On Some Deformed Canonical Commutation Relations: The Role of Distributions

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1 Introduction

According to Gianni's view of research, the role of Mathematics is essential in many fields of science, and in Theoretical Physics in particular. The literature is full of claims which look reasonable, but are not based on theorems or rigorous proofs. And this is true in Condensed Matter, Statistical Physics, Quantum Field Theory, Elementary Particles and so on. The content of this chapter, I believe, could have been of some interest for Gianni, even if, as far as I know, he never worked on what I will discuss here. Therefore, with this in mind, let me start my scientific contribution to this volume.

Among the various tools which play a relevant role in quantum mechanics, ladder operators are quite interesting, and useful. We all know the bosonic annihilation and creation operators, mainly because they are found already very early, while studying the spectrum of a simple quantum harmonic oscillator. But these operators are then used in a different, many-body, context: they describe *bosonic modes*, needed, for instance, in the analysis of quantum fields describing interactions. Fermionic ladder operators are also well known: they are used to model easily the Pauli exclusion principle, but they appear also in some quantum fields describing matter.

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These two classes of ladded operators are usually defined in terms of suitable commutation rules between a fixed (annihilation) operator and its adjoint, the creation operator. For instance, $[c, c^\dagger] = \mathbb{1}_b$ and $\{d, d^\dagger\} = \mathbb{1}_f$, with $d^2 = 0$, are respectively the canonical commutation and the canonical anti-commutation relations. Here $\mathbb{1}_b$ and $\mathbb{1}_f$ are the identity operators in $\mathcal{H}_b$ and $\mathcal{H}_f$ respectively, the *bosonic* and the *fermionic* Hilbert spaces. It is well known that, for a single mode. $\dim(\mathcal{H}_f) = 2$ and $\dim(\mathcal{H}_b) = \infty$. So the two spaces are truly different. Moreover, $c^\dagger$ is the adjoint of c, and $d^\dagger$ that of d. Of course, these adjoints should be computed with respect to the scalar product in the related Hilbert space. This is an easy task for fermions, since the scalar product is just the one in $\mathbb{C}^2$, while it is not entirely trivial in $\mathcal{H}_b$ which is quite often identified with $\mathcal{L}^2(\mathbb{R})$, due to the fact that c is an unbounded operator. This creates a lot of subtle points to consider, of course, since unbounded operators are not as easy as bounded operators: they have, in particular, domain issues that should be considered to avoid mistakes. Hence, writing $[c, c^\dagger] = \mathbb{1}_b$ is just a formal relation which needs to be made more precise, for instance by making explicit the vectors of $\mathcal{H}_b$ on which this formula is well defined. And we can easily imagine that this problem becomes even more complicated when $[c, c^\dagger] = \mathbb{1}_b$ is replaced by $[a, b] = \mathbb{1}_b$, for some pair of operators a and b with $b \neq a^\dagger$. The analysis of this latter situation is indeed the core of this chapter. This is both because the mathematical properties of these operators can be rather interesting, but also because they appear, in a somehow hidden way, in several applications considered in recent years in the physical literature, mainly in connection with manifestly non self-adjoint Hamiltonians. In particular, as we will show later, removing the constraint that the commutation rule is given between an operator (c) and its adjoint ($c^\dagger$), gives us the possibility to extend the functional framework from $\mathcal{L}^2(\mathbb{R})$ to the space of tempered distributions $S'(\mathbb{R})$. Of course, this extension opens the problem of a correct interpretation of the results from a physical side. This is because the usual probabilistic interpretation of quantum mechanics can be lost. Still, some interesting physically relevant operators appear strongly connected to what we will discuss later and, in this perspective, we believe our framework may have some intriguing consequences.

This chapter is organized as follows: we propose our special deformation of the canonical commutation relations and we discuss some of the mathematical consequences of our definition. Section 2.1 is devoted to a brief list of quantum mechanical systems, considered in the literature in recent years, which can be analyzed in terms of our ladder operators, named *pseudo-bosonic*. In particular we show that a and b, together with $a^\dagger$ and $b^\dagger$, behave as ladder operators and allow the construction of two different families of vectors in $\mathcal{L}^2(\mathbb{R})$ which are biorthonormal and are eigenstates of the pseudo-bosonic number operator $N = ba$, and of its adjoint $N^\dagger$. These two sets are not necessarily bases in $\mathcal{L}^2(\mathbb{R})$, but they are usually total sets. In Sect. 3 we take advantage of the fact that in our pseudo-bosonic commutation rule $[a, b] = \mathbb{1}_b$ a and b can be, in general, quite unrelated to propose a generalized version of the Hilbertian settings proposed in Sect. 2. Hence we construct a general settings for what we call *weak pseudo-bosons* (WPBs). Several appearance of these WPBs are described in the second part of Sect. 3. In particular, in Sect. 3.1 we show how the position and the momentum operators $\hat{x}$ and $\hat{p}$ can be seen as weak pseudo-bosonic ladder operators, and we show that an extended scalar product can be introduced to prove the biorhogonality of the generalized eigenstates of the pseudo-bosonic number operators. In Sect. 3.2 we discuss the role of WPBs in the context of the so-called inverted quantum harmonic oscillator (IQHO), while in Sect. 3.3 we propose a rather general family of pseudo-bosons (PBs) which can be defined in or out of $\mathcal{L}^2(\mathbb{R})$. Section 4 contains our conclusions.

2 Pseudo-Bosons

We begin our analysis by recalling few well known facts on bosonic operators. This is important to fix the notation and later to stress the differences between PBs and ordinary bosons.

Let c be an operator on an Hilbert space[1] $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ satisfying the canonical commutation relation (CCR) $[c, c^\dagger] = \mathbb{1}$, $c^\dagger$ being the adjoint of c and $\mathbb{1}$ the identity operator on $\mathcal{H}$. Notice that, using for c the representation $c = \frac{1}{\sqrt{2}}\big(\hat{x} + \frac{d}{dx}\big)$, where $\hat{x}$ is the multiplication operator and $\frac{d}{dx}$ is the derivative operator[2], the set of the test functions on $\mathbb{R}$, $\mathcal{S}(\mathbb{R})$, i.e. of all those C^∞ functions which go to zero, together with their derivatives, faster than any inverse power, is stable under the action of c and $c^\dagger$: if $f(x) \in \mathcal{S}(\mathbb{R})$, then $cf(x), c^\dagger f(x) \in \mathcal{S}(\mathbb{R})$. Now we replace $[c, c^\dagger] = \mathbb{1}$ with its *more complete* version, writing

$$[c, c^\dagger] f(x) = f(x), \tag{1}$$

for all $f \in \mathcal{S}(\mathbb{R})$. If we consider a vector $e_0(x) \in \mathcal{S}(\mathbb{R})$ which is annihilated by c, $c\, e_0(x) = 0$, it is clear that all the vectors $e_n(x) = \frac{1}{\sqrt{n!}} {c^\dagger}^n e_0(x)$, $n \geq 0$, belong to $\mathcal{S}(\mathbb{R})$. The set $\mathcal{F}_e = \{e_n(x),\ n \geq 0\}$ is an orthonormal basis for $\mathcal{H}$:

$$\langle e_n, e_m \rangle = \delta_{n,m}, \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \langle e_n, f \rangle\, e_n(x),$$

$\forall f(x) \in \mathcal{L}^2(\mathbb{R})$, so that $\mathcal{L}_e = l.s.\{e_n(x)\}$, the linear span of the $e_n(x)$'s, is dense in $\mathcal{L}^2(\mathbb{R})$. The following Parseval equality holds:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \langle f, e_n \rangle \langle e_n, g \rangle, \tag{2}$$

$\forall f(x), g(x) \in \mathcal{H}$. The explicit form of $e_n(x)$ is well known:

$$e_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x)\, e^{-\frac{x^2}{2}}, \tag{3}$$

[1] From now on we will simply use $\mathcal{H}$ rather than $\mathcal{H}_b$, since $\mathcal{H}_f$ will have no role in the rest of this chapter.

[2] We recall that this is proportional to the momentum operator $\hat{p}$, $\hat{p} = -i\frac{d}{dx}$.

where $H_n(x)$ is the n-th Hermite polynomial. It is evident now that $e_n(x) \in \mathcal{S}(\mathbb{R})$, for all $n \geq 0$, so that the (strict) inclusion $\mathcal{L}_e \subset \mathcal{S}(\mathbb{R})$ holds.

The set $\mathcal{F}_e$ has interesting features, when considered in connection with c and $c^\dagger$. Indeed we have the following

$$c\, e_n = \sqrt{n}\, e_{n-1}, \quad c^\dagger e_n = \sqrt{n+1}\, e_{n+1}, \tag{4}$$

with the agreement that $e_{-1} = 0$. An immediate consequence of these ladder equations is the following eigenvalue equation

$$N_0\, e_n = n\, e_n, \tag{5}$$

$n \geq 0$, where $N_0 = c^\dagger c$ is called the *number* operator. Because of (4), c is a *lowering* or an *annihilation* operator, while $c^\dagger$ is a *raising* or a *creation* operator. Together they are called *ladder operators*. N_0, c and $c^\dagger$ are all unbounded. In particular, N_0 is symmetric, since $\langle N_0 f, g\rangle = \langle f, N_0 g\rangle$ $\forall f, g \in D(N_0)$, and is positive: $\langle N_0 f, f\rangle = \langle cf, cf\rangle = \|cf\|^2 \geq 0$, $\forall f \in D(N_0)$. Hence, N_0 admits a Friedrichs extension, which we still denote with N_0, which is self-adjoint.

Summarizing, if c satisfies the CCR (1), then we can build up an interesting functional settings: a family of vectors, the $e_n(x)$'s, which are eigenvectors of the self-adjoint operator N_0 with eigenvalues $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, see (5), which obey some relevant ladder conditions, see (4), and which, all together, produce a set of functions $\mathcal{F}_e$ which is an orthonormal basis for $\mathcal{H}$.

During the past few decades, many physicists realized that some non self-adjoint operators can play a significant role in the analysis of various physical systems, [1–5], since there exist quantum mechanical situations in which the dynamics is better described by Hamiltonians (and other *observables*) which are not self-adjoint. This evidence has produced a huge interest in an extended version of quantum mechanics, where self-adjointness of the observables is not a key aspect. This suggested to consider ladder operators of different kind, not necessarily linked by the usual adjoint operation, and their connected number-like operators. We refer to [6] for some preliminary results and to [7] for a more recent monograph on these topics. This chapter is intended to be a review of some recent results on these generalized ladder operators, and to their weak[3] version in particular. For readers' convenience, we begin our analysis by proposing first our definitions and their consequences in a purely Hilbertian settings, postponing their distributional counterparts to Sect. 3.

Let a and b be two operators on $\mathcal{H}$, with domains $\mathcal{D}(a)$ and $\mathcal{D}(b)$ respectively, $a^\dagger$ and $b^\dagger$ their adjoint, and let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, where $x^\sharp$ is either x or $x^\dagger$: $\mathcal{D}$ is assumed to be stable under the action of a, b, $a^\dagger$ and $b^\dagger$. Notice that we are not requiring here that $\mathcal{D}$ coincides with, e.g. $\mathcal{D}(a)$ or $\mathcal{D}(b)$. However due to the fact that $a^\sharp f$ is well defined, and belongs to $\mathcal{D}$ for all $f \in \mathcal{D}$, it is clear that $\mathcal{D} \subseteq \mathcal{D}(a^\sharp)$. Analogously, we can also conclude that $\mathcal{D} \subseteq \mathcal{D}(b^\sharp)$. The stability of $\mathcal{D}$ implies that both $a(bf)$ and $b(af)$ are well defined, $\forall f \in \mathcal{D}$.

[3] In the sense of distributions!

Definition 1 The operators (a, b) are $\mathcal{D}$-pseudo-bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

$$a\,b\,f - b\,a\,f = f. \tag{6}$$

Sometimes, to simplify the notation, rather than (6) we will simply write $[a, b] = \mathbb{1}$. Of course, when $b = a^\dagger$ we go back to CCR, and $a, b \notin \mathcal{B}(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$. a and b are unbounded also when $a \neq b^\dagger$, and this is the reason why the role of $\mathcal{D}$ is so relevant.

Our working assumptions, based on several existing systems in quantum mechanics, are the following:

Assumption $\mathcal{D}$-pb 1 there exists a non-zero $\varphi_0 \in \mathcal{D}$ such that $a\,\varphi_0 = 0$.

Assumption $\mathcal{D}$-pb 2 there exists a non-zero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger\,\Psi_0 = 0$.

It is clear that, if $b = a^\dagger$, these two assumptions collapse into a single one and (6) becomes the ordinary CCR, for which the existence of a vacuum which belongs to an invariant set ($S(\mathbb{R})$, for instance) is guaranteed. On the other hand, if a and b are uncorrelated, it might easily happen that Assumptions $\mathcal{D}$-pb 1 or $\mathcal{D}$-pb 2, or one of the two, are not satisfied. One important example of this situation will be discussed at length in Sect. 3.1.

The stability of $\mathcal{D}$ under the action of b and $a^\dagger$ implies, in particular, that $\varphi_0 \in D^\infty(b) := \cap_{k\geq 0} D(b^k)$ and that $\Psi_0 \in D^\infty(a^\dagger)$. Here $D^\infty(X)$ is the domain of all the powers of the operator X. Hence

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \quad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \tag{7}$$

$n \geq 0$, are well defined vectors in $\mathcal{D}$ and, therefore, they belong to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$, where $N = ba$ and $N^\dagger$ is the adjoint of N. We introduce the sets $\mathcal{F}_\Psi = \{\Psi_n,\, n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$.

It is now simple to deduce the following lowering and raising relations:

$$\begin{cases} b\,\varphi_n = \sqrt{n+1}\,\varphi_{n+1}, & n \geq 0, \\ a\,\varphi_0 = 0, \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}, & n \geq 1, \\ a^\dagger\Psi_n = \sqrt{n+1}\,\Psi_{n+1}, & n \geq 0, \\ b^\dagger\Psi_0 = 0, \quad b^\dagger\Psi_n = \sqrt{n}\,\Psi_{n-1}, & n \geq 1, \end{cases} \tag{8}$$

as well as the following eigenvalue equations: $N\varphi_n = n\varphi_n$ and $N^\dagger\Psi_n = n\Psi_n$, $n \geq 0$. Hence, despite of the fact that N and $N^\dagger$ are manifestly non self-adjoint, in general, their eigenvalues are real and, actually, coincide with those of the operator $N_0 = c^\dagger c$. We call $\mathcal{D}$-pseudo-bosons ($\mathcal{D}$-PBs) the *excitations* described by φ_n and Ψ_n, in the same way we call bosons those described by the vectors e_n in (3).

As a consequence of these equations, choosing the normalization of φ_0 and Ψ_0 in such a way $\langle\varphi_0, \Psi_0\rangle = 1$, it can be shown that

$$\langle\varphi_n, \Psi_m\rangle = \delta_{n,m}, \tag{9}$$

for all $n, m \geq 0$. The conclusion is, therefore, that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of N and $N^\dagger$, respectively. The properties we have deduced for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ does not allow us to conclude anything about the fact that they are also (Riesz) bases for $\mathcal{H}$. In fact, it is well known that, in some relevant concrete examples, this is not the case, while in other situations this is true. We will return on this aspect in Sect. 2.1, where several counterexamples will be given. With this in mind, we introduce the following (not always satisfied, in view of what just observed) assumption:

Assumption $\mathcal{D}$-pb 3 $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

This is equivalent to assume that $\mathcal{F}_\Psi$ is a basis as well, [8]. Since this assumption is not always true, it is more reasonable to replace Assumption $\mathcal{D}$-pb 3 with a weaker version, which thought being weaker, still produces several interesting results and, maybe more relevant, is satisfied even when Assumption $\mathcal{D}$-pb 3 does not hold. We ask the following:

Assumption $\mathcal{D}$-pbw 3 $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-quasi bases, for some subspace $\mathcal{G}$ dense[4] in $\mathcal{H}$.

This means that, $\forall f, g \in \mathcal{G}$, the following identities hold

$$\langle f, g\rangle = \sum_{n\geq 0}\langle f, \varphi_n\rangle\langle\Psi_n, g\rangle = \sum_{n\geq 0}\langle f, \Psi_n\rangle\langle\varphi_n, g\rangle, \tag{10}$$

which, as it is clear, extend the standard closure relation in $\mathcal{H}$, also known as *Parseval identity*.

While Assumption $\mathcal{D}$-pb 3 implies (10), the reverse is false. However, if $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ satisfy (10), we still have some (weak) form of resolution of the identity, and we can deduce several useful consequences. For instance, just to state a simple result, if $f \in \mathcal{G}$ is orthogonal to all the Ψ_n's (or to all the φ_n's), then f is necessarily zero: $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$ are total in $\mathcal{G}$.

For completeness we briefly discuss the role of two intertwining operators which are intrinsically related to our $\mathcal{D}$-PBs. More details can be found in [6].

Assumption $\mathcal{D}$-pb 3 means that

$$f = \sum_{n=0}^{\infty}\langle\varphi_n, f\rangle\,\Psi_n = \sum_{n=0}^{\infty}\langle\Psi_n, f\rangle\,\varphi_n, \tag{11}$$

[4] $\mathcal{G}$ does not need to coincide with $\mathcal{D}$.

$\forall f \in \mathcal{H}$. Then, it is natural to ask if sums like $S_\varphi f = \sum_{n=0}^{\infty}\langle\varphi_n, f\rangle\,\varphi_n$ or $S_\Psi f = \sum_{n=0}^{\infty}\langle\Psi_n, f\rangle\,\Psi_n$ make some sense, or for which vectors they do converge, if any. It is clear that, if $b = a^\dagger$, then $\mathcal{F}_\varphi = \mathcal{F}_\Psi$ and $S_\varphi = S_\Psi = \mathbb{1}$: in this case not only the series for S_φ and S_Ψ converge, but they converge to the identity operator.

If, in particular, $\mathcal{F}_\varphi$ is a Riesz basis, [8], then $\mathcal{F}_\Psi$ is a Riesz basis too, and we know that an orthonormal basis $\mathcal{F}_c = \{c_n\}$ exists, together with a bounded operator R with bounded inverse, such that $\varphi_n = Rc_n$ and $\Psi_n = (R^{-1})^\dagger c_n$, $\forall n$. It is clear that, if $R = \mathbb{1}$, the sums for $S_\varphi f$ and $S_\Psi f$ collapse and converge to f. But, what if $R \neq \mathbb{1}$? In this case, let us take $f \in D(S_\varphi)$, which for the moment we do not assume to be coincident with $\mathcal{H}$. Then

$$S_\varphi f := \sum_n \langle\varphi_n, f\rangle\,\varphi_n = \sum_n \langle Rc_n, f\rangle\,Rc_n = R\left(\sum_n \left\langle c_n, R^\dagger f\right\rangle c_n\right) = RR^\dagger f,$$

where we have used the facts that $\mathcal{F}_c$ is an orthonormal basis and that R is bounded and, therefore, continuous. Of course $RR^\dagger$ is bounded as well and the above equality can be extended to all of $\mathcal{H}$. Therefore we conclude that $S_\varphi = RR^\dagger$. In a similar way we can deduce that $S_\Psi = (R^\dagger)^{-1}R^{-1} = S_\varphi^{-1}$, which is also bounded. In fact, using the C*-property for $B(\mathcal{H})$, we deduce that $\|S_\varphi\| = \|R\|^2$ and $\|S_\Psi\| = \|R^{-1}\|^2$. In this situation, our $\mathcal{D}$-PBs are called *regular*.

Similar results can also be deduced without introducing the operator R, but simply using the biorthonormality of $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$:

$$S_\varphi \Psi_n = \varphi_n, \quad S_\psi \varphi_n = \Psi_n, \tag{12}$$

for all $n \geq 0$. These equalities together imply that $\Psi_n = (S_\Psi\, S_\varphi)\Psi_n$ and $\varphi_n = (S_\varphi\, S_\Psi)\varphi_n$, for all $n \geq 0$. Now, since $S_\varphi, S_\Psi \in B(\mathcal{H})$, we can extend these identities to all of $\mathcal{H}$, and we conclude that

$$S_\Psi\, S_\varphi = S_\varphi\, S_\Psi = \mathbb{1} \quad \Rightarrow \quad S_\Psi = S_\varphi^{-1}. \tag{13}$$

In other words, both S_Ψ and S_φ are invertible and one is the inverse of the other. It is also clear that S_φ and S_Ψ are positive operators, and it is interesting to check that they obey the following intertwining relations:

$$S_\Psi N\varphi_n = N^\dagger S_\Psi \varphi_n, \quad NS_\varphi \Psi_n = S_\varphi N^\dagger \Psi_n, \tag{14}$$

Indeed we have, recalling that $N\varphi_n = n\varphi_n$ and $N^\dagger\Psi_n = n\Psi_n$, $S_\Psi N\varphi_n = n(S_\Psi\varphi_n) = n\Psi_n$, as well as $N^\dagger S_\Psi\varphi_n = N^\dagger\Psi_n = n\Psi_n$. The second equality in (14) follows from the first one, simply by left-multiplying $S_\Psi N\varphi_n = N^\dagger S_\Psi\varphi_n$ with S_φ, and using (12). These relations are not surprising, since intertwining relations can be often deduced between operators sharing the same eigenvalues.

The situation is mathematically much more complicated, in particular, for $\mathcal{D}$-PBs which are not regular. This is connected to the fact that S_φ and S_Ψ are not bounded, so that the series $\sum_{n=0}^{\infty}\langle\varphi_n, f\rangle\,\varphi_n$ and $\sum_{n=0}^{\infty}\langle\Psi_n, f\rangle\,\Psi_n$ do not converge uniformly on $\mathcal{H}$. This case, together with many other details on PBs, can be found in [6] and in references therein.

2.1 Few Appearances of PBs

During the last few decades a lot of physical systems have been considered, mostly in connection with PT-Quantum Mechanics, [1–5], driven by manifestly non self-adjoint Hamiltonians which can be rewritten in terms of PBs. We briefly list here some of these Hamiltonians, and we refer to [6, 7] for many more mathematical details and physical applications. It is useful to remark that, in what follows, we will be extremely concise, since *ordinary* PBs are not the main object of our review here, but are only needed to provide a better setup for WPBs.

The Extended Quantum Harmonic Oscillator

We begin our list of models with the following Hamiltonian, proposed in [9]

$$H_\nu = \frac{\nu}{2}\big(\hat{p}^2 + \hat{x}^2\big) + i\sqrt{2}\,\hat{p},$$

where ν is a strictly positive parameter and $[\hat{x}, \hat{p}] = i\mathbb{1}$. H_ν is manifestly non self-adjoint. However, with some algebra, it can be easily diagonalized in terms of PBs.

For that, we start introducing the (standard) bosonic operators $c = \frac{1}{\sqrt{2}}\big(\hat{x} + \frac{d}{dx}\big)$, $c^\dagger = \frac{1}{\sqrt{2}}\big(\hat{x} - \frac{d}{dx}\big)$, $[c, c^\dagger] = \mathbb{1}$, and the related operators $A_\nu = c - \frac{1}{\nu}$, and $B_\nu = c^\dagger + \frac{1}{\nu}$. Then we can rewrite $H_\nu = \nu(B_\nu A_\nu + \gamma_\nu \mathbb{1})$, where $\gamma_\nu = \frac{2+\nu^2}{2\nu^2}$. It is clear that, for all $\nu > 0$, $A_\nu^\dagger \neq B_\nu$ and that $[A_\nu, B_\nu] = \mathbb{1}$. Hence we are dealing, at least formally, with pseudo-bosonic operators. Indeed, we can check that Assumptions $\mathcal{D}$-pb1, $\mathcal{D}$-pb2 and $\mathcal{D}$-pbw3, are satisfied, while Assumption $\mathcal{D}$-pb3 is not, see [6, 10].

The Swanson Model

The starting point is here the non self-adjoint Hamiltonian,

$$H_\theta = \frac{1}{2}\big(\hat{p}^2 + \hat{x}^2\big) - \frac{i}{2}\tan(2\theta)\big(\hat{p}^2 - \hat{x}^2\big),$$

where θ is a real parameter taking value in $\big(-\frac{\pi}{4}, \frac{\pi}{4}\big) \setminus \{0\} =: I$, [9, 11]. As before, $[\hat{x}, \hat{p}] = i\mathbb{1}$. Of course, $\theta = 0$ is excluded from I just to avoid going back to the standard, self-adjoint, harmonic oscillator, which is not so interesting for us. Notice also that H_θ can be rewritten as

$$H_\theta = \frac{1}{2\cos(2\theta)}\big(\hat{p}^2 e^{-2i\theta} + \hat{x}^2 e^{2i\theta}\big) = \frac{e^{-2i\theta}}{2\cos(2\theta)}\big(\hat{p}^2 + \hat{x}^2 e^{4i\theta}\big),$$

which has, except for an unessential overall complex constant, the same form considered in [12], $H = -\frac{d^2}{dx^2} + z^4\hat{x}^2$, $z \in \mathbb{C}$, taking $z = e^{i\theta}$.

Introducing now the bosonic annihilation and creation operators c, $c^\dagger$, and their linear combinations

$$\begin{cases} A_\theta = \cos(\theta)\, c + i\, \sin(\theta)\, c^\dagger = \frac{1}{\sqrt{2}}\big(e^{i\theta}\,\hat{x} + e^{-i\theta}\,\frac{d}{dx}\big), \\ B_\theta = \cos(\theta)\, c^\dagger + i\, \sin(\theta)\, c = \frac{1}{\sqrt{2}}\big(e^{i\theta}\,\hat{x} - e^{-i\theta}\,\frac{d}{dx}\big), \end{cases}$$

we can write $H_\theta = \omega_\theta\big(B_\theta\, A_\theta + \frac{1}{2}\mathbb{1}\big)$, where $\omega_\theta = \frac{1}{\cos(2\theta)}$ is well defined because $\cos(2\theta) \neq 0$ for all $\theta \in I$. It is clear that, for θ in this set, $A_\theta^\dagger \neq B_\theta$ and that $[A_\theta, B_\theta] = \mathbb{1}$. Again, we have rewritten the Hamiltonian in terms of PBs, and again, we can check that Assumptions $\mathcal{D}$-pb1, $\mathcal{D}$-pb2 and $\mathcal{D}$-pbw3, are satisfied, while Assumption $\mathcal{D}$-pb3 is not, see [6, 10].

Two Coupled Oscillators

The next example we want to briefly mention was originally introduced by Carl Bender and Hugh Jones in [13] and then considered further in [14]. The starting point is the following, manifestly non self-adjoint, Hamiltonian:

$$H = (\hat{p}_1^2 + \hat{x}_1^2) + (\hat{p}_2^2 + \hat{x}_2^2 + 2i\,\hat{x}_2) + 2\epsilon\hat{x}_1\hat{x}_2, \tag{15}$$

where ϵ is a real constant, with $\epsilon \in]-1, 1[$. Here the following commutation rules are assumed: $[\hat{x}_j, \hat{p}_k] = i\,\delta_{j,k}\mathbb{1}$, $\mathbb{1}$ being the identity operator on $\mathcal{L}^2(\mathbb{R}^2)$. All the other commutators are zero.

In order to rewrite H in a more convenient form it is possible to perform some changes of variables, [14], starting by introducing the operators P_j, X_j, $j = 1, 2$, via

$$P_1 := \frac{1}{2a}(\hat{p}_1 + \xi\,\hat{p}_2), \quad P_2 := \frac{1}{2b}(\hat{p}_1 - \xi\,\hat{p}_2),$$
$$X_1 := a(\hat{x}_1 + \xi\hat{x}_2), \quad X_2 := b(\hat{x}_1 - \xi\hat{x}_2),$$

where ξ can be ± 1, while a and b are real, non zero, arbitrary constants. These operators satisfy the same canonical commutation rules as the original ones: $[X_j, P_k] = i\,\delta_{j,k}\mathbb{1}$. Next we put

$$\Pi_1 = P_1, \quad \Pi_2 = P_2, \quad q_1 = X_1 + i\,\frac{a\xi}{1+\epsilon\,\xi}, \quad q_2 = X_2 - i\,\frac{b\xi}{1-\epsilon\,\xi},$$

and it is clear that $q_j^\dagger \neq q_j$, $j = 1, 2$. However, the commutation rules are preserved: $[q_j, \Pi_k] = i\,\delta_{j,k}\mathbb{1}$. Finally, we introduce the operators:

$$\begin{cases} a_1 = \frac{a}{\sqrt[4]{1+\epsilon\,\xi}}\left(i\,\Pi_1 + \frac{\sqrt{1+\epsilon\,\xi}}{2a^2}\,q_1\right), \\ a_2 = \frac{a}{\sqrt[4]{1-\epsilon\,\xi}}\left(i\,\Pi_2 + \frac{\sqrt{1-\epsilon\,\xi}}{2b^2}\,q_2\right), \end{cases} \tag{16}$$

and

$$\begin{cases} b_1 = \frac{a}{\sqrt[4]{1+\epsilon\,\xi}}\left(-i\,\Pi_1 + \frac{\sqrt{1+\epsilon\,\xi}}{2a^2}\,q_1\right), \\ b_2 = \frac{a}{\sqrt[4]{1-\epsilon\,\xi}}\left(-i\,\Pi_2 + \frac{\sqrt{1-\epsilon\,\xi}}{2b^2}\,q_2\right). \end{cases} \tag{17}$$

It may be worth remarking that $b_j \neq a_j^\dagger$, since the q_j's are not self-adjoint. These operators satisfy, at least formally, the pseudo-bosonic commutation rules

$$[a_j, b_k] = \delta_{j,k}\mathbb{1}, \tag{18}$$

the other commutators being zero.

Going back to H, and introducing the operators $N_j := b_j a_j$, we can write

$$H = H_1 + H_2 + \frac{1}{1-\epsilon^2}\,\mathbb{1}, \quad H_1 = \sqrt{1+\epsilon\,\xi}(2N_1+\mathbb{1}), \quad H_2 = \sqrt{1-\epsilon\,\xi}(2N_2+\mathbb{1}). \tag{19}$$

In [6, 15] it has been proved that these operators provide a two-dimensional version of the general framework described in Sect. 2: we are dealing with PBs, but in 2D.

Another 2D Example

The last quantum mechanical model of this short (and very minimal!) list was originally introduced, in our knowledge, in [16]. The starting point is the following manifestly non self-adjoint Hamiltonian,

$$H = \frac{1}{2}(\hat{p}_1^2 + \hat{x}_1^2) + \frac{1}{2}(\hat{p}_2^2 + \hat{x}_2^2) + i\,[A(\hat{x}_1 + \hat{x}_2) + B(\hat{p}_1 + \hat{p}_2)], \tag{20}$$

where A and B are real constants, while $\hat{x}_j$ and $\hat{p}_j$ are the self-adjoint position and momentum operators, satisfying $[\hat{x}_j, \hat{p}_k] = i\,\delta_{j,k}\mathbb{1}$. Notice that in [6] a noncommutative version of this system has also been considered.

Let us introduce the shifted operators

$$P_1 = \hat{p}_1 + iB, \quad P_2 = \hat{p}_2 + iB, \quad X_1 = \hat{x}_1 + iA, \quad X_2 = \hat{x}_2 + iA,$$

and then

$$a_j = \frac{1}{\sqrt{2}}(X_j + iP_j), \quad b_j = \frac{1}{\sqrt{2}}(X_j - iP_j), \tag{21}$$

$j = 1, 2$. It is easy to check that $[X_j, P_k] = i\,\delta_{j,k}\mathbb{1}$, $[a_j, b_k] = \delta_{j,k}\mathbb{1}$, and that, since (if $A \neq 0$ or $B \neq 0$) $X_j^\dagger \neq X_j$ and $P_j^\dagger \neq P_j$, $b_j \neq a_j^\dagger$. Introducing further $N_j = b_j a_j$ we can rewrite H as follows: $H = N_1 + N_2 + (A^2 + B^2 + 1)\mathbb{1}$. Also in this case, we can rewrite H in diagonal form in terms of pseudo-bosonic number operators. We refer to [6] to see the details of our computations, and for the mathematical subtleties connected with the Hamiltonian in (20).

We conclude here this list of concrete appearances of PBs in some quantum mechanical models already existing in the literature, before the analysis given in Sect. 2 was undertaken. It is useful to add that PBs have shown to be useful in the analysis of many other models, and in connection with other interesting situations. We refer to [7], in particular, for some applications of PBs to path integrals.

3 Weak PBs

From now on we will concentrate on a specific class of PBs, the so-called weak PBs, WPBs. These are ladder operators acting on distributions, rather than on square-integrable functions. They have, as we will see, similar properties as those of ordinary PBs, but are maybe more intriguing for their mathematical properties.

We start introducing here, as before, two operators a and b which, together with their adjoints $a^\dagger$ and $b^\dagger$, map a certain dense subset of $\mathcal{H}$, $\mathcal{D}$, into itself. Then we assume that a and b can be extended to a larger set, $\mathcal{E} \supset \mathcal{H}$, which is again stable under their action, and under the action of their adjoints. The existence of such a set $\mathcal{E}$ is, of course, very much model-dependent. Some explicit example will be discussed later in Sect. 3. With this in mind, we propose the following

Definition 2 The operators a and b are *weak* $\mathcal{E}$-pseudo bosonic if

$$[a, b]\, F = F, \tag{22}$$

for all $F \in \mathcal{E}$. When the role of $\mathcal{E}$ is clear we will simply call a and b *weak* pseudo bosonic operators.

As in Sect. 2, the commutator in (22) is just the starting point to construct an interesting mathematical framework. This is exactly what we will do here. In particular, the following two assumptions reflect Assumptions $\mathcal{D}$-pb 1 and $\mathcal{D}$-pb 2:

Assumption $\mathcal{E}$-wpb 1 there exists a non-zero $\varphi_0 \in \mathcal{E}$ such that $a\,\varphi_0 = 0$.

Assumption $\mathcal{E}$-wpb 2 there exists a non-zero $\Psi_0 \in \mathcal{E}$ such that $b^\dagger\,\Psi_0 = 0$.

As before, the invariance of $\mathcal{E}$ under the action of the operators a, b, $a^\dagger$ and $b^\dagger$ implies that $\varphi_0 \in D^\infty(b) := \cap_{k\geq 0} D(b^k)$ and $\Psi_0 \in D^\infty(a^\dagger)$, in the sense of generalized domains, so that the vectors

$$\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad \Psi_n := \frac{1}{\sqrt{n!}} {a^\dagger}^n \Psi_0, \tag{23}$$

$n \geq 0$, can be defined and they all belong to $\mathcal{E}$. Defining now the sets $\mathcal{F}_\psi = \{\psi_n,\, n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$, from (22) and from the definition in (23) we easily deduce the same raising and lowering relations as in (8), together with the eigenvalue equations $N\varphi_n = n\varphi_n$ and $N^\dagger\Psi_n = n\Psi_n$, $n \geq 0$. In the attempt to generalize what we have proved for PBs, it is now natural to assume that, with a suitable choice of the normalization of φ_0 and Ψ_0 which implies that $\langle\varphi_0, \Psi_0\rangle = 1$, then

$$\langle\varphi_n, \Psi_m\rangle = \delta_{n,m}, \tag{24}$$

for all $n, m \geq 0$. This means that $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$ are requested to be biorthonormal, with respect to a bilinear form $\langle ., .\rangle$ which extends the ordinary scalar product to $\mathcal{E}$, and which needs to be identified in concrete situations.

Of course, since the vectors of $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$ are not, in general, in $\mathcal{H}$, it makes not much sense to require any strong version of the basis property for $\mathcal{F}_\Psi$ or $\mathcal{F}_\varphi$. On the other hand, what seems natural to require is that a set $C \subseteq \mathcal{E}$ exists, *consisting of "sufficiently many" functions*, such that

$$\langle F, G\rangle = \sum_{n=0}^{\infty}\langle F, \psi_n\rangle\langle\varphi_n, G\rangle = \sum_{n=0}^{\infty}\langle F, \varphi_n\rangle\langle\psi_n, G\rangle, \tag{25}$$

for all $F, G \in C$. A pragmatic view on C is that it should contains all those (generalized) functions which are *interesting* for us, for some specific physical or mathematical reason.

As in Sect. 2, we can use $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ to introduce two operators, S_φ and S_ψ, as follows: let

$$D(S_\varphi) = \{F \in \mathcal{E}: S_\varphi F \in \mathcal{E}\}, \quad D(S_\psi) = \{F \in \mathcal{E}: S_\psi F \in \mathcal{E}\}.$$

These are to be understood as generalized domains of S_φ and S_ψ, respectively. Most of the properties found for ordinary PBs are recovered. Calling $\mathcal{L}_\varphi$ and $\mathcal{L}_\psi$ respectively the linear spans of the vectors φ_n and ψ_n, we see that $\mathcal{L}_\varphi \subseteq D(S_\psi)$, $\mathcal{L}_\psi \subseteq D(S_\varphi)$, $S_\varphi: \mathcal{L}_\psi \to \mathcal{L}_\varphi$, and $S_\psi: \mathcal{L}_\varphi \to \mathcal{L}_\psi$. In particular we have

$$S_\varphi\left(\sum_{k=0}^{N} c_k\psi_k\right) = \sum_{k=0}^{N} c_k\varphi_k, \quad S_\psi\left(\sum_{k=0}^{N} c_k\varphi_k\right) = \sum_{k=0}^{N} c_k\psi_k, \tag{26}$$

as well as

$$S_\varphi S_\psi F = F, \quad S_\psi S_\varphi G = G, \tag{27}$$

and

$$NS_\varphi G = S_\varphi N^\dagger G, \quad N^\dagger S_\psi F = S_\psi NF. \tag{28}$$

Moreover

$$a\,F = S_\varphi b^\dagger S_\psi F, \quad b\,F = S_\varphi a^\dagger S_\psi F, \quad a^\dagger G = S_\psi b S_\varphi G, \quad b^\dagger G = S_\psi a S_\varphi G, \tag{29}$$

for all $F \in \mathcal{L}_\varphi$ and $G \in \mathcal{L}_\psi$. Using the same notation proposed in [6], the operators a and $b^\dagger$ could be called S_ψ-conjugate. Conjugate operators are sometimes considered in a Hilbertian context, and produce several interesting results. For this reason, a deeper investigation of these similarities conditions in the distributional sense could be interesting, and this analysis is in progress.

We can then conclude that there is no particular obstacle, in principle, in extending the main ideas and results deduced for PBs to WPBs. Of course, the rather abstract construction proposed so far in the section becomes more interesting if it can be really used, that is if there are (physical) systems which can be analysed in terms of WPBs. This is indeed the case, as we will show in the rest of this section.

3.1 Weak PBs for $\hat{x}$ and $\hat{p}$

Let us consider the following operators defined on $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$: $\hat{x} f(x) = x f(x)$, $(\hat{D} g)(x) = g'(x)$, the derivative of $g(x)$, for all $f(x) \in D(\hat{x}) = \{h(x) \in \mathcal{L}^2(\mathbb{R})$: $xh(x) \in \mathcal{L}^2(\mathbb{R}\}$ and $g(x) \in D(\hat{D}) = \{h(x) \in \mathcal{L}^2(\mathbb{R}: h'(x) \in \mathcal{L}^2(\mathbb{R})\}$. Of course, the set of test functions $S(\mathbb{R})$ is a subset of both sets above: $S(\mathbb{R}) \subset D(\hat{x})$ and $S(\mathbb{R}) \subset D(\hat{D})$. The adjoints of $\hat{x}$ and $\hat{D}$ in $\mathcal{H}$ are $\hat{x}^\dagger = \hat{x}$, $\hat{D}^\dagger = -\hat{D}$. We have $[\hat{D}, x] f(x) = f(x)$, for all $f(x) \in S(\mathbb{R})$. This suggests that $\hat{x}$ and $\hat{D}$ could be thought as $S(\mathbb{R})$-pseudo bosons, since they satisfy Definition 1 and since $S(\mathbb{R})$ is stable under their action, and the action of their adjoints. However, if we look for the vacua of $a = \hat{D}$ and $b = \hat{x}$, we easily find that $\varphi_0(x) = 1$ and $\psi_0(x) = \delta(x)$, with a suitable choice of the *normalizations*[5]. It is clear, therefore, that neither $\varphi_0(x)$ nor $\psi_0(x)$ belong to $S(\mathbb{R})$. Also, they not even belong to $\mathcal{L}^2(\mathbb{R})$. Nonetheless, it is interesting to see what can be recovered of the framework proposed in Sect. 2, or if it can be extended, and how. In fact, we will show how the general settings proposed in the first part of Sect. 3 work for our operators $\hat{x}$ and $\hat{p}$.

First of all, let us check if (7) still makes some sense. We have

$$\varphi_n(x) = \frac{b^n}{\sqrt{n!}}\,\varphi_0(x) = \frac{x^n}{\sqrt{n!}}, \quad \psi_n(x) = \frac{(a^\dagger)^n}{\sqrt{n!}}\,\psi_0(x) = \frac{(-1)^n}{\sqrt{n!}}\,\delta^{(n)}(x), \tag{30}$$

for all $n = 0, 1, 2, 3, \ldots$ Here $\delta^{(n)}(x)$ is the n-th weak derivative of the Dirac delta function. We can check that $\varphi_n(x), \psi_n(x) \in S'(\mathbb{R})$, the set of the tempered distributions, [17], that is the continuous linear functionals on $S(\mathbb{R})$. This suggests to consider $a^\dagger$ and b as linear operators acting on $S'(\mathbb{R})$. This is possible since the action of $\hat{x}$ and $\hat{D}$ can be extended outside $\mathcal{L}^2(\mathbb{R})$, to $S'(\mathbb{R})$, which is stable under the action of these operators. In other words: $a, b, a^\dagger$ and $b^\dagger$ all map $S'(\mathbb{R})$ into itself. This is exactly what required before Definition 2, with $S'(\mathbb{R})$ playing the role of $\mathcal{E}$. Then we can extend the pseudo-bosonic commutation relation, originally defined

[5] In fact, to talk of normalization we should have a scalar product but, for the moment, it is not clear what such a scalar product could be in the present context. This will be clarified later in this section.

as $[D, x]f(x) = f(x)$, for all $f(x) \in S(\mathbb{R})$, to the space of tempered distributions:

$$[a, b]\varphi(x) = \varphi(x), \tag{31}$$

for all $\varphi(x) \in S'(\mathbb{R})$.

From (30) it follows that b and $a^\dagger$ act as raising operators, respectively on the sets $\mathcal{F}_\varphi = \{\varphi_n(x)\}$ and $\mathcal{F}_\psi = \{\psi_n(x)\}$:

$$b\varphi_k(x) = \sqrt{k+1}\varphi_{k+1}(x), \quad a^\dagger\psi_k(x) = \sqrt{k+1}\psi_{k+1}(x), \tag{32}$$

$k = 0, 1, 2, 3, \ldots$ Moreover, from (31), we deduce that $b^\dagger$ and a act as lowering operators on these sets:

$$a\varphi_k(x) = \sqrt{k}\varphi_{k-1}(x), \quad b^\dagger\psi_k(x) = \sqrt{k}\psi_{k-1}(x), \tag{33}$$

$k = 0, 1, 2, 3, \ldots$, with the understanding that $a\varphi_0(x) = b^\dagger\psi_0(x) = 0$. It is now clear that, calling $N = ba = \hat{x}\hat{D}$, $N\varphi_k(x) = k\varphi_k(x)$, for all $k = 0, 1, 2, 3, \ldots$ This is because $N\varphi_k(x) = b(a\varphi_k(x)) = \sqrt{k}\, b\varphi_{k-1}(x) = k\varphi_k(x)$. But the same result can also be found in a different, more explicit, way:

$$N\varphi_k(x) = \hat{x}\,\hat{D}\,\frac{x^k}{\sqrt{k!}} = \hat{x}\,\frac{kx^{k-1}}{\sqrt{k!}} = \frac{kx^k}{\sqrt{k!}} = k\varphi_k(x).$$

The distributions $\psi_k(x)$ are also (generalized) eigenstates of a number-like operator. In fact, calling $N^\dagger = a^\dagger b^\dagger$, and using formulas (32) and (33), one proves that $N^\dagger\psi_k(x) = k\psi_k(x)$. Again, this can be checked explicitly by computing

$$N^\dagger\psi_k(x) = -\hat{D}\hat{x}\left(\frac{(-1)^k}{\sqrt{k!}}\,\delta^{(k)}(x)\right) = \frac{(-1)^{k+1}}{\sqrt{k!}}\,(x\delta^{(k)}(x))' = k\psi_k(x),$$

since the weak derivative of $x\delta^{(k)}(x)$ can be easily computed and we have $(x\delta^{(k)}(x))' = -k\delta^{(k)}(x)$, for all $k = 0, 1, 2, 3, \ldots$ Summarizing, we have that

$$N\varphi_k(x) = k\varphi_k(x), \quad N^\dagger\psi_k(x) = k\psi_k(x), \tag{34}$$

for all $k = 0, 1, 2, 3, \ldots$ This formula, together with (32) and (33), are analogous to those deduced in Sect. 2. Hence, this suggests that a framework close to that of PBs can be extended, for $\hat{x}$ and $\hat{D}$, from the Hilbert space $\mathcal{L}^2(\mathbb{R})$ to the set of tempered distributions.

The next step consists in checking, if possible, the biorthogonality of the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$, and their basis properties, if any. In other words, we are interested in understanding whether equations (24) and (25), or some similar expressions, can be deduced for our families of tempered distributions.

Of course, to talk of biorthogonality, we should first define some sort of scalar product. But this is impossible for distributions, in general. However, there are pairs of distributions for which such an operation can be defined, as we will discuss now. We should also stress that this *extended scalar product* is not unique: other choices are possible, and a different choice was recently proposed in [18].

First we observe that the scalar product between two *good* functions, for instance $f(x), g(x) \in S(\mathbb{R})$, can be written in terms of a convolution between $\overline{f(x)}$ and the function $\tilde{g}(x) = g(-x)$. Indeed we have $\langle f, g\rangle = (\overline{f} * \tilde{g})(0)$. In the same way we define the scalar product between two elements $F(x), G(x) \in S'(\mathbb{R})$ as the following convolution:

$$\langle F, G\rangle = (\overline{F} * \tilde{G})(0), \tag{35}$$

whenever this convolution exists. This existence issue is discussed, for instance, in [19]. As we will see, this will not be a problem for us. In order to compute $\langle F, G\rangle$, it is necessary to compute $(\overline{F} * \tilde{G})[f]$, $f(x) \in S(\mathbb{R})$, and this can be computed by using the equality[6] $(\overline{F} * \tilde{G})[f] = \langle F, G * f\rangle$.

In our situation we have $F(x) = x^n$ and $G(x) = \delta^{(m)}(x)$, $n, m = 0, 1, 2, 3, \ldots$ Hence $(G * f)(x) = \int_{\mathbb{R}} \delta^{(m)}(y) f(x-y)dy = f^{(m)}(x)$, where $f^{(m)}(x)$ is the ordinary m-th derivative of the test function $f(x)$. Then we have

$$(\overline{F} * \tilde{G})[f] = \langle F, G * f\rangle = \int_{\mathbb{R}} \overline{F(x)} f^{(m)}(x)\,dx = \int_{\mathbb{R}} x^n \frac{d^m f(x)}{dx^m}\,dx$$
$$= (-1)^m \int_{\mathbb{R}} \frac{d^m x^n}{dx^m} f(x)\,dx.$$

But

$$\frac{d^m x^n}{dx^m} = \begin{cases} 0 & \text{if } m > n \\ n! & \text{if } m = n \\ \frac{n!}{(n-m)!}\, x^{n-m} & \text{if } m < n, \end{cases}$$

and therefore

$$(\overline{F} * \tilde{G})[f] = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! \int_{\mathbb{R}} f(x)\,dx & \text{if } m = n \\ (-1)^m \frac{n!}{(n-m)!} \int_{\mathbb{R}} x^{n-m} f(x)\,dx & \text{if } m < n. \end{cases}$$

Hence

$$(\overline{F} * \tilde{G})(x) = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! & \text{if } m = n \\ (-1)^m \frac{n!}{(n-m)!} x^{n-m} & \text{if } m < n, \end{cases}$$

and therefore that $(\overline{F} * \tilde{G})(0) = (-1)^n n! \delta_{n,m}$. Putting all these results together, we conclude that not only $\langle \varphi_n, \psi_m\rangle$ exists, but also that

$$\langle \varphi_n, \psi_m\rangle = \delta_{n,m}, \tag{36}$$

[6] We stress once more that $(\overline{F} * \tilde{G})[f]$ is not always defined, but there exist useful situations when it is. This is the case when $\langle F, G * f\rangle$ exists. It is maybe useful to stress that $(\overline{F} * \tilde{G})[f]$ represents the action of $(\overline{F} * \tilde{G})(x)$ on the function $f(x)$.

as claimed before. Notice that our original choice of normalization for $\varphi_0(x)$ and $\psi_0(x)$ guarantees the biorthonormality (and not only the biorthogonality) of the families $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$.

Remark It is clear that $\langle .,. \rangle$ cannot satisfy all the properties of an *ordinary* scalar product. In particular, it could be impossible to check that $\langle F, F\rangle \geq 0$ for all tempered distributions F, and that $\langle F, F\rangle = 0$ if, and only if, $F = 0$. The reason is simple: there is no guarantee that $\langle F, F\rangle$ does even exist, indeed. However, $\langle .,. \rangle$ has all the properties of an ordinary scalar product when restricted, for instance, to $\mathcal{S}(\mathbb{R})$ since, in this case, $\langle .,. \rangle$ coincides with the ordinary scalar product in $\mathcal{L}^2(\mathbb{R})$.

It is clear that it makes no much sense to check if $\mathcal{F}_\varphi$ or $\mathcal{F}_\psi$, or both, are bases in $\mathcal{H}$. This is because none of the $\varphi_n(x)$ and $\psi_n(x)$ even belongs to $\mathcal{L}^2(\mathbb{R})$. However, the pair $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ can still be used to expand a certain class of functions, those which admit expansion in Taylor series. In fact we have

$$\sum_{n=0}^{\infty}\langle\psi_n, f\rangle\varphi_n(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle\delta^{(n)}, f\rangle x^n = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)\,x^n = f(x),$$

for all $f(x)$ admitting this kind of expansion. However, if we invert the role of $\mathcal{F}_\psi$ and $\mathcal{F}_\varphi$, the result is more complicated:

$$\sum_{n=0}^{\infty}\langle\varphi_n, f\rangle\psi_n(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle x^n, f\rangle\delta^{(n)}(x).$$

This is, in principle, an infinite series of derivatives of delta, called *dual Taylor series*, see [20, 21], for instance. It is known that the series does not define in general an element of $D'(\mathbb{R})$, a distribution, (hence it cannot define a tempered distribution) except when the number of non zero moments of $f(x)$, $\langle x^n, f\rangle$, is finite, since, in this case, the series above returns a finite sum, which is indeed a (tempered) distribution.

This preliminary analysis shows that the pair $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ obeys a sort of *weak basis property*, at least for very special functions or distributions. What we will do next is to check if, and for which objects, a formula like that in (25) can be written. In this perspective, let us introduce the following set of functions:

$$\mathcal{D} = \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R}) \cap A(\mathbb{R}), \tag{37}$$

where $A(\mathbb{R})$ is the set of entire real analytic functions, which admit expansion in Taylor series, everywhere convergent in $\mathbb{R}$. It might be useful to notice that $\mathcal{D}$ contains many functions of $\mathcal{S}(\mathbb{R})$, but not all.

Let now $f(x), g(x) \in \mathcal{D}$, and let us consider the following sequence of functions: $R_N(x) = \overline{f(x)}\,\sum_{n=0}^{N}\frac{g^{(n)}(0)}{n!}\,x^n$. It is clear, first of all, that $R_N(x)$ converges to $\overline{f(x)}\,g(x)$ almost everywhere (a.e.) in $\mathbb{R}$. Of course, it also converges with respect to stronger topologies, but this is not relevant for us. The second useful property is that $R_N(x)$ can be estimated as follows:

$$|R_N(x)| \leq R(x) \equiv |f(x)|(M + \|g\|_\infty), \tag{38}$$

for some fixed $M > 0$ and for all N large enough. It is clear that $R(x) \in \mathcal{L}^1(\mathbb{R})$. To prove the estimate in (38) it is enough to observe that, a.e. in x,

$$|R_N(x)| \leq |f(x)| \left(\left| \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} x^n - g(x) \right| + |g(x)| \right) \leq |f(x)|(M + \|g\|_\infty),$$

where M surely exists (independently of x) due to the uniform convergence of $\sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} x^n$ to $g(x)$. Then we can apply the Lebesgue dominated convergence theorem to conclude that

$$\lim_{N,\infty} \int_{\mathbb{R}} R_N(x)dx = \int_{\mathbb{R}} \overline{f(x)}\, g(x)dx = \langle f, g\rangle.$$

Incidentally we observe that, since $f, g \in \mathcal{D}$, $|\langle f, g\rangle| \leq \|f\|_1 \|g\|_\infty$, which ensures that $\langle f, g\rangle$ is well defined. Now,

$$\langle f, g\rangle = \lim_{N,\infty} \int_{\mathbb{R}} R_N(x)dx = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(0) \langle f, x^n\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle f, x^n\rangle \langle \delta^{(n)}, g\rangle =$$
$$= \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \langle \psi_n, g\rangle.$$

In a similar way we can also check that, for the same $f(x)$ and $g(x)$,

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \langle f, \psi_n\rangle \langle \varphi_n, g\rangle.$$

Hence we conclude that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ are $\mathcal{D}$-quasi bases. It should be stressed that it is not clear if $\mathcal{D}$ is dense or not in $\mathcal{H}$, but this is not particularly relevant in the present context, where the role of the Hilbert space is only marginal. Moreover, there are also distributions which satisfy (half of) formula (25). For instance, if $f(x) = \sum_{k=0}^{M} a_k \psi_k(x)$ for some complex a_k and fixed M, the equality $\langle f, g\rangle = \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \langle \psi_n, g\rangle$ is automatically satisfied, while it is not even clear if $\sum_{n=0}^{\infty} \langle f, \psi_n\rangle \langle \varphi_n, g\rangle$ does converge or not. Similarly, if we take $g(x) = \sum_{k=0}^{L} b_k \varphi_k(x)$ for some complex b_k and fixed L, $\langle f, g\rangle = \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \langle \psi_n, g\rangle$ is true, while $\sum_{n=0}^{\infty} \langle f, \psi_n\rangle \langle \varphi_n, g\rangle$ could be not even convergent.

In analogy with what we have done in Sect. 2, we can use $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ to introduce two operators, S_φ and S_ψ, which we formally write, for the moment,

$$S_\varphi = \sum_n |\varphi_n >< \varphi_n|, \quad S_\psi = \sum_n |\psi_n >< \psi_n|. \tag{39}$$

We have seen that these operators have interesting properties, and it makes sense to understand if they can be extended, and in which sense, to the present distributional context. In particular, it is interesting to check formulas (26)-(29).

First of all, we introduce the following subsets of $S'(\mathbb{R})$:

$$D(S_\varphi) = \{F(x) \in S'(\mathbb{R}) : (S_\varphi F)(x) \in S'(\mathbb{R})\}$$

and

$$D(S_\psi) = \{F(x) \in S'(\mathbb{R}) : (S_\psi F)(x) \in S'(\mathbb{R})\}.$$

As always, we call these sets the *generalized domains* of S_φ and S_ψ, respectively. It is easy to see that $\mathcal{L}_\varphi \subseteq D(S_\psi)$ and $\mathcal{L}_\psi \subseteq D(S_\varphi)$ and that $S_\varphi : \mathcal{L}_\psi \to \mathcal{L}_\varphi$, while $S_\psi : \mathcal{L}_\varphi \to \mathcal{L}_\psi$. In particular we have

$$S_\varphi\left(\sum_{k=0}^{N} c_k \psi_k\right) = \sum_{k=0}^{N} c_k \varphi_k, \quad S_\psi\left(\sum_{k=0}^{N} c_k \varphi_k\right) = \sum_{k=0}^{N} c_k \psi_k, \tag{40}$$

as well as

$$S_\varphi S_\psi F = F, \quad S_\psi S_\varphi G = G, \tag{41}$$

and

$$N S_\varphi G = S_\varphi N^\dagger G, \quad N^\dagger S_\psi F = S_\psi N F, \tag{42}$$

for $F(x) \in \mathcal{L}_\varphi$, $G(x) \in \mathcal{L}_\psi$. Furthermore, it is possible to see that $\mathcal{L}_\psi \neq D(S_\varphi)$. In fact, for F to belong to $D(S_\varphi)$, it is sufficient that the series $\sum_{n=0}^{\infty} \langle \varphi_n, F\rangle \varphi_n(x) = \sum_{n=0}^{\infty} \alpha_n x^n$, $\alpha_n = \frac{1}{n!}\langle x^n, F\rangle$, converges. For instance, if $F(x)$ is equal to 1 for $x \in [0, 1]$ and zero otherwise, the series converges for all $x \in \mathbb{R}$, even if $F(x) \notin \mathcal{L}_\psi$.

We refer to [22] for more results on this specific example of WPBs.

3.2 Weak PBs for the Inverted Quantum Harmonic Oscillator

This section is devoted to another appearance of WPBs. In this case, this will occur while studying a particular Hamiltonian which looks like a rotated version of the harmonic oscillator. Once again, we will see that distributions are relevant for our system.

We start considering the Hamiltonian

$$H_\theta = \frac{1}{2}\left(\hat{p}^2 + e^{2i\theta}\Omega^2 \hat{x}^2\right), \tag{43}$$

for $\theta \in [-\pi, \pi]$, for the moment, and $\Omega > 0$. Here, as usual, $[\hat{x}, \hat{p}] = i\mathbb{1}$, $\hat{x} = \hat{x}^\dagger$ and $\hat{p} = \hat{p}^\dagger$. It is clear that, if $\theta = \pm\frac{\pi}{2}$, H_θ becomes the Hamiltonian of the IQHO, $H_+ = H_- = \frac{1}{2}\left(\hat{p}^2 - \Omega^2 \hat{x}^2\right) =: H$, which is what we are really interested in.

Let us introduce the operators

$$A_\theta=\frac{1}{\sqrt{2\Omega}}\big(e^{i\theta/2}\Omega\,\hat{x}+i\,e^{-i\theta/2}\hat{p}\big),\quad B_\theta=\frac{1}{\sqrt{2\Omega}}\big(e^{i\theta/2}\Omega\,\hat{x}-i\,e^{-i\theta/2}\hat{p}\big),\tag{44}$$

for all admissible θ. It is clear that A_θ and B_θ are densely defined in $\mathcal{L}^2(\mathbb{R})$, since in particular any test function $f(x)\in\mathcal{S}(\mathbb{R})$ belongs to the domains of both these operators: $\mathcal{S}(\mathbb{R})\subseteq D(A_\theta)$ and $\mathcal{S}(\mathbb{R})\subseteq D(B_\theta)$, for all θ. It is also clear that $A_\theta^\dagger\neq B_\theta$. Indeed we can check that, for instance on $\mathcal{S}(\mathbb{R})$,

$$A_\theta^\dagger=\frac{1}{\sqrt{2\Omega}}\big(e^{-i\theta/2}\Omega\,\hat{x}-i\,e^{i\theta/2}\hat{p}\big),\quad B_\theta^\dagger=\frac{1}{\sqrt{2\Omega}}\big(e^{-i\theta/2}\Omega\,\hat{x}+i\,e^{i\theta/2}\hat{p}\big).\tag{45}$$

The set $\mathcal{S}(\mathbb{R})$ is stable under the action of all these operators. Formulas (45) show that

$$A_\theta^\dagger=B_{-\theta},\quad B_\theta^\dagger=A_{-\theta}.\tag{46}$$

Moreover, it is easy to see that these operators obey pseudo-bosonic commutation rules, [6]:

$$[A_\theta,B_\theta]f(x)=f(x)\tag{47}$$

for all $f(x)\in\mathcal{S}(\mathbb{R})$, and for all values of $\theta\in[-\pi,\pi]$. This is in agreement with the fact that, if $\theta=0$, we go back to the ordinary bosonic operators $d=\frac{\Omega\hat{x}+i\hat{p}}{\sqrt{2\Omega}}$ and $d^\dagger=\frac{\Omega\hat{x}-i\hat{p}}{\sqrt{2\Omega}}$, $[d,d^\dagger]=1\!\!1$. Indeed we have

$$A_0=B_0^\dagger=d,\quad B_0=A_0^\dagger=d^\dagger.$$

In terms of the operators in (44) H_θ can be rewritten as

$$H_\theta=\Omega e^{i\theta}\left(B_\theta A_\theta+\frac{1}{2}1\!\!1\right).\tag{48}$$

Then, because of (46), we have that

$$H_\theta^\dagger=\Omega e^{-i\theta}\left(A_\theta^\dagger B_\theta^\dagger+\frac{1}{2}1\!\!1\right)=H_{-\theta},\tag{49}$$

on $\mathcal{S}(\mathbb{R})$. Now the eigensystems of H_θ and $H_\theta^\dagger$ can be constructed by using the strategy adopted for PBs, see Sect. 2, and for WPBs, as shown in the first part of Sect. 3: we should first look for the ground state of the two annihilation operators A_θ and $B_\theta^\dagger$. But, since $B_\theta^\dagger=A_{-\theta}$, it is sufficient to solve the differential equation $A_\theta\varphi_0^{(\theta)}(x)=0$, since the solution of $B_\theta^\dagger\psi_0^{(\theta)}(x)=0$ is simply $\psi_0^{(\theta)}(x)=\varphi_0^{(-\theta)}(x)$. Hence, recalling that $\hat{p}=-i\frac{d}{dx}$, we find:

$$\varphi_0^{(\theta)}(x)=N^{(\theta)}e^{-\frac{1}{2}\Omega e^{i\theta}x^2},\quad \psi_0^{(\theta)}(x)=N^{(-\theta)}e^{-\frac{1}{2}\Omega e^{-i\theta}x^2},\tag{50}$$

where $N^{(\pm\theta)}$ are normalization constants which will be fixed later. From (50) we see that the vacua are in $\mathcal{L}^2(\mathbb{R})$ if $\Re(e^{\pm i\theta}) = \cos(\theta) > 0$. For this reason, from now on, we will restrict to $\theta \in I = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. This constraint reminds very much the similar one for the Swanson model, where it was needed both for ensuring square-integrability of the eigenstates of the Hamiltonian, but also to work with a well defined Hamiltonian, [9, 10].

With this in mind, and using again the usual pseudo-bosonic approach, we can construct two families of functions, $\mathcal{F}_\varphi^{(\theta)} = \{\varphi_n^{(\theta)}(x),\, n = 0, 1, 2, \ldots\}$ and $\mathcal{F}_\psi^{(\theta)} = \{\psi_n^{(\theta)}(x),\, n = 0, 1, 2, \ldots\}$, where

$$\varphi_n^{(\theta)}(x) = \frac{B_\theta^n}{\sqrt{n!}}\,\varphi_0^{(\theta)}(x) = \frac{N^{(\theta)}}{\sqrt{2^n\, n!}}\, H_n\left(e^{i\theta/2}\sqrt{\Omega}\, x\right) e^{-\frac{1}{2}\,\Omega e^{i\theta} x^2}, \tag{51}$$

$$\psi_n^{(\theta)}(x) = \frac{A_\theta^{\dagger\, n}}{\sqrt{n!}}\,\psi_0^{(\theta)}(x) = \varphi_n^{(-\theta)}(x) = \frac{N^{(-\theta)}}{\sqrt{2^n\, n!}}\, H_n\left(e^{-i\theta/2}\sqrt{\Omega}\, x\right) e^{-\frac{1}{2}\,\Omega e^{-i\theta} x^2}. \tag{52}$$

Here $H_n(x)$ is the n-th Hermite polynomial. The proof of these formulas is given in [26].

It is clear that, for $\theta \in I$, $\varphi_n^{(\theta)}(x), \psi_n^{(\theta)}(x) \in \mathcal{L}^2(\mathbb{R})$, for all $n \geq 0$. Also, these functions belong to the domain of A_θ, B_θ and of their adjoints, and we have ladder and eigenvalue equations as those in Sect. 2, see (8) in particular:

$$\begin{cases} B_\theta\, \varphi_n^{(\theta)}(x) = \sqrt{n+1}\, \varphi_{n+1}^{(\theta)}(x), & n \geq 0, \\ A_\theta\, \varphi_0^{(\theta)}(x) = 0, \quad A_\theta\, \varphi_n^{(\theta)}(x) = \sqrt{n}\, \varphi_{n-1}^{(\theta)}(x), & n \geq 1, \\ A_\theta^\dagger\, \psi_n^{(\theta)}(x) = \sqrt{n+1}\, \psi_{n+1}^{(\theta)}(x), & n \geq 0, \\ B_\theta^\dagger\, \psi_0^{(\theta)}(x) = 0, \quad B_\theta^\dagger\, \psi_n^{(\theta)}(x) = \sqrt{n}\, \psi_{n-1}^{(\theta)}(x), & n \geq 1, \\ N^{(\theta)}\varphi_n^{(\theta)}(x) = n\, \varphi_n^{(\theta)}(x), & n \geq 0, \\ N^{(\theta)\dagger}\psi_n^{(\theta)}(x) = n\, \psi_n^{(\theta)}(x), & n \geq 0, \end{cases} \tag{53}$$

where $N^{(\theta)} = B_\theta A_\theta$ and $N^{(\theta)\dagger}$ is its adjoint. Then, using (48) and (49), we conclude that

$$H_\theta \varphi_n^{(\theta)}(x) = E_n^{(\theta)} \varphi_n^{(\theta)}(x), \quad H_\theta^\dagger \psi_n^{(\theta)}(x) = E_n^{(-\theta)} \psi_n^{(\theta)}(x), \tag{54}$$

where $E_n^{(\theta)} = \omega e^{i\theta}\left(n + \frac{1}{2}\right)$. Notice that $E_n^{(-\theta)} = \overline{E_n^{(\theta)}}$. Hence the eigenvalues of H_θ and $H_\theta^\dagger$ have, for generic $\theta \in I$, a non zero real and a non zero imaginary part.

Remark If $\theta = 0$ everything collapses to the usual quantum harmonic oscillator, as it is clear from (43). In this case, if we take $N^{(0)} = \left(\frac{\Omega}{\pi}\right)^{1/4}$,

$$\varphi_n^{(0)}(x) = \psi_n^{(0)}(x) = e_n(x) = \frac{1}{\sqrt{2^n\, n!}}\left(\frac{\Omega}{\pi}\right)^{1/4} H_n\left(\sqrt{\Omega}\, x\right) e^{-\frac{1}{2}\,\Omega x^2}, \tag{55}$$

which is the well known n-th eigenstate of the quantum harmonic oscillator, as expected.

Another, also expected, feature of the families $\mathcal{F}_\varphi^{(\theta)}$ and $\mathcal{F}_\psi^{(\theta)}$ is that, with a proper choice of normalization, their vectors are mutually biorthonormal. Indeed if we fix

$$N^{(\theta)} = \left(\frac{\Omega}{\pi}\right)^{1/4} e^{i\theta/4}, \tag{56}$$

we can check that

$$\langle \varphi_n^{(\theta)}, \psi_m^{(\theta)} \rangle = \delta_{n,m}, \tag{57}$$

for all $n, m \geq 0$ and for all $\theta \in I$. Incidentally we observe that (56) gives back the right normalization of $e_n(x)$ when $\theta = 0$.

It is interesting to observe that the functions $\varphi_n^{(\theta)}(x)$ and $\psi_n^{(\theta)}(x)$ are essentially the rotated versions of the eigenstates $e_n(x)$ in (55):

$$\varphi_n^{(\theta)}(x) = e^{i\theta/4} e_n(e^{i\theta/2}x), \quad \psi_n^{(\theta)}(x) = e^{-i\theta/4} e_n(e^{-i\theta/2}x), \tag{58}$$

for all $n \geq 0$. This is in agreement with (57):

$$\langle \varphi_n^{(\theta)}, \psi_m^{(\theta)} \rangle = \int_{\mathbb{R}} \overline{\varphi_n^{(\theta)}(x)}\, \psi_m^{(\theta)}(x) dx = \int_{\Gamma_\theta} e_n(z) e_m(z) dz$$
$$= \int_{\mathbb{R}} e_n(x) e_m(x) dx = \langle e_n, e_m \rangle = \delta_{n,m},$$

using well known results in complex integration, see [26] for the details.

Next we can check that $\mathcal{F}_\varphi^{(\theta)}$ and $\mathcal{F}_\psi^{(\theta)}$ are complete (or, as some authors prefer, total) in $\mathcal{L}^2(\mathbb{R})$. This follows from a standard argument adopted in several papers, see [6] for instance, and originally proposed, in our knowledge, in [27]: if $\rho(x)$ is a Lebesgue-measurable function which is different from zero almost everywhere (a.e.) in $\mathbb{R}$, and if there exist two positive constants δ, C such that $|\rho(x)| \leq C\, e^{-\delta|x|}$ a.e. in $\mathbb{R}$, then the set $\{x^n \rho(x)\}$ is complete in $\mathcal{L}^2(\mathbb{R})$. We refer to [6] for some physical applications of this result. Because of their completeness, the sets $\mathcal{L}_\varphi^{(\theta)} = l.s.\{\varphi_n^{(\theta)}(x)\}$ and $\mathcal{L}_\psi^{(\theta)} = l.s.\{\psi_n^{(\theta)}(x)\}$, i.e. the linear spans of the functions in $\mathcal{F}_\varphi^{(\theta)}$ and in $\mathcal{F}_\psi^{(\theta)}$, are both dense in $\mathcal{L}^2(\mathbb{R})$. Now, (57) implies that

$$\sum_{n=0}^{\infty} \langle f, \varphi_n^{(\theta)} \rangle \langle \psi_n^{(\theta)}, g \rangle = \langle f, g \rangle, \tag{59}$$

$\forall f(x) \in \mathcal{L}_\psi^{(\theta)}$ and $\forall g(x) \in \mathcal{L}_\varphi^{(\theta)}$, which is our usual weak version of the resolution of the identity.

Incidentally we observe that what we have discussed here is, in fact, another concrete example of PBs, not particularly different from the Swanson model briefly described in Sect. 2.1.

We refer to [26] for more result on coherent states and for the analysis of a similarity operator which can be used in the analysis of the Hamiltonian in (43). Here we are more interested in discussing how to connect what we have deduced for H_θ to similar results for the IQHO.

From $\mathcal{L}^2(\mathbb{R})$ to Distributions

The Hamiltonian we want to consider in this section is the following:

$$H = \frac{1}{2}\left(\hat{p}^2 - \Omega^2 \hat{x}^2\right), \tag{60}$$

where, as in (43), $\Omega > 0$. This is what, in the literature, is called an inverted harmonic oscillator: we have a quadratic potential that, rather being convex, is concave, see, e.g., [23–25]. Hence it is reasonable to expect that there are no bound, square integrable, eigenstates. This is, indeed, what we are going to deduce here. We have already seen that H can be formally deduced by H_θ fixing θ either to $\frac{\pi}{2}$ or to $-\frac{\pi}{2}$. For this reason it is natural to define, see (58),

$$\varphi_n^{(\pm)}(x) = \varphi_n^{(\pm\frac{\pi}{2})}(x) = \frac{e^{\pm i\pi/8}}{\sqrt{2^n\, n!}}\left(\frac{\Omega}{\pi}\right)^{1/4} H_n\left(e^{\pm i\pi/4}\sqrt{\Omega}\, x\right) e^{\mp\frac{i}{2}\Omega\, x^2} \tag{61}$$

and

$$\begin{aligned}\psi_n^{(\pm)}(x) &= \psi_n^{(\pm\frac{\pi}{2})}(x) = \varphi_n^{(\mp)}(x)\\ &= \frac{e^{\mp i\pi/8}}{\sqrt{2^n\, n!}}\left(\frac{\Omega}{\pi}\right)^{1/4} H_n\left(e^{\mp i\pi/4}\sqrt{\Omega}\, x\right) e^{\pm\frac{i}{2}\Omega\, x^2}.\end{aligned} \tag{62}$$

It is clear that

$$\|\varphi_n^{(\pm)}\| = \|\psi_n^{(\pm)}\| = \infty,$$

so that none of these functions is square-integrable. However, even if they are not in $\mathcal{L}^2(\mathbb{R})$, they are connected to the operators $A_\pm$, $B_\pm$ and their adjoints, where

$$\begin{aligned}A_\pm &= A_{\pm\frac{\pi}{2}} = \frac{1}{\sqrt{2\Omega}}\left(e^{\pm i\pi/4}\Omega\,\hat{x} + i\, e^{\mp i\pi/4}\hat{p}\right),\\ B_\pm &= B_{\pm\frac{\pi}{2}} = \frac{1}{\sqrt{2\Omega}}\left(e^{\pm i\pi/4}\Omega\,\hat{x} - i\, e^{\mp i\pi/4}\hat{p}\right),\end{aligned} \tag{63}$$

and

$$B_\pm^\dagger = A_\mp, \quad A_\pm^\dagger = B_\mp. \tag{64}$$

These operators can all be written in terms of the ordinary bosonic operators d and $d^\dagger$ introduced before as follows:

$$A_\pm = \frac{d \pm i d^\dagger}{\sqrt{2}}, \quad B_\pm = \frac{d^\dagger \pm i d}{\sqrt{2}}, \tag{65}$$

with $A_\pm^\dagger$ and $B_\pm^\dagger$ deduced as in (64). All these operators leave $\mathcal{S}(\mathbb{R})$ stable. Then we have

$$[A_\pm, B_\pm] f(x) = f(x), \tag{66}$$

for all $f(x) \in \mathcal{S}(\mathbb{R})$. Moreover, these operators can also be applied to functions which are outside $\mathcal{S}(\mathbb{R})$, and even outside $\mathcal{L}^2(\mathbb{R})$. In fact, these operators can also act on $\varphi_n^{(\pm)}(x)$ and $\psi_n^{(\pm)}(x)$ and satisfy ladder equations of the same kind as those given in (53):

$$\begin{cases} A_\pm \, \varphi_0^{(\pm)}(x) = 0, \quad A_\pm \, \varphi_n^{(\pm)}(x) = \sqrt{n}\, \varphi_{n-1}^{(\pm)}(x), & n \geq 1, \\ B_\pm \, \varphi_n^{(\pm)}(x) = \sqrt{n+1}\, \varphi_{n+1}^{(\pm)}(x), & n \geq 0, \end{cases} \tag{67}$$

and

$$\begin{cases} B_\pm^\dagger \, \psi_0^{(\pm)}(x) = 0, \quad B_\pm^\dagger \, \psi_n^{(\pm)}(x) = \sqrt{n}\, \psi_{n-1}^{(\pm)}(x), & n \geq 1, \\ A_\pm^\dagger \, \psi_n^{(\pm)}(x) = \sqrt{n+1}\, \psi_{n+1}^{(\pm)}(x), & n \geq 0. \end{cases} \tag{68}$$

Hence the set $\mathcal{E}$ in Definition 2 surely contains $\mathcal{S}(\mathbb{R})$ and the set of all the finite linear combinations of the functions $\psi_n^{(\pm)}(x)$ and $\varphi_n^{(\pm)}(x)$.

Some easy computations show that H in (60) can be written in terms of these ladder operators. To simplify the notation we give the results in an operatorial form[7]. Specializing H_θ in (43) by taking $\theta = \pm\frac{\pi}{2}$ we put

$$H_\pm = \pm i\,\Omega\left(B_\pm A_\pm + \frac{1}{2}\mathbb{1}\right). \tag{69}$$

We have, as expected,

$$H = H_+ = H_-. \tag{70}$$

Using (64) we conclude that $H_+ = H_+^\dagger$, at least formally. Furthermore

$$H_\pm \varphi_n^{(\pm)}(x) = \pm i\,\Omega\left(n + \frac{1}{2}\right)\varphi_n^{(\pm)}(x), \tag{71}$$

$\forall n \geq 0$. Hence the eigenvalues of the IQHO are purely imaginary with both a positive and a negative imaginary part. Of course the functions $\psi_n^{(\pm)}(x)$, which are usually the eigenstates of the adjoint of the *original* Hamiltonian, see (54), are not

[7] All the operators we are considering in this section can be applied to functions of $\mathcal{S}(\mathbb{R})$, but not necessarily: they can also act on $\varphi_n^{(\pm)}(x)$ and $\psi_n^{(\pm)}(x)$, and to their linear combinations.

so relevant here since the adjoint of H_+ is H_+ itself. This is not surprising since, see (62), $\psi_n^{(\pm)}(x) = \varphi_n^{(\mp)}(x)$.

To put the eigenfunctions of H in a more interesting mathematical settings we start defining the following quantities:

$$\Phi_n^{(\pm)}[f] = \langle \varphi_n^{(\pm)}, f \rangle, \quad \Psi_n^{(\pm)}[g] = \langle \psi_n^{(\pm)}, g \rangle, \tag{72}$$

$\forall f(x), g(x) \in S(\mathbb{R})$ and $\forall n \geq 0$. Here $\langle .,. \rangle$ is the form with extend the ordinary scalar product to *compatible pairs*, i.e. to pairs of functions which are, when multiplied together, integrable, but separately they are not (or, at least, one is not). Compatible pairs have been considered in several contributions in the literature. We refer to [28] for their appearance in *partial inner product spaces*, and to [7] for some consideration closer (in spirit) to what we are doing here.

It is not hard to prove that $\Phi_n^{(\pm)}[f]$ and $\Psi_n^{(\pm)}[g]$ are well defined, linear, and continuous in the natural topology τ_S in $S(\mathbb{R})$. In few words, they are tempered distributions, $\Phi_n^{(\pm)}, \Psi_n^{(\pm)} \in S'(\mathbb{R})$. We will only prove this claim for $\Phi_n^{(+)}$, since for $\Phi_n^{(-)}$ and for $\Psi_n^{(\pm)}$ not many differences appear.

To check that $\Phi_n^{(+)}[f]$ is well defined, we observe that

$$\left|\Phi_n^{(+)}[f]\right| \leq \frac{(\Omega/\pi)^{1/4}}{\sqrt{2^n\, n!}} \int_{\mathbb{R}} \left| H_n\left(e^{i\pi/4}\sqrt{\Omega}\, x\right) f(x) \right| dx \leq M_n \sup_{x\in\mathbb{R}} (1+|x|)^{n+2} |f(x)|. \tag{73}$$

Here we have defined

$$M_n = \frac{(\Omega/\pi)^{1/4}}{\sqrt{2^n\, n!}} \int_{\mathbb{R}} \frac{|H_n(e^{i\pi/4}\sqrt{\Omega}\, x)|}{(1+|x|)^{n+2}}\, dx.$$

As we see, in this computation we have multiplied and divided the original integrand function $\left| H_n\left(e^{i\pi/4}\sqrt{\Omega}\, x\right) f(x) \right|$ for $(1+|x|)^{n+2}$. In this way, since the ratio $\frac{|H_n(e^{i\pi/4}\sqrt{\Omega}\, x)|}{(1+|x|)^{n+2}}$ has no singularity and decreases to zero for $|x|$ divergent as $|x|^{-2}$, we can conclude that M_n is finite (and positive). Moreover,

$$\sup_{x\in\mathbb{R}} (1+|x|)^{n+2}|f(x)| = \sup_{x\in\mathbb{R}} \sum_{k=0}^{n+2} \binom{n+2}{k} |x|^k |f(x)| = \sum_{k=0}^{n+2} \binom{n+2}{k} p_{k,0}(f),$$

where $p_{k,0}(.)$ is one of the seminorms defining the topology τ_S, see [29] for instance: $p_{k,l}(f) = \sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)|$, $k, l = 0, 1, 2, \ldots$ Of course, all these seminorms are finite for all $f(x) \in S(\mathbb{R})$.

Summarizing we have

$$\left|\Phi_n^{(+)}[f]\right| \leq M_n \sum_{k=0}^{n+2} \binom{n+2}{k} p_{k,0}(f),$$

so that $\Phi_n^{(+)}[f]$ is well defined for all $f(x) \in S(\mathbb{R})$, as we had to check.

The linearity of $\Phi_n^{(+)}$ is clear: $\Phi_n^{(+)}[\alpha f + \beta g] = \alpha\Phi_n^{(+)}[f] + \beta\Phi_n^{(+)}[g]$, for all $f(x), g(x) \in S(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$.

To conclude that $\Phi_n^{(+)} \in S'(\mathbb{R})$ we still have to prove that $\Phi_n^{(+)}$ is continuous. For that we have to consider a sequence of functions $\{f_k(x) \in S(\mathbb{R})\}$, τ_S-convergent to $f(x) \in S(\mathbb{R})$, and check that $\Phi_n^{(+)}[f_k] \to \Phi_n^{(+)}[f]$ for $k \to \infty$ in $\mathbb{C}$, for all fixed n. The proof of this fact is based on the following lemma, whose proof can be found in [26].

Lemma 1 Given a sequence of functions $\{f_k(x) \in S(\mathbb{R})\}$, τ_S-convergent to $f(x) \in S(\mathbb{R})$, it follows that $|x|^l|f_k(x)|$ converges, in the norm $\|.\|$ of $\mathcal{L}^2(\mathbb{R})$, to $|x|^l|f(x)|$, $\forall l \geq 0$.

Then we have

$$\left|\Phi_n^{(+)}[f_k - f]\right| = \left|\langle\varphi_n^{(+)}, f_k - f\rangle\right| = \left|\left\langle\frac{\varphi_n^{(+)}}{(1+|x|)^{n+1}}, (1+|x|)^{n+1}(f_k - f)\right\rangle\right|,$$

with an obvious manipulation. Now, since both $\frac{\varphi_n^{(+)}(x)}{(1+|x|)^{n+1}}$ and $(1+|x|)^{n+1}(f_k(x) - f(x))$ are in $\mathcal{L}^2(\mathbb{R})$, for all n, k, we can use the Schwarz inequality and we get

$$\left|\Phi_n^{(+)}[f_k - f]\right| \leq \left\|\frac{\varphi_n^{(+)}}{(1+|x|)^{n+1}}\right\| \left\|(1+|x|)^{n+1}(f_k - f)\right\| \to 0$$

when $k \to \infty$, for all fixed $n \geq 0$, because of Lemma 1.

The role of tempered distributions in the context of the IQHO is further clarified by the following result.

Theorem 1 For each fixed $n \geq 0$ the vector $\varphi_n^{(\pm)}(x)$ is a weak limit of $\varphi_n^{(\theta)}(x)$, for $\theta \to \pm\frac{\pi}{2}$:

$$\varphi_n^{(\pm)}(x) = w - \lim_{\theta,\pm\frac{\pi}{2}} \varphi_n^{(\theta)}(x). \tag{74}$$

Analogously,

$$\psi_n^{(\pm)}(x) = w - \lim_{\theta,\pm\frac{\pi}{2}} \psi_n^{(\theta)}(x). \tag{75}$$

Proof It is sufficient to prove that $\varphi_n^{(+)}(x) = w - \lim_{\theta,+\frac{\pi}{2}} \varphi_n^{(\theta)}(x)$, i.e. that

$$\langle\varphi_n^{(+)} - \varphi_n^{(\theta)}, f\rangle \to 0 \tag{76}$$

when $\theta \to \frac{\pi}{2}$, for all fixed $n \geq 0$ and for all $f(x) \in \mathcal{S}(\mathbb{R})$. First of all we observe that,

$$|\varphi_n^{(+)}(x) - \varphi_n^{(\theta)}(x)| \leq \frac{(\Omega/\pi)^{1/4}}{\sqrt{2^n\, n!}}\left(\left|H_n(e^{i\pi/4}\sqrt{\Omega}\,x)\right| + \left|H_n(e^{i\theta/2}\sqrt{\Omega}\,x)\right|\right)$$
$$\leq \frac{(\Omega/\pi)^{1/4}}{\sqrt{2^n\, n!}}\, p_n(x),$$

where $p_n(x)$ is a suitable polynomial in $|x|$ of degree n, independent of θ, whose expression is not particularly relevant[8]. This estimate implies that the function

$$\chi_n^{(\theta)}(x) = \frac{\varphi_n^{(+)}(x) - \varphi_n^{(\theta)}(x)}{(1+|x|)^{n+1}}$$

is square integrable for all fixed n and for all $\theta \in I$. Therefore, since $(1 + |x|)^{n+1} f(x) \in \mathcal{L}^2(\mathbb{R})$ as well, due to the fact that $f(x) \in \mathcal{S}(\mathbb{R})$, we have

$$\left|\langle \varphi_n^{(+)} - \varphi_n^{(\theta)}, f\rangle\right| = \left|\left\langle \frac{\varphi_n^{(+)} - \varphi_n^{(\theta)}}{(1+|x|)^{n+1}}, (1+|x|)^{n+1} f\right\rangle\right| \leq \|\chi_n^{(\theta)}\|\, \|(1+|x|)^{n+1} f\|,$$

using the Schwarz inequality. Now, to conclude as in (76), it is sufficient to show that $\|\chi_n^{(\theta)}\| \to 0$ when $\theta \to \frac{\pi}{2}$, i.e. that

$$\lim_{\theta,\frac{\pi}{2}} \int_{\mathbb{R}} |\chi_n^{(\theta)}(x)|^2\, dx = 0.$$

This is a consequence of the Lebesgue dominated convergence theorem, since it is clear first that $\lim_{\theta,\frac{\pi}{2}} \chi_n^{(\theta)}(x) = 0$ a.e. in x and since $|\chi_n^{(\theta)}(x)|^2$ is bounded by an $\mathcal{L}^1(\mathbb{R})$ function, in view of what we have shown before. Indeed we have

$$|\chi_n^{(\theta)}(x)|^2 = \frac{|\varphi_n^{(+)}(x) - \varphi_n^{(\theta)}(x)|^2}{(1+|x|)^{2n+2}} \leq \frac{(\Omega/\pi)^{1/2}}{2^n\, n!}\frac{p_n^2(x)}{(1+|x|)^{2n+2}},$$

which goes to zero for $|x|$ divergent as $|x|^{-2}$. $\square$

Summarizing the results proved so far we can write that *the eigenstates of the IQHO are not square integrable. They define tempered distributions and can be obtained as weak limits of the eigenstates of the Swanson-like Hamiltonian introduced in (43).*

We refer to [26] for more results also on coherent states associated to the IQHO.

[8] To clarify this aspect of the proof, let us consider, for instance $H_3(x) = 8x^3 - 12x$. Hence $|H_3(x)| \leq 8|x|^3 + 12|x|$ and, therefore $|H_3(e^{i\theta/2}\sqrt{\Omega}\,x)| \leq 8(\Omega)^{3/2}|x|^3 + 12\sqrt{\Omega}|x| = p_3(x)$, for instance.

3.3 A General Class of Pseudo-Bosonic Operators

Another interesting class of first order differential operators connected to PBs and to WPBs are of the form

$$a = \alpha_a(x)\,\frac{d}{dx} + \beta_a(x), \quad b = -\frac{d}{dx}\,\alpha_b(x) + \beta_b(x), \tag{77}$$

for some suitable functions $\alpha_j(x)$ and $\beta_j(x)$, $j = a, b$, which, for convenience, will be assumed to be C^∞ functions. This is what happens in concrete models: for ordinary bosons, for instance, we have $\alpha_a(x) = \alpha_b(x) = \frac{1}{\sqrt{2}}$, and $\beta_a(x) = \beta_b(x) = \frac{1}{\sqrt{2}}\,x$. For the shifted harmonic oscillator, see [6] and references therein, we have $a = c + \alpha\mathbb{1}$ and $b = c^\dagger + \beta\mathbb{1}$, for some complex α and β with $\alpha \neq \overline{\beta}$, and therefore $\alpha_a(x) = \alpha_b(x) = \frac{1}{\sqrt{2}}$ as before, while $\beta_a(x) = \frac{1}{\sqrt{2}}\,x + \alpha$ and $\beta_b(x) = \frac{1}{\sqrt{2}}\,x + \beta$. For the Swanson model, see again [6] and Sect. 2.1, $\alpha_a(x) = \alpha_b(x) = \frac{e^{-i\theta}}{\sqrt{2}}$, while $\beta_a(x) = \beta_b(x) = \frac{e^{i\theta}x}{\sqrt{2}}$.

More recently, [7, 30], a rather general class of pseudo-bosonic operators A and B have been considered, where $A = \frac{d}{dx} + w_A(x)$ and $B = -\frac{d}{dx} + w_B(x)$. In this case $\alpha_a(x) = \alpha_b(x) = 1$, while $w_A(x)$ and $w_B(x)$ have been called *pseudo-bosonic superpotentials* (PBSs) and they must satisfy $(w_A(x) + w_B(x))' = 1$, where the prime is the first x-derivative. In particular, in this last example, different choices of C^∞ functions $w_A(x)$ and $w_B(x)$ give rise to different families of functions, $\varphi_n(x)$ and $\Psi_n(x)$, constructed as in Sect. 2, which may, or may not, be square-integrable. However, see [30], we have proven the following result:

Proposition 1 If $w_A(x)$ and $w_B(x)$ are C^∞ PBSs, then $\varphi_n(x)\,\overline{\Psi_m(x)} \in \mathcal{L}^1(\mathbb{R})$ and $\langle \Psi_m, \varphi_n \rangle = \delta_{n,m}$, for all $n, m \geq 0$. □

This is another case, see also (72), in which the functions $\varphi_n(x)$ and $\Psi_n(x)$ are called *compatible*, in the sense of PIP-spaces, [28]. In this perspective it is useful to recall that two functions $h_1(x) \in \mathcal{L}^p(\mathbb{R})$ and $h_2(x) \in \mathcal{L}^q(\mathbb{R})$ can be multiplied producing a third function $h(x) = h_1(x)h_2(x)$ which is integrable, $h(x) \in \mathcal{L}^1(\mathbb{R})$, if $\frac{1}{p} + \frac{1}{q} = 1$. Hence, a *compatibility form* between $h_1(x)$ and $h_2(x)$ can be introduced, whose functional expression is the same as a scalar product in $\mathcal{L}^2(\mathbb{R})$, to which it reduces if $p = q = 2$. It is clear that, for those functions which are compatible, a generalized notion of biorthonormality can also be introduced.

In what follows, we are interested in extending all the particular cases listed above using the general forms of the operators in (77). Of course, our results will be strongly connected to the functions $\alpha_j(x)$ and $\beta_j(x)$.

To proceed in this direction we first compute the commutator $[a, b]$ on some sufficiently regular function $f(x)$. In particular, if not explicitly said, we will assume $f(x)$ to be at least C^2, while we will not insist much on $f(x)$ being or not square-integrable. Of course, this requirement could be relaxed if we interpret $\frac{d}{dx}$ as the weak derivative, but this will not be done here. An easy computation shows that, under this mild condition on $f(x)$, $[a,b]f(x)$ does make sense, and $[a,b]f(x)=f(x)$ if $\alpha_j(x)$ and $\beta_j(x)$, $j=a,b$, satisfy the following equalities

$$\begin{cases}\alpha_a(x)\alpha_b'(x)=\alpha_a'(x)\alpha_b(x),\\ \alpha_a(x)\beta_b'(x)+\alpha_b(x)\beta_a'(x)=1+\alpha_a(x)\alpha_b''(x).\end{cases} \tag{78}$$

It is easy to check that all the examples listed at the beginning of this section satisfy indeed these two conditions, in agreement with their nature of pseudo-bosonic operators. In particular the first equation in (78) is clearly satisfied by any constant choice of $\alpha_a(x)$ and $\alpha_b(x)$. Moreover, in this case, the second equation in (78) can be rewritten as $(\alpha_a\beta_b(x)+\alpha_b\beta_a(x))'=1$, which implies that $\alpha_a\beta_b(x)+\alpha_b\beta_a(x)=x+k$, for some constant k. This is essentially the situation described in terms of the PBSs $w_A(x)$ and $w_B(x)$ in [30]. Incidentally it is also clear that, if $\alpha_a(x)=\alpha_a\neq 0$, constant, then (78) implies that $\alpha_a(x)\alpha_b'(x)=\alpha_a\alpha_b'(x)=0$, which means that $\alpha_b(x)$ must also be constant. For this reason, to avoid going back to PBSs, in the rest of this section we will mainly focus on the situation in which both $\alpha_a(x)$ and $\alpha_b(x)$ depend on x in a non trivial way. Moreover, it is convenient for what follows to assume that they are never zero: $\alpha_j(x)\neq 0$, $\forall x\in\mathbb{R}$, $j=a,b$.

Under this assumption it is easy to deduce the vacua of a and of $b^\dagger$, as in Sects. 2 and 3. In what follows the following expressions are used for the adjoint in $\mathcal{H}$ of a and b:

$$a^\dagger=-\frac{d}{dx}\,\overline{\alpha_a(x)}+\overline{\beta_a(x)},\quad b^\dagger=\overline{\alpha_b(x)}\,\frac{d}{dx}+\overline{\beta_b(x)}. \tag{79}$$

The vacua of a and $b^\dagger$ are the solutions of $a\varphi_0(x)=0$ and $b^\dagger\psi_0(x)=0$, which turn out to be:

$$\varphi_0(x)=N_\varphi\exp\left\{-\int\frac{\beta_a(x)}{\alpha_a(x)}\,dx\right\},\quad \psi_0(x)=N_\psi\exp\left\{-\int\frac{\overline{\beta_b(x)}}{\overline{\alpha_b(x)}}\,dx\right\}, \tag{80}$$

and are well defined under our assumptions on $\alpha_j(x)$ and $\beta_j(x)$. Here N_φ and N_ψ are normalization constants which will be fixed later. If we now introduce $\varphi_n(x)$ and $\psi_n(x)$ as in (7),

$$\varphi_n(x)=\frac{1}{\sqrt{n!}}\,b^n\varphi_0(x),\quad \psi_n(x)=\frac{1}{\sqrt{n!}}\,a^{\dagger^n}\psi_0(x), \tag{81}$$

$n\geq 0$, we can prove the following, see [31]:

Proposition 2 Calling $\theta(x)=\alpha_a(x)\beta_b(x)+\alpha_b(x)\beta_a(x)$ we have

$$\varphi_n(x)=\frac{1}{\sqrt{n!}}\,\pi_n(x)\varphi_0(x),\quad \psi_n(x)=\frac{1}{\sqrt{n!}}\,\sigma_n(x)\psi_0(x), \tag{82}$$

$n \geq 0$, where $\pi_n(x)$ and $\sigma_n(x)$ are defined recursively as follows:

$$\pi_0(x) = \sigma_0(x) = 1, \tag{83}$$

and

$$\pi_n(x) = \left(\frac{\theta(x)}{\alpha_a(x)} - \alpha_b'(x)\right)\pi_{n-1}(x) - \alpha_b(x)\pi_{n-1}'(x), \tag{84}$$

$$\sigma_n(x) = \overline{\left(\frac{\theta(x)}{\alpha_b(x)} - \alpha_a'(x)\right)}\,\sigma_{n-1}(x) - \overline{\alpha_a(x)}\,\sigma_{n-1}'(x), \tag{85}$$

$n \geq 1$. □

A Special Case: Constant $\boldsymbol{\alpha_j(x)}$

We have already commented that taking $\alpha_a(x) = \alpha_a$ and $\alpha_b(x) = \alpha_b$ is not new, compared to what was done in [30]. However, it is still an interesting exercise, and for this reason we briefly discuss this case first. In this situation, $\alpha_a(x)$ and $\alpha_b(x)$ are always different from zero, at least if $\alpha_a\alpha_b \neq 0$. Formulas (84) and (85) simplify significantly now since, in particular, as we have already deduced before, $\theta(x) = \alpha_a\beta_b(x) + \alpha_b\beta_a(x) = x + k$. Hence we find

$$\begin{aligned}\pi_n(x) &= \frac{1}{\alpha_a}(x+k)\,\pi_{n-1}(x) - \alpha_b\,\pi_{n-1}'(x),\\ \sigma_n(x) &= \frac{1}{\overline{\alpha}_a}(x+\overline{k})\,\sigma_{n-1}(x) - \overline{\alpha}_a\,\sigma_{n-1}'(x),\end{aligned} \tag{86}$$

The case $\alpha_a = \alpha_b = 1$ has been considered in [30], while $\alpha_a = \alpha_b = \frac{1}{\sqrt{2}}$ is discussed in [7]. If α_a is not necessarily equal to α_b, similar conclusions can still be deduced. In particular from (86) we find that

$$\pi_n(x) = \sqrt{\left(\frac{\alpha_b}{2\alpha_a}\right)^n} H_n\left(\frac{x+k}{\sqrt{2\alpha_a\alpha_b}}\right), \quad \sigma_n(x) = \sqrt{\left(\frac{\overline{\alpha}_b}{2\overline{\alpha}_a}\right)^n} H_n\left(\frac{x+\overline{k}}{\sqrt{2\overline{\alpha}_a\overline{\alpha}_b}}\right). \tag{87}$$

Here $H_n(x)$ is the n-th Hermite polynomial, and the square root of the complex quantities are taken to be their principal determinations.

As for the functions in (80) we get $\varphi_0(x) = N_\varphi \exp\left\{-\frac{1}{\alpha_a}\int \beta_a(x)\,dx\right\}$, and $\psi_0(x) = N_\psi \exp\left\{-\frac{1}{\overline{\alpha}_b}\int \overline{\beta_b(x)}\,dx\right\}$, where $\beta_a(x)$ and $\beta_b(x)$ are only required to satisfy the condition $\alpha_a\beta_b(x) + \alpha_b\beta_a(x) = x + k$. Now, it is easy to show that $\varphi_n(x)\,\overline{\Psi_m(x)} \in \mathcal{L}^1(\mathbb{R})$, for all $n, m \geq 0$, as in Proposition 1 above, if $\alpha_a\alpha_b > 0$. The proof is based on the fact that $\varphi_n(x)\,\overline{\Psi_m(x)}$ is (a part some normalization constants), the product of a polynomial of degree $n+m$ times the following exponential

$$\exp\left\{-\int\left(\frac{\beta_a(x)}{\alpha_a}+\frac{\beta_b(x)}{\alpha_b}\right)dx\right\}=\exp\left\{-\frac{1}{\alpha_a\alpha_b}\int\theta(x)\,dx\right\}$$
$$=\exp\left\{-\frac{1}{\alpha_a\alpha_b}\int(x+k)\,dx\right\}$$
$$=\exp\left\{-\frac{1}{\alpha_a\alpha_b}\left(\frac{x^2}{2}+kx+\tilde{k}\right)\right\},$$

for some integration constant $\tilde{k}$. Notice that this is a gaussian term under our assumption on $\alpha_a\alpha_b$. We refer to [30] for the analysis of the biorthonormality (with a little abuse of language) of $\mathcal{F}_\varphi=\{\varphi_n(x)\}$ and $\mathcal{F}_\psi=\{\psi_n(x)\}$ in this specific case of constant $\alpha_j(x)$.

A General Example

The situation we will now consider is when $\alpha_a(x)=\alpha_b(x)=\alpha(x)$, where $\alpha(x)\neq 0$ for all $x\in\mathbb{R}$. In this case the first equation in (78) is automatically true, independently of the particular form of $\alpha(x)$. The second equation becomes $(\beta_a(x)+\beta_b(x))'=\frac{1}{\alpha(x)}+\alpha''(x)$, so that

$$\beta_a(x)+\beta_b(x)=\int\frac{dx}{\alpha(x)}+\alpha'(x). \tag{88}$$

From now on we will identify $\beta_a(x)$ and $\beta_b(x)$ as follows:

$$\beta_a(x)=\int\frac{dx}{\alpha(x)},\qquad \beta_b(x)=\alpha'(x). \tag{89}$$

Of course, other possible choices exist, like that in which the role of $\beta_a(x)$ and $\beta_b(x)$ are simply exchanged. But we could also consider $\beta_a(x)=\int\frac{dx}{\alpha(x)}+\Phi(x)$ and $\beta_b(x)=\alpha'(x)-\Phi(x)$, for some fixed, sufficiently regular, $\Phi(x)$. This can produce interesting results, depending on how $\Phi(x)$ is fixed. However, to simplify our analysis here, we will take $\Phi(x)=0$ in what follows. Similarly, we will also fix to zero all the integration constants, except when explicitly stated. The function $\theta(x)$ introduced in Proposition 2 becomes $\theta(x)=\alpha(x)(\beta_a(x)+\beta_b(x))$, so that

$$\theta(x)=\alpha(x)\left(\int\frac{dx}{\alpha(x)}+\alpha'(x)\right), \tag{90}$$

which, when replaced in (84), produces the following sequence of functions: $\pi_0(x)=1$ and

$$\pi_n(x)=\left(\int\frac{dx}{\alpha(x)}\right)\pi_{n-1}(x)-\alpha(x)\pi'_{n-1}(x). \tag{91}$$

Calling $\rho(x) = \int \frac{dx}{\alpha(x)}$ we can rewrite (91) in the following alternative way:

$$\pi_n(x) = \rho(x)\pi_{n-1}(x) - \frac{1}{\rho'(x)}\pi'_{n-1}(x), \tag{92}$$

$n \geq 1$, which can be used to deduce the following expression for $\pi_n(x)$:

$$\pi_n(x) = \frac{1}{\sqrt{2^n}} H_n\left(\frac{\rho(x)}{\sqrt{2}}\right), \tag{93}$$

for all $n \geq 0$, [31].

Remarks

(1) Equation (93) returns the first equation in (87) if $\alpha(x) = \alpha$, constant in x, as it should.
(2) If $\alpha(x)$ is real then, using (89), $\beta_b(x)$ is also real. Also, $\beta_a(x)$ is real if the integration constant is chosen to be real, as we will do always here[9]. In these conditions, $\sigma_n(x) = \pi_n(x)$, $\forall n \geq 0$.
(3) We believe (but we don't have a rigorous result for that) that Hermite polynomials of some "complicated" argument always appear in connection with PBs and WPBs because these are connected to deformed CCR, and CCR gives rise to Hermite polynomials. This is indeed what we have observed along the years, in all the models we have analysed so far.

As for the vacua in (80), using the fact that $\alpha_a(x) = \alpha_b(x) = \alpha(x)$, together with formulas (89), we deduce that

$$\varphi_0(x) = N_\varphi \exp\left\{-\frac{1}{2}(\rho(x))^2\right\}, \quad \psi_0(x) = \frac{N_\psi}{\overline{\alpha}(x)}, \tag{94}$$

or simply $\psi_0(x) = \frac{N_\psi}{\alpha(x)}$ if $\alpha(x)$ is real, as we will assume from now on, to simplify the notation. Putting all together we conclude that

$$\varphi_n(x) = \frac{N_\varphi}{\sqrt{2^n n!}} H_n\left(\frac{\rho(x)}{\sqrt{2}}\right) e^{-\left(\frac{\rho(x)}{\sqrt{2}}\right)^2}, \quad \psi_n(x) = \frac{N_\psi}{\sqrt{2^n n!}} H_n\left(\frac{\rho(x)}{\sqrt{2}}\right) \frac{1}{\alpha(x)}. \tag{95}$$

These formulas suggest that, for many possible choices of $\alpha(x)$, it is quite easy that $\psi_n(x) \notin \mathcal{L}^2(\mathbb{R})$, even if maybe not for all the values of n. On the contrary, we could easily imagine that, for the same choice of $\alpha(x)$, $\varphi_n(x) \in \mathcal{L}^2(\mathbb{R})$.

It is now very easy to prove that, under very mild assumption on $\alpha(x)$, the families $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are compatible and biorthonormal (in our slightly extended meaning), even when the functions $\varphi_n(x)$ or $\psi_n(x)$ do not both belong to $\mathcal{L}^2(\mathbb{R})$. To prove this claim, it is useful to assume that $\rho(x)$ is increasing in x and that, calling

[9] Actually, as already stated, we will often fix to zero this integration constant.

$s=\frac{\rho(x)}{\sqrt{2}}$, $s\to\pm\infty$ when $x\to\pm\infty$. It is clear then that ρ can be inverted, and that $x=\rho^{-1}(\sqrt{2}s)$. Since $\rho'(x)=\frac{1}{\alpha(x)}$, it follows that $\rho(x)$ is always increasing if $\alpha(x)>0$. However, this is not enough to ensure that s diverges with x, and therefore this must also be required.

Now, to prove that $\varphi_n(x)$ and $\psi_m(x)$ are compatible (and biorthonormal), we compute the compatibility form:

$$\langle\psi_m,\varphi_n\rangle=\frac{\overline{N_\psi}N_\varphi}{\sqrt{2^{n+m}\,n!\,m!}}\int_{-\infty}^{\infty}H_m\left(\frac{\rho(x)}{\sqrt{2}}\right)H_n\left(\frac{\rho(x)}{\sqrt{2}}\right)e^{-\left(\frac{\rho(x)}{\sqrt{2}}\right)^2}\frac{dx}{\alpha(x)}.$$

This integral can be easily rewritten in terms of s. In fact, recalling the definition of $\rho(x)$, we first observe that $\frac{ds}{dx}=\frac{1}{\sqrt{2}\alpha(x)}$, so that $\frac{dx}{\alpha(x)}=\sqrt{2}\,ds$. Hence we have

$$\langle\psi_m,\varphi_n\rangle=\frac{\overline{N_\psi}N_\varphi}{\sqrt{2^{n+m-1}\,n!\,m!}}\int_{-\infty}^{\infty}H_m(s)H_n(s)e^{-s^2}ds=\sqrt{2\pi}\,\overline{N_\psi}N_\varphi\,\delta_{n,m},$$

which returns

$$\langle\psi_m,\varphi_n\rangle=\delta_{n,m},\quad\text{if}\quad\overline{N_\psi}N_\varphi=\frac{1}{\sqrt{2\pi}},\tag{96}$$

as will be assumed in the rest of this section. This is what we had to prove.

An Example Let us fix $\alpha(x)=\frac{1}{1+x^2}$. This function is always strictly positive, and produces, using (89) and the definition of $\rho(x)$, the functions $\beta_a(x)=\rho(x)=x+\frac{x^3}{3}$ and $\beta_b(x)=\frac{-2x}{(1+x^2)^2}$. We see that $\rho(x)\to\pm\infty$ when $x\to\pm\infty$. Also, the inverse of ρ exists and can be computed explicitly looking for the only real solution of the equation $\sqrt{2}s=x+\frac{x^3}{3}$. We get

$$x=\rho^{-1}(\sqrt{2}\,s)=\left(\frac{2}{-3\sqrt{2}s+\sqrt{2}\sqrt{2+9s^2}}\right)^{1/3}-\left(\frac{-3\sqrt{2}s+\sqrt{2}\sqrt{2+9s^2}}{2}\right)^{1/3}.$$

The functions in (94) turn out to be

$$\varphi_0(x)=N_\varphi\exp\left\{-\frac{1}{2}(x+x^3/3)^2\right\},\quad\psi_0(x)=N_\psi\,(1+x^2).\tag{97}$$

It is clear that $\varphi_0(x)\in\mathcal{L}^2(\mathbb{R})$, while $\psi_0(x)$ is not square-integrable. Furthermore, see (92), we have

$$\pi_n(x)=\left(x+\frac{x^3}{3}\right)\pi_{n-1}(x)-\frac{1}{(1+x^2)}\,\pi_{n-1}'(x),$$

with $\pi_0(x) = 1$, and a similar expression for $\sigma_n(x)$. More explicitly we get

$$\pi_n(x) = \sigma_n(x) = \frac{1}{\sqrt{2^n}} H_n\left(\frac{x + x^3/3}{\sqrt{2}}\right),$$

and

$$\varphi_n(x) = \frac{N_\varphi}{\sqrt{2^n n!}} H_n\left(\frac{x + x^3/3}{\sqrt{2}}\right) e^{-\frac{1}{2}(x+x^3/3)^2},$$
$$\psi_n(x) = \frac{N_\psi}{\sqrt{2^n n!}} H_n\left(\frac{x + x^3/3}{\sqrt{2}}\right)(1 + x^2), \tag{98}$$

$n \geq 0$. Hence $\varphi_n(x) \in \mathcal{L}^2(\mathbb{R})$, while $\psi_n(x) \notin \mathcal{L}^2(\mathbb{R})$, for all $n = 0, 1, 2, \dots$

The fact that these functions are compatible follows from the speed of decay of $\varphi_n(x)$, when compared with the speed of divergence of $\psi_m(x)$. In particular, formula (96) shows that these functions are biorthonormal if $N_\psi N_\varphi = \frac{1}{\sqrt{2\pi}}$: $\langle \psi_m, \varphi_n \rangle = \delta_{n,m}$, $\forall n, m \geq 0$.

Notice that, for our particular operators in (77), there is no need to move to $S'(\mathbb{R})$. However, we see that $\mathcal{L}^2(\mathbb{R})$ is not enough, in general, and we have to use *compatible spaces*, with a compatibility form which extends the ordinary scalar product in $\mathcal{L}^2(\mathbb{R})$. This is different from what we have seen in Sects. 3.1 and 3.2, where the role of $S'(\mathbb{R})$ was more relevant, if not essential. In other words, WPBs are not intrinsically connected with distributions; they can appear when $\mathcal{L}^2(\mathbb{R})$ is not sufficient in the analysis of our pseudo-bosonic operators.

Since, as the example above shows, the functions $\varphi_n(x)$ and $\psi_n(x)$ are not necessarily square-integrable, it is clear that there is no reason for $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ to be bases for $\mathcal{L}^2(\mathbb{R})$. However, despite of the fact that $\varphi_n(x)$ and $\psi_n(x)$ are not necessarily square-integrable, we will show that a set $\mathcal{W}$, dense in $\mathcal{L}^2(\mathbb{R})$, does indeed exist such that $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are $\mathcal{W}$-quasi bases.

Let us introduce the set

$$\mathcal{W} = \left\{h(s) \in \mathcal{L}^2(\mathbb{R}) \colon h_-(s) := h(\rho^{-1}(\sqrt{2}s))\, e^{s^2/2} \in \mathcal{L}^2(\mathbb{R})\right\} \tag{99}$$

This set is dense in $\mathcal{L}^2(\mathbb{R})$, since it contains the set $\mathcal{D}(\mathbb{R})$ of all the compactly supported C^∞ functions, [31]. It is useful to observe that, if $h(x) \in \mathcal{W}$, then the function $h_+(s) := h(\rho^{-1}(\sqrt{2}s))\, \alpha(\rho^{-1}(\sqrt{2}s))\, e^{-s^2/2} \in \mathcal{L}^2(\mathbb{R})$ as well, at least under very general conditions on $\alpha(x)$. This is because $|h_+(s)|^2 = |h_-(s)|^2 |g(s)|^2$, where $g(s) = \alpha(\rho^{-1}(\sqrt{2}s))\, e^{-s^2}$. Now, it is sufficient that $g(s) \in \mathcal{L}^\infty(\mathbb{R})$ to conclude that $h_+(s) \in \mathcal{L}^2(\mathbb{R})$. But, because of the presence of e^{-s^2} in $g(s)$, this is true for many choices of $\alpha(x)$, like for instance the one proposed in the previous example, $\alpha(x) = \frac{1}{1+x^2}$. However, even if $\alpha(x)$ diverges very fast, if $h(x) \in \mathcal{D}(\mathbb{R})$ then $h_+(s) \in \mathcal{L}^2(\mathbb{R})$ anyhow, which is what we will use in the following.

Theorem 2 $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ are $\mathcal{W}$-quasi bases.

Proof Let us take $f(x), g(x) \in \mathcal{W}$. It is possible to check that the following equalities hold:

$$\langle f, \varphi_n\rangle = N_\varphi\, \pi^{1/4}\sqrt{2}\langle f_+, e_n\rangle, \quad \langle \psi_n, g\rangle = \overline{N}_\psi\, \pi^{1/4}\sqrt{2}\langle e_n, g_-\rangle. \tag{100}$$

Here $e_n(s) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} H_n(s)e^{-s^2/2}$ is the n-th eigenstate of the quantum harmonic oscillator already considered several times in this chapter, while $f_+(s)$ and $g_-(s)$ should be constructed from $f(s)$ and $g(s)$ as shown before. The equalities in (100) show, in particular, that the pairs $(f(x), \varphi_n(x))$ and $(g(x), \psi_n(x))$ are compatible, $\forall n \geq 0$, since all the functions involved in the right-hand sides of the equalities in (100), $e_n(s)$, $f_+(s)$ and $g_-(s)$, are square integrable. The proof of these identities is based on the change of variable $s = \frac{\rho(x)}{\sqrt{2}}$, which has already been used before, to prove (96). Now, since $\mathcal{F}_e = \{e_n(s),\ n \geq 0\}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$, we have

$$\sum_{n=0}^{\infty}\langle f, \varphi_n\rangle\langle \psi_n, g\rangle = \overline{N}_\psi\, N_\varphi\, 2\sqrt{\pi}\sum_{n=0}^{\infty}\langle f_+, e_n\rangle\langle e_n, g_-\rangle = \sqrt{2}\langle f_+, g_-\rangle,$$

using (96) and the Parseval identity for $\mathcal{F}_e$. Next we have

$$\begin{aligned}\langle f_+, g_-\rangle &= \int_{-\infty}^{\infty}\overline{f_+(s)}\, g_-(s)\, ds \\ &= \int_{-\infty}^{\infty}\overline{f(\rho^{-1}(\sqrt{2}s))}\,\alpha(\rho^{-1}(\sqrt{2}s))e^{-s^2/2}g(\rho^{-1}(\sqrt{2}s))e^{s^2/2}\, ds \\ &= \int_{-\infty}^{\infty}\overline{f(\rho^{-1}(\sqrt{2}s))}\,\alpha(\rho^{-1}(\sqrt{2}s))g(\rho^{-1}(\sqrt{2}s))\, ds = \frac{1}{\sqrt{2}}\langle f, g\rangle,\end{aligned}$$

introducing the new variable $x = \rho^{-1}(\sqrt{2}s)$ in the integral. Summarizing,

$$\sum_{n=0}^{\infty}\langle f, \varphi_n\rangle\langle \psi_n, g\rangle = \langle f, g\rangle,$$

and, with similar computations, $\sum_{n=0}^{\infty}\langle f, \psi_n\rangle\langle \varphi_n, g\rangle = \langle f, g\rangle$. □

The conclusion is therefore that, even if $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ are not necessarily made of functions in $\mathcal{L}^2(\mathbb{R})$, they can be used, together, to deduce a resolution (better, two resolutions) of the identity on $\mathcal{W}$.

More results and explicit examples of these WPBs, together with some application to bi-coherent states, can be found in [31].

4 Conclusions

In this chapter we have reviewed some general aspects and applications of WPBs, and we have discussed how distribution theory and compatible spaces are relevant in this context, and how ladder operators can be extended outside a purely Hilbertian settings. We have not discussed here several aspects of this general framework. In particular, we have not considered the role of coherent states in connection with lowering operators of pseudo-bosonic type. We refer to [6, 7] for many results on this, but more recent results can also be found, for instance in [22].

As we have noticed, biorthonormality of the eigenstates of our number-like operators refers to some bilinear form which cannot be the ordinary scalar product in $\mathcal{L}^2(\mathbb{R})$. In particular, the one proposed in Sect. 3.1 is only one possibility, among many. In [32] we have proposed a new extension of the scalar product not related to convolutions, and we proved that this class of multiplications can be flexible enough to succeed where the convolution cannot really be useful. More on this new definition of multiplication, and its role in connection with the properties of the adjoint of an operator and with the consequences of its definition, is work in progress.

Our approach, thought being mathematically already interesting by itself (in our opinion, at least!), needs some extra effort in the attempt of connecting it with physics, and in particular with the probabilistic interpretation of the wave function. This is another open aspect of our approach, and surely deserve further investigation. This is also an active line of research.

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References

1. Mostafazadeh, A.: Pseudo-hermitian quantum mechanics. Int. J. Geom. Methods Mod. Phys. **7**, 1191–1306 (2010)
2. Bender, C., Fring, A., Günther, U., Jones, H. (eds.): Special issue on quantum physics with non-Hermitian operators. J. Phys. A: Math. and Ther., **45** (2012)
3. Bagarello, F., Gazeau, J.P., Szafraniec, F.H., Znojil, M. (eds.): Non-selfadjoint operators in quantum physics: Mathematical aspects. John Wiley & Sons (2015)
4. Bagarello, F., Passante, R., Trapani, C.: Non-Hermitian Hamiltonians in Quantum Physics. Selected Contributions from the 15th International Conference on Non-Hermitian Hamiltonians in Quantum Physics, Palermo, Italy, 18–23 May 2015. Springer (2016)
5. Bender, C.M.: PT Symmetry in quantum and classical physics. World Scientific (2019)
6. Bagarello, F.: Deformed canonical (anti-)commutation relations and non hermitian Hamiltonians. In: Bagarello, F., Gazeau, J.P., Szafraniec, F.H., Znojil, M. (eds.) Non-selfadjoint operators in quantum physics: Mathematical aspects. John Wiley & Sons (2015)
7. Bagarello, F.: Pseudo-bosons and their coherent states. Springer (2022)
8. Heil, C.: A basis theory primer: expanded edition. Springer, New York (2010)
9. da Providência, J., Bebiano, N., da Providência, J.P.: Non hermitian operators with real spectrum in quantum mechanics. ELA, vol. 21., pp. 98–109 (2010)

10. Bagarello, F.: Examples of Pseudo-bosons in quantum mechanics. Phys. Lett. A. **374**, 3823–3827 (2010)
11. Swanson, M.S.: Transition elements for a non-Hermitian quadratic hamiltonian. J. Math. Phys. **45**, 585–601 (2004)
12. Davies, E.B., Kuijlaars, B.J.: Spectral asymptotics of the non-self-adjoint harmonic oscillator. J. London Math. Soc. **70**, 420–426 (2004)
13. Bender, C.M., Jones, H.F.: Interactions of Hermitian and non-Hermitian Hamiltonians. J. Phys. A **41**, 244006 (2008)
14. Li, J.-Q., Li, Q.Y.-G.M.: Investigation of PT-symmetric Hamiltonian Systems from an Alternative Point of View. Commun. Theor. Phys. **58**, 497 (2012)
15. Bagarello, F., Lattuca, M.: $\mathcal{D}$ pseudo-bosons in quantum models. Phys. Lett. A. **377**, 3199–3204 (2013)
16. Li, J.-Q., Miao, Y.-G.Z.X.: A Possible Method for Non-Hermitian and Non-PT-Symmetric Hamiltonian Systems. PLoS ONE **9**(6), e97107 (2014)
17. Gelf́and, I.M., Shilov, G.E.: Generalized Functions, vol. I. Academic Press, New York and London (1964)
18. Bagarello, F.: Multiplication of distributions in a linear gain and loss system. ZAMP **74**, 136 (2023)
19. Vladimirov, V.S.: Le distribuzioni nella fisica matematica. MIR, Moscow (1981)
20. Kanwal, R.P.: Delta series solutions of differential and integral equations. Int. Transf. Spec. Funct. **6**(1–4), 49–62 (1998)
21. Estrada, R., Kanwal, R.P.: A Distributional Approach to Asymptotics Theory and Applications. Birkhäuser, Boston (2002)
22. Bagarello, F., Gargano, F.: Bi-coherent states as generalized eigenstates of the position and the momentum operators. ZAMP **73**, 119 (2022)
23. Barton, G.: Quantum mechanics of the inverted oscillator potential. Ann. Phys. **166**, 322–363 (1986)
24. Krason, P., Milewski, J.: On eigenproblem for inverted harmonic oscillators. Banach Cent. Publ. **124**, 61–73 (2021)
25. Subramanyan, V., Hegde, S.S., Vishveshwara, S., Bradlyn, B.: Physics of the inverted harmonic oscillator: from the lowest Landau level to event horizons. Ann. Phys. **435**, 168470 (2021)
26. Bagarello, F.: A Swanson-like Hamiltonian and the inverted harmonic oscillator. J. Phys. A **55**, 225204 (2022)
27. Kolmogorov, A., Fomine, S.: Eléments de la théorie des fonctions et de l'analyse fonctionelle. Mir (1973)
28. Antoine, J.-P., Trapani, C.: Reproducing pairs of measurable functions and partial inner product spaces. Adv. Op. Th. **2**, 126–146 (2017)
29. Reed, S., Simon, B.: Methods of modern mathematical physics, Vol I: Functional analysis. Academic Press, New York (1972)
30. Bagarello, F.: Pseudo-bosons and bi-coherent states out of $\mathcal{L}^2(\mathbf{R})$. J. Phys.: Conf. Ser. **2038**, 12001 (2021)
31. Bagarello, F.: A class of weak pseudo-bosons and their bi-coherent states. JMAA **516**(2), 126531 (2022)
32. Bagarello, F.: Multiplication of distributions in a linear gain and loss system. ZAMP **74**, 136 (2023)