ_n weakly-* converges to ξ if for

(continued)

Box 2.2 (continued)
every $x \in \mathcal{X}$ we have $\langle \xi_n, x \rangle \to \langle \xi, x \rangle$, and we write $\xi_n \overset{*}{\rightharpoonup} \xi$. Note that this is a priori different than the notion of weak convergence on $\mathcal{X}'$, which involves the bidual $\mathcal{X}''$.

Theorem (Banach–Alaoglu) – If $\mathcal{X}$ is a separable normed space and ξ_n is a bounded sequence in $\mathcal{X}'$, then there exists a subsequence ξ_{n_k} and a vector $\xi \in \mathcal{X}'$ such that $\xi_{n_k} \overset{*}{\rightharpoonup} \xi$.

Box 2.3 Memo—*Reflexive Banach Spaces*

A normed vector space $\mathcal{X}$ can always be embedded in its bidual $\mathcal{X}''$ since we can associate with every $x \in \mathcal{X}$ the linear form ι_x on $\mathcal{X}'$ defined via $\langle \iota_x, \xi \rangle := \langle \xi, x \rangle$. In general not all elements of $\mathcal{X}''$ are of the form ι_x for $x \in \mathcal{X}$. When $\iota : \mathcal{X} \to \mathcal{X}''$ is surjective we say that $\mathcal{X}$ is reflexive. In this case it is isomorphic to its bidual which, as a dual space, is a Banach space.

All L^p spaces for $p \in (1, \infty)$ are reflexive as their dual is $L^{p'}$ and their bidual is again L^p. In contrast, L^∞ is the dual of L^1 but L^1 is not the dual of L^∞. The space $C(X)$ of continuous functions on a compact metric space X is not reflexive either, since its dual is $\mathcal{M}(X)$, the space of measures, and the dual of $\mathcal{M}(X)$ is not $C(X)$. Of course all Hilbert spaces are reflexive, as they coincide with their own dual. Sobolev spaces $W^{k,p}$ for $p \in (1, \infty)$ are also reflexive.

Theorem (Eberlein–Smulian) *A Banach space $\mathcal{X}$ is reflexive if and only if any bounded sequence in $\mathcal{X}$ is weakly compact.*

In optimization the most important part of this theorem is that it provides weak compactness of bounded sequences. If $\mathcal{X}$ is separable this is a consequence of the Banach–Alaoglu theorem, as this convergence coincides with the weak-* convergence on the dual of $\mathcal{X}'$. Yet, separability is not necessary since one can restrict to the closure of the vector space generated by the sequence: this space is separable, closed subspaces of reflexive spaces are reflexive, and a space is reflexive and separable if and only if its dual is reflexive and separable.

Many variants of the above result are possible. A first possibility is to replace the positivity assumption on F with a lower bound of the form $F(x, u) \geq -a(x) - C|u|^r$ for $a \in L^1(\Omega)$ and $r < p$. In this case, we cannot immediately deduce a bound on $||\nabla u_n||_{L^p}$ from the bound on $J(u_n)$. Yet, we can act as follows:

$$J(u_n) \leq C \Rightarrow ||\nabla u_n||^p_{L^p} \leq C + C||u_n||^r_{L^r} \leq C + C||u_n||^r_{L^p} \leq C + C||\nabla u_n||^r_{L^p}$$

and the condition $r < p$ allows us to prove that we cannot have $||\nabla u_n||_{L^p} \to \infty$. This is enough to obtain the desired compactness but the lower semicontinuity of the functional J is still to be proven, since we cannot directly apply Fatou's lemma to F, which is no longer non-negative. The idea is then to define

$$\tilde{F}(x,u) := F(x,u) + C|u|^r$$

and write

$$J(u) = \tilde{J}(u) - C||u||^r_{L^r}, \quad \text{where } \tilde{J}(u) := \int_\Omega \left(\tilde{F}(x,u(x)) + |\nabla u(x)|^p\right) \mathrm{d}x.$$

The semicontinuity of $\tilde{J}$ can be handled exactly as in Theorem 2.1, while on the L^r part we need, because of the negative part, continuity (upper semicontinuity of the norm would be enough, but we already know that norms are lower semicontinuous). This is easy to handle because of the compact injection $W^{1,p} \hookrightarrow L^r$ (an injection which is compact also for $r = p$ and even for some $r > p$).

The reader may be disappointed that we need to require $r < p$, while we know that $W^{1,p}$ implies a summability better than L^p and that the injection is still compact. Indeed, it would be possible to handle the case where the functional includes a positive part of the form $||\nabla u_n||^p_{L^p}$ and a negative part of the form $||u_n||^{r_2}_{L^{r_1}}$ for $r_1 < p^*$, but we would still need $r_2 < p$ if we want to prove that minimizing sequences are bounded. Since only $r_1 = r_2$ gives a classical integral function, we only consider $r_1 = r_2 = r < p$.

Another variant concerns the Dirichlet boundary data. In dimension one we considered the case where each endpoint was either penalized or fixed, independently. A first natural question is how to deal with a Dirichlet boundary condition on a part of $\partial\Omega$ and not on the full boundary. In order to deal with the values on the boundary we recall the notion of the trace of a Sobolev function.

Box 2.4 Important Notion—*Traces of Sobolev Functions*

If Ω is a smooth open domain in $\mathbb{R}^d$ (we assume that its boundary has Lipschitz regularity, i.e. Ω can locally be written as the epigraph of a Lipschitz function) and $p > 1$, there exists an operator $\mathrm{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ (where the boundary $\partial\Omega$ is endowed with the natural surface measure $\mathcal{H}^{d-1}$) with the following properties

- Tr is a linear continuous and compact operator when $W^{1,p}(\Omega)$ and $L^p(\partial\Omega)$ are endowed with their standard norms;
- for every Lipschitz function $u : \Omega \to \mathbb{R}$ we have $\mathrm{Tr}[u] = u_{|\partial\Omega}$;
- the kernel of Tr coincides with $W^{1,p}_0(\Omega)$, i.e. with the closure in $W^{1,p}(\Omega)$ of $C^\infty_c(\Omega)$;

(continued)

Box 2.4 (continued)

- for every Lipschitz function $h : \mathbb{R} \to \mathbb{R}$ we have $\mathrm{Tr}[h(u)] = h(\mathrm{Tr}[u])$;
- the operator Tr actually takes values in a space $L^q(\partial\Omega)$ with $q > p$ and is continuous when valued in this space: we can take $q = (d-1)p/(d-p)$ when $p < d$, $q = \infty$ when $p > d$ and any $q \in (p, \infty)$ when $p = d$; when $p > d$ the operator is also valued and continuous in $C^{0,\alpha}$ with $\alpha = 1-d/p$.

For the theory of traces of Sobolev functions we refer to [84, Chapter 4].

We need the following Poincaré-like inequality, that we state in the most general form so as to be able to re-use it when dealing with boundary penalization instead of Dirichlet boundary data. We will summarize in the same statement both a result involving the trace and a result involving the behavior inside the domain.

Proposition 2.2 *Given a connected open domain Ω, for every $\varepsilon > 0$ there exist two constants C_1, C_2 such that we have*

1. *if $u \in W^{1,p}(\Omega)$ satisfies $\mathcal{H}^{d-1}(\{x \in \partial\Omega : \mathrm{Tr}[u](x) = 0\}) \geq \varepsilon$ then we have $||u||_{L^p} \leq C_1||\nabla u||_{L^p}$;*
2. *if $u \in W^{1,p}(\Omega)$ satisfies $|\{x \in \Omega : u(x) = 0\}| \geq \varepsilon$, then we have $||u||_{L^p} \leq C_2||\nabla u||_{L^p}$.*

The proof of the above proposition will be done by contradiction, which provides a useful technique to prove similar inequalities, even if the constants are in general neither explicit nor quantified.

Proof We suppose by contradiction that one of these constants, for a certain $\varepsilon > 0$, does not exist. This means that for every n we find a function $u_n \in W^{1,p}(\Omega)$ which violates the claimed inequality with $C = n$. Up to multiplying this function by a multiplicative constant we can assume $||u_n||_{L^p} = 1$ and $||\nabla u_n||_{L^p} < 1/n$. The sequence u_n is hence bounded in $W^{1,p}$ and a weakly convergent subsequence can be extracted. We call u the weak limit and the compact injection of $W^{1,p}$ into L^p provides $u_n \to u$ in L^p and hence $||u||_{L^p} = 1$ as well. We also have $||\nabla u||_{L^p} \leq \liminf_n ||\nabla u_n||_{L^p} = 0$, so that u is a constant function. The condition on its L^p norm implies $u = c$ with $c \neq 0$. Moreover, the convergence $u_n \to u = c$ is actually strong in $W^{1,p}$ since we also have $||\nabla u_n - \nabla u||_{L^p} = ||\nabla u_n||_{L^p} \to 0$.

We now distinguish the two cases. For the sake of the constant C_1 related to the behavior on the boundary, we observe that the strong convergence in $W^{1,p}$ implies the convergence of the trace, so that $\mathrm{Tr}[u_n]$ strongly converges to $\mathrm{Tr}[u]$ and we have $\mathrm{Tr}[u] = c$. Then we write

$$||\mathrm{Tr}[u_n] - \mathrm{Tr}[u]||^p_{L^p(\partial\Omega)} \geq c^p\mathcal{H}^{d-1}(\{x \in \partial\Omega : \mathrm{Tr}[u_n](x) = 0\}) \geq c^p\varepsilon,$$

which is a contradiction since the limit of the left-hand side is 0.

For the sake of the constant C_2 related to the behavior inside Ω, the situation is easier, since we already have strong convergence $u_n \to u$ in $L^p(\Omega)$. Yet, we have

$$||u_n - u||^p_{L^p(\partial\Omega)} \geq c^p |(\{x \in \Omega \,:\, u_n(x) = 0\})| \geq c^p \varepsilon,$$

and, again, this provides a contradiction when $n \to \infty$. □

The idea behind the inequality proven above is that the L^p norm of a function is bounded by the L^p norm of its gradient as soon as we impose sufficient conditions which prevent the function from being a non-zero constant. We note that the very same technique could also be used to prove another inequality of this form, which is the one concerning zero-mean functions, and is known as *Poincaré–Wirtinger* inequality.

Proposition 2.3 *Given a connected, Lipschitz, and open domain Ω there exists a constant C such that we have $||u||_{L^p} \leq C||\nabla u||_{L^p}$ for all functions $u \in W^{1,p}(\Omega)$ satisfying $⨍_\Omega u = 0$.*

Proof We suppose by contradiction that such a constant does not exist. For every n we find a function $u_n \in W^{1,p}(\Omega)$ with $||u_n||_{L^p} = 1$ and $||\nabla u_n||_{L^p} < 1/n$. As before, we extract a weakly convergent subsequence and call u the weak limit. The compact injection of $W^{1,p}$ into L^p provides again $u_n \to u$ in L^p and hence $||u||_{L^p} = 1$ as well. We also have $||\nabla u||_{L^p} \leq \liminf_n ||\nabla u_n||_{L^p} = 0$, so that u is a constant function because Ω is connected. We also have $⨍ u_n \to ⨍ u = 0$ so that $⨍ u = 0$. We have a contradiction since u is a constant with zero mean (hence $u = 0$) but $||u||_{L^p} = 1$. □

The above result on zero-mean functions was presented for the sake of completeness. We now come back to the notion of trace, which we are able to use in order to provide a more general existence result.

Theorem 2.4 *Given two measurable functions $\psi : \partial\Omega \times \mathbb{R} \to [0, +\infty]$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}_+$ such that $\psi(x, \cdot)$ is lower semicontinuous for $\mathcal{H}^{d-1}$-a.e. x and $F(x, \cdot)$ is lower semicontinuous for a.e. x, let us consider the problem*

$$\min \left\{ J(u) \,:\, u \in W^{1,p}(\Omega) \right\}, \tag{2.2}$$

where

$$J(u) := \int_\Omega \left(F(x, u(x)) + |\nabla u(x)|^p\right) \mathrm{d}x + \int_{\partial\Omega} \psi(x, \mathrm{Tr}[u](x))\, \mathrm{d}\mathcal{H}^{d-1}(x).$$

This minimization problem admits a solution provided one of the following conditions is satisfied for a function $g : \mathbb{R} \to \mathbb{R}_+$ with $\lim_{s\to\pm\infty} g(s) = +\infty$

1. *either there exists an $A \subset \partial\Omega$ with $\mathcal{H}^{d-1}(A) > 0$ and such that $\psi(x, s) \geq g(s)$ for all $x \in A$;*

2. *or there exists a* $B \subset \Omega$ *with positive Lebesgue measure* $|B| > 0$ *and such that* $F(x, s) \geq g(s)$ *for all* $x \in B$.

Proof Take a minimizing sequence u_n with $J(u_n) \leq C$.

We first analyze the case (1). From the positivity of all the terms in the optimization problem we can deduce that the set $\{x \in A : g(\mathrm{Tr}[u_n](x)) > \ell\}$ has measure at most $\mathcal{H}^{d-1}(A) - C/\ell$ so that, taking $\ell = 2C/\mathcal{H}^{d-1}(A)$, we have $g(\mathrm{Tr}[u_n](x)) \leq \ell$ on a set $A_n \subset A$ with $\mathcal{H}^{d-1}(A_n) \geq \mathcal{H}^{d-1}(A)/2$. Using the condition on g we deduce that there exists an M independent of n such that $|(\mathrm{Tr}[u_n](x)| \leq M$ on A_n. If instead, we are in the situation described in case 2), we see analogously that we have a set $B_n \subset B$ with $|B_n| \geq |B|/2$ and $|u| \leq M$ on B_n.

In both cases, defining the function $h : \mathbb{R} \to \mathbb{R}$ as

$$h(s) = \begin{cases} s - M & \text{if } s > M, \\ 0 & \text{if } |s| \leq M, \\ s + M & \text{if } s < -M, \end{cases}$$

we can apply the bounds provided by Proposition 2.2 to $v_n = h(u_n)$. Using $|h'| \leq 1$ the norm $||\nabla v_n||_{L^p}$ is bounded, and so is the L^p norm of v_n. Using $|v_n - u_n| \leq M$ this implies a bound on $||u_n||_{L^p}$ and then on $||u_n||_{W^{1,p}}$.

We can then extract a subsequence which weakly converges in $W^{1,p}$ to a function u. We have $u_n \to u$ in $L^p(\Omega)$ as well as $\mathrm{Tr}[u_n] \to \mathrm{Tr}[u]$ in $L^p(\partial\Omega)$ (since both the injection of $W^{1,p}$ into L^p and the trace operator are compact). This means, up to a subsequence, a.e. convergence $u_n(x) \to u(x)$ in Ω and $\mathrm{Tr}[u_n](x) \to \mathrm{Tr}[u](x)$ a.e. for the $\mathcal{H}^{d-1}$ measure on $\partial\Omega$. Fatou's lemma provides the semi-continuity of both the integral term $\int_\Omega F(x, u(x))\,\mathrm{d}x$ and the boundary integral $\int_{\partial\Omega} \psi(x, \mathrm{Tr}[u](x))\,\mathrm{d}\mathcal{H}^{d-1}(x)$, and the gradient term is treated by semicontinuity of the norm as usual. This shows that u is a minimizer. □

It is important to observe that the assumption on the boundary penalization ψ in the above theorem allows us to consider the case

$$\psi(x, s) = \begin{cases} 0 & \text{if } x \notin A, \\ 0 & \text{if } x \in A, s = h(x), \\ +\infty & \text{if } x \in A, s \neq h(x), \end{cases}$$

which is equivalent to imposing the Dirichlet boundary condition $u = h$ on $A \subset \partial\Omega$. In particular, the function ψ satisfies $\psi(x, s) \geq g(s) := (s - M)_+$ for all $x \in A$ with $|h(x)| \leq M$, a set of points x which is of positive measure for large M.

2.2 The Multi-Dimensional Euler–Lagrange Equation

This section is just a translation to higher dimensions of Sect. 1.3. We consider an open domain $\Omega \subset \mathbb{R}^d$ and an integrand $L : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, where the variables will be called x (the position, which is now the independent variable, replacing time), s (the value of the function) and v (its variation, which here is a gradient vector). The function L will be assumed to be C^1 in (u, v) and its partial derivatives w.r.t. s and v_i will be denoted by $\partial_s L$ and $\partial_{v_i} L$, respectively (and the gradient vector with all the corresponding partial derivatives w.r.t. v, by $\nabla_v L$).

As we did in Chap. 1, we write the optimality conditions for multi-dimensional variational problems starting from the case where the boundary data are fixed. For simplicity, in particular in order to handle the boundary, we will stick to the case where the functional space X is a Sobolev space $W^{1,p}(\Omega)$. In 1D we presented a more abstract situation where X was a generic functional space contained in C^0 but in higher dimensions few spaces are at the same time a reasonable choice for the most classical optimization problems and are made of continuous functions, so we prefer to directly choose the Sobolev formalism, and the boundary data will be imposed using the space $W_0^{1,p}$. We then face

$$\min \left\{ J(u) := \int_\Omega L(x, u(x), \nabla u(x))\,\mathrm{d}x \ : \ u \in W^{1,p}(\Omega),\ u - g \in W_0^{1,p}(\Omega) \right\}.$$

We assume a lower bound of the form $L(x, u, v) \geq -a(x) - c(|u|^p + |v|^p)$ for $a \in L^1$, so that for every $u \in W^{1,p}(\Omega)$ the negative part of $L(\cdot, u, \nabla u)$ is integrable,[1] thus J is a well-defined functional from $W^{1,p}(\Omega)$ to $\mathbb{R} \cup \{+\infty\}$. We assume that u is a solution of such a minimization problem. Since all functions in $u + C_c^\infty(\Omega)$ are admissible competitors, for every $\varphi \in C_c^\infty(\Omega)$ we have $J(u) \leq J(u + \varepsilon\varphi)$ for small ε.

We now fix the minimizer u and a perturbation φ and consider the one-variable function

$$j(\varepsilon) := J(u_\varepsilon), \quad \text{where} \quad u_\varepsilon := u + \varepsilon\varphi,$$

which is defined in a neighborhood of $\varepsilon = 0$, and minimal at $\varepsilon = 0$. We compute exactly as in Chap. 1 the value of $j'(0)$.

The assumption to justify the differentiation under the integral sign will be the same as in Chap. 1: we assume that for every $u \in W^{1,p}(\Omega)$ with $J(u) < +\infty$ there exists a $\delta > 0$ such that we have

$$x \mapsto \sup\left\{ |\partial_s L(x, s, v)| + |\nabla_v L(x, s, v)| \ : \ |s - u(x)| < \delta,\ v \in B(\nabla u(x)), \delta) \right\} \in L^1(\Omega). \tag{2.3}$$

[1] Actually, exploiting $W^{1,p} \subset L^{p^*}$, we could accept a weaker lower bound in terms of u, i.e. $L(x, u, v) \geq -a(x) - c(|u|^{p^*} + |v|^p)$.

Examples of sufficient conditions on L in order to guarantee (2.3) are the same as in 1D. Under these conditions, we can differentiate w.r.t. ε the function $\varepsilon \mapsto L(t, u_\varepsilon, \nabla u_\varepsilon)$, and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(t, u_\varepsilon, \nabla u_\varepsilon) = \partial_s L(x, u_\varepsilon, \nabla u_\varepsilon)\varphi + \nabla_v L(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\varphi.$$

Then, using the fact that both φ and $\nabla\varphi$ are bounded, we can apply the assumption in (2.3) in order to obtain domination in L^1 of the pointwise derivatives and thus, for small ε, we have

$$j'(\varepsilon) = \int_\Omega (\partial_s L(x, u_\varepsilon, \nabla u_\varepsilon)\varphi + \nabla_v L(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\varphi)\, \mathrm{d}x$$

and

$$j'(0) = \int_\Omega (\partial_s L(x, u, \nabla u)\varphi + \nabla_v L(x, u, \nabla u) \cdot \nabla\varphi)\, \mathrm{d}x.$$

Imposing $j'(0) = 0$, which comes from the optimality of u, means precisely that we have, in the sense of distributions, the following partial differential equation, also known as the *Euler–Lagrange* equation:

$$\nabla \cdot (\nabla_v L(x, u, \nabla u)) = \partial_s L(x, u, \nabla u).$$

This second-order partial differential equation (PDE) is usually of elliptic type, because of the standard assumption (which is satisfied in all the examples that we saw for the existence of a solution and will be natural from the lower semicontinuity theory that will be developed in Chap. 3) that L is convex in v, so that $\nabla_v L(x, u, \nabla u)$ is a monotone function of ∇u. It is formally possible to expand the divergence and indeed obtain

$$\begin{aligned}
\nabla \cdot (\nabla_v L(x, u, \nabla u)) &= \sum_j \frac{d}{\mathrm{d}x_j} \left(\partial_{v_j} L(x, u, \nabla u)\right) \\
&= \sum_j \left(\partial^2_{x_j, v_j} L(x, u, \nabla u) + \partial^2_{u, v_j} L(x, u, \nabla u) u_j\right) \\
&\quad + \sum_{j,k} \partial^2_{v_k, v_j} L(x, u, \nabla u) u_{j,k},
\end{aligned}$$

where u_j and $u_{j,k}$ stand for the various partial derivatives of the solution u. This computation shows that the second-order term in this PDE is ruled by the matrix $A^{j,k}(x) := \partial^2_{v_k, v_j} L(x, u(x), \nabla u(x))$, which is (semi-)positive-definite if L is convex in v.

It is important to stress the exact meaning of the Euler–Lagrange equation.

Box 2.5 Memo—*Distributions*

We call a distribution on an open domain Ω any linear operator T on $C_c^\infty(\Omega)$ with the following continuity property: for every compact subset $K \subset \Omega$ and every sequence $\phi_n \in C_c^\infty(\Omega)$ with $\operatorname{spt}\phi_n \subset K$ and such that all derivatives $D^\alpha\phi_n$ of any order of ϕ_n uniformly converge on K to the corresponding derivative $D^\alpha\phi$ we have $\langle T, \phi_n\rangle \to \langle T, \phi\rangle$.

On the set of distributions we define a notion of convergence: we say $T_n \to T$ if for every $\phi \in C_c^\infty(\Omega)$ we have $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$.

L^1_{loc} functions are examples of distributions in the sense that for any $f \in L^1_{loc}(\Omega)$ we can define a distribution T_f by $\langle T_f \phi\rangle := \int_\Omega f\phi$. Locally finite measures are also distributions (with $\langle \mu, \phi\rangle := \int \phi \, d\mu$).

Some operations are possible with distributions: the derivative $\partial_{x_i} T$ is defined via the integration-by-parts formula $\langle \partial_{x_i} T, \phi\rangle := -\langle T, \partial_{x_i}\phi\rangle$; the product with a C^∞ function u by $\langle uT, \phi\rangle := \langle T, u\phi\rangle$, and the convolution (for simplicity with an even function η with compact support) by $\langle \eta * T, \phi\rangle := \langle T, \eta * \phi\rangle$. It is possible to show that, whenever $\eta \in C_c^\infty$, then $\eta * T$ is a distribution associated with a locally integrable function which is C^∞. If η is not C^∞ but less smooth, then the regularity of $\eta * T$ depends both on that of η and of T.

A distribution is said to be of order m on a compact set K whenever $|\langle T, \phi\rangle| \le C||\phi||_{C^m(K)}$ for all functions $\phi \in C_c^\infty(\Omega)$ with $\operatorname{spt}\phi \subset K$. When convolving a function $\eta \in C^k$ with a convolution T of order m we have $\eta * T \in C^{k-m}$ if $k \ge m$ while, if $k < m$, we obtain a distribution $\eta * T$ of order $m - k$.

Whenever a function $f \in L^1_{loc}$ belongs to a Sobolev space $W^{1,p}$ it is easy to see that the derivative in terms of distributions coincides with that in Sobolev spaces. The notion of convergence is also essentially the same: for every space $W^{k,p}$ with $1 < p < \infty$ a sequence $u_n \rightharpoonup u$ weakly converges in $W^{k,p}$ to u if and only if it is bounded in $W^{k,p}$ and converges in the sense of distributions to u (in the sense that the associated distributions converge).

It is easy to see that $T_n \to T$ in the sense of distributions implies the convergence of the derivatives $\partial_{x_i} T_n \to \partial_{x_i} T$. The convergence in the sense of distributions is essentially the weakest notion of convergence that is used in functional analysis and is implied by all other standard notions of convergence.

Indeed, in the expression that we obtained by expanding the divergence, the terms $\partial^2_{v_k,v_j} L(x, u, \nabla u) u_{j,k}$ are the product of a measurable function (whose regularity is the same as that of ∇u, which is just summable) and the Hessian of a Sobolev function, which is in general a distribution but not a function. Since distributions can only be multiplied by C^∞ functions, our Euler–Lagrange equation cannot be written in the distributional sense in this way. Indeed, the precise meaning of the PDE under the current assumptions is the following: the vector-valued function $\nabla_v L(x, u, \nabla u)$ is an L^1 function whose distributional divergence equals $\partial_s L(x, u, \nabla u)$ since it satisfies

$$\int_\Omega (\partial_s L(x, u, \nabla u)\varphi + \nabla_v L(x, u, \nabla u) \cdot \nabla\varphi)\, dx = 0$$

for every $\varphi \in C_c^\infty(\Omega)$. In dimension one, it was easy to deduce from this condition extra regularity for u, at least on some examples, since, for instance, having a C^0 distributional derivative is equivalent to being C^1. In higher dimensions we do not have an equation on each partial derivative of $\nabla_v L(x, u, \nabla u)$, but only on its divergence, which makes it difficult to deduce regularity.

A classical example is the case $L(x, u, v) = \frac{1}{2}|v|^2$, i.e. the minimization of

$$H^1(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^2$$

with prescribed boundary conditions. In this case we have $\nabla_v L(x, u, \nabla u) = \nabla u$ and the equation is $\nabla \cdot \nabla u = 0$. This corresponds to the Laplace equation $\Delta u = 0$ which would mean, in the distributional sense, $\int_\Omega u\Delta\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$. Here, instead, since we know $u \in H^1$ and hence ∇u is a well-defined integrable function, we require $\int_\Omega \nabla u \cdot \nabla\varphi = 0$. We will see anyway that these two conditions coincide and both imply $u \in C^\infty$ and $\Delta u = 0$ in the most classical sense (see Sect. 2.3, devoted to the study of harmonic functions and distributions). On the other hand, the situation could be trickier when the equation is non-linear, or if there is an explicit dependence on x and/or some low-regularity lower order terms, and in this case the regularity of the optimal solution u is usually obtained from the Euler–Lagrange equation by choosing suitable test functions φ. As we will see in Chap. 5, for instance, it is often useful to choose a function φ related to the solution u itself. This requires the weak formulation of the Euler–Lagrange equation to be extended to test functions which, instead of being C^∞, have the same regularity as u. This can be done via the following lemma.

Lemma 2.5 *Let $p > 1$ be a given exponent, with $p' = \frac{p}{p-1}$ its dual exponent, and let $z \in L^{p'}(\Omega; \mathbb{R}^d)$ be a vector field such that $\int_\Omega z \cdot \nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$. Then we also have $\int_\Omega z \cdot \nabla\varphi = 0$ for all $\varphi \in W_0^{1,p}(\Omega)$. This applies in particular to the case $z = \nabla_v L(x, u, \nabla u)$ whenever $u \in W^{1,p}$ and L satisfies $|\nabla_v L(x, u, v)| \leq C(1 + |v|^{p-1})$.*

Proof The condition $z \in L^{p'}$ implies that $L^p \ni v \mapsto \int z \cdot v$ is continuous in L^p and hence $W^{1,p} \ni \varphi \mapsto \int z \cdot \nabla\varphi$ is continuous in $W^{1,p}$. Thus, if this linear form vanishes on $C_c^\infty(\Omega)$, it also vanishes on its closure for the $W^{1,p}$ norm, i.e. on $W_0^{1,p}(\Omega)$.

For the second part of the claim, note that whenever $u \in W^{1,p}(\Omega)$ we have $|\nabla u|^{p-1} \in L^{\frac{p}{p-1}}$, which implies, under the growth assumption of the claim, $\nabla_v L(x, u, \nabla u) \in L^{p'}$. □

Boundary and Transversality Conditions We switch now to the case where no boundary value is prescribed. Exactly as in the 1D case we can consider $u_\varepsilon = u+\varepsilon\varphi$ for arbitrary $\varphi \in C^\infty$, without imposing a zero boundary value or compact support for φ. The same computations as before provide the necessary optimality condition

$$\int_\Omega (\partial_s L(x, u, \nabla u)\varphi + \nabla_v L(x, u, \nabla u) \cdot \nabla\varphi)\, dx = 0 \quad \text{for all } \varphi \in C^\infty(\Omega).$$

Distributionally, this corresponds to saying that the vector field $x \mapsto \nabla_v L(x, u, \nabla u)$ extended to 0 outside Ω has a distributional divergence equal to the scalar function $\partial_s L(x, u, \nabla u)$, also extended to 0 outside Ω. A formal integration by parts would instead give

$$0 = \int_\Omega (\partial_s L(x, u, \nabla u) - \nabla \cdot (\nabla_v L(x, u, \nabla u)))\, \varphi\, dx + \int_{\partial\Omega} \nabla_v L(x, u, \nabla u) \cdot \mathbf{n}\, \varphi\, d\mathcal{H}^{d-1},$$

where $\mathbf{n}$ is the normal vector to $\partial\Omega$, so that besides the differential condition inside Ω we also obtain a Neumann-like boundary condition on $\partial\Omega$, i.e. we obtain a weak version of the system

$$\begin{cases} \nabla \cdot (\nabla_v L(x, u, \nabla u)) = \partial_s L(x, u, \nabla u), & \text{in } \Omega, \\ \nabla_v L(x, u, \nabla u)) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

In the particular case where $L(x, sv) = \frac{1}{2}|v|^2 + F(x, s)$ the boundary condition is exactly a homogeneous Neumann condition $\partial u/\partial\mathbf{n} = 0$.

It is of course possible to consider mixed cases where part of the boundary is subject to a Dirichlet boundary condition and part of the boundary is either free or penalized. We then consider the optimization problem

$$\min \left\{ \begin{array}{r} J(u) := \int_\Omega L(x, u(x), \nabla u(x))\, dx + \int_{\partial\Omega} \psi(x, \mathrm{Tr}[u])\, d\mathcal{H}^{d-1} : \\ u \in W^{1,p}(\Omega), \mathrm{Tr}[u] = g \text{ on } A \end{array} \right\}, \tag{2.4}$$

where $A \subset \partial\Omega$ is a prescribed part of the boundary (which is possibly empty) and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a prescribed penalization function, which could be zero for x in some parts of $\partial\Omega$ and is irrelevant for $x \in A$. In order to perform a differentiation under the integral sign we need to add to condition (2.3) a similar condition on the boundary, i.e.

$$x \mapsto \sup \{|\partial_s \psi(x,s)| \,:\, |s - \mathrm{Tr}[u](x)| < \delta\} \in L^1(\partial\Omega \setminus A) \tag{2.5}$$

for every $u \in W^{1,p}$ with $J(u) < +\infty$. The conditions to guarantee this bound are similar to those for the corresponding bound on L. Condition (2.5) is in particular satisfied when $p \le d$ every time that $\partial_s \psi(x,s)$ has a growth of order q in s (i.e. $|\partial_s \psi(x,s)| \le C(1+|s|)^q$) and $\mathrm{Tr}[u] \in L^q(\partial\Omega)$ (in particular, we can take $q = p$ since the trace operator is valued in $L^p(\partial\Omega)$, but also $q = (d-1)p/(d-p)$, if we use the last property of the trace operator in Box 2.4). In the case $p > d$ any local bound on $\partial_s \psi$ is enough since $\mathrm{Tr}[u]$ is bounded, in particular this can apply to the case where ψ is differentiable in s with $\partial_s \psi \in C^0(\partial\Omega \times \mathbb{R})$.

A computation which will be now standard for the reader provides the necessary optimality condition for the problem (2.4): u should satisfy

$$\int_\Omega (\partial_s L(x,u,\nabla u)\varphi + \nabla_v L(x,u,\nabla u)\cdot \nabla\varphi)\,\mathrm{d}x + \int_{\partial\Omega} \partial_s \psi(x,\mathrm{Tr}[u])\varphi\,\mathrm{d}\mathcal{H}^{d-1} = 0 \tag{2.6}$$

for all $\varphi \in C_c^\infty(\overline{\Omega} \setminus A)$.

When $A = \emptyset$, distributionally this means that the vector field $x \mapsto \nabla_v L(x,u,\nabla u)$ extended to 0 outside Ω has a distributional divergence equal to the scalar function $\partial_s L(x,u,\nabla u)$, also extended to 0 outside Ω, plus a singular measure concentrated on $\partial\Omega$ with a density w.r.t. the surface measure $\mathcal{H}^{d-1}$, this density being equal to $\partial_s \psi(x,\mathrm{Tr}[u])$. When $A \neq \emptyset$, this distributional condition is satisfied on the open set $\mathbb{R}^d \setminus \overline{A}$.

A formal integration by parts provides the Euler–Lagrange system

$$\begin{cases} \nabla\cdot(\nabla_v L(x,u,\nabla u)) = \partial_s L(x,u,\nabla u), & \text{in } \Omega, \\ \nabla_v L(x,u,\nabla u))\cdot \mathbf{n} = -\partial_s \psi(x,u) & \text{on } \partial\Omega \setminus A \\ u = g & \text{on } A, \end{cases} \tag{2.7}$$

where we dared to write u instead of $\mathrm{Tr}[u]$ since, anyway, the condition is quite formal.

2.3 Harmonic Functions

This section will be devoted to the study of harmonic functions, which are the solutions of the simplest and most classical variational problem, i.e. the minimization of the Dirichlet energy $\int |\nabla u|^2$ among functions with prescribed values on the boundary of a given domain. However, we will not make use here of the variational properties of these functions, but only of the equation they solve. Recall that we denote by Δ the differential operator given by

$$\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

We will start from the case of a C^2 function u satisfying $\Delta u = 0$ in a given open domain Ω. The first property that we consider is the mean value formula, which connects the value of u at a point x_0 to the averages of u on balls or spheres around x_0.

Proposition 2.6 *If $u \in C^2(\Omega)$ and $\Delta u = 0$ then, for every point x_0 and $R > 0$ such that $B(x_0, R) \subset \Omega$, we have the following properties*

- $(0, R) \ni r \mapsto \fint_{\partial B(x_0,r)} u \, \mathrm{d}\mathcal{H}^{d-1}$ *is constant and equal to $u(x_0)$;*
- $(0, R) \ni r \mapsto \fint_{B(x_0,r)} u \, \mathrm{d}x$ *is also constant and equal to $u(x_0)$.*

Proof We start from the function $h(r) := \fint_{\partial B(x_0,r)} u \, \mathrm{d}\mathcal{H}^{d-1}$, which we can write as $h(r) = \fint_{\partial B(0,1)} u(x_0 + re) \, \mathrm{d}\mathcal{H}^{d-1}(e)$. The smoothness of u allows us to differentiate and obtain

$$h'(r) = \fint_{\partial B(0,1)} \nabla u(x_0 + re) \cdot e \, \mathrm{d}\mathcal{H}^{d-1}(e) = \fint_{\partial B(0,1)} \nabla u(x_0 + re) \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{d-1}(e),$$

where we used that, for every point e on the boundary of the unit ball, the vector e coincides with the outward normal vector. The last integral can be then re-transformed into an integral on the sphere $\partial B(x_0, r)$, and it equals

$$\fint_{\partial B(0,r)} \nabla u \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{d-1}.$$

We can then integrate by parts

$$\int_{\partial B(0,r)} \nabla u \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{d-1} = \int_{B(0,r)} \Delta u \, \mathrm{d}x = 0,$$

which proves that h is constant as soon as $\Delta u = 0$. Since u is continuous, we have $\lim_{r \to 0} h(r) = u(x_0)$, so that we have $h(r) = u(x_0)$ for every r, which proves the first part of the claim.

We then use the polar coordinate computation

$$\fint_{B(0,r)} u \, \mathrm{d}x = \frac{\int_0^r s^{d-1} h(s) \, \mathrm{d}s}{\int_0^r s^{d-1} \, \mathrm{d}s} = u(x_0),$$

to obtain the second part of the claim. □

Remark 2.7 It is clear from the above proof that if one replaces the assumption $\Delta u = 0$ with $\Delta u \geq 0$ then the functions $r \mapsto ⨍_{\partial B(x_0,r)} u \, d\mathcal{H}^{d-1}$ and $r \mapsto ⨍_{B(x_0,r)} u \, dx$ are no longer constant but non-decreasing. Functions with non-negative Laplacian are called *sub-harmonic*.

A key point about harmonic functions is that if u is harmonic and C^∞, then all derivatives of u are also harmonic. This allows us to prove the following bounds.

Proposition 2.8 *If $u \in C^\infty(\Omega)$ and $\Delta u = 0$ then, for every point x_0 and $R > 0$ such that $B(x_0, R) \subset \Omega \subset \mathbb{R}^d$, we have*

$$|\nabla u(x_0)| \leq \frac{d}{R} ||u||_{L^\infty(B(x_0,R))}.$$

Proof We consider the vector function ∇u, which is also harmonic (each of its components is harmonic) and use

$$\nabla u(x_0) = ⨍_{B(x_0,R)} \nabla u \, dx = \frac{1}{\omega_d R^d} \int_{B(x_0,R)} \nabla u \, dx = \frac{1}{\omega_d R^d} \int_{\partial B(x_0,R)} u \mathbf{n} \, d\mathcal{H}^{d-1},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. To establish the last equality we used an integration by parts: for every constant vector e we have $\nabla \cdot (ue) = \nabla u \cdot e$ and $\int_A \nabla \cdot (ue) = \int_{\partial A} ue \cdot \mathbf{n}$ for every open set A; the vector e being arbitrary, this provides $\int_A \nabla u = \int_{\partial A} u\mathbf{n}$.

We finish the estimate by bounding $\left| \int_{\partial B(x_0,R)} u\mathbf{n} \, d\mathcal{H}^{d-1} \right|$ by the maximal norm of $u\mathbf{n}$ times the measure of the surface, i.e. by $||u||_{L^\infty(B(x_0,R))} \mathcal{H}^{d-1}(\partial B(x_0, R))$ or, equivalently, by $||u||_{L^\infty(B(x_0,R))} d\omega_d R^{d-1}$. □

In order to analyze higher-order derivatives of u we recall the notation with multi-indices: a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a vector of natural numbers indicating how many times we differentiate w.r.t. each variable, and we write

$$D^\alpha u := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}} u,$$

where $|\alpha| := \sum_{i=1}^d \alpha_i$ is the order of the multi-index α. A consequence of the previous estimate, if iteratively applied to the derivatives of u, is the following result.

Proposition 2.9 *If $u \in C^\infty(\Omega)$ and $\Delta u = 0$ then, for every point x_0 and $R > 0$ such that $B(x_0, R) \subset \Omega \subset \mathbb{R}^d$ and every multi-index α with $|\alpha| = m$, we have*

$$|D^\alpha u(x_0)| \leq \frac{d^m e^{m-1} m!}{R^m} ||u||_{L^\infty(B(x_0,R))}. \tag{2.8}$$

Proof Given x_0 and r, assume $B(x_0, mr) \subset \Omega$. We can find a finite family of harmonic functions v_k, $k = 0, \ldots, m$ such that each v_{k+1} is a derivative (with respect to a suitable variable x_i) of the previous v_k, with $v_0 = u$, and $v_m = D^\alpha u$. Applying iteratively Proposition 2.8 we obtain $|v_k(x)| \le \frac{d}{r}||v_{k-1}||_{L^\infty(B(x,r))}$, so that we have

$$|D^\alpha u(x_0)| = |v_m(x_0)| \le \frac{d}{r}||v_{m-1}||_{L^\infty(B(x,r))} \le \frac{d^2}{r^2}||v_{m-2}||_{L^\infty(B(x,2r))} \le \dots$$
$$\dots \le \frac{d^k}{r^k}||v_{m-k}||_{L^\infty(B(x,kr))} \le \frac{d^m}{r^m}||u||_{L^\infty(B(x,mr))}.$$

We then take $r = R/m$ and get

$$|D^\alpha u(x_0)| \le \frac{m^m d^m}{R^m}||u||_{L^\infty(B(x,R))}.$$

We then apply the inequality $m^m \le \mathrm{e}^{m-1}m!$ and conclude.

To prove this inequality we observe that we have $m\log m = m\log m - 1\log 1 = \int_1^m(\log s + 1)\,\mathrm{d}s \le \sum_{k=2}^m(\log k + 1) = \log(m!) + m - 1$, which gives exactly the desired inequality when taking the exponential on both sides.[2] □

With this precise estimate in mind we can also prove the following fact.

Proposition 2.10 *C^∞ harmonic functions are analytic.*

We recall that analytic functions are those functions which locally coincide with their Taylor series around each point.

Proof For a C^∞ function u we can write its Taylor expansion as

$$u(x) = u(x_0) + \sum_{k=1}^m \sum_{\alpha\,:\,|\alpha|=k} \frac{1}{\alpha!}D^\alpha u(x_0)x^\alpha + R(m+1, x),$$

where the notation $\alpha!$ stands for $\alpha_1!\alpha_2!\dots\alpha_d!$, the notation x^α for $x_1^{\alpha_1}x_2^{\alpha_2}\dots x_d^{\alpha_d}$ and the remainder $R(m+1, x)$ has the form

$$R(m+1, x) = \sum_{\alpha\,:\,|\alpha|=m+1} \frac{1}{\alpha!}D^\alpha u(\xi)x^\alpha$$

for a certain point $\xi \in [x_0, x]$. In order to prove that the function u is analytic we need to prove $R(m, x) \to 0$ as $m \to \infty$ for x in a neighborhood of x_0. Assume $B(x_0, 2R) \subset \Omega$; if $x \in B(x_0, r)$ the point ξ also belongs to the same ball and if

[2] A refinement of this argument provides the well-known Stirling formula for the factorial $n! = \sqrt{2\pi n}(n/\mathrm{e})^n(1 + O(1/n))$.

$r \le R$ we have $B(\xi, R) \subset B(x_0, 2R) \subset \Omega$. Then, we can estimate, thanks to (2.8),

$$|R(m, x)| \le \sum_{\alpha : |\alpha|=m} \frac{1}{\alpha!} r^m \frac{d^m e^{m-1} m!}{R^m} ||u||_{L^\infty(B(x_0,2R))}$$
$$= C \left(\frac{d \cdot e \cdot r}{R} \right)^m \sum_{\alpha : |\alpha=m} \frac{m!}{\alpha!}.$$

We then use the well-known formula for multinomial coefficients

$$\sum_{\alpha : |\alpha|=m} \frac{m!}{\alpha!} = d^m$$

and obtain

$$|R(m, x)| \le C \left(\frac{d^2 \cdot e \cdot r}{R} \right)^m .$$

The convergence is guaranteed as soon as $r < \frac{R}{d^2 e}$, which proves the analyticity of u. □

Another consequence of Proposition 2.9 is the following regularity result.

Proposition 2.11 *If $u \in L^1_{loc}(\Omega)$ satisfies $\Delta u = 0$ in the distributional sense, then $u \in C^\infty(\Omega)$.*

Proof Given a sequence (with $\varepsilon \to 0$) of standard convolution kernels η_ε we consider $u_\varepsilon := u * \eta_\varepsilon$. We consider a ball $B(x_0, R)$ compactly contained in Ω; we have $u_\varepsilon \in C^\infty(B(x_0, R))$ for small ε and $\Delta u_\varepsilon = 0$ in the distributional, and hence classical, sense, in the same ball. Moreover, $||u_\varepsilon||_{L^1(B(x_0,R))}$ is bounded independently of ε because of $u \in L^1_{loc}(\Omega)$. We deduce a uniform bound, independent of ε, on $||u_\varepsilon||_{L^\infty(B(x_0,R/2))}$, as a consequence of the mean formula. Indeed, for each $x \in B(x_0, R/2)$ we have

$$u_\varepsilon(x) = \fint_{B(x,R/2)} u_\varepsilon(y)\, dy = \frac{1}{\omega_d (R/2)^d} \int_{B(x,R/2)} u_\varepsilon(y)\, dy$$

and using $B(x, R/2) \subset B(x_0, R)$ we obtain $|u_\varepsilon(x)| \le C||u_\varepsilon||_{L^1(B(x_0,R))}$. Then, the estimate (2.8) allows us to bound all the norms $||D^\alpha u_\varepsilon||_{L^\infty(B(x_0,R/4))}$ by $||u_\varepsilon||_{L^\infty(B(x_0,R/2))}$. This shows that, on the ball $B(x_0, R/4)$, all derivatives of u_ε are bounded by a constant independent of ε. Using $u_\varepsilon \to u$ in the distributional sense, we obtain $u \in C^k(B(x_0, R/4))$ for every k and the same bounds on the derivatives are satisfied by u. This shows $u \in C^\infty(B(x_0, R/4))$ and, the point $x_0 \in \Omega$ being arbitrary, $u \in C^\infty(\Omega)$. □

Box 2.6 Memo—*Convolutions and Approximation*

Given two functions u, v on $\mathbb{R}^d$ we define their convolution $u * v$ via

$$(u * v)(x) := \int_{\mathbb{R}^d} u(x - h)v(h)\,\mathrm{d}h$$

when this integral makes sense, for instance when one function is L^1 and the other L^∞. Via a simple change of variables we see that we have $(u * v)(x) := \int u(x')v(x - x')$, so that the convolution is commutative and $u * v = v * u$. When $u, v \in L^1$ it is possible to prove that $u * v$ is well-defined for a.e. x and it is an L^1 function itself.

In the particular case where $v \geq 0$ and $\int v = 1$ we can write $u * v = \fint u_h\,\mathrm{d}v(h)$, where u_h is the translation of u defined via $u_h(x) := u(x - h)$ and we identify v with a probability measure. This means that $u*v$ is a suitable average of translations of u.

A very common procedure consists in using a function $\eta \in C_c^\infty$ with $\eta \geq 0$ and $\int \eta = 1$, and taking $v_n(x) := n^d\eta(nx)$, which is also a probability density, supported in a ball whose radius is $O(1/n)$. It is clear that when u is uniformly continuous we have $u * v_n \to u$ uniformly. Moreover, using the density of C_c^∞ in L^p, it is possible to prove that for any $u \in L^p$ we have $u * v_n \to u$ strongly in L^p.

Derivatives of convolutions of smooth functions can be computed very easily: we have $\partial_{x_i}(u * v) = (\partial_{x_i} u) * v$ as a consequence of the differentiation under the integral sign. Using $u * v = v * u$, the derivative can be taken on any of the two functions. If one of the functions is C^k and the other C^m the convolution is C^{k+m}, and if one is C^k and the other is only L^1 the convolution inherits the regularity of the best one. In particular, for $u \in L^1$ the sequence $u_n := u * v_n$ defined above is an approximation of u made of C^∞ functions.

A striking fact is the following: if F is any functional defined on a functional space which is convex and invariant under translations (which is the case, for instance, for all the L^p norms, but also for norms depending on the derivatives) and v is a probability measure, we necessarily have $F(u * v) \leq F(u)$.

Finally, it is also possible to define convolutions for distributions, thanks to the procedure in Box 2.5. When using the convolution kernel v_n above, for any distribution T we have $T * v_n \to T$ in the sense of distributions.

We want to go on with our analysis, proving that all harmonic distributions are actually analytic functions. In order to proceed, we first need to make a short digression about the equation $\Delta u = f$ and its fundamental solution. We introduce the fundamental solution of the Laplacian as the function Γ given by

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } d = 2, \\ -\frac{1}{d(d-2)\omega_d}|x|^{2-d} & \text{if } d > 2. \end{cases}$$

We can see that we have

- $\Gamma \in L^1_{loc}$, $\nabla\Gamma \in L^1_{loc}$, but $D^2\Gamma \notin L^1_{loc}$.
- $\int_{\partial B(0,R)} \nabla\Gamma \cdot \mathbf{n} = 1$ for every R.
- $\Gamma \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\Delta\Gamma = 0$ on $\mathbb{R}^d \setminus \{0\}$.
- $\Delta\Gamma = \delta_0$ in the sense of distributions.

As a consequence, for every $f \in C_c^\infty(\mathbb{R}^d)$, the function $u = \Gamma * f$ is a smooth function solving $\Delta u = f$ in the classical sense. It is of course not the unique solution to this equation, since we can add to u arbitrary harmonic functions.

For this solution, we have the following estimate

Proposition 2.12 *Given $f \in C_c^\infty(\mathbb{R}^d)$, let u be given by $u = \Gamma * f$. Then we have $\int_{\mathbb{R}^d} |D^2u|^2 = \int_{\mathbb{R}^d} |f|^2$, where $|D^2u|^2$ denotes the squared Frobenius norm of the Hessian matrix, given by $|A|^2 = \mathrm{Tr}(A^tA) = \sum_{i,j} A_{ij}^2$.*

Proof We consider a ball $B(0, R)$ and obtain, by integration by parts,

$$\begin{aligned}\int_{B(0,R)} f^2\,\mathrm{d}x = \int_{B(0,R)} |\Delta u|^2\,\mathrm{d}x &= \sum_{i,j}\int_{B(0,R)} u_{ii}u_{jj}\,\mathrm{d}x \\ &= -\sum_{i,j}\int_{B(0,R)} u_{iij}u_j\,\mathrm{d}x + \sum_{i,j}\int_{\partial B(0,R)} u_{ii}u_j\mathbf{n}^j\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

If the support of f is compactly contained in $B(0, R)$ then the last boundary term vanishes since it equals $\int_{\partial B(0,R)} f\nabla u \cdot \mathbf{n}$. Going on with the integration by parts we have

$$\int_{B(0,R)} f^2\,\mathrm{d}x = \sum_{i,j}\int_{B(0,R)} u_{ij}u_{ij}\,\mathrm{d}x - \sum_{i,j}\int_{\partial B(0,R)} u_{ij}u_j\mathbf{n}^i\,\mathrm{d}\mathcal{H}^{d-1},$$

hence

$$\int_{B(0,R)} f^2\,\mathrm{d}x = \int_{B(0,R)} |D^2u|^2\,\mathrm{d}x - \int_{\partial B(0,R)} D^2u\nabla u \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}.$$

We then note that, for $R \gg 1$, we have $|\nabla u(x)| \leq C|x|^{1-d}$ and $|D^2u(x)| \leq C|x|^{-d}$, as a consequence of the shape of Γ and the compact support of f, so that we have

$$\left| \int_{\partial B(0,R)} D^2 u \nabla u \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{d-1} \right| \leq C R^{d-1} \cdot R^{-d} \cdot R^{1-d} = C R^{-d} \to 0 \text{ as } R \to \infty,$$

which proves the claim. □

Using Γ, $\nabla\Gamma \in L^1_{loc}$ we see that $f \in L^2$ also implies that the L^2 norms of $\Gamma * f$ and $\nabla(\Gamma * f)$ are locally bounded by constants multiplied by the L^2 norm of f, so that we have local bounds on the full norm $||\Gamma * f||_{H^2}$ in terms of $||f||_{L^2}$.

Applying the same result to the derivatives of f we immediately obtain the following.

Corollary 2.13 *Given $f \in C_c^\infty(\Omega)$ for a bounded set Ω, let u be given by $u = \Gamma * f$. Then for every $k \geq 0$ we have*

$$||u||_{H^{k+2}(\Omega)} \leq C(k, \Omega)||f||_{H^k(\Omega)}.$$

Lemma 2.14 *Assume that $u \in C^\infty(\Omega)$ is harmonic in Ω and take $x_0, r < R$ such that we have $\overline{B(x_0, R)} \subset \Omega$. Then, for every integer $k \geq 1$, we have*

$$||u||_{H^{1-k}(B(x_0,r))} \leq C(k, r, R)||u||_{H^{-k}(B(x_0,R))}.$$

Proof We want to take $\varphi \in C^\infty$ with $\mathrm{spt}(\varphi) \subset B(x_0, r)$ and estimate $\int u\varphi$ in terms of $||\varphi||_{H^{k-1}}$ and $||u||_{H^{-k}(B(x_0,R))}$. To do this, we first consider $v = \Gamma * \varphi$ and a cutoff function $\eta \in C^\infty(\Omega)$ with $\eta = 1$ on $B(x_0, r)$ and $\eta = 0$ outside of $B_R := B(x_0, R)$. We write

$$0 = \int_{B_R} u\Delta(v\eta) \, \mathrm{d}x = \int_{B_R} u\varphi\eta \, \mathrm{d}x + \int_{B_R} uv\Delta\eta \, \mathrm{d}x + 2\int_{B_R} u\nabla v \cdot \nabla\eta \, \mathrm{d}x.$$

Using $\varphi\eta = \varphi$ (since $\eta = 1$ on $\mathrm{spt}(\varphi)$), we obtain

$$\begin{aligned} \int_{B_R} u\varphi \, \mathrm{d}x &= -\int_{B_R} uv\Delta\eta \, \mathrm{d}x - 2\int_{B_R} u\nabla v \cdot \nabla\eta \, \mathrm{d}x \\ &\leq ||u||_{H^{-k}(B(x_0,R)} \left(||v\Delta\eta||_{H^k(B(x_0,R)} + 2||\nabla v \cdot \nabla\eta||_{H^k(B(x_0,R)} \right). \end{aligned}$$

Since η is smooth and fixed, and its norms only depend on r, R, we obtain

$$||v\Delta\eta||_{H^k(B(x_0,R)}, ||\nabla v \cdot \nabla\eta||_{H^k(B(x_0,R)} \leq C(k, r, R)||\nabla v||_{H^k(B(x_0,R)}.$$

Applying the Corollary 2.13 we obtain $||\nabla v||_{H^k} \leq ||v||_{H^{k+1}} \leq C(k, r, R)||\varphi||_{H^{k-1}}$, which provides the desired result. □

We can then obtain

Proposition 2.15 *Assume that* $u \in H^{-k}_{loc}(\Omega)$ *is harmonic in* Ω *in the sense of distributions. Then* u *is an analytic function.*

Proof The proof is the same as in Proposition 2.11: we regularize by convolution and apply the bounds on the Sobolev norms. Lemma 2.14 allows us to pass from H^{-k} to H^{1-k} and, iterating, arrive at L^2. Once we know that u is in L^2_{loc} we directly apply Proposition 2.11. □

Finally, we have

Proposition 2.16 *Assume that* u *is a harmonic distribution in* Ω. *Then* u *is an analytic function.*

Proof We just need to show that u locally belongs to a space H^{-k} (said differently, every distribution is of finite order, when restricted to a compact set). This is a consequence of the definition of distributions. Indeed, we have the following: for every distribution u and every compact set $K \subset \Omega$ there exist n, C such that $\langle u, \varphi \rangle \leq C||\varphi||_{C^n}$ for every $\varphi \in C^\infty$ with $\text{spt}(\varphi) \subset K$. If we want to prove this we just need to proceed by contradiction: if it is not true, then there exists a distribution u and a compact set K such that for every n we can find a φ_n with

$$\langle u, \varphi_n \rangle = 1, \quad ||\varphi_n||_{C^n} \leq \frac{1}{n}, \quad \text{spt}(\varphi_n) \subset K.$$

Note that we define the C^n norm as the sup of all derivatives up to order n; so that $||\varphi||_{C^{n+1}} \geq ||\varphi||_{C^n}$. Yet, this is a contradiction since the sequence φ_n tends to 0 in the space of C^∞_c functions and u should be continuous for this convergence. So we have the inequality $\langle u, \varphi \rangle \leq C||\varphi||_{C^n}$ which can be turned into $< u, \varphi > \leq C||\varphi||_{H^k}$ because of the continuous embedding of Sobolev spaces into C^n spaces (take $k > n + d/2$). □

The techniques that we presented for the case $\Delta u = 0$ now allow us to discuss the regularity for the Poisson equation $\Delta u = f$ in Sobolev spaces H^k, and we can prove the following.

Proposition 2.17 *Assume that* u *is a distributional solution in* Ω *of* $\Delta u = f$, *where* $f \in H^{-k}_{loc}(\Omega)$. *Then* $u \in H^{2-k}_{loc}(\Omega)$.

Proof Let B be an open ball compactly contained in Ω and let us define a distribution $\tilde{f}$ of the form ηf, where $\eta \in C^\infty_c(\Omega)$ is such that $B \subset \{\eta = 1\}$. Since u and $\Gamma * \tilde{f}$ have the same Laplacian in B, they differ by a harmonic function, and we know that harmonic functions are locally smooth, so that we just need to prove $\Gamma * \tilde{f} \in H^{2-k}_{loc}(B)$. This result is proven in Corollary 2.13 if $k \leq 0$, so we can assume $k \geq 1$.

We take a test function $\phi \in C_c^\infty(B)$ and we use the properties of the convolution to write

$$\int_B (\Gamma * \tilde{f})\phi \, dx = \int_B \tilde{f}(\Gamma * \phi) \, dx = \int_B f\eta(\Gamma * \phi) \, dx \leq ||f||_{H^{-k}(B)} ||\eta(\Gamma * \phi)||_{H^k(B)}.$$

We then exploit the C^∞ regularity of η to obtain $||\eta(\Gamma * \phi)||_{H^k(B)} \leq C||\Gamma * \phi||_{H^k(B)}$ and, if $k \geq 2$, we have, using again Corollary 2.13, $||\eta(\Gamma * \phi)||_{H^k(B)} \leq C||\phi||_{H^{k-2}(B)}$. This proves $\int(\Gamma * \tilde{f})\phi \leq C||\phi||_{H^{k-2}(B)}$, i.e. $\Gamma * \tilde{f} \in H^{2-k}(B)$.

We are left with the case $k = 1$. Since this case is treated in a different way, we refer to a separate proposition for it. □

Proposition 2.18 *Let $B_i = B(x_0, R_i)$, $i = 1, 2$, be two concentric balls with $R_1 < R_2$. Then, given $f \in H^{-1}(B_2)$, setting $u = \Gamma * f$, we have $||u||_{H^1(B_1)} \leq C(R_1, R_2)||f||_{H^{-1}(B_2)}$.*

Proof By density, it is enough to prove this inequality if $f \in C^\infty$. We choose a smooth cut-off function χ with $0 \leq \chi \leq 1$, $\chi = 1$ on B_1 and spt $\chi \subset B_2$. We then write

$$\int_{B_2} fu\chi^2 \, dx = \int_{B_2} \Delta u(u\chi^2) \, dx = -\int_{B_2} \nabla u \nabla(u\chi^2) \, dx,$$

where in the integration by parts there are no boundary terms since χ is compactly supported. We then obtain

$$\int_{B_1} |\nabla u|^2 \, dx \leq \int_{B_2} |\nabla u|^2 \chi^2 \, dx$$

and

$$\int_{B_2} |\nabla u|^2 \chi^2 \, dx = -\int_{B_2} fu\chi^2 \, dx - 2\int_{B_2} \nabla u \cdot \nabla\chi u\chi \, dx.$$

The right-hand side above can be bounded by

$$||f||_{H^{-1}} ||\nabla(u\chi^2)||_{L^2} + C||\nabla u\chi||_{L^2} ||u\nabla\chi||_{L^2}.$$

Using $||\nabla(u\chi^2)||_{L^2} \leq C||\nabla u\chi||_{L^2} + C||u\nabla\chi||_{L^2}$ (an estimate where we use $|\chi| \leq 1$) and applying Young's inequality we obtain

$$\int_{B_2} |\nabla u|^2 \chi^2 \leq C||f||^2_{H^{-1}(B_2)} + C||f||_{H^{-1}(B_2)} ||u\nabla\chi||_{L^2(B_2)} + C||u\nabla\chi||^2_{L^2(B_2)}.$$

The same estimate as in Proposition 2.17, for $k = 2$, shows that we have $||u||_{L^2(B_2)} \le C||f||_{H^{-2}(B_2)} \le C||f||_{H^{-1}(B_1)}$, so that we can deduce from the previous inequality

$$\int_{B_2} |\nabla u|^2 \chi^2 \, dx \le C(r, R)||f||^2_{H^{-1}(B_2)}$$

and then

$$\int_{B_1} |\nabla u|^2 \, dx \le C(r, R)||f||^2_{H^{-1}(B_2)},$$

which is the claim. □

Box 2.7 Good to Know—*Manifold- and Metric-Valued Harmonic Maps*

All the regularity theory on harmonic functions which has been presented in this section relies on the linear structure of the target space (we presented the results for scalar harmonic maps, but working componentwise they are easily seen to be true for vector-valued harmonic maps).

Given a manifold $M \subset \mathbb{R}^N$, one could consider the problem

$$\min\{\int_\Omega |\nabla u|^2 \,:\, u \in H^1(\Omega; \mathbb{R}^N),\ u \in M \text{ a.e.}, u - g \in H^1_0(\Omega)\}$$

together with its optimality conditions written as a PDE. This PDE would not be $\Delta u = 0$ as it should take into account the constraint, and formally one finds $\Delta u = c(x)\mathbf{n}_M(u)$ (the Laplacian is pointwisely a vector normal to the manifold M at the point u). In this case, because of the non-convexity of the problem, the PDE would not be sufficient for the minimization, and because of the right-hand side in $\Delta u = f$ the regularity is not always guaranteed.

For instance, when $M = \mathbb{S}^2 \subset \mathbb{R}^3$ the equation is $\Delta u = -|\nabla u|^2 u$ (see Exercise 2.10) and $u(x) = x/|x|$, a function defined on $\Omega = B(0, 1) \subset \mathbb{R}^3$ which solves it, is H^1, but not continuous. This lack of regularity is not surprising considering that the right-hand side $-|\nabla u|^2 u$ is only L^1. It has been pointed out in [165] that the condition of being an H^1 distributional solution of $\Delta u = -|\nabla u|^2 u$ is so weak that for any boundary data there are infinitely many solutions. Yet, the function $u(x) = x/|x|$ is not only a solution of this PDE, but also a minimizer of the Dirichlet energy for M-valued maps which are the identity on $\partial B(0, 1)$. This nontrivial fact was first proven by Brezis, Coron and Lieb in [49] and then extended to other dimensions (and even other $W^{1,p}$ norms, for $p < d$) in [116] and [139].

(continued)

Box 2.7 (continued)
The situation is different when M has non-positive curvature, in which case it is possible to prove regularity of harmonic maps valued in M. These harmonic maps were first studied by Ells and Sampson in [80]. Then, Schoen and Uhlenbeck proved in [181], for maps minimizing the Dirichlet energy $\int |\nabla u|^2$ (and not only solving the Euler–Lagrange equation), some partial regularity results (smoothness out of a lower-dimensional set), guaranteeing in some cases that the singular set is empty. The interested reader can have a look at the book [122, Chapter 8] or at the survey by Hélein and Woods [112, Section 3.1].

Finally, it is possible to study the problem of minimizing the Dirichlet energy when replacing the Euclidean space or its submanifolds as a target space with a more abstract metric space. This requires us to give a definition of this energy, as the notion of gradient is not well-defined for maps valued in a metric space (consider that we also had to provide a suitable definition for the derivative of curves valued in a metric space, see Sect. 1.4.1). In this case the key observation to define a Dirichlet energy is the fact that, for the Euclidean case, we have for $u \in C_c^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x = c(d) \lim_{\varepsilon \to 0^+} \varepsilon^{-(d+2)} \int\int_{|x-y|<\varepsilon} |u(x) - u(y)|^2 \,\mathrm{d}x \,\mathrm{d}y,$$

where $c(d)$ is a dimensional constant. Then it is possible to replace $|u(x) - u(y)|$ with $d(u(x), u(y))$ and study the maps minimizing the limit energy. For this approach we refer to [127] and [122]. Besides the definition of the energy, these references provide a study of the harmonic maps valued in a metric space (X, d), provided it has negative curvature in a suitable sense.

2.4 Discussion: p-Harmonic Functions for $1 \leq p \leq \infty$

Section 2.3 provides a full description of the functions which minimize $||\nabla u||_{L^2}$ with given boundary conditions, which are the solutions of $\Delta u = 0$ (and actually more, since it also considers harmonic distributions, i.e. we discuss distributional solutions of $\Delta u = 0$ without requiring $u \in H^1$). A similar problem can be considered by minimizing $||\nabla u||_{L^p}$, for arbitrary p. Let us start with $p \in (1, \infty)$.

From the variational point of view, minimizing $\int_\Omega |\nabla u|^p \,\mathrm{d}x$ with given boundary conditions $u - u_0 \in W_0^{1,p}(\Omega)$ is a very classical and simple problem: not only can we easily show that the minimizers exist thanks to the techniques presented in this chapter, but they are unique (because of strict convexity), and they are characterized by the p-Laplacian equation

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0,$$

whose solutions are called *p-harmonic functions*. Moreover, the growth conditions on the integrand allow us to formulate the equation as $\int |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = 0$ for every $\varphi \in W_0^{1,p}(\Omega)$.

For $p = 2$ we have harmonic functions, and we proved that they are actually smooth and even analytic. What about p-harmonic functions? These functions have been the object of a huge number of works in elliptic PDEs: there are of course regularity results, but the situation is actually much trickier than the linear case $p = 2$. We will see in Chap. 4 some results—which are presented in this book through techniques coming from convex duality but could have been introduced using more classical PDE tools—about the Sobolev regularity of solutions of $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f$ (even for $f \neq 0$, but depending of course on the regularity of f). When we mention Sobolev regularity, considering that we already assume $u \in W^{1,p}$, we mean Sobolev regularity of the gradient, i.e. proving that ∇u, or some function of ∇u, belongs to H^1 or some other spaces H^s.

Another possible, and classical, direction of research could be the Hölder regularity of the gradient. This started with [192], which proved that, among other results, for $p \geq 2$, functions which are p-harmonic are $C^{1,\alpha}$. In dimension $d = 2$ the result can be made much stronger and sharp, as it is possible to prove (see [119]) $u \in C^{k,\alpha}$ with

$$k + \alpha = \frac{1}{6}\left(7 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right). \tag{2.9}$$

The literature on the p-Laplacian equation is huge (see, for instance, the classical references [30, 131]), and these considerations as well as those in Chap. 4 only cover a small part of the results.

It is then interesting to consider the limit cases $p = 1$ and $p = \infty$. Let us start from the latter. We can take two different approaches: either we look at the limit of the minimization problem, and we consider functions minimizing $||\nabla u||_{L^\infty}$ with prescribed boundary datum, or we look at the limit of the equation. The equation $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ can be expanded and it can be written as

$$|\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\nabla u \cdot D^2 u \nabla u = 0.$$

We can factorize $(p-2)|\nabla u|^{p-4}$ and this becomes

$$\frac{\Delta u}{p-2} + \nabla u \cdot D^2 u \nabla u = 0.$$

In the limit $p \to \infty$, only the second terms remains, and we define ∞-harmonic functions to be those which solve, in a suitable sense, the non-linear equation

$$\nabla u \cdot D^2 u \nabla u = 0.$$

Since the equation is non-linear and is not in divergence form, we cannot use distributional or weak solutions. On the other hand, the function $(p, M) \mapsto p \cdot Mp$ is non-decreasing in M, which allows us to use the notion of viscosity solutions (see below). Thus, we define ∞-harmonic functions as those which solve the above equation in the viscosity sense. An important result (many proofs are available, but we suggest the reader to look at [19], which is very elementary) states that there exists a unique viscosity solution u of $\nabla u \cdot D^2 u \nabla u = 0$ with given boundary conditions. Solutions of this equation, that we also write as $\Delta_\infty u = 0$, are said to be ∞-harmonic.

Box 2.8 Important Notion—*Viscosity Solution of Second-Order PDEs*

Let us take a function $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}$, where $\mathbb{R}^{d\times d}_{sym}$ stands for the space of $d \times d$ symmetric matrices. Let us assume that F is non-decreasing in the last variable, in the sense that $F(x, s, p, M) \le F(x, s, p, N)$ every time that $N - M$ is positive-definite. We then say that a function $u \in C^0(\Omega)$ is a viscosity solution of $F(x, u, \nabla u, D^2 u) = 0$ if it satisfies the following two properties

- for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $\varphi \ge u$ but $\varphi(x_0) = u(x_0)$ we have $F(x_0, \varphi(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \ge 0$;
- for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $\varphi \le u$ but $\varphi(x_0) = u(x_0)$ we have $F(x_0, \varphi(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \le 0$.

As a reference for the theory of viscosity solutions for second-order PDEs, see [58].

Another approach to the limit $p \to \infty$ concerns the limit of the optimization problem. Limits of optimization problems will be discussed in detail in Chap. 7, but in this case it is very easy, as one only needs to replace the L^p norm with the L^∞ norm. What can we say about minimizers of $u \mapsto ||\nabla u||_{L^\infty}$ with given boundary conditions? On convex domains, this quantity coincides with the Lipschitz constant, which makes it easy to prove the existence of a solution using the Ascoli–Arzelà theorem. More than this, it is also possible to identify the optimal Lipschitz constant, which is equal, when speaking of real-valued functions, to the Lipschitz constant of the boundary datum. Indeed, given an L-Lipschitz function on a set A (not necessarily $\partial\Omega$), it is always possible to extend it to the whole space keeping L as a Lipschitz constant. For instance, given $u : A \to \mathbb{R}$, one can define $\underline{u}(x) :=$

$\sup\{L|x-y|+u(y)\,:\,y\in A\}$. But this is only a possible choice, since we can, for instance, also choose $\overline{u}(x):=\inf\{-L|x-y|+u(y)\,:\,y\in A\}$. These two functions do not coincide in general, and they are actually the smallest and the largest possible L-Lipschitz extension of u, in the sense that any $u\in \mathrm{Lip}_L$ with prescribed value on A satisfies $\underline{u}\le u\le\overline{u}$.

As a consequence, we understand that many different functions can minimize $u\mapsto ||\nabla u||_{L^\infty}$ with given boundary conditions, but which is the one that we can find as the limit of the solutions u_p of $-\Delta_p u=0$ with the same boundary conditions? (or, equivalently, which is the limit of the minimizers of $u\mapsto ||\nabla u||_{L^p}$?). We see now that the minimization problem of the L^∞ norm lacks a property which is obviously satisfied by the minimization of the L^p norm and, more generally, of all integral functionals: when we minimize on a domain Ω an integral cost of the form $\int_\Omega L(x,u,\nabla u)\,\mathrm{d}x$, the solution u is such that, for every subdomain $\Omega'\subset\Omega$, it also minimizes $\int_{\Omega'} L(x,u,\nabla u)\,\mathrm{d}x$ among all functions defined on Ω' sharing the same value with u on $\partial\Omega'$. This is just due to the additivity of the integral, so that if a better function existed on Ω' we could use it on Ω' instead of u, and strictly improve the value of the integral. When the functional is defined as a sup, if the sup is not realized in Ω' but in $\Omega\setminus\Omega'$, most likely modifying u on Ω' does not change the value of the functional. This means that minimizers of the L^∞ norm of the gradient can, in many situations, be almost arbitrary in large portions of the domain Ω and, unlike the case of the L^p norm, cannot be characterized by any PDE. This motivates the following definition.

Definition 2.19 A Lipschitz function $u:\Omega\to\mathbb{R}$ is said to be an *absolute minimizer* of the L^∞ norm of the gradient if it satisfies the following condition: for every $\Omega'\subset\Omega$ we have

$$||\nabla u||_{L^\infty(\Omega')}=\min\{||\nabla\tilde u||_{L^\infty(\Omega')}\,:\,\tilde u\in \mathrm{Lip}(\overline{\Omega'}),\ \tilde u=u\ \text{ on }\partial\Omega'\}.$$

We then have the following results, which we will not prove.

Theorem 2.20 *A Lipschitz function $u:\Omega\to\mathbb{R}$ is an absolute minimizer if and only if it is a viscosity solution of $\Delta_\infty u=0$.*

Given a Lipschitz boundary datum $g:\partial\Omega\to\mathbb{R}$, the functions u_p defined as the unique solution of $\Delta_p u_p=0$ in Ω with $u_p=g$ on $\partial\Omega$ uniformly converge as $p\to\infty$ to the function u_∞ defined as the unique viscosity solution of $\Delta_\infty u=0$ with the same boundary data.

We now switch to the other extreme case, i.e. $p=1$. The first difficulty in solving $\min||\nabla u||_{L^1}$ with given boundary data consists in the fact that the space L^1 is neither reflexive (as is the case for $1<p<\infty$) nor it is a dual space (as is the case for $p=\infty$). So, it is in general not possible to extract weakly convergent subsequences from bounded sequences, and the minimization problem could turn out to be ill-posed if we consider $u\in W^{1,1}$. The natural framework in order to obtain the existence of a minimizer is to extend the problem to $u\in\mathrm{BV}$ and replace the L^1 norm of the gradient with the mass of the measure ∇u (see Sects 1.5 and 1.6

for the first appearance of BV functions, and Box 6.1 for the multi-dimensional case). Exactly as in the 1D case (in particular, as it happened in Sect. 1.6), boundary data are not always well-defined for BV functions, as they could jump exactly on the boundary. Hence, it is the same to solve

$$\min\{||\nabla u||_{\mathcal{M}(\overline{\Omega})} \;:\; u \in \mathrm{BV}(\Omega),\; u = g \text{ on } \partial\Omega\},$$

where the condition $u = g$ on $\partial\Omega$ can be intended as "u is a BV function on a larger domain $\tilde{\Omega}$ which contains Ω in its interior, which coincides with an extension of g on $\tilde{\Omega} \setminus \Omega$, this extension being continuous in a suitable sense in $\tilde{\Omega} \setminus \Omega$", or to solve

$$\min\{||\nabla u||_{\mathcal{M}(\Omega)} + \int_{\partial\Omega} |\mathrm{Tr}[u] - g| \, d\mathcal{H}^{d-1} \;:\; u \in \mathrm{BV}(\Omega)\}, \tag{2.10}$$

where the trace of a BV function should also be suitably defined (and corresponds in this case to the trace *coming from inside* Ω).

In particular, it is possible for certain choices of g and of Ω that the solution of this problem (which is in general not unique, because the functional which is minimized is no longer strictly convex) is a smooth function inside Ω but does not take the value g on $\partial\Omega$.

Formally, the equation solved by the minimizers is the 1-Laplacian equation, i.e.

$$\Delta_1 u := \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0$$

and functions which solve it are called, besides 1-harmonic, *least gradient* functions.

In order to treat points where $\nabla u = 0$, a possible interpretation of the above equation is the following: there exists a vector field $z : \Omega \to \mathbb{R}^d$ with $|z| \le 1$ and $z \cdot \nabla u = |\nabla u|$ (i.e. $z = \nabla u / |\nabla u|$ whenever $\nabla u \neq 0$) such that $\nabla \cdot z = 0$. Since the condition $|z| \le 1$ already implies $z \cdot \nabla u \le |\nabla u|$, the equality $z \cdot \nabla u = |\nabla u|$ is equivalent to $\int z \cdot \nabla u = \int |\nabla u|$, a condition which has a meaning also when ∇u is a measure.

Note that, whenever u is a smooth function with non-vanishing gradient, the quantity $\Delta_1 u(x_0)$ coincides with the curvature (in the sense of the sum of the principal curvatures) of the codimension 1 surface $\{u = u(x_0)\}$ at the point x_0. In particular, imposing that it vanishes means imposing that these surfaces all have zero mean curvature; in dimension $d = 2$ they should be locally segments. Finally, also note that the level sets are invariant if one composes u with a monotone increasing function, and this is absolutely consistent with the fact that we have $\Delta_1 u = \Delta_1(f(u))$ for every monotone increasing function f, just because the quantity $\nabla u / |\nabla u|$ is 0-homogeneous and it is not affected by the multiplication by $f'(u)$ which appears in the composition.

We do not want to give general results on least gradient functions here, but we want to underline a nice transformation which is only possible in dimension $d = 2$. Indeed, in the two-dimensional case, if we set $\mathbf{v} := R\nabla u$, where R is a 90° rotation, we have $\nabla \cdot \mathbf{v} = 0$ in Ω. Moreover, the Dirichlet condition on u, which can be seen (up to additive constants) as a condition on its tangential derivative, becomes after this rotation a condition on the normal component of $\mathbf{v}$. Imposing the divergence in the interior of Ω and $\mathbf{v}\cdot\mathbf{n}$ on $\partial\Omega$ is the same as imposing the distributional divergence (which can have a singular part on $\partial\Omega$) on the whole space of the vector field obtained by extending $\mathbf{v}$ to 0 outside Ω. Since the norms of $\mathbf{v}$ and of ∇u are the same, the optimization problem defining least gradient functions can be reformulated as the minimization of the L^1 norm (or of the mass of a vector measure) under divergence constraints. This minimal flow problem will be discusses in Chap. 4 and in particular in the discussion Sect. 4.6, and it is equivalent to an optimal transport problem from the positive to the negative part of the prescribed divergence. In this case, it consists in an optimal transport problem between measures supported on $\partial\Omega$. This reformulation, suggested in [104], made it possible in [79] to prove the following results in the 2D case:

Theorem 2.21 *Assume that Ω is a 2D strictly convex domain and assume that g is a BV function on the 1D curve $\partial\Omega$. Then the solutions of Problem (2.10) satisfy* $\mathrm{Tr}[u] = g$ *on Ω. If moreover g is continuous on $\partial\Omega$, then the solution is unique.*

If furthermore Ω is uniformly convex and $g \in W^{1,p}(\partial\Omega)$ for $p \leq 2$, then the solution u satisfies $u \in W^{1,p}(\Omega)$. If $g \in C^{1,\alpha}(\partial\Omega)$, then u satisfies $u \in W^{1,p}(\Omega)$ for $p = \frac{2}{1-\alpha}$. If $g \in C^{1,1}$, then u is Lipschitz continuous.

The very last part of the above statement (Lipschitz regularity if the boundary data are $C^{1,1}$) is a particular case of what can be proven under the so-called *bounded slope condition*, which we do not detail here and for which we refer, for instance, to [64, 186].

2.5 Exercises

Exercise 2.1 Prove the existence and uniqueness of the solution of

$$\min\left\{\int_\Omega \left(f(x)|u(x)| + |\nabla u(x)|^2\right) \mathrm{d}x \;:\; u \in H^1(\Omega),\ \int_\Omega u = 1\right\}$$

when Ω is an open, connected and bounded subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$. Where do we use connectedness? Also prove that, if Ω is not connected but has a finite number of connected components and we add the assumption $f \geq 0$, then we have existence but maybe not uniqueness, and that if we withdraw both connectedness and positivity of f, then we might not even have existence.

Exercise 2.2 Consider $\Omega = (0,1)^d \subset \mathbb{R}^d$ and $A = \{0\} \times (0,1)^{d-1} \subset \partial\Omega$. Prove that the set of functions $u \in W^{1,p}(\Omega)$ such that $\mathrm{Tr}[u] = 0$ a.e. on A coincides with the closure in $W^{1,p}(\Omega)$ of the set $\{\varphi \in C^1(\overline{\Omega}) \;:\; \mathrm{spt}(\varphi) \cap A = \emptyset\}$.

Exercise 2.3 Given a smooth and connected open domain $\Omega \subset \mathbb{R}^d$ and two exponents $\alpha, \beta > 0$, consider the following problem:

$$\min \left\{ \int_\Omega \left(-|u|^\alpha + |\nabla u|^p\right) \mathrm{d}x + \int_{\partial\Omega} |\mathrm{Tr}[u]|^\beta \, \mathrm{d}\mathcal{H}^{d-1} \; : \; u \in W^{1,p}(\Omega) \right\}.$$

Prove that the above problem has a solution if $\alpha < \min\{\beta, p\}$ and that it has no solution if $\alpha > \min\{\beta, p\}$.

Exercise 2.4 Given a measurable function $c : \Omega \to \mathbb{R}$ such that $c_0 \le c \le c_1$ a.e., for two strictly positive constants c_0, c_1, a function $g \in W^{1,p}(\Omega)$, and a continuous function $F : \Omega \times \mathbb{R} \to \mathbb{R}_+$, prove that the following problem has a solution:

$$\min \left\{ \int_\Omega \left(c(x)|\nabla u|^p + F(x, u(x))\right) \mathrm{d}x \; : \; u - g \in W_0^{1,p}(\Omega) \right\}.$$

Exercise 2.5 Fully solve

$$\min \left\{ \int_Q \left(|\nabla u(x, y)|^2 + u(x, y)^2\right) \mathrm{d}x \, \mathrm{d}y \; : \; u \in C^1(Q), \; u = \phi \text{ on } \partial Q \right\},$$

where $Q = [-1, 1]^2 \subset \mathbb{R}^2$ and $\phi : \partial Q \to \mathbb{R}$ is given by

$$\phi(x, y) = \begin{cases} 0 & \text{if } x = -1, \; y \in [-1, 1] \\ 2(\mathrm{e}^y + \mathrm{e}^{-y}) & \text{if } x = 1, \; y \in [-1, 1] \\ (x + 1)(\mathrm{e} + \mathrm{e}^{-1}) & \text{if } x \in [-1, 1], \; y = \pm 1. \end{cases}$$

Find the minimizer and the value of the minimum.

Exercise 2.6 Which among the functions u which can be written in polar coordinates as $u(\rho, \theta) = \rho^\alpha \sin(k\theta)$ are harmonic on the unit ball? (i.e. for which values of $\alpha, k \in \mathbb{R}$).

Exercise 2.7 Let $u \in \mathcal{D}'(\mathbb{R}^d)$ be a distributional solution of $\Delta u = b(x) \cdot \nabla u + f(x)u$, where $f : \mathbb{R}^d \to \mathbb{R}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are given C^∞ functions. Prove $u \in C^\infty(\mathbb{R}^d)$.

Exercise 2.8 Prove, using the mean property, that any bounded harmonic function u on the whole space $\mathbb{R}^d$ is constant. Also prove that any harmonic function u on the whole space $\mathbb{R}^d$ satisfying a growth bound of the form $|u(x)| \le C(1 + |x|)^p$ for some exponent p is actually a polynomial function.

Exercise 2.9 Prove that any smooth solution of $\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $d \geq 3$, satisfies

$$\fint_{\partial B(x_0,R)} u \, dx = u(x_0) + \frac{1}{d(d-1)\omega_d} \int_{B(x_0,R)} f(x) \left(|x - x_0|^{1-d} - R^{1-d} \right) dx.$$

Find an analogous formula for the case $d = 2$ and for the average on the ball instead of the sphere.

Exercise 2.10 Let $M \subset \mathbb{R}^N$ be the zero set of a smooth function $h : \mathbb{R}^N \to \mathbb{R}$ with $\nabla h \neq 0$ on M. Prove that if a function $u : \Omega \to M$ (with $\Omega \subset \mathbb{R}^d$) satisfies $\Delta u(x) = c(x)\nabla h(u(x))$ for a scalar function c, then we have

$$c = -\frac{\sum_{i=1}^d D^2 h(u(x)) u_i(x) \cdot u_i(x)}{|\nabla h(u(x))|^2}.$$

Also prove that any smooth function $u : \Omega \to M$ satisfying $\Delta u(x) = c(x)\nabla h(u(x))$ for the above expression of c is a minimizer of $v \mapsto \int_\Omega |\nabla v|^2$ with prescribed boundary datum as soon as Ω is contained in a small enough ball.

Exercise 2.11 Determine for which values of $p > 1$ the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x_1, x_2) := |x_1|^p - |x_2|^p$ is ∞-harmonic, and use this information to give bounds on the optimal $C^{1,\alpha}$ regularity of ∞-harmonic functions. Compare to Formula 2.9.

Exercise 2.12 For $p \in (1, \infty)$, let u be a solution of $\Delta_p u = 0$ on Ω. Let M be a constant such that $\mathrm{Tr}[u] \leq M$ a.e. on $\partial\Omega$. Prove $u \leq M$ a.e. in Ω.

Hints

Hint to Exercise 2.1 Minimizing sequences are bounded in H^1 because of the Poincaré–Wirtinger inequality. This cannot be applied when Ω is non-connected. In this case we use $f \geq 0$ and the term with $\int f |u|$ on each component to bound a suitable L^1 norm of u (distinguishing the components where f is identically 0, where we can replace u with a constant).

Hint to Exercise 2.2 We approximate u with $u(1-\eta_n)$ where η_n is a cutoff function with $\eta_n = 1$ on A and $\mathrm{spt}\,\eta_n \subset [0, 1/n] \times [0, 1]^{d-1}$. We need to prove $u\eta_n \to 0$ at least weakly in $W^{1,p}$. The key point is bounding the norm $||u\nabla\eta_n||_{L^p}$ using the trace of u and suitable Poincaré inequalities.

Hint to Exercise 2.3 If α is large the infimum is $-\infty$ using either large constants if $\alpha > \beta$ or nu_0 with $u_0 \in W_0^{1,p}$ if $\alpha > p$. If α is small we can prove compactness.

Hint to Exercise 2.4 For a minimizing sequence, prove that ∇u_n is bounded and hence weakly convergent, both in $L^p(\,\mathrm{d}x)$ and in $L^p(c(x)\,\mathrm{d}x)$.

Hint to Exercise 2.5 Write and solve the Euler–Lagrange equation (by guessing the form of the solution).

Hint to Exercise 2.6 For smoothness, we need $\alpha \geq 0$ and $k \in \mathbb{Z}$. When $\alpha^2 = k^2$ we have the real part of a holomorphic function. In general, compute the gradient and its divergence in polar coordinates.

Hint to Exercise 2.7 Start from a distribution in a certain H^{-k} and prove that it actually belongs to H^{1-k}.

Hint to Exercise 2.8 Use Proposition 2.8 to prove $\nabla u = 0$. For the polynomial growth, prove $|\nabla u| \leq C(1+|x|)^{p-1}$ and iterate.

Hint to Exercise 2.9 Re-perform the proof of the mean property keeping into account the non-vanishing Laplacian.

Hint to Exercise 2.10 Differentiate $h \circ u$ twice. For the optimality, act as we did for geodesics in Sect. 1.1.2.

Hint to Exercise 2.11 Check that only $p = 4/3$ works. This also coincides with the limit regularity for p-harmonic functions when $p \to \infty$, which gives $8/6$.

Hint to Exercise 2.12 Use the variational interpretation of p-harmonic functions as minimizers of the L^p norm of the gradient, and truncate them at M obtaining a better competitor.