# A Proof-Theoretic Analysis of the Meaning of a Formula in a Combination of Intuitionistic and Classical Propositional Logic

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**Abstract.** This paper provides a proof-theoretic analysis of the meaning of a formula in a combination of intuitionistic and classical propositional logic, based on the analysis proposed by Restall (2009). Restall showed that his analysis is applicable to both intuitionistic and classical propositional logic separately, but this paper shows that it is also applicable to a combination of the two logics called $\mathbf{C}+\mathbf{J}$. In addition, two points of improvement of Restall's analysis are mentioned, and they are overcome by employing the method provided by Takano (2018). Moreover, this paper explains how the analysis of $\mathbf{C}+\mathbf{J}$, which is based on Restall's analysis and improved by Takano's method, is related to the bilateralism-unilateralism debate. It is shown that a unilateral approach is possible for $\mathbf{C}+\mathbf{J}$, although Restall's original analysis is based on bilateralism.

**Keywords:** Sequent Calculus · Combination of Logics · Intuitionistic Logic · Classical Logic · Semantic Completeness · Bilateralism

## 1 Introduction and Motivation

### 1.1 Introduction

This paper provides a proof-theoretic analysis of the meaning of a formula in a combination of intuitionistic and classical propositional logic. A proof-theoretic analysis of meaning is an analysis explaining the meaning of a formula by the notion of arguments, proofs, or inference rules, not by the notion of truth, models, or validity. Such analyses are studied in, for example, [15,30,34,38,39,43]. The analysis presented in this paper is based on the one proposed by Restall [39], which uses a sequent calculus. A sequent calculus is a proof theory dealing with an object called "a sequent," which has the following form: $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas. The derivability of $\Gamma \Rightarrow \Delta$ is usually interpreted as follows: if all of the formulas in $\Gamma$ hold, then some of the formulas in $\Delta$ hold. The central idea of Restall's analysis is to interpret inference rules in a sequent calculus by the notions of assertion and denial and to obtain a model from the admissibility of these inference rules. Accordingly, in addition to the usual interpretation described above,


D. Bekki et al. (Eds.): LENLS 2019, LNCS 14213, pp. 100–119, 2023.
https://doi.org/10.1007/978-3-031-43977-3_7

Restall [38,39] provides the following interpretation of the derivability of a sequent by the notions of assertion and denial: it is incoherent to assert all the formulas in $\Gamma$ and to deny all the formulas in $\Delta$. Corresponding to this interpretation of the derivability of a sequent, inference rules in a sequent calculus are also interpreted by the notions of assertion and denial. Based on these interpretations, the notion of a model is obtained successfully in Restall's analysis.

On the other hand, various combinations of intuitionistic and classical logic are studied in [6,8–11,18,21,22,25,26,31–33,35,49,50]. In this paper, we regard a logic as a combination of intuitionistic and classical logic if the language of the logic has both intuitionistic and classical operators and if it is a conservative extension of both logics. Although various combinations of intuitionistic and classical logic exist, this paper analyzes the one studied in [10,11,18,22,49,50], because for this logic a sequent calculus using an ordinary notion of a sequent was already proposed in [49,50]. As is noted above, since Restall's original analysis employs a sequent calculus, the existence of a sequent calculus enables us to apply the analysis straightforwardly. In the following, this combination and the sequent calculus for this combination are called $\mathbf{C}+\mathbf{J}$ and $\mathsf{G}(\mathbf{C}+\mathbf{J})$, respectively. The idea of constructing $\mathbf{C}+\mathbf{J}$ is easy to see in the Kripke semantics provided in [11,18]. This Kripke semantics is obtained by adding to the Kripke semantics for intuitionistic propositional logic the satisfaction relation for a formula whose main connective is classical negation, denoted by "$\neg_{\mathtt{c}}$", described as follows:

$$w \models_M \neg_{\mathtt{c}} A \text{ iff } w \not\models_M A,$$

where $M = \langle W, R, V\rangle$ is a Kripke model for intuitionistic propositional logic and $w$ is a state in $W$. The sequent calculus for $\mathbf{C}+\mathbf{J}$ is proposed in [49]. This calculus is obtained by adding the right and left rules for intuitionistic implication, denoted by "$\rightarrow_{\mathtt{i}}$," to the propositional fragment of the classical sequent calculus **LK**. The left rule for "$\rightarrow_{\mathtt{i}}$" added to the propositional fragment of **LK** is the one in the intuitionistic multi-succedent sequent calculus **mLJ**, proposed by Maehara [23], but the right rule should be restricted as follows:

$$\frac{A, C_1{\rightarrow_{\mathtt{i}}}D_1, \ldots, C_m{\rightarrow_{\mathtt{i}}}D_m, p_1, \ldots, p_n \Rightarrow B}{C_1{\rightarrow_{\mathtt{i}}}D_1, \ldots, C_m{\rightarrow_{\mathtt{i}}}D_m, p_1, \ldots, p_n \Rightarrow A{\rightarrow_{\mathtt{i}}}B}\ (\Rightarrow \rightarrow_{\mathtt{i}}).$$

The sequent calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$ is sound and complete to the Kripke semantics described above, which was shown in [50].

This paper basically applies Restall's analysis to $\mathbf{C}+\mathbf{J}$ in terms of $\mathsf{G}(\mathbf{C}+\mathbf{J})$. However, two points of improvement and one open problem exist in Restall's analysis. The first point of improvement concerns the relationship between the admissibility of an inference rule and the corresponding satisfaction relation. The second point of improvement concerns the admissibility of the rule ($Cut$). These two points are overcome by employing the method provided by Takano [48] for fifteen modal logics. The open problem is the following one: is it possible to analyze the meaning of a formula in a combination of intuitionistic and classical propositional logic? This paper solves this open problem positively by showing that Restall's analysis, improved by Takano's method, is applicable to $\mathbf{C}+\mathbf{J}$.

It is also shown that an analysis based on unilateralism, which is the opposite position of bilateralism, is possible for $\mathbf{C}+\mathbf{J}$. Bilateralism is a position claiming that two linguistic acts are primitive when the meaning of a formula or a statement is considered, whereas unilateralism is a position claiming that only one linguistic act is primitive. Since Restall's analysis introduces the notions of assertion and denial as primitive, it is categorized as bilateralism, and the analysis improved by Takano's method may also be categorized as bilateralism. However, this paper shows that a unilateral approach is also possible for $\mathbf{C}+\mathbf{J}$.

The outline of this paper is as follows. Section 2 and Sect. 3 review Restall's analysis and $\mathbf{C}+\mathbf{J}$, respectively. Section 4 applies Takano's [48] method to $\mathbf{C}+\mathbf{J}$. Section 5 explains how the analysis in this paper is related to the bilateralism-unilateralism debate and shows that unilateral analysis is possible for $\mathbf{C}+\mathbf{J}$.

### 1.2 Motivation for Analyzing the Meaning of a Formula in a Combination of Intuitionistic and Classical Logic

Before proceeding to Sect. 2, let us see why an analysis of the meaning of a formula in a combination of intuitionistic and classical logic should be provided. Since Restall's analysis is possible for both intuitionistic and classical propositional logic separately, it may be thought that giving the meaning of a formula in a combination of both logics is not needed. This section provides an argument claiming that an analysis of the meaning of a formula in a combination of intuitionistic and classical logic is necessary.

By combining intuitionistic and classical logic, we can tackle the following question: how do advocates of intuitionistic/classical logic understand the meaning of a formula in the other logic?[1] As Quine [37] pointed out, intuitionistic and classical connectives can be regarded as denoting different subjects. By analyzing the meaning of a formula in a combination of intuitionistic and classical logic, in which connectives of both logics exist, we can codify how advocates of intuitionistic/classical logic understand the meaning of a formula in the other logic. For example, we can explain how an advocate of intuitionistic logic understands the meaning of a formula $\neg_{\mathsf{c}} p \vee p$, where "$\neg_{\mathsf{c}}$" denotes classical negation. Being an advocate of intuitionistic logic, he/she basically uses negation in the intuitionistic way. However, in order to give the analysis of the meaning of $\neg_{\mathsf{c}} p \vee p$ as Restall did for classical and intuitionistic logic, the advocates of intuitionistic logic also need to appeal to the inference rules for the classical negation, since Restall's analysis is based on the inference rules for a connective. Therefore, in order to explain how the advocates of intuitionistic logic understand the meaning of $\neg_{\mathsf{c}} p \vee p$, we should provide an analysis of the meaning of a formula in a combination of intuitionistic and classical logic whose proof theory contains the inference rules for both intuitionistic and classical connectives.[2]

[1] This question is not a new one. Similar questions were already mentioned in [26,35].

[2] Some may consider ordinary intuitionistic logic itself to be a combination, since Kolmogorov-Gödel-Gentzen translation exists. A combination based on this view is studied in [31–33,35]. However, such a view is criticized in [12].

Some may disagree with this argument claiming that there is no need to codify how advocates of intuitionistic/classical logic understand the meaning of a formula in the other logic, since they understand the meaning of a formula in the other logic by seeing a proof theory or semantics for it. For example, they may claim that advocates of classical logic understand the meaning of a formula in intuitionistic logic by seeing Kripke semantics for intuitionistic logic, and there is no need to appeal to a logic having both intuitionistic and classical connectives.

However, the codification is necessary. If we accept Quine's view that intuitionistic and classical connectives denote different subjects, it is admitted that the discussion between advocates of intuitionistic logic and those of classical logic is not about a valid logical law but about the use of connectives. For example, advocates of intuitionistic logic do not accept law of excluded middle. They do not accept $\neg_{\mathtt{i}} A \vee A$ generally, where "$\neg_{\mathtt{i}}$" denotes intuitionistic negation. On the other hand, advocates of classical logic accept law of excluded middle. They accept $\neg_{\mathtt{c}} A \vee A$ generally. There is no disagreement about law of excluded middle between advocates of intuitionistic logic and those of classical logic, since it is possible that $\neg_{\mathtt{c}} A \vee A$ is valid while $\neg_{\mathtt{i}} A \vee A$ is not. Thus, the discussion between advocates of intuitionistic logic and those of classical logic is the following one: how should negation be used?, or what kind of meaning should be attached to negation? In this discussion, advocates of intuitionistic/classical logic should take a connective in the other logic into consideration. For example, if advocates of classical logic attempt to claim that negation should be used as in classical logic and that the use of law of excluded middle should be permitted, they must give an argument claiming that a formula containing intuitionistic implication, such as $\neg_{\mathtt{c}}(p \rightarrow_{\mathtt{i}} q) \vee (p \rightarrow_{\mathtt{i}} q)$, is also to be admitted. The reason for this is that if an argument does not take such a formula into consideration, it is clearly begging the question. In order to formulate the discussion between advocates of intuitionistic and those of classical logic in this way, a formula such as $\neg_{\mathtt{c}}(p \rightarrow_{\mathtt{i}} q) \vee (p \rightarrow_{\mathtt{i}} q)$ should be expressed and considered. Therefore, the explanation of the meaning of a formula consisting of intuitionistic and classical connectives is necessary.

Some may think that a combination does not contribute to such an argument, because since a combination of intuitionistic and classical logic is a conservative extension of both intuitionistic and classical logic, all the theorems in the ordinary intuitionistic and classical logics are also theorems in the combination. However, it is not guaranteed that all the theorems in the ordinary intuitionistic and classical logics are also theorems in a combination by the fact that it is a conservative extension of both logics. For example, $A \rightarrow_{\mathtt{i}} (B \rightarrow_{\mathtt{i}} A)$ is no longer a theorem in $\mathbf{C} + \mathbf{J}$, the combination dealt with in this paper, since $\neg_{\mathtt{c}} p \rightarrow_{\mathtt{i}} (q \rightarrow_{\mathtt{i}} p)$ is not derivable. This may imply that advocates of intuitionistic logic cannot claim that all of the intuitionistic theorems should be admitted. This is because the addition of classical negation in the way of $\mathbf{C} + \mathbf{J}$ leads to an instance of this theorem that is not derivable in a proof theory of $\mathbf{C} + \mathbf{J}$.[3]

[3] It is noted that the results in $\mathbf{C} + \mathbf{J}$ may not be conclusive for deciding whether an intuitionistic or classical theorem should be admitted. The results in another combination also need to be considered.

# 2 Restall's Analysis, Two Points of Improvement, and One Open Problem

## 2.1 Restall's Analysis

This section reviews Restall's analysis proposed in [39]. As is noted in Sect. 1.1, the central idea of this analysis is to interpret inference rules in a sequent calculus by the notions of assertion and denial and to obtain the notion of a model from the admissibility of these inference rules. Restall regards inference rules in a sequent calculus as "normative constraints" on assertion and denial. Although this analysis is applied to classical propositional logic, intuitionistic propositional logic, and modal logic **S5**, only the case of classical propositional logic is described here.

We define the syntax of classical logic as consisting of a countably infinite set of propositional variables and the following logical connectives: falsum $\bot$, conjunction $\wedge$, disjunction $\vee$, and negation $\neg_{\mathsf{c}}$.[4] As far as classical propositional logic is concerned, the subscript "$\mathsf{c}$" for the negation is not necessary, but since intuitionistic negation is introduced in Sect. 3, we use this subscript from this section onward to avoid confusion. Classical implication $\rightarrow_{\mathsf{c}}$ is not introduced as a primitive symbol, since it can be defined as follows: $A \rightarrow_{\mathsf{c}} B := \neg_{\mathsf{c}} A \vee B$. When classical propositional logic is analyzed, the propositional fragment of the sequent calculus **LK** is used. In the rest of this paper, the expression **LK** denotes only the propositional fragment. This calculus deals with an object called "a sequent," which has the following form: $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas. The sequent calculus **LK** consists of the axioms and rules in Table 1.[5]

**Table 1.** Sequent Calculus **LK**

**Axioms**

$$\frac{}{A \Rightarrow A}\ (Id) \qquad \frac{}{\bot \Rightarrow}\ (\bot)$$

**Structural Rules**

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (Cut)$$

**Propositional Logical Rules**

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\Rightarrow \wedge) \qquad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow_1) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow_2)$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\Rightarrow \vee_2) \qquad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_{\mathsf{c}} A}\ (\Rightarrow \neg_{\mathsf{c}}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\neg_{\mathsf{c}} A, \Gamma \Rightarrow \Delta}\ (\neg_{\mathsf{c}} \Rightarrow)$$

[4] Note that $\bot$ is not considered in [39], because $\bot$ is definable by negation and conjunction. However, the addition of $\bot$ as a primitive symbol creates no problem.

[5] Since the antecedent and succedent of a sequent are defined as sets, contraction and exchange rules are not necessary. It is noted that although a sequent calculus that does not contain $(w \Rightarrow)$ or $(\Rightarrow w)$ is used in [39], this difference creates no problem.

The derivability of a sequent $\Gamma \Rightarrow \Delta$ in **LK** is defined by the existence of a finite tree consisting only of axioms and rules in **LK** whose root is the sequent. The derivability of $\Gamma \Rightarrow \Delta$ is usually interpreted as follows: if all of the formulas in $\Gamma$ hold, then some of the formulas in $\Delta$ hold.

Restall starts his analysis by defining the notion of a "position."

**Definition 1 (Position [39, Definition 1].)** *A pair* $(\Gamma : \Delta)$ *of finite sets of formulas is a* position *if* $\Gamma \Rightarrow \Delta$ *is not derivable in* **LK**.

In the rest of this paper, $(\Gamma \cup \{A\} : \Delta \cup \{B\})$ is abbreviated as $(\Gamma, A : \Delta, B)$, for any finite sets $\Gamma \cup \{A\}, \Delta \cup \{B\}$ of formulas. The antecedent and succedent of a position are regarded as the set of asserted and denied formulas, respectively. A position expresses a coherent situation with respect to assertion and denial. Consider the pairs $(p : p \wedge q)$ and $(p : p \vee q)$. The former is a position, since $p \Rightarrow p \wedge q$ is not derivable in **LK**. This implies that to assert $p$ and to deny $p \wedge q$ is coherent in classical logic. However, the latter is not a position, since $p \Rightarrow p \vee q$ is derivable in **LK**. This implies that to assert $p$ and to deny $p \vee q$ is incoherent in classical logic. Accordingly, the derivability of $\Gamma \Rightarrow \Delta$ is interpreted by the notions of assertion and denial, as follows: it is incoherent to assert all the formulas in $\Gamma$ and to deny all the formulas in $\Delta$. An inference rule in **LK** is also interpreted by the notions of assertion and denial. For example, $(\wedge \Rightarrow_1)$ is interpreted by reading the rule from the lower sequent to the upper sequent, as follows: if it is coherent to assert $A \wedge B$ and all the formulas in $\Gamma$ and to deny all the formulas in $\Delta$, then it is also coherent to assert $A$ and all the formulas in $\Gamma$ and to deny all the formulas in $\Delta$. The other rules in **LK** are interpreted in the same way. Since inference rules in **LK** govern assertion and denial, they are considered to be "normative constraints" on assertion and denial.

Based on the notion of a position, the notion of a "limit position" is defined.

**Definition 2 (Limit Position [39, Definition 4]).** *A pair* $(\Gamma : \Delta)$ *of sets of formulas is a* limit position *if it satisfies the following:*

- *For any finite sets* $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ *of formulas, the pair* $(\Gamma' : \Delta')$ *is a position.*
- *The union of* $\Gamma$ *and* $\Delta$ *contains all the formulas in classical logic,*

A limit position expresses an ideal situation with respect to assertion and denial, in which any formula in classical logic is either asserted or denied. Thus, a limit position does not express the actual linguistic situation, as was noted in [39, pp. 249-252]. Technically, the antecedent of a limit position corresponds to the notion of a maximal consistent set, the notion used to show the semantic completeness (cf. [5, Definition 4.15]).

**Fact 1. [39, Fact 4]** *For any position* $(\Gamma : \Delta)$*, there is a limit position* $(\Gamma^* : \Delta^*)$ *such that* $\Gamma \subseteq \Gamma^*$ *and* $\Delta \subseteq \Delta^*$*.*

This fact is shown by making use of $(Cut)$. This rule ensures the following: if $(\Gamma : \Delta)$ is a position, then either $(\Gamma : \Delta, A)$ or $(A, \Gamma : \Delta)$ is also a position. The proof is almost the same as the one of extension lemma, the lemma used to show the semantic completeness (cf. [5, Lemma 4.17]).

**Fact 2.** [**39**, **Fact 5**] *For any limit position* $(\Gamma : \Delta)$*, all of the following hold:*

1. $A \wedge B \in \Gamma$ *iff* $A \in \Gamma$ *and* $B \in \Gamma$,
2. $A \wedge B \in \Delta$ *iff* $A \in \Delta$ *or* $B \in \Delta$,
3. $A \vee B \in \Gamma$ *iff* $A \in \Gamma$ *or* $B \in \Gamma$,
4. $A \vee B \in \Delta$ *iff* $A \in \Delta$ *and* $B \in \Delta$,
5. $\neg_{\mathsf{c}} A \in \Gamma$ *iff* $A \in \Delta$,
6. $\neg_{\mathsf{c}} A \in \Delta$ *iff* $A \in \Gamma$,
7. $\bot \in \Delta$.

This fact is shown by reading the rules in **LK** from the lower sequent to the upper sequent(s) and by appealing to the fact that the union of $\Gamma$ and $\Delta$ is the set of all the formulas in classical logic. As is seen in Fact 2, if the formulas in the antecedent of a limit position are regarded as true, the truth conditions are obtained, while if the formulas in the succedent of a limit position are regarded as false, the false conditions are obtained. Therefore, starting from the admissibility of the inference rules in **LK**, the notion of a model for classical logic is obtained. Although the notion of a model is considered, it is not introduced as given but obtained by analyzing the rules in **LK**.[6] Thus, this analysis is proof-theoretic.

### 2.2 Two Points of Improvement and One Open Problem

Although Restall's analysis is very refined and explains the relationship between an inference rule in a sequent calculus and the corresponding satisfaction relation for a formula, two points of improvement and one open problem exist.

The first point of improvement concerns on the relationship between the admissibility of an inference rule and the corresponding satisfaction relation for a formula. Restall's analysis obtains the satisfaction relation for a formula from the admissibility of the corresponding inference rule. In other words, Restall's analysis explains the following: if an inference rule is admissible in a sequent calculus, then the corresponding satisfaction relation for a formula will be obtained. However, in addition to this, if it is possible to establish the other direction, the tighter relation between the admissibility of an inference rule and the corresponding satisfaction relation for a formula is obtained. The other direction tells us what kind of inference rule is admissible if we choose some satisfaction relation for a formula, which enables us to describe in detail the relation between the admissibility of an inference rule and the corresponding satisfaction relation for a formula.

The second point of improvement concerns the rule $(Cut)$. Restall's analysis, especially Fact 1, depends on this rule. In terms of assertion and denial, $(Cut)$ expresses the following normative constraint: if it is coherent to assert all the formulas in $\Gamma$ and to deny all the formulas in $\Delta$, then either the assertion of $A$ or the denial of $A$ is also coherent. However, it is far from trivial to accept this normative constraint, and some may refuse to accept this normative constraint.[7]

[6] Proof-theoretic semantics, the representative of the proof-theoretic analyses of meaning, explains the meaning of a formula by using purely syntactical objects, such as arguments or proofs directly (cf. [15,34,43]). On the other hand, Restall's analysis introduces the notion of a model. Thus, these two analyses are different on this point.

[7] For example, Ripley [40, Section 3.2] argues that there is no reason to postulate it. Even Restall [38, footnote 5] himself admits that the account of assertion and denial

If an analysis with no dependence on $(Cut)$ is obtained, such an analysis will be acceptable for those who refuse to accept the normative constraint expressed by $(Cut)$. Moreover, if the semantic condition corresponding to $(Cut)$ is obtained, we can treat this rule in the same way as the rules for connectives, not as given.

The open problem is whether Restall's analysis is possible for a combination of intuitionistic and classical propositional logic. Although Restall's analysis is applicable to both of intuitionistic and classical propositional logic separately, this does not imply that it is applicable to a combination of both logics.

In Sect. 4, these two points of improvement are overcome by employing the method proposed by Takano [48]. Moreover, the open problem is solved positively by carrying out the analysis on a combination $\mathbf{C} + \mathbf{J}$ of intuitionistic and classical propositional logic. It should be noted that the analysis of this paper, which is based on Takano's method, does not appeal to König's infinite lemma or Zorn's lemma, the latter being appealed to in Restall's analysis. As noted in [1], it is controversial whether the axiom of choice is acceptable for advocates of intuitionistic logic. Since Zorn's lemma is equivalent to the axiom of choice and König's infinite lemma is weaker than Zorn's lemma, the fact that these lemmas are dispensable implies that our analysis does not presuppose any position about whether advocates of intuitionistic logic accept the axiom of choice. Thus, our analysis is acceptable for an advocate of intuitionistic logic, independently of whether he/she admits the axiom of choice.[8] Before proceeding to Takano's method, the combination $\mathbf{C} + \mathbf{J}$ is reviewed briefly in Sect. 3.

## 3 Combination of Intuitionistic and Classical Propositional Logic $\mathbf{C} + \mathbf{J}$

The combination $\mathbf{C} + \mathbf{J}$ is provided by Humberstone [18], and he proposed a natural deduction system. A Hilbert system for this logic was first proposed by del Cerro and Herzig [11], and De and Omori [10] proposed another Hilbert system by expanding a subintuitionistic logic. The single-succedent structured sequent calculus for this logic was proposed by Lucio [22]. The multi-succedent sequent calculus $\mathsf{G}(\mathbf{C} + \mathbf{J})$ was provided in [49,50]. A first-order expansion of $\mathbf{C} + \mathbf{J}$ was studied in [22,50]. Although many proof theories exist for $\mathbf{C} + \mathbf{J}$, we use the sequent calculus $\mathsf{G}(\mathbf{C} + \mathbf{J})$, because Restall's original analysis employs a sequent calculus.

The syntax of $\mathbf{C} + \mathbf{J}$ is obtained by adding intuitionistic implication to that of classical logic. Intuitionistic negation $\neg_{\mathtt{i}}$ can be defined as follows: $\neg_{\mathtt{i}} A := A \to_{\mathtt{i}} \bot$. Since "$\to_{\mathtt{c}}$" is definable, $\mathbf{C} + \mathbf{J}$ has two types of implication and negation.

recorded in $(Cut)$ is a subtle one for advocates of intuitionistic logic. This paper does not discuss whether the normative constraint expressed by $(Cut)$ is acceptable. Thus, this paper does not claim that it is unacceptable. What is shown in this paper is that it is not necessary to postulate the rule $(Cut)$ in order to carry out a proof-theoretic analysis of the meaning of a formula.

[8] Clearly, this point holds only in the propositional setting. Therefore, if we try to expand the analysis in this paper to the first-order setting, we need to appeal to either König's infinite lemma or Zorn's lemma.

Let us proceed to the semantics for $\mathbf{C}+\mathbf{J}$. We introduce a Kripke semantics for $\mathbf{C}+\mathbf{J}$, provided in [11,18]. The Kripke semantics is obtained by adding the satisfaction relation for a formula whose main connective is classical negation to the Kripke semantics for intuitionistic propositional logic (cf. [7, Section 6.3]).

**Definition 3 (Kripke Model [11]).** *A* Kripke model *is a tuple* $M = \langle W, R, V\rangle$*, where*

- $W$ *is a non-empty set of states,*
- $R$ *is a preorder on* $W$*, i.e.,* $R$ *satisfies reflexivity and transitivity,*[9]
- $V : \mathsf{Prop} \to \mathcal{P}(W)$ *is a valuation function satisfying the following* heredity *condition:* $w \in V(p)$ *and* $wRv$ *jointly imply* $v \in V(p)$ *for all states* $w, v \in W$.

**Definition 4.** [11] *Given a Kripke model* $M = \langle W, R, V\rangle$*, a state* $w \in W$*, and a formula* $A$*, the* satisfaction relation $w \models_M A$ *is defined inductively as follows:*

$$\begin{array}{ll} w \models_M p & \text{iff } w \in V(p), \\ w \not\models_M \bot, & \\ w \models_M A \wedge B & \text{iff } w \models_M A \textit{ and } w \models_M B, \\ w \models_M A \vee B & \text{iff } w \models_M A \textit{ or } w \models_M B, \\ w \models_M \neg_{\mathsf{c}} A & \text{iff } w \not\models_M A, \\ w \models_M A \to_{\mathtt{i}} B & \text{iff } \textit{for all } v \in W, wRv \textit{ and } v \models_M A \textit{ jointly imply } v \models_M B. \end{array}$$

The notion of a semantic consequence is defined by the truth preservation on an arbitrary state $w \in W$. A formula $A$ is valid if $A$ is a semantic consequence of $\emptyset$.

Let us proceed to the sequent calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$ for $\mathbf{C}+\mathbf{J}$.

**Definition 5 (Sequent Calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$ [49]).** *The sequent calculus* $\mathsf{G}(\mathbf{C}+\mathbf{J})$ *is obtained by adding to* **LK***, consisting of the rules in Table 1, the right and left rules for intuitionistic implication, formulated as follows:*

$$\frac{A, C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow A \to_{\mathtt{i}} B}\ (\Rightarrow \to_{\mathtt{i}})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to_{\mathtt{i}} B, \Gamma \Rightarrow \Delta}\ (\to_{\mathtt{i}} \Rightarrow).$$

The left rule is the same as the one in the intuitionistic multi-succedent sequent calculus **mLJ**, proposed by Maehara [23]. However, the right rule should be restricted to the form described above. If this restriction were not imposed on the

[9] Although $R$ is defined as a preorder on $W$ in [11], it is defined as a partial order on $W$ in the Kripke semantics provided in [18]. It is noted that both definitions are possible for a Kripke semantics for $\mathbf{C}+\mathbf{J}$ (cf. [5, Section 4.5]).

right rule for intuitionistic implication, intuitionistic implication would collapse into classical implication, as was pointed out in [4,17,38,52].[10]

**Fact 3 (Soundness and Completeness [50, Theorems 1 and 3]).** *The sequent calculus* $\mathsf{G}(\mathbf{C}+\mathbf{J})$ *is sound and complete to the Kripke semantics, defined in Definitions 3 and 5.*

**Fact 4 (Cut-Elimination and Subformula Property [50, Theorem 2]).** *The sequent calculus* $\mathsf{G}(\mathbf{C}+\mathbf{J})$ *is cut-free and satisfies the subformula property.*

## 4 Applying Takano's Method to $\mathbf{C}+\mathbf{J}$

This section applies the method proposed by Takano [48] to $\mathbf{C}+\mathbf{J}$ and overcomes two points of improvement of Restall's original analysis. As a result, the open problem is solved positively.

**Stipulation 1 (Sequent Calculus [48, Stipulation 1]).** *A sequent calculus is a calculus having* $A \Rightarrow A$ *as an axiom for any* $A$ *and having weakening rules.*

By Stipulation 1, a sequent calculus that has only some rules in $\mathbf{C}+\mathbf{J}$ can be discussed. Note that the existence of $(Cut)$ is not assumed in this stipulation. In the following, let **GL** be a sequent calculus in the sense of Stipulation 1.

**Definition 6.** *Let* $\Gamma$ *be a finite set of formulas. Then, we define* $\mathsf{Sub}(\Gamma)$ *as the set of all subformulas of some formulas in* $\Gamma$*. A set* $\Gamma$ *of formulas is* subformula-closed (sf-closed) *if* $\mathsf{Sub}(\Gamma) \subseteq \Gamma$ *and* $\bot \in \Gamma$*.*

In this paper, the definition of an sf-closed set of formulas is slightly different from the ordinary definition, since the condition $\bot \in \Gamma$ is required in Definition 6. This condition is necessary for dealing with the rule $(\bot)$.

In the following, an sf-closed finite set $\Xi$ of formulas is considered, while it is not considered in [48]. However, such a set is considered in [20,28,42,47], and a finite model will be obtained by considering it. The notion of derivability can be defined relative to $\Xi$.

**Definition 7 ($\Xi$-derivability).** *Let* $\Xi$ *be an sf-closed finite set of formulas and* $\Gamma \cup \Delta \subseteq \Xi$*. A sequent* $\Gamma \Rightarrow \Delta$ *is* $\Xi$-derivable *in* **GL** *if it has a derivation in* **GL** *consisting solely of formulas in* $\Xi$*.*

In the following, when it is said that a sequent $\Gamma \Rightarrow \Delta$ is $\Xi$-derivable or $\Xi$-underivable, it is presupposed that $\Gamma \cup \Delta \subseteq \Xi$ holds.

[10] The reason why this restriction on the right rule for intuitionistic implication enables us to avoid collapsing is explained in [50]. Since the right rule for intuitionistic implication is restricted compared with the original rule in **mLJ**, some might wonder whether the semantic completeness of $\mathbf{C}+\mathbf{J}$ fails. However, the semantic completeness holds, and the detailed proof is described in [50, Section 4]. Moreover, the rule $(\Rightarrow \to_{\mathtt{i}})$ in $\mathsf{G}(\mathbf{C}+\mathbf{J})$ can be regarded as the core of the ordinary right rule for implication in **mLJ**, as noted in [50, p.32].

**Definition 8 ($\Xi$-underivable pair).** *Let $\Xi$ be an sf-closed finite set of formulas. A pair $(\Gamma : \Delta)$ of finite sets of formulas is a* $\Xi$-underivable pair *in a sequent calculus* **GL** *if $\Gamma \Rightarrow \Delta$ is not derivable in* **GL**.

The notion of a $\Xi$-underivable pair plays almost the same role as the notion of a position in Restall's analysis. It is noted that in Definition 8, the notion of derivability is defined relative to an sf-closed set $\Xi$ and a sequent calculus **GL**. The antecedent and succedent of a $\Xi$-underivable pair can be regarded as the set of asserted and denied formulas, respectively, as is done in Restall's analysis. In the following, instead of the notion of a limit position, the notion of a $\Xi$-*analytically saturated pair* is introduced. This notion is obtained by modifying the notion of an *analytically saturated sequent*, defined in [48, Definition 1.1].

**Definition 9 ($\Xi$-analytically saturated pair).** *Let $\Xi$ be an sf-closed finite set of formulas. A pair $(\Gamma : \Delta)$ of finite sets of formulas is* $\Xi$-analytically saturated *in a sequent calculus* **GL** *if it satisfies all of the following:*

1. *$\Gamma \Rightarrow \Delta$ is not $\Xi$-derivable in* **GL**.
2. *For any formula $A \in \Xi$,*
   - *$A \in \Gamma$ if $A, \Gamma \Rightarrow \Delta$ is not $\Xi$-derivable in* **GL**,
   - *$A \in \Delta$ if $\Gamma \Rightarrow \Delta, A$ is not $\Xi$-derivable in* **GL**,

The first condition of this definition is almost the same as the first condition of the definition of a limit position (Definition 2). The important difference from the notion of a limit position is contained in the second condition. In the second condition of the definition of a limit position (Definition 2), any formula $A$ must be an element of either $\Gamma$ or $\Delta$. However, in the second condition of the definition of a $\Xi$-analytically saturated pair (Definition 9), this is not required.

**Lemma 1.** *Let $\Xi$ be an sf-closed finite set of formulas and $(\Gamma : \Delta)$ be a $\Xi$-underivable pair in* **GL**. *Then, there exists a $\Xi$-analytically saturated pair $(\Gamma^* : \Delta^*)$ in* **GL** *such that $\Gamma \subseteq \Gamma^*$, $\Delta \subseteq \Delta^*$, and $\Gamma^* \cup \Delta^* \subseteq \Xi$.*

This lemma is shown in almost the same way as [48, Lemma 1.3]. Lemma 1 ensures that any $\Xi$-underivable pair of sets of formulas in **GL** can be extended to some $\Xi$-analytically saturated pairs in **GL**. This lemma corresponds to extension lemma of cut-free semantic completeness (cf. [27, Lemma 10]).

**Definition 10.** *For any sf-closed finite set $\Xi$ of formulas, $W^{\Xi}$ is defined as the set of all $\Xi$-analytically saturated pairs in* **GL**.

**Definition 11.** *For any $(\Gamma : \Delta), (\Pi : \Sigma) \in W^{\Xi}$, $(\Gamma : \Delta) R^{\Xi} (\Pi : \Sigma)$ if the following hold:*

- *For any propositional variable $p \in \Xi$, if $p \in \Gamma$, then $p \in \Pi$,*
- *For any formulas $A \rightarrow_{\mathtt{i}} B \in \Xi$, if $A \rightarrow_{\mathtt{i}} B \in \Gamma$, then $A \rightarrow_{\mathtt{i}} B \in \Pi$.*

This definition of $R^{\Xi}$ is imported from [50, Definition 11].

**Definition 12.** *A valuation* $V^{\Xi}$ *is defined as follows for any propositional variable* $p \in \Xi$ *and any* $(\Gamma : \Delta) \in W^{\Xi}$*:*

$$(\Gamma : \Delta) \in V^{\Xi}(p) \text{ iff } p \in \Gamma.$$

The obtained tuple $\langle W^{\Xi}, R^{\Xi}, V^{\Xi} \rangle$ is a well-defined Kripke model described in Sect. 3. Since $\Xi$ is finite, $W^{\Xi}$ is finite. Thus, $\langle W^{\Xi}, R^{\Xi}, V^{\Xi} \rangle$ is a finite model.

Based on the notion of $\Xi$-derivability, we define the notion of $\Xi$-admissibility, the notion of admissibility relative to an sf-closed finite set $\Xi$ of formulas.

**Definition 13 ($\Xi$-admissibility).** *An inference rule is* $\Xi$-admissible *in* **GL** *if whenever all of the upper sequents are* $\Xi$*-derivable in* **GL**, *then the lower sequent is also* $\Xi$*-derivable in* **GL**.

**Definition 14.** *If the side condition* $A \in \mathsf{Sub}(\Gamma \cup \Delta)$ *is imposed on* $(Cut)$*, the restricted rule is defined as* $(Cut)^a$.

**Theorem 1.** *For any sf-closed finite set* $\Xi$ *of formulas, all of the following hold:*

1. *The left rule for "$\wedge$" is $\Xi$-admissible in* **GL** *iff $A \wedge B \in \Gamma$ implies $A \in \Gamma$ and $B \in \Gamma$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
2. *The right rule for "$\wedge$" is $\Xi$-admissible in* **GL** *iff $A \wedge B \in \Delta$ implies $A \in \Delta$ or $B \in \Delta$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
3. *The left rule for "$\vee$" is $\Xi$-admissible in* **GL** *iff $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
4. *The right rule for "$\vee$" is $\Xi$-admissible in* **GL** *iff $A \vee B \in \Delta$ implies $A \in \Delta$ and $B \in \Delta$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
5. *The left rule for "$\neg_{\mathsf{c}}$" is $\Xi$-admissible in* **GL** *iff $\neg_{\mathsf{c}}A \in \Gamma$ implies $A \in \Delta$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
6. *The right rule for "$\neg_{\mathsf{c}}$" is $\Xi$-admissible in* **GL** *iff $\neg_{\mathsf{c}}A \in \Delta$ implies $A \in \Gamma$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
7. *The left rule for "$\to_{\mathtt{i}}$" is $\Xi$-admissible in* **GL** *iff for any $(\Gamma : \Delta) \in W^{\Xi}$, $A \to_{\mathtt{i}} B \in \Gamma$ implies $A \in \Sigma$ or $B \in \Pi$ for any $(\Pi : \Sigma) \in W^{\Xi}$ such that $(\Gamma : \Delta) R^{\Xi} (\Pi : \Sigma)$,*
8. *The right rule for "$\to_{\mathtt{i}}$" is $\Xi$-admissible in* **GL** *iff for any $(\Gamma : \Delta) \in W^{\Xi}$, $A \to_{\mathtt{i}} B \in \Delta$ implies $A \in \Pi$ and $B \in \Sigma$ for some $(\Pi : \Sigma) \in W^{\Xi}$ such that $(\Gamma : \Delta) R^{\Xi} (\Pi : \Sigma)$,*
9. *The rule for "$\bot$" is $\Xi$-admissible in* **GL** *iff $\bot \notin \Gamma$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
10. *The rule $(Cut)$ is $\Xi$-admissible in* **GL** *iff $A \in \Xi$ implies $A \in \Gamma$ or $A \in \Delta$ for any $(\Gamma : \Delta) \in W^{\Xi}$,*
11. *The rule $(Cut)^a$ is $\Xi$-admissible in* **GL** *iff $A \in \mathsf{Sub}(\Gamma \cup \Delta)$ implies $A \in \Gamma$ or $A \in \Delta$ for any $(\Gamma : \Delta) \in W^{\Xi}$.*

*Proof.* We show only (8) and (10) here. Fix any sf-closed finite set $\Xi$ of formulas.

(8)($\Rightarrow$) Fix any $(\Gamma : \Delta) \in W^{\Xi}$ and suppose $A \to_{\mathtt{i}} B \in \Delta$. Our goal is to show that there is some $(\Pi : \Sigma) \in W^{\Xi}$ such that $(\Gamma : \Delta) R^{\Xi} (\Pi : \Sigma)$, $A \in \Pi$ and $B \in \Sigma$. Let $\Theta = \{p \mid p \in \Gamma\} \cup \{C \to_{\mathtt{i}} D \mid C \to_{\mathtt{i}} D \in \Gamma\}$. By the first condition of the definition of a $\Xi$-analytically saturated pair (Definition 9), $\Gamma \Rightarrow \Delta, A \rightarrow_{\mathtt{i}} B$ is not $\Xi$-derivable in **GL**. Thus, $\Theta \Rightarrow A \rightarrow_{\mathtt{i}} B$ is also not $\Xi$-derivable in **GL**. By the $\Xi$-admissibility of $(\Rightarrow \rightarrow_{\mathtt{i}})$, $A, \Theta \Rightarrow B$ is not $\Xi$-derivable in **GL**. By Lemma 1, there is $(\Pi : \Sigma) \in W^{\Xi}$ such that $\Theta \cup \{A\} \subseteq \Pi$, $\{B\} \subseteq \Sigma$, and $\Pi \cup \Sigma \subseteq \Xi$. It suffices to show $(\Gamma : \Delta) R^{\Xi} (\Pi : \Sigma)$, but this is ensured by $\Theta \subseteq \Pi$, the construction of $\Theta$, and the definition of $R^{\Xi}$ (Definition 11).

($\Leftarrow$) Let $\Theta$ be a finite set of propositional variables and formulas whose main connective is "$\rightarrow_{\mathtt{i}}$". Suppose $\Theta \Rightarrow A \rightarrow_{\mathtt{i}} B$ is not $\Xi$-derivable in **GL**. Our goal is to show that $A, \Theta \Rightarrow B$ is not $\Xi$-derivable in **GL**. By Lemma 1, there is $(\Gamma : \Delta) \in W^{\Xi}$ such that $\Theta \subseteq \Gamma$, $\{A \rightarrow_{\mathtt{i}} B\} \subseteq \Delta$, and $\Gamma \cup \Delta \subseteq \Xi$. By the assumed semantic condition, there is $(\Pi : \Sigma) \in W^{\Xi}$ such that $(\Gamma : \Delta) R^{\Xi} (\Pi : \Sigma)$, $A \in \Pi$, and $B \in \Sigma$. By the first condition of the definition of a $\Xi$-analytically saturated pair (Definition 9), $A, \Pi \Rightarrow B, \Sigma$ is not $\Xi$-derivable in **GL**. By $\Theta \subseteq \Gamma$, $\Gamma R^{\Xi} \Pi$, and the definition of $R^{\Xi}$ (Definition 11), $\Theta \subseteq \Pi$. Therefore, $A, \Theta \Rightarrow B$ is not $\Xi$-derivable in **GL**.

(10)($\Rightarrow$) Fix any $(\Gamma : \Delta) \in W^{\Xi}$ and any $A \in \Xi$. Our goal is to show $A \in \Gamma$ or $A \in \Delta$. By the first condition of the definition of a $\Xi$-analytically saturated pair (Definition 9), $\Gamma \Rightarrow \Delta$ is not $\Xi$-derivable in **GL**. By the $\Xi$-admissibility of $(Cut)$, either $A, \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, A$ is not $\Xi$-derivable in **GL**. By the second condition of the definition of a $\Xi$-analytically saturated pair (Definition 9), $A \in \Gamma$ or $A \in \Delta$, as desired.

($\Leftarrow$) Suppose $\Gamma \Rightarrow \Delta$ is not $\Xi$-derivable in **GL**. Our goal is to show that either $\Gamma \Rightarrow \Delta, A$ or $A, \Gamma \Rightarrow \Delta$ is not $\Xi$-derivable in **GL**. By Lemma 1, there is $(\Gamma^* : \Delta^*) \in W^{\Xi}$ such that $\Gamma \subseteq \Gamma^*$, $\Delta \subseteq \Delta^*$, and $\Gamma^* \cup \Delta^* \subseteq \Xi$. By the assumed semantic condition, $A \in \Delta^*$ or $A \in \Gamma^*$. By the first condition of the definition of a $\Xi$-analytically saturated pair (Definition 9), $\Gamma^* \Rightarrow \Delta^*, A$ or $A, \Gamma^* \Rightarrow \Delta^*$ is not $\Xi$-derivable in **GL**. Since $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$, either $\Gamma \Rightarrow \Delta, A$ or $A, \Gamma \Rightarrow \Delta$ is not $\Xi$-derivable in **GL**.

This theorem shows that the two points of improvement of Restall's analysis are overcome. Firstly, in this theorem, the equivalence between the admissibility of an inference rule in **GL** and the corresponding satisfaction relation for a formula is shown. Secondly, the admissibility of $(Cut)$ or $(Cut)^a$ is not presupposed in this analysis, and the semantic conditions corresponding to the admissibility of $(Cut)$ and $(Cut)^a$ are identified.

**Theorem 2.** *Let $\Xi$ be an sf-closed finite set of formulas and $W^{\Xi}_{\mathbf{C+J}}$ be the set of all $\Xi$-analytically saturated pairs in $\mathsf{G}(\mathbf{C}+\mathbf{J})$. Then, for any $(\Gamma : \Delta) \in W^{\Xi}_{\mathbf{C+J}}$ and any formula $C \in \Xi$, the following holds:*

$$C \in \Gamma \textit{ implies } (\Gamma : \Delta) \models C \textit{ and } C \in \Delta \textit{ implies } (\Gamma : \Delta) \not\models C.$$

Theorem 2 is shown by induction on the construction of a formula $C$, as is done in [48]. This theorem corresponds to a lemma called "partial truth lemma" (cf. [27, Lemma 11]), which is established to show cut-free semantic completeness.

Theorem 2 ensures that Restall's analysis, improved by Takano's method, is carried out successfully for $\mathbf{C}+\mathbf{J}$. This is because the formulas in the antecedent and succedent of a $\Xi$-analytically saturated pair can be regarded as true and false in the state described by the pair. Thus, as is done in Sect. 2, the notion of a Kripke model is obtained from the admissibility of inference rules in $\mathsf{G}(\mathbf{C}+\mathbf{J})$. This means that the open problem of Restall's analysis is solved positively.

## 5 Analysis of C + J Based on Unilateralism

This section explains how the analysis presented in Sect. 4 is connected to the bilateralism-unilateralism debate and shows that a unilateral approach is also possible for $\mathbf{C}+\mathbf{J}$. As far as the author knows, the bilateralism-unilateralism debate occurs mainly in the field of philosophy of logic and philosophy of language. Bilateralism is the position claiming that two linguistic acts are primitive when the meaning of a formula or a statement is considered, whereas unilateralism is the position claiming that only one linguistic act is primitive. The representatives of unilateralism are Frege [16] and Dummett [13,14], while bilateralism is studied in [3,19,36,41,44].[11] Since the notions of assertion and denial are used, both Restall's analysis and the analysis presented in Sect. 4 are based on bilateralism.[12] In the following, it is argued that an analysis based on unilateralism is also possible for $\mathbf{C}+\mathbf{J}$. Classical negation plays a central role in this analysis.

The most straightforward way to choose unilateralism is to interpret the derivability of a sequent $\Gamma \Rightarrow \Delta$ only by the notion of assertion, as follows: it is incoherent to assert all the formulas in $\Gamma$ but to assert no formulas in $\Delta$. However, as was pointed out by Restall [38, pp. 4-5], this interpretation contains a too strong requirement.[13] Generally, we do not know every consequence of the assumptions. Therefore, if we assert some formulas, there is a possibility of not asserting a consequence of the formulas. Thus, when we interpret the derivability of a sequent in terms of linguistic acts, the notion of denial seems necessary.

[11] It is usually said that unilateralism fits intuitionistic logic (cf. [15]) and bilateralism fits classical logic (cf. [41]). The reason why unilateralism seems to fit intuitionistic logic but does not seem to fit classical logic lies in the fact that standard proof-theoretic semantics seems possible for the former but impossible for the latter. The reason why bilateralism seems to fit classical logic lies in the fact that by introducing the notion of denial, proof-theoretic semantics for classical logic seems possible, as Rumfitt [41] did.

[12] It is noted that Steinberger [45] claims that Restall's position is crucially different from the positions of Smiley [44] and Rumfitt [41].

[13] In [38], Restall argues against the following view: if $A$ entails $B$, then it ought to be the case that if you accept $A$, then you accept $B$. If we consider an interpretation of the derivability of $\Gamma \Rightarrow \Delta$ based on this view, we can obtain the following interpretation: it ought to be the case that if you accept all the formulas in $\Gamma$, then you accept some formulas in $\Delta$. It is noted that if this interpretation is employed, the argument described here also works.

Another way of defending unilateralism is to claim that the notion of denial is not primitive, although it is necessary. In other words, it is claimed that the notion of denial is conceptually reduced to that of assertion. The most basic strategy of doing this is to define the denial of a formula as the assertion of the negation of the formula. However, this strategy does not work for every logic, since the denial of a formula and the assertion of the negation of a formula seem different in some logics, as was already pointed out in [38, pp. 2-3].

However, this strategy of defending unilateralism works for classical logic. The reason why this strategy works is the formulation of rules for classical negation in **LK** (cf. Table 1). The rules ($\Rightarrow \neg_{\mathtt{c}}$) and ($\neg_{\mathtt{c}} \Rightarrow$) imply that the denial and the assertion of $\neg_{\mathtt{c}} A$ can be replaced with the assertion and the denial of $A$, respectively. By these rules, we can regard the denial of a formula as the assertion of the classical negation of the formula. This fact implies that advocates of classical logic who defend unilateralism can make use of the notion of denial, since it can be reduced to that of assertion.

On the other hand, this strategy does not work for intuitionistic logic. The reason for this is that the denial of $A$ cannot be replaced with the assertion of $\neg_{\mathtt{i}} A$, since the right rule for the intuitionistic multi-succedent sequent calculus **mLJ** is restricted to the following one:

$$\frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg_{\mathtt{i}} A} \ .$$

Thus, the notion of denial cannot be reduced to that of assertion by using only intuitionistic negation. This implies that advocates of intuitionistic logic who defend unilateralism, such as Dummett [13,14], face a difficulty. Since they cannot use the notion of denial, they have to interpret the derivability of a sequent $\Gamma \Rightarrow \Delta$ only by the notion of assertion, but the resulting interpretation contains a too strong requirement, as noted above.[14]

As noted above, advocates of classical logic who defend unilateralism do not fall into this problem, since they can use the strategy of reducing the notion of denial to that of assertion because of the existence of "$\neg_{\mathtt{c}}$." However, this strategy is possible not only in classical logic but also in $\mathbf{C} + \mathbf{J}$. This implies that advocates of classical logic can view intuitionistic logic based on unilateralism. The rest of this section briefly sketches the unilateral analysis for $\mathbf{C} + \mathbf{J}$ based on this strategy.

**Proposition 1.** *For any formula $A$ and any set $\Gamma \cup \Delta$ of formulas, $\Gamma \Rightarrow \Delta, A$ is derivable in $\mathsf{G}(\mathbf{C} + \mathbf{J})$ iff $\neg_{\mathtt{c}} A, \Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{C} + \mathbf{J})$.*

[14] This problem also holds when another proof theory is considered. For example, if a natural deduction system is considered, an interpretation of the derivability of a formula from a set of assumptions using only the notion of assertion should contain a too strong requirement. Thus, advocates of intuitionistic logic who defend unilateralism should propose an interpretation of the derivability that does not contain a too strong requirement, although it is usually said that unilateralism fits intuitionistic logic and bilateralism fits classical logic, as noted in footnote 11.

The direction from the left to the right of this proposition is shown by applying $(\neg_{\mathtt{c}} \Rightarrow)$. The other direction is shown by induction on the construction of a derivation, as is done in [24, 51].[15] Note that this proposition no longer holds if classical negation is replaced with intuitionistic negation. Based on this equivalence, we can transform $\mathsf{G}(\mathbf{C}+\mathbf{J})$ to a one-sided calculus by transmitting succedent to antecedent.[16] For example, $(Id)$, $(Cut)$, and $(\to_{\mathtt{i}} \Rightarrow)$ are transformed to the following rules, respectively:

$$\frac{}{A, \neg_{\mathtt{c}} A \Rightarrow} \quad \frac{\Gamma, \neg_{\mathtt{c}} A \Rightarrow \quad A, \Gamma \Rightarrow}{\Gamma \Rightarrow} \quad \frac{\Gamma, \neg_{\mathtt{c}} A \Rightarrow \quad B, \Gamma \Rightarrow}{A \to_{\mathtt{i}} B, \Gamma \Rightarrow} \quad .$$

A finite set $\Gamma$ of formulas expresses a coherent situation with respect to assertion if $\Gamma \Rightarrow$ is not derivable in this one-sided calculus. Thus, the derivability of $\Gamma \Rightarrow$ is interpreted as follows: it is incoherent to assert all the formulas in $\Gamma$. The notion of denial does not exist in this interpretation. Accordingly, inference rules in the one-sided calculus are regarded as normative constraints on assertion. In the following, this one-sided calculus is called $\mathsf{GS}(\mathbf{C}+\mathbf{J})$. In order to carry out the analysis presented in Sect. 4, the notion of a subformula should be expanded to that of an *extended subformula*.

**Definition 15.** *The set* $\mathsf{Esub}(A)$ *of all* extended subformulas *of a formula* $A$ *is defined inductively as follows:*

- $\mathsf{Esub}(p) := \{p\}$,
- $\mathsf{Esub}(\bot) := \{\bot\}$,
- $\mathsf{Esub}(A \Box B) := \{A \Box B\} \cup \mathsf{Esub}(A) \cup \mathsf{ESub}(B) (\Box \in \{\wedge, \vee\})$,
- $\mathsf{Esub}(A \to_{\mathtt{i}} B) := \{A \to_{\mathtt{i}} B\} \cup \mathsf{Esub}(\neg_{\mathtt{c}} A) \cup \mathsf{Esub}(B)$,
- $\mathsf{Esub}(\neg_{\mathtt{c}} p) := \{\neg_{\mathtt{c}} p\}$,
- $\mathsf{Esub}(\neg_{\mathtt{c}} \bot) := \{\neg_{\mathtt{c}} \bot\}$,
- $\mathsf{Esub}(\neg_{\mathtt{c}}(A \Box B)) := \{\neg_{\mathtt{c}}(A \Box B)\} \cup \mathsf{Esub}(\neg_{\mathtt{c}} A) \cup \mathsf{ESub}(\neg_{\mathtt{c}} B) (\Box \in \{\wedge, \vee\})$,
- $\mathsf{Esub}(\neg_{\mathtt{c}}(A \to_{\mathtt{i}} B)) := \{\neg_{\mathtt{c}}(A \to_{\mathtt{i}} B)\} \cup \mathsf{Esub}(A) \cup \mathsf{Esub}(\neg_{\mathtt{c}} B)$.

**Definition 16.** *Let* $\Gamma$ *be a finite set of formulas. Then, we define* $\mathsf{ESub}(\Gamma)$ *as the set of all extended subformulas of some formulas in* $\Gamma$. *A set* $\Gamma$ *of formulas is* extended subformula-closed (esf-closed) *if* $\mathsf{ESub}(\Gamma) \subseteq \Gamma$ *and* $\bot \in \Gamma$.

We can define the notion of $\Xi$-derivability in the one-sided calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$ as in Definition 7. Based on this notion and the notion of an extended subformula, we can define the notion of a $\Xi$-analytically saturated set in the one-sided calculus $\mathsf{GS}(\mathbf{C}+\mathbf{J})$, which plays the same role as the notion of a $\Xi$-analytically saturated pair in the bilateral analysis provided in Sect. 4.

[15] The direction from the right to the left of Proposition 1 is the inversion of $(\neg_{\mathtt{c}} \Rightarrow)$. Inversion of rules for logical connectives is shown by induction on the construction of a derivation in [24, Theorem 3.1.1] and [51, Proposition 3.5.4]. Although rules for classical negation are not dealt with in [24, 51], we can apply this induction to show this direction. Note that although the height-preserving inversion is shown in [24, 51], the direction from the right to the left of Proposition 1 is not height-preserving.

[16] Similar transformations are carried out in [2, 29, 46, 51], but one-sided calculi in [2, 29, 46, 51] are obtained by transmitting antecedent to succedent. Thus, the directions of transformation are different.

**Definition 17 ($\Xi$-analytically saturated set).** *Let $\Xi$ be an esf-closed finite set of formulas. A finite set $\Gamma$ of formulas is $\Xi$-*analytically saturated *in the one-sided calculus* $\mathsf{GS}(\mathbf{C}+\mathbf{J})$ *if it satisfies all of the following:*

1. *$\Gamma \Rightarrow$ is not $\Xi$-derivable in* $\mathsf{GS}(\mathbf{C}+\mathbf{J})$.
2. *For any formula $A \in \Xi$, if $A, \Gamma \Rightarrow$ is not $\Xi$-derivable in* $\mathsf{GS}(\mathbf{C}+\mathbf{J})$*, $A \in \Gamma$,*

By this definition, the unilateral analysis becomes available by almost the same method as presented in Sect. 4, employing the one-sided calculus $\mathsf{GS}(\mathbf{C}+\mathbf{J})$.

Once $\mathbf{C}+\mathbf{J}$ is explained, this result may not be very surprising, because in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, the left and right rules for classical negation are formulated as in **LK**. However, it is far from trivial that this unilateral approach is possible not only for ordinary classical logic but also for a combination of intuitionistic and classical logic, since this approach is impossible for ordinary intuitionistic logic. This result implies that advocates of classical logic can obtain the meaning of not only formulas in classical logic but also formulas in intuitionistic logic, based on unilateralism. This is because, once classical negation is accepted, the analysis of $\mathbf{C}+\mathbf{J}$ is possible independently of the choice between bilateralism and unilateralism. On the other hand, advocates of intuitionistic logic cannot carry out the analysis for $\mathbf{C}+\mathbf{J}$ based on unilateralism. Thus, they have to accept classical negation in some way if they intend to give a unilateral analysis of the meaning of a formula. Finally, it should be noted that it is not ensured that this unilateral analysis is possible for another combination of intuitionistic and classical logic.

**Acknowledgment.** This paper is based on a discussion with Katsuhiko Sano (Hokkaido University), for which I thank him. I also thank an anonymous referee for giving very helpful comments. Shunsuke Yatabe (Kyoto University) asked very interesting questions and gave very helpful comments at LENLS19, and Koji Mineshima (Keio University) informed me about [52]. Moreover, Takuro Onishi (Kyoto University) asked very interesting questions and gave very helpful comments at the 55th Annual Meeting of Philosophy of Science Society, Japan. In spite of their help, I take full responsibility for the content of this paper. This research is partially supported by Grant-in-Aid for JSPS Fellows (Grant Number JP22J20341).

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