



Efficient Parameterized Pattern Matching in Sublinear Space

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Abstract. The parameterized matching problem is a variant of string matching, which is to search for all *parameterized* occurrences of a pattern P in a text T . In considering matching algorithms, the combinatorial natures of strings, especially *periodicity*, play an important role. In this paper, we analyze the properties of periods of parameterized strings and propose a generalization of Galil and Seiferas's exact matching algorithm (1980) into parameterized matching, which runs in $O(\pi|T| + |P|)$ time and $O(\log|P| + |\Pi|)$ space in addition to the input space, where Π is the parameter alphabet and π is the number of parameter characters appearing in P plus one.

Keywords: Parameterized matching · String matching · Sublinear space · Combinatorics on words

1 Introduction

String matching is a problem to search for all occurrences of a pattern P in a text T . Since it is one of the most important computer applications, many efficient algorithms for the problem have been proposed. Let us denote the length of T and P by n and m , respectively. While a naive algorithm takes $O(nm)$ time to solve the problem, Knuth, Morris, and Pratt [13] gave an algorithm which runs in only $O(n+m)$ time by constructing auxiliary arrays called *border arrays*. After that, various algorithms to solve the problem in linear time have been proposed, which use auxiliary data structures, such as suffix trees [19], suffix arrays [15], LCP arrays [15]. All of those algorithms outperform the naive algorithm in terms of time complexity. They require additional space to store their auxiliary data, whose sizes are typically $\Theta(n)$ or $\Theta(m)$. On the other hand, studies for reducing such extra space were conducted. Firstly, Galil and Seiferas reduced extra space usage to $O(\log m)$ [11], and later several time-space-optimal, $O(n+m)$ time and $O(1)$ extra-space algorithms were devised [5, 6, 12].

In this paper, we consider a variant of string matching: *parameterized matching*. It is a pattern matching paradigm in which two strings are considered a *match* if we can map some characters (*parameter characters*) in one string to

characters in another string. This paradigm was first introduced by Baker [4] for use in software maintenance by the ability to detect ‘identical’ computer programs renaming their variables. For solving the parameterized matching problem, a number of linear-time algorithms have been proposed that extend algorithms for exact matching [2, 7, 8, 10, 14, 17, 18]. See also [16] for a survey. However, we know of no previous attempt to reduce extra space usage to sublinear for time-efficient parameterized matching algorithms, although one can solve the problem in constant extra space if the time efficiency does not matter.

The main contribution of this paper is to give a sublinear-extra-space algorithm for the parameterized matching problem by extending Galil and Seiferas’s exact matching algorithm [11]. It runs in $O(|\Pi_P|n + m)$ time and $O(\log m + |\Pi|)$ space in addition to the input space, where Π is the set of parameter characters and Π_P is the non-empty¹ set of parameter characters appearing in P .

In order to provide the basis for our algorithm, we also investigate the properties of periodicity of parameterized strings in this paper. It is widely known that periods of strings are useful for exact matching algorithms [5, 6, 11–13], which is also the case for parameterized matching [2]. We extend previous work on parameterized periods by Apostolico and Giancarlo [3] and derive several properties for our algorithm. In particular, we focus on ‘sufficiently short’ periods of parameterized strings having properties useful for matching algorithms. Those results contain a parameterized version of Fine and Wilf’s periodicity lemma [9].

Remark 1. The time and space complexities of our algorithm stated above are based on a computing model in which functions $\Pi \rightarrow \mathbb{N}$ can be stored as arrays. If not, one can use AVL trees [1] instead of arrays to store such functions. Then, our algorithm runs in $O((|\Pi_P|n + m) \log |\Pi_P|)$ time and $O(\log m + |\Pi_P|)$ extra space.

2 Preliminaries

Let \mathbb{N} and \mathbb{N}^+ be the set of natural numbers including and excluding 0, respectively. For $x, y \in \mathbb{N}$, we denote by $x \mid y$ that y is a multiple of x .

For $n \in \mathbb{N}$ and a function f whose domain and codomain are the same, we denote by f^n the composite of the function n times.

2.1 Parameterized Matching Problem

In parameterized matching, we consider two disjoint alphabets: the *constant alphabet* Σ and the *parameter alphabet* Π . A string over $\Sigma \cup \Pi$ is called a *parameterized string* or a *p-string*. Consider a p-string $w \in (\Sigma \cup \Pi)^*$. We denote the length of w by $|w|$. For $0 \leq i < |w|$, let us denote i -th letter of w by $w[i]$, where the index i is 0-based. For $0 \leq i \leq j \leq |w|$, we denote the substring $w[i]w[i+1] \cdots w[j-1]$ by $w[i : j]$. (Note that $w[i : j]$ does not contain $w[j]$.)

¹ We can assume $\Pi_P \neq \emptyset$ without loss of generality. See Remark 2.

We denote the set of permutations of Π by S_Π . Throughout this paper, for a permutation $f \in S_\Pi$ and a constant character $c \in \Sigma$, let $f(c) = c$. Then, the map f is naturally expanded as a bijection over p-strings: $(\Sigma \cup \Pi)^* \rightarrow (\Sigma \cup \Pi)^*$.

Definition 1 (Baker [4]). *Two p-strings x and y are called a parameterized-match or a p-match if and only if there exists a permutation $f \in S_\Pi$ such that $f(x) = y$. Denote this relation by $x \equiv y$.*

Example 1. Let $\Sigma = \{a, b, c\}$ and $\Pi = \{A, B, C\}$. We have $ABaCBCa \equiv BCaACAa$ with a permutation f such that $f(A) = B$, $f(B) = C$, and $f(C) = A$.

Clearly, the relation \equiv is an equivalence relation over $(\Sigma \cup \Pi)^*$. Note that if $x \equiv y$, we have $|x| = |y|$ and $x[i : j] \equiv y[i : j]$ for any $0 \leq i \leq j \leq |x|$. By this relation, the problem we consider in this paper, the *parameterized matching problem*, is defined as follows.

Problem 1 ([4]). *Given two p-strings T (text) and P (pattern), find all $0 \leq i \leq |T| - |P|$ such that $T[i : i + |P|] \equiv P$.*

Remark 2. For Problem 1, we can assume that P contains at least one parameter character without loss of generality. If $P \in \Sigma^*$, choose any $c \in \Sigma$ appearing in P and let constant and parameter alphabets be $\Sigma \cup \Pi \setminus \{c\}$ and $\{c\}$, respectively. Our algorithm presented in Sect. 4 is based on this assumption.

2.2 Periodicity of Parameterized Strings

Periodicity is one of the most fundamental concepts in combinatorics of strings and a wealth of applications. In exact matching, the Knuth-Morris-Pratt algorithm and various algorithms based on it rely on the properties of periods [5, 6, 11–13]. It is also the case for parameterized matching [2], where periods of parameterized strings are defined as follows:

Definition 2 (Apostolico and Giancarlo [3]). *Consider $w \in (\Sigma \cup \Pi)^*$ and $p \in \mathbb{N}^+$ with $p \leq |w|$. Then, p is called a period of w if and only if $w[0 : |w| - p] \equiv w[p : |w|]$.*

If p is a period of w , there exists $f \in S_\Pi$ satisfying $f(w[0 : |w| - p]) = w[p : |w|]$ by definition. We denote this relation by $p \parallel_f w$ or simply by $p \parallel w$ when f is not specified.

In general, a p-string w can have multiple periods. We denote the shortest period of w as *period*(w). It is clear that a period p of a p-string w is also a period of any substring w' of w such that $|w'| \geq p$.

Example 2. Let $\Sigma = \{a, b, c\}$ and $\Pi = \{A, B, C\}$. For $w := ABaCBCaACAa$, we have $4 \parallel_f w$ as $ABaCBCa \equiv BCaACAa$ with $f(A) = B$, $f(B) = C$, and $f(C) = A$.

Instead of Definition 2, one can use the following equivalent definition for periods, which is a more intuitive representation of the repetitive structure of strings:

Lemma 1 ([3]). Consider $w \in (\Sigma \cup \Pi)^*$, $p \in \mathbb{N}^+$, and $f \in S_\Pi$. Then, $p \parallel_f w$ holds if and only if w can be written as

$$w = f^0(v) \cdot f^1(v) \cdot f^2(v) \cdots f^{\lfloor \rho \rfloor - 1}(v) \cdot f^{\lfloor \rho \rfloor}(v'),$$

where $\rho = \frac{|w|}{p}$, $v = w[0 : p]$ and v' is a prefix of v (allowing the case v' is empty).

The following lemma has important applications for various matching algorithms. Particularly, it is used to shift the pattern string safely in the Knuth-Morris-Pratt algorithm and variants [2, 13].

Lemma 2. Consider $x, y \in (\Sigma \cup \Pi)^*$ with $x \equiv y$. For any $0 < \delta < \text{period}(y)$, we have $x[\delta : |x|] \not\equiv y[0 : |y| - \delta]$.

Proof. We give a proof by contraposition. Suppose $x[\delta : |x|] \equiv y[0 : |y| - \delta]$. Then we have $y[0 : |y| - \delta] \equiv x[\delta : |x|] \equiv y[\delta : |y|]$, which means $\delta \parallel y$. Hence, $\delta \geq \text{period}(y)$ holds. \square

One of the main interest regarding string periodicity is what holds when a string w has two different periods p and q . For ordinary strings, Fine and Wilf’s periodicity lemma [9] gives an answer: $\text{gcd}(p, q)$ is also a period when $|w| \geq p + q - \text{gcd}(p, q)$, where $\text{gcd}(p, q)$ is the greatest common divisor of p and q . Apostolico and Giancarlo showed a similar property for parameterized strings.

Lemma 3 ([3]). For $w \in (\Sigma \cup \Pi)^*$, $p, q \in \mathbb{N}^+$, and $f, g \in S_\Pi$, assume that $p \parallel_f w$ and $q \parallel_g w$. If $|w| \geq p + q$ and $fg = gf$, we have $\text{gcd}(p, q) \parallel w$.

It is known that the length $|w| = p + q - \text{gcd}(p, q)$ is not sufficient for this lemma unlike in the case of ordinary strings [3].

3 Properties of Parameterized Periods

In this section, we show some properties of periods of parameterized strings. They play an important role in our algorithm presented in Sect. 4.

3.1 Alternative Periodicity Lemma

The requirements of Lemma 3 are slightly different from Fine and Wilf’s lemma for ordinary strings. Particularly, the commutativity of f and g is essential (Lemma 5 in [3]). In this section, we show a new periodicity lemma for parameterized strings which does not assume the commutativity.

Firstly, we focus on parameter characters contained in a given p-string and its substrings. For $w \in (\Sigma \cup \Pi)^*$, we denote by Π_w the set of parameter characters appearing in w .

Example 3. Let $\Sigma = \{a, b, c\}$ and $\Pi = \{A, B, C\}$. For $w := \text{ABabAca}$, we have $\Pi_w = \{A, B\}$.

Lemma 4. Consider $w \in (\Sigma \cup \Pi)^*$ and any of its substrings w' and w'' . Then, the following hold:

- If $|w'| \geq \text{period}(w) \cdot (|\Pi_w| - 1)$, we have $|\Pi_{w'}| \geq |\Pi_w| - 1$.
- If $|w''| \geq \text{period}(w) \cdot |\Pi_w|$, we have $\Pi_{w''} = \Pi_w$.

Proof. The case $\Pi_w = \emptyset$ is trivial. Suppose $\Pi_w \neq \emptyset$. Let $p := \text{period}(w)$ and f be a permutation of Π such that $p \parallel_f w$. It suffices to show the lemma for the cases $|w'| = p \cdot (|\Pi_w| - 1)$ and $|w''| = p \cdot |\Pi_w|$. By Lemma 1, w' and w'' can be written as $w' = v' \cdot f(v') \cdots f^{|\Pi_w|-2}(v')$ and $w'' = v'' \cdot f(v'') \cdots f^{|\Pi_w|-1}(v'')$, where v' and v'' are the prefixes of w' and w'' of length p , respectively. Now, we consider the cyclic decomposition of f .

Suppose the characters in Π_w make one cyclic permutation in f . Let a be any parameter character contained in v' . Note that $a, f(a), \dots, f^{|\Pi_w|-2}(a)$ are all different characters and all appear in w' . Therefore, we have $|\Pi_{w'}| \geq |\Pi_w| - 1$. The analogous argument shows $|\Pi_{w''}| = |\Pi_w|$.

Suppose the characters in Π_w make two or more cyclic permutations in f . Then, those cyclic permutations are all of length $|\Pi_w| - 1$ or less. For $0 \leq i < |w|$, there exists an integer k such that $w[i + kp], w[i + (k + 1)p], \dots, w[i + (k + |\Pi_w| - 2)p]$ are all contained in w' . Then, those characters can be represented as $f^k(w[i]), f^{k+1}(w[i]), \dots, f^{k+|\Pi_w|-2}(w[i])$, and by the assumption about f , at least one of them is equal to $w[i]$. Therefore, we have $w[i] \in \Pi_{w'}$. Since i is arbitrary, we end up with $\Pi_w \subseteq \Pi_{w'}$, as required. \square

Now, we show a variant of Lemma 3. It does not require any assumption on the permutations, in exchange of a stricter requirement for the length of strings.

Lemma 5. Suppose $w \in (\Sigma \cup \Pi)^*$ with $\Pi_w \neq \emptyset$ has periods p and q . If $|w| \geq p + q + \min(p, q) \cdot (|\Pi_w| - 1)$, we have $\text{gcd}(p, q) \parallel w$.

Proof. Let f and g be permutations of Π such that $p \parallel_f w$ and $q \parallel_g w$. Without loss of generality, we suppose $f(a) = a$ and $g(a) = a$ for any $a \in \Pi \setminus \Pi_w$. By Lemma 3, it suffices to show that $fg = gf$. Let $w' := w[0 : |w| - p - q]$. Then, notice that $fg(w') = f(w[q : |w| - p]) = w[p + q : |w|] = g(w[p : |w| - q]) = gf(w')$, which claims $fg(a) = gf(a)$ for any $a \in \Pi_{w'}$. Moreover, given $|w'| = |w| - p - q \geq \min(p, q) \cdot (|\Pi_w| - 1) \geq \text{period}(w) \cdot (|\Pi_w| - 1)$, we have $|\Pi_{w'}| \geq |\Pi_w| - 1$ by Lemma 4. Hence, the permutations fg and gf behave the same for at least $|\Pi| - 1$ parameter characters. This implies $fg = gf$. \square

Corollary 1. Suppose $w \in (\Sigma \cup \Pi)^*$ with $\Pi_w \neq \emptyset$ has a period q . If $q \leq \frac{|w|}{|\Pi_w|+1}$, then $\text{period}(w) \mid q$.

Proof. Let $p := \text{period}(w)$. By $p \leq q \leq \frac{|w|}{|\Pi_w|+1}$, we have $p \cdot |\Pi_w| + q \leq q \cdot (|\Pi_w| + 1) \leq \frac{|w|}{|\Pi_w|+1} \cdot (|\Pi_w| + 1) = |w|$. Hence, we can use Lemma 5 to obtain $\text{gcd}(p, q) \parallel w$. Then, since p is the smallest period of w , we have $\text{gcd}(p, q) \geq p$, which means $\text{gcd}(p, q) = p$ i.e. $p \mid q$, as required. \square

Table 1. Let $\Pi = \{A, B\}$. A p-string $w := ABABBABAABABBABAABBA$ has prefix periods 1 and 4. Circled numbers in the table below are prefix periods of w with $w[0 : i + 1]$ as witnesses. For instance, 4 is a prefix period of w with $w[0 : 18]$ as a witness because $period(w[0 : 18]) = 4$ and $4 \leq \frac{|w[0:18]|}{k}$. (Note that $k = |\Pi_w| + 2 = 4$.)

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$w[i]$	A	B	A	B	B	A	B	A	A	B	A	B	B	A	B	A	A	B	B	A
$period(w[0 : i + 1])$	1	1	1	①	4	4	4	4	4	4	4	4	4	4	4	④	④	④	18	18
	$reach_w(1) = 4$								$reach_w(4) = 18$											

3.2 Prefix Periods

Galil and Seiferas’s exact matching algorithm [11] can be regarded as an extension of the Knuth-Morris-Pratt algorithm [13]. The main idea of their algorithm is to deal with only periods of pattern prefixes which are ‘short enough.’ They pointed out that periods shorter than $\frac{1}{k}$ times the length of the string have useful properties for saving space usage in exact string matching for an arbitrarily fixed $k \geq 3$. We show, in this section, that similar properties hold for parameterized strings as well when k is set to be $|\Pi_w| + 2$. Most of those properties come from Lemma 5 we proved in the previous section.

Lemma 6. *Suppose $w \in (\Sigma \cup \Pi)^*$ has a period p . If $p \leq \frac{|w|}{|\Pi_w|+1}$, there exists only one character $a \in \Sigma \cup \Pi$ such that $p \parallel wa$.*

Proof. Consider the prefix $w' := w[0 : |w| - p]$. By $p \leq \frac{|w|}{|\Pi_w|+1}$, we have $|w| - p \geq p|\Pi_w| \geq period(w)|\Pi_w|$. By Lemma 4, $\Pi_{w'} = \Pi_w$. Therefore, $w[|w| - p]$ already appears in w' as $w[i] = w[|w| - p]$ for some $i < |w| - p$. Hence, for any f such that $p \parallel_f w$, it holds that $p \parallel_f wa$ if and only if $a = w[i + p]$. \square

Corollary 2. *Suppose $w \in (\Sigma \cup \Pi)^*$ has a period p . For any $\ell \in \mathbb{N}^+$ such that $\ell p \leq \frac{|w|}{|\Pi_w|+1}$, we have $p \parallel wa \iff \ell p \parallel wa$ for any $a \in \Sigma \cup \Pi$.*

Proof. By Lemma 6, the characters a_1 and a_2 such that $p \parallel wa_1$ and $\ell p \parallel wa_2$ are unique respectively. Then, since $p \parallel wa_1 \implies \ell p \parallel wa_1$ (shown immediately by Lemma 1), we get $a_1 = a_2$, as required. \square

Now, we introduce the key concept for our algorithm: *prefix periods*. This is a natural extension of the one introduced in [12] for parameterized strings. Hereafter in this section, we consider a fixed p-string $w \in (\Sigma \cup \Pi)^*$ with $\Pi_w \neq \emptyset$ and let $k := |\Pi_w| + 2$.

Definition 3. *A positive integer $p \in \mathbb{N}^+$ is called a prefix period of w if and only if there exists a prefix w' of w such that $period(w') = p$ and $p \leq \frac{|w'|}{k}$.*

We give an example of prefix periods in Table 1. For a fixed p , only prefixes w' of w satisfying $|w'| \geq kp$ can be a witness for p being a prefix period. We show in the following lemmas that it suffices to consider only one prefix $w' = w[0 : kp]$ for checking whether p is a prefix period.

Lemma 7. *For any $a \in \Sigma \cup \Pi$, if $\overline{\text{period}(wa)} \neq \text{period}(w)$, we have $\text{period}(wa) > \frac{|w|}{|\Pi_w|+1}$.*

Proof. We show the lemma by contraposition. Suppose $\text{period}(wa) \leq \frac{|w|}{|\Pi_w|+1}$. Since $\text{period}(wa)$ is also a period of w , we can use Corollary 1 to obtain $\text{period}(w) \mid \text{period}(wa)$. Therefore, we get $\text{period}(w) \parallel wa$ by Corollary 2, which implies $\text{period}(w) \geq \text{period}(wa)$. On the other hand, we have $\text{period}(w) \leq \text{period}(wa)$ by definition. Thus $\text{period}(w) = \text{period}(wa)$ holds. \square

Lemma 8. *Consider any $0 < p \leq \frac{|w|}{k}$. Then, p is a prefix period of w if and only if $\text{period}(w') = p$ where $w' := w[0 : kp]$.*

Proof. (\Leftarrow) Immediate by the definition of prefix periods.
 (\Rightarrow) Let v be a prefix of w that witnesses p being a prefix period, i.e., $|v| \geq kp$ and $\text{period}(v) = p$. If $|v| = kp$, we are done. Suppose $|v| > kp$ and let $u := v[0 : |v| - 1]$. Then, $\text{period}(v) = p < \frac{|v|}{k} \leq \frac{|v|}{|\Pi_u|+2} < \frac{|u|}{|\Pi_u|+1}$. By Lemma 7, we have $\text{period}(u) = \text{period}(v) = p$. By repeatedly applying this discussion, we can shorten the witness up to length kp . \square

Next, we introduce an auxiliary function reach_w .

Definition 4. *For any $0 < p \leq |w|$, let*

$$\text{reach}_w(p) := \max\{r \in \mathbb{N} : r \leq |w| \text{ and } p \parallel w[0 : r]\}.$$

Note that $p \parallel w[0 : r] \iff \text{reach}_w(p) \geq r$ holds by definition. Using reach_w , we get an equivalent definition of prefix periods as follows, which is directly used in our searching algorithm.

Lemma 9. *Consider any $0 < p \leq \frac{|w|}{k}$. Then, p is a prefix period of w if and only if all the following hold:*

- (1) $\text{reach}_w(p) \geq kp$,
- (2) $\text{reach}_w(q) < \text{reach}_w(p)$ for any $0 < q < p$.

Proof. (\Rightarrow) (1) is by definition. We show (2). By Lemma 8, $\text{period}(w[0 : kp]) = p$. Thus, $q < p$ is not a period of $w[0 : kp]$, i.e., $\text{reach}_w(q) < kp \leq \text{reach}_w(p)$ by (1).

(\Leftarrow) Let $w' := w[0 : \text{reach}_w(p)]$. (2) implies $\text{period}(w') = p$ since any q satisfying $0 < q < p$ is not a period of w' . Additionally, we have $p \leq \frac{|w'|}{k}$ by (1). Thus p is a prefix period of w with w' as a witness. \square

Galil and Seiferas [11] in Corollary 1 pointed out that the number of prefix periods of a word w is $O(\log |w|)$. We show in the following lemma that it is the case for parameterized strings. It contributes directly to reducing the space complexity of our algorithm.

Lemma 10. *Suppose w has prefix periods p and q . If $p < q$, then $2p \leq q$.*

Proof. We prove the lemma by contradiction. Suppose $p < q < 2p$. By definition, $p \parallel w[0 : kp]$ and $q \parallel w[0 : kq]$ hold. Let $w' := w[0 : kp]$. By Lemma 8, p is the shortest period of w' . Since both p and q are periods of w' and $p \cdot |\Pi_{w'}| + q < p \cdot |\Pi_w| + 2p = kp = |w'|$, we get $\gcd(p, q) \parallel w'$ by Lemma 5. Hence, we have $\gcd(p, q) \geq \text{period}(w') = p$, which claims $\gcd(p, q) = p$ i.e. $p \mid q$. However, this contradicts to the assumption $p < q < 2p$. \square

Corollary 3. *The number of prefix periods of $w \in (\Sigma \cup \Pi)^*$ is at most $\log_2 |w|$.*

4 Proposed Algorithm

In this section, we propose a sublinear-extra-space algorithm for the parameterized matching problem. Throughout this section, let T and P be p-strings whose lengths are n and m respectively, and let $k := |\Pi_P| + 2$. Besides, we suppose $\Pi_P \neq \emptyset$. Our algorithm is an extension of Galil and Seiferas’s exact string matching algorithm [11] and runs in $O(|\Pi_P|n + m)$ time and $O(\log m + |\Pi|)$ extra space. When $|\Pi| = |\Pi_P| = 1$, our algorithm behaves exactly as theirs.

Firstly, we introduce a method for testing whether two p-strings match. While it is common to use the *prev-encoding* [4] for this purpose, it is not suitable for our goal since it requires additional space proportional to the input size. Thus we use an alternative method as follows, which requires only $O(|\Pi|)$ extra space.

Lemma 11. *Consider a prefix x of P and $y \in (\Sigma \cup \Pi)^*$ with $x \equiv y$ and any $a, b \in \Sigma \cup \Pi$. We have $xa \equiv yb$ if and only if one of the following holds:*

1. $a \in \Sigma$ and $a = b$,
2. $a \in \Pi$ and $\text{first}_P(a) \geq |x|$ and $b \in \Pi$ and $\text{count}_y(b) = 0$,
3. $a \in \Pi$ and $\text{first}_P(a) < |x|$ and $y[\text{first}_P(a)] = b$,

where $\text{first}_P : \Pi \rightarrow \mathbb{N}$ and $\text{count}_y : \Pi \rightarrow \mathbb{N}$ are defined as follows:

$$\text{first}_P(c) = \begin{cases} \min\{i \in \mathbb{N} : i < |P| \text{ and } P[i] = c\} & \text{if } c \in \Pi_P, \\ |P| & \text{if } c \in \Pi \setminus \Pi_P, \end{cases}$$

$$\text{count}_y(c) = |\{i \in \mathbb{N} : i < |y| \text{ and } y[i] = c\}|$$

Proof. By definition, we have $xa \equiv yb$ if and only if $b = f(a)$, where f satisfies $y = f(x)$. If a is a constant character or appears in x , the value $f(a)$ is determined (Cases 1 and 3). Otherwise, b must be a parameter character not appearing in y (Case 2). \square

Let $\text{MATCH}(x, y, a, b, \text{first}_P, \text{count}_y)$ be the function which returns whether $xa \equiv yb$ under the condition $x \equiv y$ using Lemma 11. Clearly, one can compute it in constant time if first_P and count_y are given as arrays. Note that first_P can be computed in $O(m)$ time and $O(|\Pi|)$ space.

Algorithm 1: PREFIX_PERIODS

Input: $P \in (\Sigma \cup \Pi)^*$
Output: a list of all prefix periods of P and their reaches

```

1 begin
2    $k \leftarrow |\Pi_P| + 2$ 
3    $first \leftarrow first_P$ 
4    $PP \leftarrow$  empty list //  $PP$  is a list of pairs (val, reach)
5    $idx \leftarrow -1$ 
6    $(p, r) \leftarrow (1, 1)$ 
7   foreach  $a \in \Pi$  do  $count[a] \leftarrow 0$ 
8    $max\_reach \leftarrow 0$ 
9   while  $kp \leq |P|$  do
10    while MATCH( $P[0 : r - p], P[p : r], P[r - p], P[r], first, count$ ) do
11      Increment  $count[P[r]]$ 
12       $r \leftarrow r + 1$ 
13      if  $idx + 1 < |PP|$  and  $PP[idx + 1].val \leq \frac{r-p}{k}$  then Increment  $idx$ 
14
15    if  $r \geq kp$  and  $r > max\_reach$  then
16      Push  $(p, r)$  into  $PP$ 
17       $max\_reach \leftarrow \max\{max\_reach, r\}$ 
18
19    if  $0 \leq idx < |PP|$  and  $PP[idx].reach \geq r - p > 0$  then
20      for  $p \leq i < p + PP[idx].val$  do Decrement  $count[P[i]]$ 
21       $p \leftarrow p + PP[idx].val$ 
22
23    else
24      for  $p \leq i < r$  do Decrement  $count[P[i]]$ 
25       $p \leftarrow p + \lfloor \frac{r-p}{k} \rfloor + 1$ 
26       $r \leftarrow p$ 
27
28    until  $PP[idx].val \leq \frac{r-p}{k}$  or  $idx = -1$  do Decrement  $idx$ 
29
30  return  $PP$ 

```

4.1 Pattern Preprocessing

In this section, we show the preprocessing for the pattern P for our matching algorithm. The output of the preprocessing is the list of pairs of a prefix period of P (in ascending order) and its reach, just like Galil and Seiferas [11] introduced for exact string matching. The list plays a similar role to the *border array* in the parameterized Knuth-Morris-Pratt algorithm [2]. While border array uses $\Theta(m)$ space to memorize the shortest periods of all prefixes of P , the prefix period list requires only $O(\log m)$ space by Corollary 3.

We present the preprocess in Algorithm 1. The algorithm finds prefix periods and their reaches in order from the smallest to the largest and put them into the list PP . By $PP[idx].val$ and $PP[idx].reach$, we denote the idx -th prefix period and its reach in PP , respectively. Starting with $p = 1$, it monotonically

increases p and checks whether an integer p is a prefix period based on Lemma 9. Throughout the algorithm run, we maintain the invariant

$$p \parallel P[0 : r], \text{ i.e., } P[0 : r - p] \equiv P[p : r] \quad (\spadesuit)$$

We calculate $\text{reach}_P(p)$ by increasing r as long as $P[0 : r - p] \equiv P[p : r]$ holds (Lines 10–13). To let the function `MATCH` decide $P[0 : r - p] \equiv P[p : r]$, we use two auxiliary arrays first and count that satisfy $\text{first}[a] = \text{first}_P(a)$ and $\text{count}[a] = \text{count}_{P[p:r]}(a)$, defined in Lemma 11. Moreover, we maintain the variable max_reach to be the largest reach calculated so far. By Lemma 9, the condition of Line 14 is satisfied if and only if p is a prefix period. One can construct the list PP by incrementing p one by one, but it takes too much time. Instead, we use a more efficient way explained later to make the algorithm run in linear time.

The following lemmas justify the behavior of our algorithm.

Lemma 12. *Throughout Algorithm 1, the value of the variable idx is always the upper bound that satisfies $PP[\text{idx}].\text{val} \leq \frac{r-p}{k}$. If there exists no such index, we have $\text{idx} = -1$.*

Proof. The variable idx is updated in conjunction with p and r to preserve the condition. See Lines 13 and 24. \square

Lemma 13. *Let \spadesuit hold at Line 17 in Algorithm 1. If $\text{period}(P[0 : r - p]) \leq \frac{r-p}{k}$, we have $PP[\text{idx}].\text{val} = \text{period}(P[0 : r - p])$.*

Proof. Let $w' := P[0 : r - p]$, $p' := \text{period}(w')$, $p'' := PP[\text{idx}].\text{val}$, and $w'' = P[0 : kp'']$. By the assumption, p' is a prefix period of P . Additionally, we have $p' \leq p$ since $p \parallel w'$. Thus p' is in the list PP , and thus we have $p' \leq p''$ by Lemma 12. On the other hand, we have $\text{period}(w'') = p''$ by Lemma 8. Since $|w''| = kp'' \leq r - p = |w'|$, we have $\text{period}(w'') \leq \text{period}(w')$, i.e. $p'' \leq p'$. Hence we get $p' = p''$. \square

Lemma 14. *Let \spadesuit hold at Line 17 in Algorithm 1. We have $PP[\text{idx}].\text{reach} \geq r - p \iff \text{period}(P[0 : r - p]) \leq \frac{r-p}{k}$.*

Proof. Let $w' := P[0 : r - p]$ and $p' := PP[\text{idx}].\text{val}$.

(\implies) We have $p' \parallel w'$ by the assumption. Then $\text{period}(w') \leq p' \leq \frac{r-p}{k}$ holds by Lemma 12.

(\impliedby) By Lemma 13, we have $p' = \text{period}(w')$. Then $PP[\text{idx}].\text{reach} = \text{reach}_P(p') = \text{reach}_P(\text{period}(w')) \geq |w'| = r - p$. \square

Now, we show that the invariant \spadesuit always holds.

Lemma 15. *Throughout Algorithm 1, we have $P[0 : r - p] \equiv P[p : r]$.*

Proof. One must see the condition preserved at the lines in which p or r is updated. The update at Lines 22–23 is trivial. Line 12 preserves the condition, ensured by the condition of Line 10. For Line 19, let $q := PP[\text{idx}].\text{val}$. Since $q =$

$period(P[0 : r - p])$ by Lemma 13, we have $P[0 : r - (p + q)] \equiv P[q : r - p] \equiv P[p + q : r]$. Note that Lemma 13 requires ♠ only at Line 17, so the argument does not circulate. \square

The following lemma plays a key role to avoid incrementing p one by one.

Lemma 16. *Consider $P \in (\Sigma \cup \Pi)^*$, $p \in \mathbb{N}^+$ and let $r := reach_P(p)$. Then, no prefix period q of P exists such that $p < q < p + period(P[0 : r - p])$.*

Proof. We use Lemma 2 for $x := P[p : r]$, $y := P[0 : r - p]$, $\delta := q - p$ to obtain $P[q : r] \not\equiv P[0 : r - q]$, which means $q \nparallel P[0 : r]$. Thus we have $reach_P(q) < r = reach_P(p)$, which implies that q is not a prefix period of P by Lemma 9. \square

We now present the way to compute the list of prefix periods efficiently, in which we skip calculating $reach_P(p)$ if we are sure that p is not a prefix period. For realizing an efficient shift, we maintain a variable idx so that it points at the largest index of PP such that $PP[idx].val \leq \frac{r-p}{k}$ (Lemma 12). The shift amount is determined in the following manner. If $PP[idx].reach \geq r - p > 0$ at Line 17, Lemmas 14 and 13 imply $PP[idx].val = period(P[0 : r - p])$. Hence, Lemma 16 justifies the shift amount $PP[idx].val$ of p at Line 19. On the other hand, if $PP[idx].reach < r - p$, by Lemma 14, we have $period(P[0 : r - p]) > \frac{r-p}{k}$. This justifies the shift $\lfloor \frac{r-p}{k} \rfloor + 1$ of p at Line 22 again by Lemma 16. If $r - p = 0$, then p is incremented by just one.

Now, we show that the algorithm runs in $O(m)$ time. Firstly, notice that the while loops at Line 9 and 10 are repeated only $O(m)$ times in total, since the quantity $kp + r$ keeps increasing and $kp + r \leq k \cdot \frac{m}{k} + m = O(m)$. Hence, the fact we must show is that decrementing $count$ and idx at Line 18, 21, and 24 takes $O(m)$ time in total. As their values are always greater than or equal to their initial values, the number of decrements does not exceed the number of increments, which is $O(m)$ since they are in Line 11–13.

Theorem 1. *All prefix periods of P and their reaches can be calculated in $O(m)$ time and $O(\log m + |\Pi|)$ extra space.*

4.2 Searching for Parameterized Matches

Our matching algorithm is shown in Algorithm 2. As it is the case for the Galil-Seiferas algorithm, it resembles the preprocess. Now, the invariants in Algorithm 2 are obtained by replacing p , r , and $P[p : r]$ in Lemma 12–15 with i , j , and $T[i : j]$, respectively. Particularly, by the invariant that $P[0 : j - i] \equiv T[i : j]$, one can find matching positions i when $j = i + |P|$ (Line 13). The shift amounts are also justified by using Lemma 2 for $x := T[i : j]$ and $y := P[0 : j - i]$, whose conclusion $T[i + \delta : j] \not\equiv P[0 : j - i - \delta]$ implies $T[i + \delta : i + \delta + |P|] \not\equiv P$ for any δ smaller than the shift by the algorithm. We can show that the searching phase (Line 8–22) runs in $O(|\Pi_P|n)$ time in the same way as for the preprocess with the increasing quantity $ki + j$.

Theorem 2. *The parameterized matching problem can be solved in $O(|\Pi_P|n + m)$ time and $O(\log m + |\Pi|)$ extra space.*

Algorithm 2: SEARCH

```

Input:  $T, P \in (\Sigma \cup \Pi)^*$ 
Output: all  $0 \leq i \leq |T| - |P|$  such that  $T[i : i + |P|] \equiv P$ 
1 begin
2    $k \leftarrow |\Pi_P| + 2$ 
3    $first \leftarrow first_P$ 
4    $PP \leftarrow PREFIX\_PERIODS(P)$ 
5    $idx \leftarrow -1$ 
6    $(i, j) \leftarrow (0, 0)$ 
7   foreach  $a \in \Pi$  do  $count[a] \leftarrow 0$ 
8   while  $i < |T| - |P|$  do
9     while  $MATCH(P[0 : j - i], T[i : j], P[j - i], T[j], first, count)$  do
10      Increment  $count[T[j]]$ 
11       $j \leftarrow j + 1$ 
12      if  $idx + 1 < |PP|$  and  $PP[idx + 1].val \leq \frac{j-i}{k}$  then Increment  $idx$ 
13
14      if  $j - i = |P|$  then
15        output  $i$ 
16
17      if  $0 \leq idx < |PP|$  and  $PP[idx].reach \geq j - i > 0$  then
18        for  $i \leq u < i + PP[idx].val$  do Decrement  $count[T[u]]$ 
19         $i \leftarrow i + PP[idx].val$ 
20      else
21        for  $i \leq u < j$  do Decrement  $count[T[u]]$ 
22         $i \leftarrow i + \lfloor \frac{j-i}{k} \rfloor + 1$ 
23         $j \leftarrow i$ 
24      until  $PP[idx].val \leq \frac{j-i}{k}$  or  $idx = -1$  do Decrement  $idx$ 

```

5 Conclusion and Future Work

We studied the periodicity of parameterized strings and extended the Galil-Seiferas algorithm [11] for parameterized matching. The proposed algorithm requires only sublinear extra space. The properties of periods of parameterized strings we presented in this paper may be used to design more space-efficient algorithms for parameterized matching, as Galil and Seiferas [12] used prefix periods to design a constant-extra-space algorithm for exact matching.

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