



On the Graph Theory of Majority Illusions

Maaïke Venema-Los¹, Zoé Christoff¹, and Davide Grossi^{1,2}

¹ University of Groningen, Groningen, The Netherlands

{m.d.los,z.l.christoff,d.grossi}@rug.nl

² University of Amsterdam, Amsterdam, The Netherlands

Abstract. The popularity of an opinion in one's direct circles is not necessarily a good indicator of its popularity in one's entire community. For instance, when confronted with a majority of opposing opinions in one's circles, one might get the impression that one belongs to a minority. From this perspective, network structure makes local information about global properties of the group potentially inaccurate. However, the way a social network is wired also determines what kind of information distortion can actually occur. In this paper, we discuss which classes of networks allow for a majority of agents to have the wrong impression about what the majority opinion is, that is, to be in a 'majority illusion'.

Keywords: majority illusion · social networks · graph colorings

1 Introduction

When making decisions, people often use information from the decisions of others in their circles and are influenced by those. For instance, if a lot of people around you buy a specific brand, vote for a specific political party, or have a specific opinion, you are more likely to buy, vote, or think the same (see e.g. [14, 17]).

However, one's view of 'the world around' might be distorted by the social network one is in, and one might as a result misrepresent one's situation with respect to the overall population. A well-known example of this is the so-called 'friendship paradox' [9]: agents in a network are likely to get the impression that their popularity is lower than average because their friends have more friends than they do. Similarly, on the basis of what they can see of others around them, agents might get the wrong impression with respect to how popular their opinions are in the entire population. Indeed, the proportions of opinions an agent observes in its neighborhood are not necessarily a representative sample of their overall distribution in the population.

From this local sampling, one could for instance get the impression that they disagree with the majority of the population on a particular opinion and get influenced by this impression when taking their decisions. As a consequence, one could in principle influence what people will decide by changing the network structure to tweak the distribution of opinions/behavior agents see locally. Figure 1 (a), (b), and (c) illustrate this: rewiring a few edges is sufficient to make all nodes observe a different majority in their neighborhood.

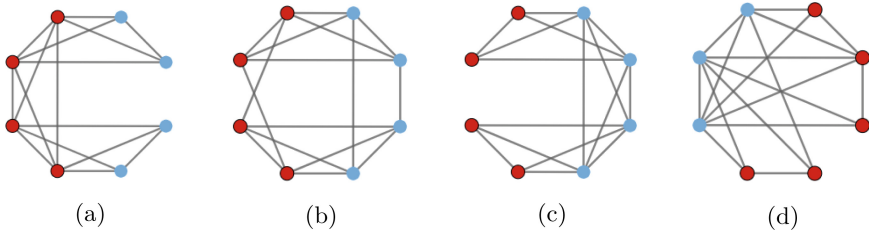


Fig. 1. (a), (b), and (c) have the same proportion of blue and red nodes but nodes see different distributions in their neighborhood: in (a), all nodes have a majority of red neighbors; changing two edges results in all nodes seeing a tie (b); changing two more edges makes nodes have a majority of blue neighbors (c). (d) is an example of a majority-majority-illusion: a majority (the red nodes) have more blue than red neighbors, while red is the actual global ‘winner’. (Color figure online)

A paradigmatic example is polling bandwagon effects [19] in political decision making: if, for any reason, citizens prefer not to vote on a losing party, a party could increase the chances of *actually* winning by making a lot of voters *think* it is winning. The idea of manipulating the network towards this end, so-called ‘information gerrymandering’, is introduced in [4, 20] and shown to potentially lead to undemocratic decisions. Similar phenomena have been observed in a variety of social networks [15].

In this paper, we examine which networks allow for which types of distortion between local and global opinion representations. We focus on a specific type of network distortion, ‘majority illusion’ as introduced by [15], where an agent observes that more than half its neighbors are in a certain state, while in the total network, less than half of the agents are in this state. An example is shown in Fig. 1d: if all nodes were to believe that their neighborhood was representative of the entire network, the red nodes would believe that the majority of nodes in the network is blue, while in reality it is red. Whether it is possible for such illusions to exist, and if so, for how many agents, depends on the network structure. On the one hand, not all networks allow for most agents to be under this type of majority illusions. On the other hand, there exist network structures in which even all agents can be under a majority illusion. In this paper we focus on the possibility of a *majority* of agents being in a majority illusion. The network in Fig. 1d is an example of such a ‘majority-majority illusion’.

Related Work. The concept of majority illusion was first introduced in [15], to show how network structures can distort individual observations. Computational simulations are used to study to which extent the majority illusion can occur in scale-free, Erdős-Rényi networks, and several real-world networks, and show that networks in which high-degree nodes tend to connect to low-degree nodes are most susceptible to this illusion.

In [20] a voter game is modeled with two competing parties to show that majority illusions can be used for the purposes of ‘information gerrymandering’,

that is, influencing people’s votes by misrepresenting the opinion of the group to them. The authors predict this by a mathematical model and confirm their results with a social network experiment with human participants. They find that information gerrymandering can even take place when all agents have the same amount of influence (the same degree).

In [10], the computational complexity of majority illusions is studied. A q -majority illusion is defined, where at least a q fraction of agents are under majority illusion, and it is shown that the problem of deciding whether or not a given network can be colored into a q -majority illusion is NP-complete for $q > \frac{1}{2}$. Whether it also holds for $q \leq \frac{1}{2}$ is left as an open question. Since majority illusions can in some situations have detrimental effects and are generally regarded as undesirable, they also study the problem of eliminating an illusion from a network by adding or deleting edges. The problem to identify whether it is possible to change the network in such a way that the number of agents with a q -majority illusion is below a given bound is also shown to be NP-complete for $q > \frac{1}{2}$.

Contributions. We study the possibility of majority illusion in its weak and strict versions, in different classes of graphs. Section 2 introduces our framework, definitions and terminology. In Sect. 3, we prove that a weak version of the majority illusion can occur on *all* network structures. In Sect. 4, we provide some stronger results on specific classes of networks: graphs with odd degrees, properly 2-colorable graphs, and regular graphs. Table 2 gives a summary of which graphs allow for which type of majority illusions.

2 Preliminaries

Binary Opinion Networks and 2-Colored Graphs. A social network (a simple graph) $G = \langle V, E \rangle$ consists of a finite set V agents (nodes/vertices), and a set E of (undirected irreflexive) edges between agents. If two agents are connected by an edge, we call them neighbors. We assume that no agent is a neighbor of itself. We write N_i for the set of neighbors of i and d_i for its degree $|N_i|$. Each agent holds a binary opinion on a single issue. Since binary single-issue opinion distributions can be seen as graph 2-colorings, we will borrow the terminology of vertex colorings, and use the terms ‘color’ and ‘opinion’ interchangeably. We write c_i to refer to agent i ’s color and c for the 2-coloring of the graph ($c : V \rightarrow \{red, blue\}$). A 2-colored graph is a triple $C = \langle V, E, c \rangle$. Throughout the paper, the term ‘colored graph’ refers to such 2-colored graph.

Majority Illusion and Opposition: Intuitions. In such opinion networks (or 2-colored graphs), we are concerned with three types of information: individual opinions, local majority opinions, and global majority opinions. Any two of these three types of opinions can be in agreement or not. We systematize and illustrate all possible relations between the above types of opinions in Table 1.

Different fields have been studying disagreement between the different types of information mentioned above. On the one hand, in social network science and

social choice theory, an agent is under *majority illusion* when its neighborhood majority disagrees with the global majority [10, 15]. On the other hand, graph theory has concerned itself with the disagreement between a node’s color and the color of its neighbors: a proper coloring requires that no two adjacent nodes are of the same color, that is, that everybody disagrees with all of their neighbors. A generalisation of that concept is that of majority-coloring [1, 5, 13], where no agent agrees with most of its neighbors. We call the local disagreement faced by an agent in a majority coloring *majority opposition*. In such a situation, one might have the impression that they belong to a global minority. For instance, in Fig. 1d all nodes might have this impression, while it is only true for the blue ones. When all agents are under this impression, then some of them must be mistaken, it must be some sort of illusion. Clearly, the two concepts of majority illusion and opposition are related. In this paper, we explore this relation to get results about majority illusions.

Formal Definitions. We start by introducing some notions to be able to talk about which opinion is prevalent in a network, be it locally or globally. Given a set S of agents such that $|S| = n$ and a coloring c , a color x is a *majority winner* of S (we write $M_S = x$) if $|\{i \in S \mid c_i = x\}| > \frac{n}{2}$. When neither color is a majority winner among S (there is a tie), we will write $M_S = \text{tie}$. We say that an agent is under *majority illusion* if both the agent’s neighbourhood and the entire network have a majority winner (no tie) but they are different. This definition is equivalent to that in [10].

Definition 1 (majority illusion). Given a colored graph $C = \langle V, E, c \rangle$, an agent $i \in V$ is *under majority illusion* if $M_{N_i} \neq \text{tie}$ and $M_V \neq \text{tie}$ and $M_{N_i} \neq M_V$. A graph is in a *majority-majority illusion* if more than half of agents are under majority illusion.










We can generalise this strict definition to weaker cases. First, there exists a weaker type of disagreement between local and global majorities: the cases where exactly one of the two is a tie. Second, the majority of agents under an illusion can also be weak, when exactly half of the agents are under illusion. The corresponding generalisations of majority illusion includes both types of weakening:

Definition 2 (weak versions of majority illusion). Given a colored graph $C = \langle V, E, c \rangle$, agent $i \in V$ is under *weak-majority illusion* if $M_{N_i} \neq M_V$. A graph is in a *majority-weak-majority illusion* if more than half of the agents are under weak-majority illusion. A graph is in a *weak-majority-(weak-)majority illusion* if at least half of the agents are under (weak-)majority illusion.

As indicated in Table 1, while it is the strict version of the majority illusion that is studied by [15]¹, and by [10], it is the weak version of majority opposition

¹ They use the strict version throughout the paper, except for Fig. 1 where the illusion can be weak.

Table 1. Possible combinations of local and global majority winners, and presence or absence of majority opposition and majority illusion. We assume w.l.o.g. that the color of the relevant individual (highlighted in the exemplary illustrations) is red, otherwise just swap ‘red’ and ‘blue’ everywhere. \times indicates absence of the opposition/illusion, \checkmark indicates presence of the opposition/illusion, ‘weak’ indicates the presence of a weak-majority opposition or a weak-majority illusion.

	local majority	majority opposition	global majority	majority illusion
	red	\times	red	\times
	red	\times	tie	weak
	red	\times	blue	\checkmark [15,10]
	tie	weak [1,5,13]	red	weak
	tie	weak [1,5,13]	tie	\times
	tie	weak [1,5,13]	blue	weak
	blue	\checkmark	red	\checkmark [15,10]
	blue	\checkmark	tie	weak
	blue	\checkmark	blue	\times

that is studied by [1, 5], and [13]. As far as we know, the strict majority opposition and the weak majority illusions have not been studied before. Furthermore, note that, in the same network, different agents can be under a weak-majority illusion with respect to different opinions, since it is possible that exactly half of the nodes in the network are of one color and half of the nodes of the other color.

Before proceeding, we introduce some extra terminology. When an agent is under majority illusion and all its neighbors all have the same color, we say that agent is in *unanimity illusion*. Similarly, when all agents are under a (weak)-majority illusion, we will call it a *unanimity-(weak-)majority illusion*. We say that *an illusion is possible* for a graph $G = \langle V, E \rangle$ if there exists a coloring c such that $C = \langle V, E, c \rangle$ is in the respective illusion.

3 Illusions in Arbitrary Networks

Our overall goal is to discover which social networks allow for majority illusions to occur. Since this is equivalent to asking which graphs can be colored in some specific way, we build on existing results from vertex colorings to obtain results about majority illusions. Recall that a coloring is called *proper* when no two neighbors are assigned the same color. The weaker notion of majority coloring [1, 13] is immediately relevant to us. In a majority coloring, each vertex is in what we described as majority opposition: at least half of its neighbors are of a different color than its own. For coherence with the rest of the paper, we call this a *weak majority coloring* here:

Definition 3 (weak majority 2-coloring). A weak majority 2-coloring of a graph $G = \langle V, E \rangle$ is a 2-coloring c such that, for each $i \in V$: $M_{N_i} \neq c(i)$.

A graph is called weak majority 2-colorable if there exists a weak majority 2-coloring of it. Given a colored graph, we call *monochromatic* the edges between nodes of the same color, and *dichromatic* the ones between nodes of different colors.

Remark 1. The main result involving majority colorings is credited to [16] in the literature [1, 5]: every graph is weak majority 2-colorable. The proof strategy for this result is commonly described as easy and relying on a simple ‘color swapping mechanism’ that can only reduce the total number of monochromatic edges in the network. However, [16] itself focuses on multigraphs and is of a much wider scope. Therefore, to make the paper self-contained, we provide both a proof of the general result in Appendix 1 and below a proof of the related lemma, crucial to our main result, Theorem 1.

Lemma 1. *Let $G = \langle V, E \rangle$ be a graph, and let c be a 2-coloring of G that minimizes the number of monochromatic edges. Then, c is a weak majority 2-coloring of G .*

Proof. Let E_M be the set of monochromatic edges and $E_D = E \setminus E_M$ the set of dichromatic edges in graph G colored by c . Assume for contradiction that there is a node $i \in V$ that is an endpoint of strictly more monochromatic edges (we write E_{M_i} for the set of such edges) than dichromatic edges (E_{D_i}): $|E_{M_i}| > |E_{D_i}|$. Consider now a second 2-coloring c' of G that only differs from c with respect to i 's color, i.e., c' assigns the same color as c did to all nodes except for i : $c_i \neq c'_i$. Let us write E'_M for the new set of monochromatic edges, and E'_{M_i} and E'_{D_i} for the new sets of monochromatic and dichromatic edges from i . Given that $|E'_{M_i}| = |E_{D_i}|$ and $|E'_{D_i}| = |E_{M_i}|$, we now have $|E'_{D_i}| > |E'_{M_i}|$ and $|E_{M_i}| > |E'_{M_i}|$. Given that no other edge of the graph is affected by this change, the total number of monochromatic edges is smaller with coloring c' than it was with c : $|E_M| > |E'_M|$. But since we started by assuming that c was such that $|E_M|$ was minimal, this is a contradiction. \square

We now use the existence of such a majority coloring to prove the following general result:

Theorem 1. *For any graph $G = \langle V, E \rangle$, a majority-weak-majority illusion is possible.*

Proof. Let $G = \langle V, E \rangle$ be a graph and let c be a 2-coloring of G that minimizes the total number of monochromatic edges. By Lemma 1, c is a weak majority 2-coloring of G . Two cases:

- $M_V \neq \text{tie}$. Assume w.l.o.g. that $M_V = \text{red}$. Since c is a weak majority coloring, for any red vertex i , $M_{N_i} \in \{\text{blue}, \text{tie}\}$, and therefore $M_{N_i} \neq M_V$. Hence, a majority of the nodes (the red ones) is under (possibly weak) majority illusion: we have a majority-weak-majority illusion.
- $M_V = \text{tie}$. Two cases:
 - If $|\{i \in V : M_{N_i} \in \{\text{blue}, \text{red}\}\}| > \frac{|V|}{2}$, we have a majority-weak-majority illusion.
 - Otherwise (if $|\{i \in V : M_{N_i} = \text{tie}\}| \geq \frac{|V|}{2}$) choose a node j with $M_{N_j} = \text{tie}$ and define a new coloring c' that is equal to c for all nodes except for j : $c'_j \neq c_j$. Since j has as many blue as red neighbors, this does not change the total number of monochromatic edges in the graph. Therefore, c' is also a coloring that minimizes this number. Hence, by Lemma 1, c' is also a weak majority 2-coloring of G . Now, we have $M_V = c'_j$, and we can apply the logic of the first case: Assume w.l.o.g. that $c'_j = \text{red}$. Since c' is a weak majority coloring, for any red vertex i , $M_{N_i} \in \{\text{blue}, \text{tie}\}$. It follows that a majority of the nodes has $M_{N_i} \neq M_V$: we have a majority-weak-majority illusion. \square

One of the results in [10] is that checking whether or not a network allows for a *majority-majority* illusion is NP-complete². Here, in stark contrast, we see that there is no need for checking whether a network allows for a *majority-weak-majority* illusion, since Theorem 1 shows that it is always the case.

4 Illusions in Specific Network Classes

While the above solves the question of the existence of *weak* majority illusions, we now aim to understand when the *strict* version of the illusion can occur. In order to obtain results in that direction, we turn to some classes of graphs with well-known global properties. Note that the results from this section are not intended to describe realistic classes of social networks but can instead be seen as a starting point for the systematic analysis of the types of graphs that allow for majority illusions. We focus on graphs with only odd-degree nodes, properly 2-colorable graphs, and regular graphs.

² [10] does not actually speak about a majority-majority illusion, but about ‘at least a fraction of q agents is under majority illusion’, with $q > \frac{1}{2}$. The fact that it then also holds for majority-majority illusion follows from that ‘more than half’ is equivalent to ‘at least some fraction q where q is more than half’ since we only have to consider rational numbers.

4.1 Graphs with Odd Degrees

Theorem 2. *For any graph $G = \langle V, E \rangle$ such that for all $i \in V$, d_i is odd, a majority-weak-majority illusion is either a unanimity-weak-majority illusion or a majority-majority illusion.*

Proof. Let $G = \langle V, E \rangle$ be such that for all $i \in V$, d_i is odd. By Theorem 1, there exists a coloring of G that induces a majority-weak-majority illusion. Consider any such coloring c . Two cases:

- $M_V = tie$. For all $i \in V$, since d_i is odd, $M_{N_i} \neq tie$ and therefore $M_{N_i} \neq M_V$: we have unanimity-weak-majority illusion.
- $M_V \neq tie$. Assume w.l.o.g. that $M_V = red$. Since G is in a majority-weak-majority illusion, $|\{i \in V : M_{N_i} \in \{blue, tie\}\}| > \frac{|V|}{2}$, but since for all i , d_i is odd, this implies that all those vertices cannot have a tie: we have $|\{i \in V : M_{N_i} = blue\}| > \frac{|V|}{2}$, a majority-majority illusion.

□

The intuition is that an agent with odd degree cannot see a tie in its neighborhood, which causes either *all* agents to be in weak-majority illusion if there is a global tie, or, if there is no global tie, a majority of agents to be in a majority illusion.

Given a graph coloring we can define a ‘swappable node’ as a node whose neighbors all have at least 2 (so 3 for odd degree) more nodes of one color than nodes of the other color. Then, a corollary of Theorem 2 is the following:

Corollary 1. *For a graph $G = \langle V, E \rangle$ such that for all $i \in V$, d_i is odd, if the coloring c witnessing that majority-weak-majority illusion is possible induces that $M_v = tie$ and that there is at least one $j \in V$ that is ‘swappable’, a weak-majority-majority illusion is possible.*

Proof. W.l.o.g. assume $c_j = red$ and define c' which is equal to c for all nodes except that $c'_j = blue$. Since c was a majority 2-coloring, all red nodes had more than half blue neighbors. Since j 's neighbors all had a margin of at least 2 and nothing except j 's color changed, all red nodes except j still have more than half blue neighbors in c' . Hence, half of the nodes are under majority illusion. □

4.2 2-Colorable Graphs

In the same way as we used the existence of a majority coloring to obtain results about the existence of majority illusions we can also use the existence of a special type of weak majority colorings, the proper colorings, to obtain results about majority illusions in 2-colorable graphs.

Lemma 2. *Any proper 2-coloring of a graph $G = \langle V, E \rangle$ is either a majority-majority illusion or a unanimity-weak-majority illusion.*

The idea of the proof is similar to that of Theorem 2: no node can see a tie among its neighbors.

Proof. Let c be a proper 2-coloring of G . Two cases:

- If $M_V \neq tie$, then w.l.o.g. assume that $M_V = red$. Since more than half the nodes are red and all red nodes have a majority of blue neighbors, we have a majority-majority illusion.
- If $M_V = tie$, then all the nodes are under weak-majority illusion, since for all nodes, all neighbors are the other color. We have a unanimity-weak-majority illusion.

□

Both cases used in the above proof are cases of majority-weak-majority illusions (which were already guaranteed to exist by 1), but we can also show the existence of the strict majority illusion in two different cases. First, when the number of nodes is odd, there cannot be a global tie, so by using the first case in Lemma 2 we get the following proposition :

Proposition 1. *For any properly 2-colorable graph $G = \langle V, E \rangle$ with $|V|$ odd, a majority-majority illusion is possible.*

Proof. Let c be a proper 2-coloring of G . Since $|V|$ is odd, $M_V \in \{red, blue\}$, we can use the first case of the proof of Lemma 2: W.l.o.g. assume $M_V = red$. Since more than half of the nodes are red and all red nodes have only blue neighbors (since c was a proper 2-coloring), we have a majority-majority-illusion. □

Second, when the color of a node can be swapped if needed, we can solve a tie:

Proposition 2. *For any properly 2-colorable graph $G = \langle V, E \rangle$ with some $i \in V$ such that for all $j \in N_i$ $d_j > 2$, a weak-majority-majority illusion is possible.*

Proof. Let c be a proper coloring of G . Two cases:

- If $M_V \neq tie$, then conformingly to Lemma 2, we have a majority-majority illusion.
- If $M_V = tie$, then swap the color of node i : let c' assign the same colors as c to all other nodes but $c'_i \neq c_i$. Now $M'_V = c'i$. W.l.o.g. say $c_i = blue$ and $c'_i = red$. All of i 's neighbors are also red and have now exactly one red neighbor (i), and more than one blue neighbor. Therefore, all red nodes except for i have more than half of their neighbors blue. Therefore, exactly $\frac{|V|}{2}$ of the nodes are in a situation of majority illusion. □

In [10], the complexity of checking whether a network admits (what we call) a weak-majority-majority illusion was left as an open problem. Propositions 1 and 2 show that by checking whether a graph is properly 2-colorable (which can be done in polynomial time [6]) and whether there exists a node whose neighbors all have degree larger than 2 or whether the number of nodes is odd, we can know that a graph admits a (weak-)majority-majority illusion. Still this does not give us the complexity of checking whether a network admits a weak-majority-majority-illusion: while this is a sufficient condition for a graph to allow for a (weak-)majority-majority illusion, it is not a necessary condition.

4.3 Regular Graphs

In [2], theoretical analysis and experiments where human subjects were asked to perform estimation tasks are used to study the influence of network structure on the wisdom of crowds. The authors find a remarkable difference between centralized networks, where the degree distribution varies a lot between nodes, and decentralized (regular) networks, in which all nodes have the same degree, regarding what social influence does to the accuracy of the estimates of individuals (when individual's estimates are based on a weighted average of their own belief and the beliefs of their neighbors). They show that in decentralized networks, social influence significantly improves individual accuracy and the group mean estimate. Furthermore, an overview of research about collective intelligence by Centola [7] mentions several studies about decentralized networks in practical applications: in decentralized networks, political polarization and biases about climate change and immigration are reduced [3,11], and social influence reduced biases about the risk of smoking [12], as well as (implicit) race and gender biases in clinical settings [8]. Since decentralized/regular networks seem to be beneficial for group accuracy and bias reduction, we wonder whether they also are 'good networks' in terms of the distortion we study: to what extent they allow for majority illusions. According to [15], differences between the degrees of nodes and their neighbors are one of the main factors enabling majority illusion. Therefore, one would expect that regular networks, where all nodes have the same degree, make majority illusions less likely. Nevertheless, we show that majority illusions (beyond the ones given by Theorem 1) are also possible in regular networks.

A k -regular network is a network in which all nodes have degree k . We start by considering the simplest cases of regular network: simple cycles, where $k = 2$, and complete networks, where $k = |V| - 1$.

Proposition 3 (simple cycles). *For any 2-regular graph $G = \langle V, E \rangle$, a (weak-)majority-majority illusion is not possible.*

Proof. Let $G = \langle V, E \rangle$ be any 2-regular graph. A node can only be under majority illusion if both of its neighbours are of the minority color. Every minority-colored node can serve as a neighbour for at most two nodes. Thus, to give at least half of the nodes a majority illusion, there must be at least $\frac{|V|}{2}$ nodes in the minority color, which is a contradiction with being a strict minority. \square

However, a majority-weak-majority illusion is possible, according to Theorem 1, and it is easy to find one (which we leave as an exercise to the reader).

Proposition 4 (weak-majority-majority illusion in complete graphs). *For any complete graph $G = \langle V, E \rangle$ (i.e. a k -regular graph with $k = |V| - 1$), a (weak-)majority-majority illusion is not possible.*

Proof. This is a corollary of Proposition 8 in Appendix 2. If $|V| = n$, according to Proposition 8, a weak- p - q illusion can occur iff there is an integer x such that $q(n-1) < x < qn$ and $n-x \geq pn$. This implies that a majority-weak-majority

illusion can occur iff there is an integer x such that $\frac{n-1}{2} < x < \frac{n}{2}$ and $n-x \geq \frac{n}{2}$. Clearly, there is no x that satisfies the first requirement. \square

We know (by Theorem 1) that a majority-*weak*-majority illusion is always possible on complete graphs too. We can go further and specify the types of colorings under which these graphs are in such an illusion.

Proposition 5 (majority-weak-majority illusion in complete graphs).

A complete 2-colored graph $G = \langle V, E, c \rangle$ is under majority-weak-majority illusion if and only if:

- *the difference in numbers of nodes of each color is one; or*
- *the number of nodes of each color is equal, then it is under unanimity-weak-majority illusion.*

Proof. If the difference in numbers of nodes of each color is one, assume w.l.o.g. that $M_V = \text{red}$. Then for all red nodes r , $M_r = \text{tie}$, so we have a majority-weak-majority illusion. If the number of nodes of each color is equal, we have $M_V = \text{tie}$ but every node will observe a majority of the other color: we have a unanimity-weak-majority illusion. \square

We return to the analysis of general regular graphs. The number of minority-colored neighbours needed for an illusion gives a restriction on the possible values of k depending on $|V|$:

Proposition 6 ((weak-)majority-majority illusion in k -regular graphs).

Let $G = \langle V, E \rangle$ be a k -regular graph with $|V| = n$. If a (weak-)majority-majority illusion is possible on G , then n and k must satisfy:

- $k \leq n - 4$ if n and k are even;
- $k \leq n - 3$ if one of n and k is even and one is odd.

Proof. This is a direct corollary of Proposition 9 in Appendix 2. \square

Example 1. Consider a k -regular graph $G = \langle V, E \rangle$ with $|V| = 6$ and $k = 4$. For any node to be in a majority illusion, at least 3 of its neighbours have to have a different color than the global majority color. Assume that the global majority color is red. Then there are at least 4 red nodes and therefore only 2 nodes can be blue. Therefore, no node can have 3 or more blue neighbours.

The number of available edges of the minority color brings another requirement on the relative values of $|V|$ and k for the strictest version.

Proposition 7 (majority-majority illusion in k -regular graphs).

Let $G = \langle V, E \rangle$ be a k -regular graph with $|V| = n$. If a majority-majority illusion is possible on G , then n and k must satisfy:

- $n \geq \frac{2(3k+2)}{k-2}$ (assuming $k \geq 3$) if n and k are even;
- $n \geq \frac{2(3k+1)}{k-1}$ (assuming $k \geq 2$) if n is even and k is odd;
- $n \geq \frac{3k+2}{k-2}$ (assuming $k \geq 3$) if k is even and n is odd.

Proof. If G is in a majority-majority illusion, there are more than half of the nodes of one color. W.l.o.g., assume that this majority color is red, and that the minority color is blue.

- When n and k both are even, in order for a majority-majority illusion to exist, at least $\frac{n}{2} + 1$ nodes have to be red. Nodes with an illusion have to have at least $\frac{k+2}{2}$ blue neighbours. Then, there have to be at least $\frac{n}{2} + 1$ such nodes with an illusion. Thus there have to be at least $\frac{k+2}{2}(\frac{n}{2} + 1) = \frac{(k+2)(n+2)}{4}$ edges to a blue node. Hence, there must be at least $\frac{(k+2)(n+2)}{4k}$ blue nodes because every blue node can have at most k edges. Since at least $\frac{n}{2} + 1$ nodes were red, there are at most $\frac{n}{2} - 1$ left over to be blue, so this means that $\frac{(k+2)(n+2)}{4k}$ must be at most $\frac{n}{2} - 1$. This is equivalent to $n \geq \frac{2(3k+2)}{k-2}$ assuming that $k > 2$;
- When n is even and k odd, in order for a majority-majority illusion to exist, at least $\frac{n}{2} + 1$ nodes have to be red. Nodes with an illusion have to have at least $\frac{k+1}{2}$ blue neighbours. Then, there have to be at least $\frac{n}{2} + 1$ such nodes with an illusion. Thus there have to be at least $\frac{k+1}{2}(\frac{n}{2} + 1) = \frac{(k+1)(n+2)}{4}$ edges to a blue node. Hence, there must be at least $\frac{(k+1)(n+2)}{4k}$ blue nodes because every blue node can have at most k edges. Since at least $\frac{n}{2} + 1$ nodes were red, there are at most $\frac{n}{2} - 1$ left over to be blue, so this means that $\frac{(k+1)(n+2)}{4k}$ must be at most $\frac{n}{2} - 1$. This is equivalent to $n + 2 \leq k(n - 6)$, which means $n \geq \frac{2(3k+1)}{k-1}$ assuming that $k > 1$;
- When k is even and n odd, in order for a majority-majority illusion to exist, at least $\frac{n+1}{2}$ nodes have to be red. Nodes with an illusion have to have at least $\frac{k+2}{2}$ blue neighbours. Then, there have to be at least $\frac{n+1}{2}$ such nodes with an illusion. Thus there have to be at least $\frac{k+2}{2} \cdot \frac{n+1}{2}$ edges to a blue node. Hence, there must be at least $\frac{(k+2)(n+1)}{4k}$ blue nodes. Since at least $\frac{n+1}{2}$ nodes were red, there are at most $\frac{n-1}{2}$ left over to be blue, so this means that $\frac{(k+2)(n+1)}{4k}$ must be at most $\frac{n-1}{2}$. This is equivalent to $n \geq \frac{3k+2}{k-2}$ assuming that $k > 2$.

□

Example 2. Consider a k -regular network with $|V| = 6$ and $k = 3$. Let us assume that red is the global majority color, so we have at least 4 red nodes and at most 2 blue nodes. Then any node with a majority illusion has at least 2 blue neighbours. Since for a majority-majority illusion there have to be at least 4 nodes with an illusion, there are at least $4 \cdot 2 = 8$ edges to blue nodes necessary. However, since we have at most 2 blue nodes that each have only 3 edges, this is not possible.

For any n and k (with $k > 2$ and n or k even) satisfying the above constraints, we can find a k -regular graph of size n that has a majority-majority illusion. Note that this does not mean that for any k -regular graph of size n we can find a coloring that gives a majority-majority illusion, because there exist many different regular graphs with the same n and k . We only show that, for all

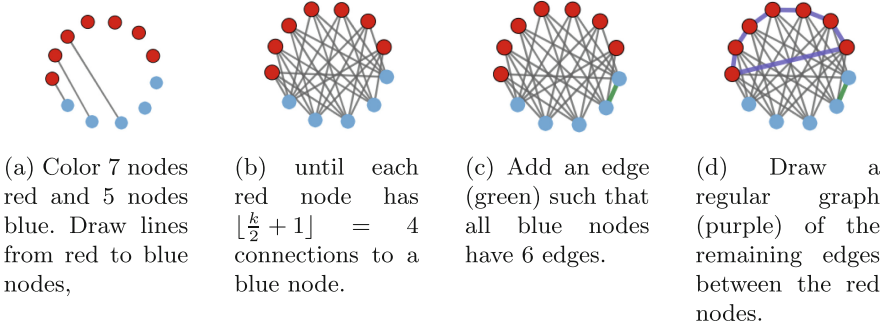


Fig. 2. Example of the algorithm for Theorem 3, with $n = 12, k = 6$.

Table 2. The (im)-possibility of majority illusions on different classes of graphs. \checkmark indicates that the illusion is possible on all graphs of the class, \times indicates that the illusion is not possible on any graph in the class, \checkmark / \times indicates that the illusion is possible on some but not all graphs of the class. For the majority-weak-majority illusion, some stronger results are shown. References to the relevant results are given.

Class of graphs	majority-weak-majority		weak-maj-majority	majority-majority
All graphs				
Graph with only odd-degree nodes	\checkmark (Thm. 1)	majority-majority or unanimity-weak-majority	(Thm. 2) \checkmark / \times	
2-colorable graphs			(Lem. 2) \checkmark (Prop. 1)	
2-colorable graphs with $ V $ odd		\checkmark (Prop. 2)		\checkmark / \times
2-colorable graphs with $i \in V : \forall j \in N_i : d_j > 2$		\times (Prop. 3)		
2-regular graphs	unanimity-weak-majority (Prop. 5)			
Complete graphs with $ V $ even			\times (Prop. 4)	
Complete graphs with $ V $ odd				

combinations of n and k not excluded by our previous results, there exists at least one such graph with the illusion, and that we know how to find it.

Theorem 3. *Let n and k be any two integers such that $k > 2$ and k or n is even. If the requirements of Propositions 6 and 7 are met, there exists a k -regular graph $G = \langle V, E \rangle$ with $|V| = n$ on which a majority-majority illusion is possible.*

Proof sketch. We prove this by construction: we give an algorithm that takes as input n and k and returns a regular graph with n nodes of degree k that has a majority-majority illusion. The algorithm and a proof that the algorithm outputs the desired graph are given in Appendix 3. See Fig. 2 for an example with 12 nodes of degree 6.

Propositions 6 and 7 and Theorem 3 together give necessary and sufficient conditions for n and k for the existence of a k -regular graph with $|V| = n$ nodes on which a majority-majority illusion is possible.

5 Conclusion and Outlook

We studied weak and strong versions of the majority illusion. Using results about majority-colorings, we proved that no network is immune to majority-weak-majority illusion, and that some classes of graphs are not immune to stronger types of illusions either. The results are summarized in Table 2. We also provided an algorithm to find a k -regular graph of size n with a majority-majority illusion, when it exists.

The most natural direction for further research is to broaden the scope of our study: first, as we have initiated with Appendix 2, by considering proportions other than majority; and second, by considering classes of graphs that are more realistic as social networks, including not necessarily irreflexive ones. Moreover, beyond verifying their sheer possibility, we keep measuring the likelihood of such illusions for future work.

A different direction is to investigate the relation between majority illusions and *majority logic* [18], which can ‘talk about’ local majorities, but not about global majority. We propose to enrich the logic with a global majority operator, to express the results from this paper. The idea is elaborated on in Appendix 4.

Last but not least, it would be interesting to measure the impact of proportional illusions on specific social phenomena. For instance, how do they affect opinion diffusion dynamics in a population? How do they interact with polling effects? And how do they relate to better known types of illusions, such as the above-mentioned ‘friendship paradox’ [9]?

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