Lorenzen-Style Strategies as Proof-Search Strategies

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Abstract. Dialogical logic, originated in the work of Lorenzen and his student Lorenz, is an approach to logic in which the validity of a certain formula is defined as the existence of a winning strategy for a particular kind of turn-based two-players games. This paper studies the relationship between winning strategies for Lorenzen-style dialogical games and sequent calculus derivations. We define three different classes of dialogical logic games for the implicational fragment of intuitionistic logic, showing that winning strategies for such games naturally correspond to classes of derivations defined by uniformly restraining the rules of the sequent calculus.

Keywords: Dialogical Logic · Sequent Calculus · Game Semantics

1 Introduction

Dialogical logic is an approach to the study of logical reasoning, introduced by Lorenzen and his student Lorenz [21,22], in which the validity of a formula is defined as the existence of a winning strategy for a turn-based two-player game. These games are articulated as argumentative dialogues in which the *Proponent* player **P** (she/her) aims at showing that a given formula is valid, while the *Opponent* player **O** (he/him) aims at finding possible fallacies disproving it. More precisely, each play starts with **P** asserting a certain formula. **O** takes his turn and attacks the claim made by **P** according to its logical form. The player **P** can, either, defend his previous claim or counter-attack. The debate evolves following this pattern. The player **P** wins whenever she has the last word, i.e., when **O** cannot attack anymore without violating the game's rules.

Dialogical logic was initially conceived as a foundation for the meaning of the connectives and quantifiers of *intuitionistic logic*, and it has gradually become detached from its connection with intuitionism over the years, becoming a subject of research in philosophical logic [5,10,23,28], in the formal semantics of natural language [8,9], in proof theory [3,13,14,17,25,29,30] and inspiring a series of work in formal argumentation theory and multi-agent systems [6,20,24,26,27]. In proof theory, the soundness and completeness of a dialogical system is proved by establishing the equivalence between the existence of a winning strategy in specific games and the notion of validity in a given logic. This result is typically attained by defining a procedure that reconstructs a formal derivation from a winning strategy (and vice versa) in a sound and complete system for a given logic [3,12,13]. In this paper, we study the correspondence between

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certain classes of winning strategies for a given dialogic system and the structure of the corresponding formal derivations in the sequent calculus. We study winning strategies in which **P** moves are restricted according to **O** precedent moves (e.g., if **O** plays a move $A \to B$ as a response to a move of **P** of a special kind, then the **P** has to immediately reply to this move). We prove that for each of the classes of winning strategies we consider, we have a correspondence with a proof-search strategy in the sequent calculus GKi for the $\to$-fragment of intuitionistic logic [33]. This latter result is obtained by showing that it is possible to narrow the proof-search space in sequent calculus without losing the soundness and completeness of the sequent system (as, e.g., in focusing [4]) and that there is a straightforward correspondence between such focused proofs and winning strategies.

This work shows how interesting results on the combinatorics of proofs can be obtained using dialogic logic, whose methods are not as well known as the ones from more widely used proof systems such as analytic tableaux, sequent calculus and natural deduction. In fact, certain intuitive restrictions on the behavior of the players in dialogical games allows us to express proof search strategies allowing us to reduce the proof search space, without requiring convoluted definitions in sequent calculus. The techniques developed in this work pave the way for further investigations on the use of dialogical logic methods in designing proof systems with restricted research space.

Related Work. Various definitions of Lorenzen-style dialogue games have been proposed over the years; the definitions that have a more direct relevance to our work are those of *Felscher's E-dialogues* [12] and *Fermüller's E-dialogues* [3,13]. In an E-dialogue, each **O** move is either a challenge to the immediately preceding **P** move or a defense from it. In Felscher's E-dialogues there are no challenges directed toward atomic formulas, and **P** cannot assert an atomic formula unless **O** has already asserted it. On the contrary, in Fermüller's E-dialogue atomic formulas can be attacked, but only by **O** and both players can freely assert them without restrictions. In our definition, we choose a hybrid approach in which **P** can assert an atomic proposition freely as long as that assertion is a challenge against a previous **O** assertion, and in which **P** cannot assert an atomic proposition as a defense to a previous attack unless that proposition has already been asserted by **O**. The assertion of an atomic proposition can be attacked, but only by **O** and only if that assertion is a challenge.

Herbelin noted the formal correspondence between winning strategies for dialogical games and sequent calculus focusing proofs in his doctoral dissertation [17]. In the fifth chapter of the dissertation, Herbelin shows that winning strategies for E-dialogues (defined by Felscher in [12]), are in bijective correspondence with proofs of the LGQ sequent calculus. Herbelin's technique to transform winning strategies into sequent calculus proofs is very elegant and will be used by us (with slight modifications) to achieve the same result.

Another work that, in spirit, is closer to ours is the one presented by Stitch in [30]. In this work, the author studies a multi-agents variant of dialogical logic games. Such games are turn-based games in which a *coalition* of Proponents plays against an Opponent: when it is their turn, each of the Proponent can make a different move. The play is won by the Proponents if the Opponent cannot react to any of the Proponents's moves of the previous round. Stitch shows that Proponents winning strategies for such games correspond to derivations in a multi-conclusion variant of the already cited LGQ sequent calculus. Plays are formalized by Stitch as paths in a peculiar sequent calculus, and strategies as derivations of this sequent calculus. While there may be some similarities between Stitch's works and ours, it is essential to note the significant differences in the details. We here consider "traditional" dialogical games played by two players, and we obtain the correspondence with restricted sequent calculus proof by restricting the way in which the Proponent can play in a game. Moreover, we show how to transform winning strategies into derivations (and vice versa) directly without the need of defining an ad-hoc sequent calculus formalism.

Outline of the Paper. The paper is organized as follows: in Sect. 2 we state definitions on dialogical logic, defining different classes of plays and strategies. In Sect. 3 we introduce different sequent calculi for intuitionistic logic, obtained by restricting the rules of the sequent calculus GKi [33]. In Sect. 4 we show how to sequentialize winning strategy, that is, how to define a sequent calculus derivation associated to a given winning strategy, and we prove the correspondence between classes of winning strategies and classes of GKi derivations. In Sect. 5 we show the converse. In Sect. 6 we discuss the obtained results and future works.

2 Dialogical Logic

In this section we fix the notation and terminology, as well as the formal definitions on dialogical logic we use in this paper.

2.1 Notation and Terminology

We denote by $|\sigma|$ the **length** of a countable[1] **sequence** $\sigma = \sigma_1, \sigma_2, \dots$, by $\sigma_{\leq i}$ the **prefix** $\sigma_1, \dots, \sigma_i$. The **parity** of an element σ_i in σ is the parity of i. It is **even** or **odd** iff i is. Given two sequences σ and τ, we write $\sigma \sqsubseteq \tau$ if $\sigma = \tau_{\leq i}$ for a given $i \leq |\tau|$ and we denote by $\sigma \cdot \tau$ their concatenation.

A *tree* $\mathcal{T} = \langle \mathcal{N}, \mathcal{E} \rangle$ is a connected directed graph with a countable set of **nodes** $\mathcal{N}$ containing a special node $r \in \mathcal{N}$ called **root**, and such that the set of **edges** $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ contains a unique edge $\langle x, y \rangle$ for every non-root node $y \in \mathcal{N}$. If $\langle x, y \rangle \in \mathcal{E}$, we say that x is the **parent** of y and that y is a **child** of x. A **path** (in $\mathcal{T}$) is a sequence of nodes $\mathcal{P} = x_1, x_2 \dots$ such that x_1 is the root of $\mathcal{T}$ and x_{i+1} is a child of x_i for all $i > 0$.

A **branch** is a maximal path. Given two nodes x and y, x is an **ancestor** of y and y is a **descendant** of x if there is a path containing x whose last element is y (note that every node is an ancestor and a descendant of itself). The **height** $|x|$ of a node x is the length of the (unique) path from the root to x. Thus, the root has height 1, a child of the root has height 2 and so on. The **height** of a tree is the maximal height of its nodes.

In this paper we consider **formulas** generated from a countable non-empty set of atomic propositions $\mathcal{A} = \{a, b, c, \dots\}$ and the implication connective $\to$ (and the parenthesis symbols). In the following, we may write $(A_1 \cdots A_n) \to c$ as a shortcut for

[1] We use the adjective *countable* in the standard mathematical sense: a set is countable iff it is in a one-to-one correspondence with a (finite or infinite) subset of the set of natural numbers.

$A_1 \to (\cdots \to (A_n \to c)\cdots)$. We consider the **implication fragments of intuitionistic logic** $\mathsf{IL}^{\to}$, defined as the smallest set of formulas containing each instance of the two axioms $A \to (B \to A)$ and $A \to (B \to C) \to ((A \to B) \to (A \to C))$ and closed for **modus ponens**, that is: if $A \in \mathsf{IL}^{\to}$ and $A \to B \in \mathsf{IL}^{\to}$ then $B \in \mathsf{IL}^{\to}$. We say that a formula F is **valild** if and only if $F \in \mathsf{IL}^{\to}$[2].

2.2 Dialogical Games

A **challenge** is a pair $\langle ?, s\rangle$ where s is either an occurrence of the symbol $\bullet$, in which case such a challenge is said **atomic**, or where s is formula F. A **defense** is a pair $\langle !, F\rangle$ where F is a formula. An **assertion (of** F**)** is a non-atomic challenge $\langle ?, F\rangle$ or a defense $\langle !, F\rangle$. A **move** is an assertion or an atomic-challenge. An **augmented sequence** is a pair $\langle \sigma, \phi\rangle$ where σ is a non-empty sequence of moves, and ϕ is a function mapping any σ_i with $i > 1$ to a $\sigma_j = \phi(\sigma_i)$ with opposite parity and such that $j < i$. A move σ_i in σ is called **P-move** (denoted $\sigma_i^{\mathbf{P}}$) if i is odd, and **O-move** (denoted $\sigma_i^{\mathbf{O}}$) if i is even. It is a **repetition** if there is $j < i$ such that i and j have opposite parity and σ_i and σ_j are assertions of the same formula.

Definition 1. *Let $\langle \sigma, \phi\rangle$ be an augmented sequence and $i \leq |\sigma|$.*

1. *A challenge σ_i is **justified** whenever:*
   (a) *either σ_i is an atomic-challenge and $\phi(\sigma_i)$ is an assertion of an atomic formula;*
   (b) *or $\sigma_i = \langle ?, A\rangle$ and $\phi(\sigma_i)$ is an assertion of a formula $A \to B$.*
2. *A defense is σ_i is **justified** whenever:*
   (a) *either σ_i and $\phi(\phi(\sigma_i))$ are assertions of a same atomic formula $a \in \mathcal{A}$ and $\phi(\sigma_i)$ is an atomic challenge;*
   (b) *or σ_i is an assertion of a formula B, $\phi(\sigma_i)$ is a justified challenge of the form $\langle ?, A\rangle$, and $\phi(\phi(\sigma_i))$ is an assertion of $A \to B$.*

*If σ_i is a justified move, we say $\phi(\sigma_i)$ **justifies** σ_i and that σ_i **is justified** by $\phi(\sigma_i)$. A challenge σ_i is **unanswered** if there is no defense σ_k such that σ_k is justified by σ_i. A justified challenge σ_i is a **counterattack** if $\phi(\sigma_i)$ is a challenge. A **justified sequence** is an augmented sequence in which any move except the first one is justified.*

Definition 2 (Play). *A play for F is a justified sequence $\mathsf{p} = \langle \sigma, \phi\rangle$ starting with* **P** *defending F, that is, $\sigma_1 = \langle !, F\rangle$ and such that the following holds for any $1 < i \leq |\sigma|$:*

1. *each* **O**-*move is justified by the immediately preceding* **P**-*move, that is, $\phi(\sigma_{2k}) = \sigma_{2k-1}$ for any $2k \leq |\sigma|$;*
2. *each* **P**-*move (but the first) is justified by some preceding* **O**-*move. In particular, if* **P** *states a defense, such defense is justified by the last unanswered challenge stated by* **O**, *that is, if $\sigma_{2k_+1} = \langle !, F\rangle$, then $\phi(\sigma_{2k+1}) = \sigma_{2h}$ is the unanswered challenge with maximal $2h \leq 2k$;*
3. *if* **P** *state a defense and such a defense is an assertion of an atomic formula, then there must be another preceding* **O** *assertion of the same atomic formula. That is, if $\sigma_{2k+1} = \langle !, a\rangle$ with $a \in \mathcal{A}$, then σ_{2k+1} is a repetition;*

[2] This definition of validity corresponds to the standard one i.e., valid in every Kripke model whose accessibility relation is a preorder and whose labeling is monotone. See e.g. [15,32].

4. *Only* **O** *can challenge assertions of atomic formulas and these assertions must be challenges. That is, if* $\sigma_i = \langle ?, \bullet \rangle$*, then i must be even and* $\phi(\sigma_i)$ *is a challenge.*

A play $\mathsf{p} = \langle \sigma, \phi \rangle$ *is finite if* σ *is. and its* ***length*** $|\mathsf{p}|$ *is the length* σ*. A move m is* ***legal*** *for* p *if* $\langle \sigma \cdot m, \phi \cup \{\langle m, \sigma_i \rangle\} \rangle$ *is a play for a* $i \leq |\sigma|$*.*

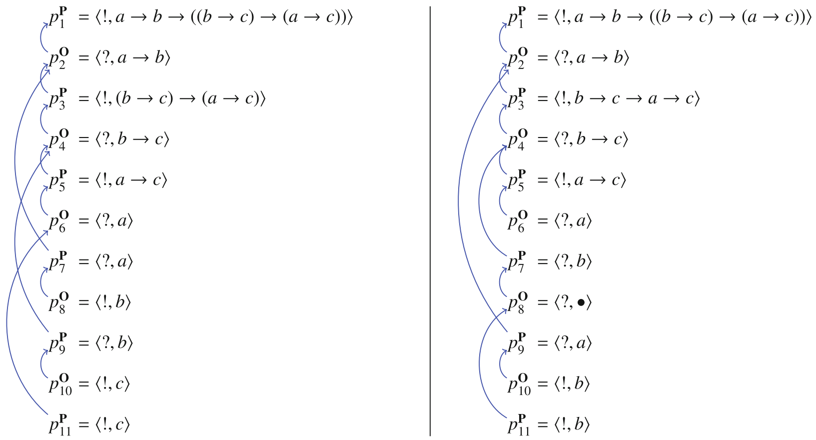
Definition 3 (Winning Condition). *The player* **P** ***wins*** *a play* $\mathsf{p} = \langle \sigma, \phi \rangle$ *if* σ *is finite and ends with a* **P***-move* $\langle !, a \rangle$ *with* $a \in \mathcal{A}$*. Otherwise,* **O** *wins.*

We now define two particular types of plays: *Lorenzen-Felscher* plays, and *Stubborn* plays. In Lorenzen-Felscher plays **P** can assert an atomic formula only if **O** has already asserted it. In Stubborn plays once **P** starts challenging an assertion of a complex formula B, she will stubbornly continue to challenge the subformulas of that formula until **O** asserts an atomic formula.

Definition 4. *Let* $\mathsf{p} = \langle \sigma, \phi \rangle$ *be a play.*

1. p *is a* ***Lorenzen-Felscher play*** *(or* LF*-**play**) if any* **P***-assertion of an atomic formula is a repetition. That is, if* $\sigma_{2k+1} \in \{\langle !, a \rangle, \langle ?, a \rangle \mid a \in \mathcal{A}\}$*, then there is* $h \leq k$ *such that* $\sigma_{2h} = \pm\langle \star, a \rangle$ *for* $\star \in \{?, !\}$
2. p *is a* ***Stubborn play*** *(or* ST*-**play**) if the following hold:*
   (a) *whenever* **O** *assert a complex formula* $A \to B$ *as a defense from a preceding challenge, then* **P***'s next move is a challenge of such a formula. That is, if* $\sigma' \cdot m^{\mathbf{O}} \sqsubseteq \sigma$ *and* $m = \langle !, A \to B \rangle$*, then* $\sigma' \cdot m^{\mathbf{O}} \cdot n^{\mathbf{P}} \sqsubseteq \sigma$ *for a* $n = \langle ?, A \rangle$ *justified by* m*.*
   (b) *whenever* **O** *assert an atomic formula c as a defense from a preceding challenge, then* **P***'s next move is a defense asserting c. That is, if* $\sigma' \cdot m^{\mathbf{O}} \sqsubseteq \sigma$ *and* $m = \langle !, c \rangle$*, then* $\sigma = \sigma' \cdot m^{\mathbf{O}} \cdot n^{\mathbf{P}}$ *where* $n = \langle !, c \rangle$*.*

Example 1. Consider the two following plays (both won by **P**). We represent a play $\langle \sigma, \phi \rangle$ as a sequence of moves. We represent the function ϕ by drawing directed edges from each move σ_i to the move $\phi(\sigma_i)$.



The play on the left is LF-play that is not a ST-play, while the one on the right is ST-play that is not an LF-play. Remark that each atomic challenge $\langle ?, \bullet \rangle$ is an **O**-move, and it is justified by a **P**-challenge asserting an atomic formula.

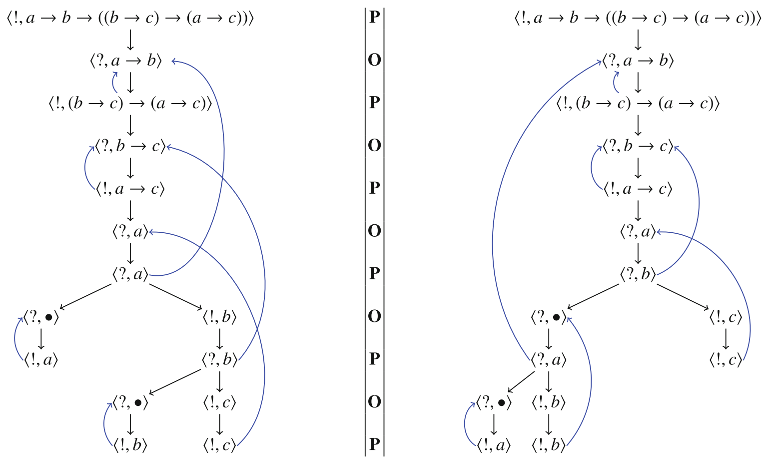
Definition 5. *Let A be a formula. The* ***game*** *for A is a pair $\mathcal{G}_A = \langle \mathcal{R}_A, \phi_A \rangle$ where $\mathcal{R}_A = \langle \mathcal{N}_A, \mathcal{E}_A \rangle$ is a tree of moves and $\phi : \mathcal{N}_A \to \mathcal{N}_A$ is a map such that:*

1. *for each path $\mathcal{P}$ of $\mathcal{R}_A$, the pair $\langle \mathcal{P}, \phi_A|_{\mathcal{P}} \rangle$ is a play for A;*
2. *for each node v of $\mathcal{R}_A$, all and only the children of v are legal move of the play in $\mathcal{G}_A$ ending with v.*

A node v of $\mathcal{G}$ is a **P*-node*** *(resp.* **O*-node****) if is its height is odd (resp. even).*

A ***strategy*** *for A is a pair $\mathcal{S} = \langle \mathcal{T}, \psi \rangle$ such that $\mathcal{T}$ is a subtree of $\mathcal{R}_A$ (and ψ is defined as the restriction of ϕ_A on the nodes in $\mathcal{T}$) in which every* **O***-node has at most one child. It is* ***winning*** *when $\mathcal{T}$ is finite and any of its branch is a play won by* **P***. A* ***Lorenzen-Felscher strategy*** *(resp.* ***Stubborn strategy****) is a strategy such that each branch of $\mathcal{S}$ is a* LF*-play (resp. a* ST*-play).*

Example 2. Below we provide a representation of Lorenzen-Felscher winning strategy (left) and of a winning Stubborn strategy (right) for the formula $a \to b \to ((b \to c) \to (a \to c))$ as tree of moves. As in Example 1 we represent the function ϕ by drawing directed edges from each move σ_i to the move $\phi(\sigma_i)$. However, we here we omit the edges with source an **O**-move because $\phi(\sigma_{2k+2}) = \sigma_{2k+1}$ for all $k \in \mathbb{N}$.



3 Sequent Calculus

In this section, we recall the definition of the sequent calculus GKi from [33] which is sound and complete for the logic $\mathsf{IL}^{\to}$. We then provide three classes of derivations obtained by imposing restrictions on rules applications, and we show that they are still sound and complete with respect to the same logic.

A **sequent** is an expression $\Gamma \vdash C$ where C is a formula and Γ is a finite (possibly empty) multiset of formulas. A derivation $\mathscr{D}$ is a finite tree of sequents constructed using the rules in Fig. 1 in which each leaf is obtained by an Ax-rule and each non-leaf sequent is obtained by $\to^{\mathsf{R}}$-rule or a $\to^{\mathsf{L}}$-rule. A sequent $\Gamma \vdash C$ is GKi-provable if it admits a derivation in the sequent calculus GKi, whose root (or conclusion) is $\Gamma \vdash C$.

$$\frac{}{\Gamma, \underline{a} \vdash a}\,\mathsf{Ax} \qquad \frac{\Gamma, A \to B \vdash \underline{A} \quad \Gamma, A \to B, \underline{B} \vdash C}{\Gamma, \underline{A \to B} \vdash C}\,\to^{\mathsf{L}} \qquad \frac{\Gamma, \underline{A} \vdash \underline{B}}{\Gamma \vdash \underline{A \to B}}\,\to^{\mathsf{R}}$$

Fig. 1. Rules for the sequent calculus GKi. In each rule we have underlined its **principal** formula in the conclusion, and the **active** formulas in each premise.

Theorem 1 [33]**.** *The sequent calculus* GKi *is sound and complete for* $\mathsf{IL}^{\to}$*, that is a formula* F *is valid if and only if* $\vdash F$ *is provable in* GKi.

We characterize derivations according to their shape.

Definition 6. *Let* $\mathscr{D}$ *be a derivation of some sequent* $\Delta \vdash F$ *in* GKi*. We say that:*

1. $\mathscr{D}$ *is a* ***strategic derivation*** *(or* ***S-derivation****) when each left-hand side premise of* $\to^{\mathsf{L}}$*-rule of the form* $\Gamma \vdash A \to B$ *is the conclusion of a* $\to^{\mathsf{R}}$*-rule;*
2. $\mathscr{D}$ *is a* ***LF-derivation*** *if the left-hand side premise of each* $\to^{\mathsf{L}}$*-rule is always the conclusion of a* $\to^{\mathsf{R}}$*-rule or an* Ax*-rule;*
3. $\mathscr{D}$ *is a* ***ST-derivation*** *if is a* ***S****-derivation and the active formula of the right-hand premise of each* $\to^{\mathsf{L}}$*-rule in* $\mathscr{D}$ *is the principal formula of this premise. That is, if* $\Gamma, A \to B, B \vdash C$ *is the right-hand premise of a* $\to^{\mathsf{L}}$*-rule, then either it is the conclusion of a* Ax *if* $B = C$ *is atomic, or it is the conclusion of a* $\to^{\mathsf{L}}$*-rule. In both cases* B *is the principal formula of* $\Gamma, A \to B, B \vdash C$

Remark 1. Every **LF**-derivation is a **S**-derivation by definition. If a sequent $\Gamma \vdash C$ occurs in a **S**-derivation $\mathscr{D}$ as conclusion of a $\to^{\mathsf{L}}$-rule and as left-hand premise of (another) $\to^{\mathsf{L}}$-rule with principal formula $A \to B$, then $A = C$ and A is an atomic formula. Similarly, if a sequent $\Gamma, A \to B, B \vdash C$ is the right-hand premise of $\to^{\mathsf{L}}$-rule in a **ST**-derivation, then C is atomic.

LF-derivations were introduced by Herbelin in the fifth chapter of his PhD thesis (where they are called LGQ-derivations [17]). Similarly, **ST**-derivations are a variant of derivations in the sequent calculus LJT, also introduced by Herbelin in [16]. The only difference is that the sequent calculus LJT contains an explicit contraction rule and operates over sequents of the form $\Gamma; A \vdash C$ or $\Gamma; \emptyset \vdash C$ with Γ set of formula occurences, and A and C formulas. The following lemma will prove useful later on.

Lemma 1 (Weakening admissibility). *If a sequent* $\Gamma \vdash C$ *admits an* ***ST****-derivation, then there is a* ***ST****-derivation* $\mathscr{D}^\star$ *of the sequent* $\Gamma, \Delta \vdash C$ *for any finite multiset* Δ*. Moreover,* $\mathscr{D}^\star$ *contains the same rules of* $\mathscr{D}$ *(with the same principal and active formulas).*

Proof. It suffices to consider the derivation $\mathscr{D}^\star$ obtained by adding Δ to any leaf of $\mathscr{D}$.

Since the sequent calculus GKi is a sound and complete with respect to $\mathsf{IL}^{\to}$, we can prove that the set of **S**-derivations and the set of **LF**-derivations are also sound and complete with respect to $\mathsf{IL}^{\to}$.

Theorem 2. *Let* $\Gamma \vdash C$ *be a sequent. It is* GKi*-provable iff it admits a* ***S****-derivation iff it admits a* ***LF****-derivation.*

Proof. The fact that any GKi-provable sequents admits a **LF**-derivation has been proved in [7]. We conclude since any **LF**-derivation is a **S**-derivation and any **S**-derivation is a derivation in GKi.

3.1 Games on Hyland-Ong Arenas

In order to prove that also the set of **ST**-derivations are sound and complete for $\mathsf{IL}^{\to}$, we establish a correspondence between *winning innocent strategies* for games on Hyland-Ong Arenas [19] and **ST**-derivations. We then conclude by levering on the result of soundness and (full-)completeness of these winning strategies with respect to $\mathsf{IL}^{\to}$.

Note 1. Both games in dialogical logic and game on Hyland-Ong arenas formalize proofs as winning strategies over games defined by a formula F. However, some terminology in these two paradigms identify objects of different nature. For this reason, we here list the main differences.

Dialogical Logic	Games on Hyland-Ong arenas
a play $\sigma_1, \sigma_2, \ldots$ starts $i = 1$ odd	a play $\tau_0, \tau_1, \ldots$ starts $i = 0$ even
a play starts with a **P**-move	a play starts with a **O**-move
a move is a subformula of F plus a polarity	a move corresponds to an atom in F

To facilitate distinguishing as much as possible the two formalisms, in games over Hyland-Ong arenas we denote the proponent **P** by $\bullet$ the opponent **O** by $\circ$.

Definition 7. *A **sink** of a directed acyclic graph $\mathcal{G} = \langle V, \to \rangle$ is a vertex v such that $\langle v, w\rangle \notin \to$ for no $w \in V$. The **arena** of a formula F is the $\mathcal{A}$-labeled directed acyclic graph $[\![F]\!] = \langle V_{[\![F]\!]}, \overset{[\![F]\!]}{\to}, \ell\rangle$ (where $\langle V_{[\![F]\!]}, \overset{[\![F]\!]}{\to}\rangle$ is a directed acyclic graph, and ℓ a labeling function associating to each $v \in V_{[}[\![F]\!]]$ an atom $\ell(v) \in \mathcal{A}$) defined as follows:*

$$[\![a]\!] = \langle \{v\}, \emptyset, \ell(v) = a\rangle \quad \text{and} \quad [\![A \to B]\!] = \langle V_{[\![A]\!]} \cup V_{[\![B]\!]}, \overset{[\![A]\!]}{\to} \cup \overset{[\![B]\!]}{\to} \cup I, \ell_{[\![A]\!]} \cup \ell_{[\![B]\!]}\rangle$$

*with $I = \{(s_{[\![A]\!]}, s_{[\![B]\!]})\}$ where $s_{[\![A]\!]}$ and $s_{[\![B]\!]}$ are the unique (by construction) sink of $[\![A]\!]$ and $[\![B]\!]$ respectively. The **arena** of a sequent $A_1, \ldots, A_n \vdash B$ is defined as the arena $[\![(A_1 \cdots A_n) \to B]\!]$.*

Definition 8. *Let F be a formula. A **justified sequence** for F is a pair $\langle \tau, f\rangle$ where $\tau = \tau_0, \ldots, \tau_n$ is a non-empty sequence of **moves** (i.e., occurrences of vertices of $[\![F]\!]$), and f is a function mapping each τ_i with $i > 0$ in its **justifier** $f(\tau_i) = \tau_j$ for a $j < i$ such that $i + j$ is odd (i.e. if i is even, then j is odd and vice versa).*

*The **pointer** of a move τ_i with $i > 0$ is the pair $\langle \tau_i, f(\tau_i)\rangle$; we identify f with the set of pointers it defines. A move τ_i is a $\circ$-move (resp. $\bullet$-move) if i is even (resp. i is odd).*

*A **view** is a justified sequence $\langle \tau, f\rangle$ such that:*

- *it is a **play**, that is, the initial move τ_0 is the sink of $[\![F]\!]$;*
- *it is $\circ$-**shortsighted**, that is, $f(\tau_i) = \tau_{i-1}$ for each non-initial $\circ$-move τ_i;*
- *it is $\bullet$-**uniform**, that is, $\ell(\tau_i) = \ell(\tau_{i-1})$ for each $\bullet$-move τ_i.*

Remark 2. By definition, it follows that each $\circ$-move (resp. $\bullet$-move) is an occurrence of a vertex v of $[\![F]\!]$ having even (resp. odd) *distance* $\mathsf{d}(v)$ from the sink $s_{[\![F]\!]}$ of $[\![F]\!]$, where the distance $\mathsf{d}(v)$ is defined as the number of vertices in a path from v to $s_{[\![F]\!]}$ minus one. The proof that each of such a path in an arena has the same length is provided in [31].

The **predecessor** of a view is the result of deleting the final move (and its pointer); the converse is the **successor** relation.

Definition 9. *Let F be a formula. A* ***winning innocent strategy*** *(or* **WIS***) Σ for F is a finite, non-empty prefix-closed set of views for F such that:*

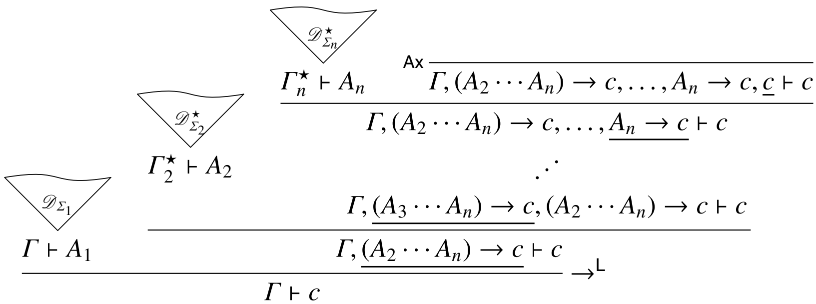
1. *The view containing a single occurrence of the sink of $[\![F]\!]$ belongs to Σ;*
2. *Σ is* ***$\circ$-complete****: if $\langle \rho \cdot v, f\rangle \in \Sigma$ with v a $\bullet$-move, then every successor of $\langle \rho \cdot v, f\rangle$ is in Σ;*
3. *Σ is* ***$\bullet$-deterministic*** *and* ***$\bullet$-total****: if $\langle \rho \cdot v, f\rangle \in \Sigma$ and v is an $\circ$-move, then exactly one successor of $\langle \rho \cdot v, f\rangle$ belongs to Σ.*

Theorem 3 [11,19]**.** *A formula F is valid iff there is a* **WIS** *for F.*

Lemma 2. *Let $\Gamma \vdash F$ be a sequent. For any* **WIS** *Σ for $\Gamma \vdash F$ there is a canonically defined* **ST***-derivation $\mathcal{D}_\Sigma$ of $\Gamma \vdash F$.*

Proof. The proof is by induction on the pair $\langle |\Sigma|, |F|\rangle$ where $|\Sigma|$ is the cardinality of Σ and $|F|$ is the height of F[3].

1. if $F = c$ is atomic, then Σ must contain the set of views $\{c^\circ, c^\circ \cdot c^\bullet\}$ where the justifier of $c^\bullet$ is c°. We have two cases
   (a) either $c^\circ \cdot c^\bullet$ is maximal in Σ, and by $\circ$-completeness we deduce that $\Gamma = \Delta, c$. In this case $\mathcal{D}_\Sigma$ is a proof of $\Delta, \underline{c} \vdash c$ obtained by an Ax-rule.
   (b) or $c^\circ \cdot c^\bullet$ is not maximal in Σ. By $\circ$-completeness, we conclude that $\Gamma = \Delta, (A_1 \cdots A_n) \to c$ for some Δ and $n \geq 1$. For each $i \leq n$ let a_i be the root of $[\![A_i]\!]$ and let Σ_i be the prefix-closed set of views containing each view of Σ that starts with a_i. We obtain that Σ_i is a **WIS** for $\Gamma \vdash A_i$ for any i and that $|\Sigma_i| < |\Sigma|$. By induction hypothesis, for each $i \leq n$ there is a canonically defined **ST**-derivation $\mathcal{D}_{\Sigma_i}$ of $\Gamma \vdash A_i$. By weakening admissibility (Lemma 1), we have a derivation $\mathcal{D}^\star_{\Sigma_i}$ of $\Gamma^\star_i \vdash A_i$ with $\Gamma^\star_i = \Gamma, (A_i \cdots A_n) \to c, \ldots, A_n \to c$ for any $i \in \{2, \ldots, n\}$. We define $\mathcal{D}_\Sigma$ as the following **ST**-derivation:



Notice that, in virtue on the restriction on the application of the $\to^{\mathsf{L}}$-rule in **ST**-derivation, this is the unique way to define $\mathcal{D}_\Sigma$ from the derivations $\mathcal{D}_{\Sigma^\star_1}, \ldots, \mathcal{D}_{\Sigma_n}$.

[3] The height of a formula is the height of its construction tree.

2. If $F = A \to B$ then Σ is also a strategy for $\Gamma, A \vdash B$. Since $|B| < |A \to B|$, by induction hypothesis there is a **ST**-derivation $\mathscr{D}_\Sigma$ of $\Gamma, A \vdash B$ and we can conclude by the application of a $\to^{\mathsf{R}}$-rule.

Theorem 4. *Let F be a formula. It is valid if and only if it admits a **ST**-derivation.*

Proof. If F is valid, then by Theorem 1 there is GKi derivation of $\vdash F$. By Theorem 3 there is a **WIS** Σ for $\vdash F$, therefore a **ST**-derivation $\mathscr{D}_\Sigma$ by Lemma 2. We conclude since the converse trivially holds because every **ST**-derivation is a GKi-derivation.

4 From Dialogical Logic Strategies to Derivations

In this section, we show how to associate to any winning dialogical strategy for a formula F a **S**-derivation of the sequent $\vdash F$. We first show how we associate a sequent to any **O**-move of a strategy.

Definition 10. *Let F be any formula and $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ be a strategy for F. Recall that each path $\mathcal{P}$ of $\mathcal{T}$ is a sequence of moves. The **O-tree** of $\mathcal{S}$ is the tree $\mathcal{T}_{\mathbf{O}}$ defined as follows:*

1. *the set of nodes of $\mathcal{T}_{\mathbf{O}}$ contains each **O** node of $\mathcal{T}$, an additional node r and nothing else;*
2. *a node v of $\mathcal{T}_{\mathbf{O}}$ is the parent of a node v' iff either $v = r$ and $v' = \mathcal{P}_2$ is the second move of a branch $\mathcal{P}$ in $\mathcal{T}$, or there is a branch $\mathcal{P}$ in $\mathcal{T}$ such that $v = \mathcal{P}_{2k}$ and $v' = \mathcal{P}_{2k+2}$.*

We recursively define the function Seq. *associating to any node v of $\mathcal{T}_{\mathbf{O}}$ a sequent* $\mathsf{Seq}(v){:=}\Gamma^v \vdash F^v$ *and it is defined as follows:*

1. *if $|v| = 1$, then v is the root r. Thus $\Gamma^v = \emptyset$ and $F^v = F$;*
2. *If $|v| = k + 1$, then there is a **P**-node t which is the parent of v in $\mathcal{S}$ and a node v' which is the parent of v in $\mathcal{T}_{\mathbf{O}}$, with associated sequent $\mathsf{Seq}(v') = \Gamma^{v'} \vdash F^{v'}$.*
   (a) if $v = \langle ?, \bullet \rangle$, then t asserts an atomic formula b. We let $\Gamma^v = \Gamma^{v'}$ and $F^v = b$;
   (b) if $v = \langle ?, A \rangle$, then t asserts a formula $A \to B$. We let $\Gamma^v = \Gamma^{v'}, A$ and $F^v = B$;
   (c) otherwise $v = \langle !, B \rangle$ and we let $\Gamma^v = \Gamma^{v'}, B$ and $F^v = F^{v'}$.

The following proposition states that the formulas asserted by **O** in the play ending with v are precisely those that are contained in Γ^v.

Proposition 1. *Let $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ be a strategy and let $\mathcal{T}_{\mathbf{O}}$ be its **O**-tree. For every node v of $\mathcal{T}_{\mathbf{O}}$ and for every formula B we have that $B \in \Gamma^v$ if and only if there is an ancestor v' of v that asserts B.*

Proof. If $B \in \Gamma^v$, we can prove that there is an ancestor v' of v that asserts B by induction on $|v|$. If v is the root of $\mathcal{T}_{\mathbf{O}}$, then the proposition is vacuously true. Otherwise we conclude by inductive hypothesis since either v is an assertion of B, and then $\Gamma^v = \Gamma^{v'}, B$ where v' is the parent of v, or $v = \langle ?, \bullet \rangle$, and then $\Gamma^v = \Gamma^{v'}$ where v' is the parent of v.

The converse implication immediately follows by the definition of Seq.

Given a winning strategy $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ for F, we can show that each leaf of $\mathcal{T}_{\mathbf{O}}$ is labeled by a sequent that is conclusion of an Ax-rule of the sequent calculus.

Proposition 2. *Let $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ be a winning strategy and m a leaf of $\mathcal{T}$. If n is the parent of m in $\mathcal{T}$ and $m = \langle !, a \rangle$, then Γ^n is of the form $\Delta, a \vdash a$.*

Proof. Since $\mathcal{S}$ is winning, then m is the last move of a play p that is won by **P**. Consequently, by Condition 2 in the definition of play, m is a repetition. Thus the atom a has already been asserted by **O** in the play. By the definition of Seq, we deduce that $a \in \Gamma^n$.

Moreover, m is justified by a **O**-challenge t. As a consequence, t is either justified by an assertion of $B \to a$ for some formula B, or by an assertion $\langle ?, a \rangle$. By the definition of Seq, we conclude the formula F^t of the sequent associated by Seq to t is a. By the Condition 2 in the definition of play, any **O**-move $t_1, \ldots, t_k$ that is after t is a defense move. This implies, by definition of Seq, that $F^{t_i} = F^t$ for all i; therefore $F^t = F^n = a$.

The two following technical propositions will be used in the proof of Lemma 3.

Proposition 3. *Let $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ be a winning strategy, $\mathcal{T}_{\mathbf{O}}$ be its **O**-tree and m a node of $\mathcal{T}_{\mathbf{O}}$. If m is the parent of a defense move m' asserting B, then $A \to B \in \Gamma^m$ for some formula A.*

Proof. Let $\mathcal{B}$ be the unique branch of $\mathcal{S}$ containing both m and m', and let t be **P**-move that is the parent of m. By the definition of strategy, $\langle B, \phi|_B \rangle$ is a play. Consequently, m is justified by t and t must be a challenge asserting some formula A. This means that the **O**-move $\phi(t)$ is an assertion of $A \to B$. Since $\phi(t)$ is an ancestor of m, we conclude that $A \to B \in \Gamma^{\mathrm{m}}$.

Proposition 4. *Let $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ be a winning strategy, $\mathcal{T}_{\mathbf{O}}$ be its **O**-tree and m a node of $\mathcal{T}_{\mathbf{O}}$. If m is the parent of m' and m' is counterattack asserting A, then $A \to B \in \Gamma^m$ for some formula B.*

Proof. The proof is entirely similar to the one of the previous proposition.

Definition 11. *Let $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$ be a winning strategy, and $\mathcal{T}_{\mathbf{O}}$ be its **O**-tree. We define a function Φ associating a tree of sequent $\mathscr{D}^v$ rooted in $\Gamma^v \vdash F^v$ to each node v of $\mathcal{T}_{\mathbf{O}}$. Such a function is defined by recursion on the number of descendants of v.*

1. *If the number of descendant of v is one, then v is a leaf of $\mathcal{T}_{\mathbf{O}}$. We associate to v a tree whose only node is $\Gamma^v \vdash F^v$.*
2. *Suppose that $\mathscr{D}^x$ is defined for all vertex of having at most $n \geq 1$ descendants and let v be a node with $k + 1$ descendants. Let t be the unique **P**-node of $\mathcal{T}$ such that v is the parent of t in $\mathcal{T}$:*
   (a) If t is a challenge asserting some formula A, then there are two cases:
      – *A is an atomic formula a, and v has (in $\mathcal{T}_{\mathbf{O}}$) two children $v_1 = \langle ?, \bullet \rangle$ and $v_2 = \langle !, B \rangle$. Then the tree of sequents $\mathscr{D}^v$ is defined as follows:*

$$\dfrac{\begin{array}{c}\mathscr{D}^{v_1}\\ \Gamma, a \to B \vdash a\end{array} \qquad \begin{array}{c}\mathscr{D}^{v_2}\\ \Gamma, a \to B, B \vdash C\end{array}}{\Gamma, a \to B \vdash C} \to^{\mathsf{L}}$$

where $\Gamma, a \to B \vdash a$ *and* $\Gamma, a \to B, B \vdash C$ *are the sequents associated to* v_1 *and* v_2 *respectively.*

– $A = A_1 \to A_2$ *and* v *has two children* $v_1 = \langle ?, A_1 \rangle$ *and* $v_2 = \langle !, B \rangle$ *(in* $\mathcal{T}_{\mathbf{O}}$*) for some formula* B*. The tree of sequents* $\mathscr{D}^v$ *is*

$$\dfrac{\dfrac{\overset{\mathscr{D}^{v_1}}{\Gamma, (A_1 \to A_2) \to B, A_1 \vdash A_2}}{\Gamma, (A_1 \to A_2) \to B, \vdash A_1 \to A_2} \to^{\mathsf{R}} \qquad \overset{\mathscr{D}^{v_2}}{\Gamma, (A_1 \to A_2) \to B, B \vdash C}}{\Gamma, (A_1 \to A_2) \to B \vdash C} \to^{\mathsf{L}}$$

where $\Gamma, (A_1 \to A_2) \to B, A_1 \vdash A_2$ *and* $\Gamma, (A_1 \to A_2), B \vdash C$ *are the sequents associated to* v_1 *and* v_2 *respectively.*

(b) *If* t *is a defense asserting* $A \to B$*, then* v *has a unique child* $v_1 = \langle ?, A \rangle$ *in* $\mathcal{T}_{\mathbf{O}}$ *and* $\mathscr{D}^v$ *is defined as:*

$$\dfrac{\overset{\mathscr{D}^{v_1}}{\Gamma, A \vdash B}}{\Gamma \vdash A \to B} \to^{\mathsf{R}}$$

Lemma 3. *For every winning strategy* $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$*, for every node* v *of* $\mathcal{T}_{\mathbf{O}}$*, the tree of sequent* $\mathscr{D}^v$ *is a* **S***-derivation of* $\Gamma^v \vdash F^v$

Proof. The proof is by induction on the height of $\mathscr{D}^v$. If the height is 1, then the lemma is immediately established in virtue of Proposition 2. The inductive cases follow by induction hypothesis, by construction of $\mathscr{D}^v$ and by Propositions 3 and 4.

Theorem 5. *For any winning strategy* $\mathcal{S} = \langle \mathcal{T}, \phi \rangle$*, the tree of sequent* $\mathscr{D}^{\mathcal{S}}$ *associated to the root-node of* $\mathcal{T}_{\mathbf{O}}$ *is a* **S***-derivation of* $\vdash F$*, moreover:*

1. *if* $\mathcal{S}$ *is a Lorenzen-Felscher winning strategy, then* $\mathscr{D}^{\mathcal{S}}$ *is a* ***LF****-derivation;*
2. *if* $\mathcal{S}$ *is a Stubborn winning strategy, then* $\mathscr{D}^{\mathcal{S}}$ *is a* ***ST****-derivation.*

Proof. The fact that $\mathscr{D}^{\mathcal{S}}$ is a **S**-derivation of $\vdash F$ follows immediately by Lemma 3. We only give a proof of (2) because the proof of (1) is easier.

Consider a sequent in $\mathscr{D}^{\mathcal{S}}$ that is obtained by an application of a $\to^{\mathsf{L}}$-rule and let $\Gamma, A \to B, B \vdash C$ be its right-hand premise. We must show that B is the principal formula of this latter sequent. Remark that the sequent $\Gamma, A \to B, B \vdash C$ is associated to a **O** move $\langle !, B \rangle$ in $\mathcal{T}_{\mathbf{O}}$. There are two cases: if $B = B \to B_1$, then since $\mathcal{S}$ is Stubborn, we must have that child t of $\langle !, B \rangle$ is a challenge $\langle ?, B_1 \rangle$. By the definition of the function Φ, the sequents associated to the child v_1 and v_2 are of the form $\Gamma, A \to B, B \vdash B_1$ and $\Gamma, A \to B, B, B_2 \vdash C$, this means that $B = B_1 \to B_2$ is the principal formula of $\Gamma, A \to B, B \vdash C$. The case in which B is an atomic formula is similar.

5 From Derivations to Dialogical Logic Strategies

In this section, we show how to transform any **S**-derivation of $\vdash F$ in a winning strategy for F. To do so, we define a function that associates to any path $\mathcal{P}$ of $\mathscr{D}$ a play for F.

Definition 12. *Let $\mathcal{P} = S_1, \ldots, S_n$ be a path in a **S**-derivation $\mathcal{D}$ of F. We associate with $\mathcal{P}$ a sequence of moves $\sigma^{\mathcal{P}}$ by induction on $|\mathcal{P}|$*

- *If $|\mathcal{P}| = 1$ then $\sigma^{\mathcal{P}} = \langle !, F\rangle$;*
- *if $|\mathcal{P}| = n$ and $\mathcal{P} = \mathcal{P}', S$ then we consider the following cases:*
  1. *If S is the conclusion of an Ax-rule whose principal formula is a, then:*
     (a) *if S is the premise of a $\rightarrow^{\mathsf{R}}$-rule then the principal formula of this last rule must be $B \rightarrow a$ for some formula B. We define $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle ?, B\rangle \cdot \langle !, a\rangle$;*
     (b) *if S is the left-hand premise of $\rightarrow^{\mathsf{L}}$-rule, then the principal formula of this last rule must be $a \rightarrow B$ for some formula B; we define $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle ?, \bullet\rangle \cdot \langle !, a\rangle$;*
     (c) *if S is the right-hand premise of a $\rightarrow^{\mathsf{L}}$-rule whose principal formula is $C \rightarrow D$, then we define $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'} \cdot \langle !, D\rangle \cdot \langle !, a\rangle$;*
  2. *If S is the conclusion of an $\rightarrow^{\mathsf{R}}$-rule whose principal formula is $A \rightarrow B$ then:*
     (a) *if S is the left-hand premise of an $\rightarrow^{\mathsf{L}}$-rule, then $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'}$*
     (b) *if S is the right-hand premise of an $\rightarrow^{\mathsf{L}}$-rule whose principal formula is $C \rightarrow D$, then $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle !, D\rangle \cdot \langle !, A \rightarrow B\rangle$;*
     (c) *if S is the premise of an $\rightarrow^{\mathsf{R}}$-rule then, the principal formula of such a rule must be $G \rightarrow (A \rightarrow B)$ for some formula G. We define $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle ?, G\rangle \cdot \langle !, A \rightarrow B\rangle$.*
  3. *If S is the conclusion of an $\rightarrow^{\mathsf{L}}$-rule whose principal formula is $A \rightarrow B$ then:*
     (a) *if S is the premise of a $\rightarrow^{\mathsf{R}}$-rule whose principal formula is $C \rightarrow D$, then $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle ?, C\rangle \cdot \langle ?, A\rangle$*
     (b) *If S is the left-hand premise of a $\rightarrow^{\mathsf{L}}$-rule whose principal formula is $C \rightarrow D$, then C must be an atomic formula. We define $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle ?, \bullet\rangle \cdot \langle ?, A\rangle$.*
     (c) *If S is the right-hand premise of $\rightarrow^{\mathsf{L}}$ whose principal formula is $C \rightarrow D$, then we define $\sigma^{\mathcal{P}} = \sigma^{\mathcal{P}'} \cdot \langle !, D\rangle \cdot \langle ?, A\rangle$.*

Proposition 5. *Let $\mathcal{D}$ be **ST**-derivation of $\vdash F$, $\mathcal{P}$ a path in $\mathcal{D}$ and $\Gamma \vdash C$ the last sequent of this path. If $B \in \Gamma$, then there is an **O**-move in $\sigma^{\mathcal{P}}$ that asserts B.*

Proof. By induction on the length of $\mathcal{P}$.

By the above proposition, if $\mathcal{P} = S_1, \ldots, S_n$ is a path of $\mathcal{D}$ and if a formula occurrence A is the principal formula of a $\rightarrow^{\mathsf{L}}$-rule in one of the S_i, then there is a **O**-move $\sigma_i^{\mathcal{P}}$ that asserts A. For any formula occurrence A, we denote by m_A the first move in $\sigma^{\mathcal{P}}$ that asserts such formula occurrence A.

Definition 13. *Let $\mathcal{D}$ be a **S**-derivation and let $\mathcal{P}$ be a path in $\mathcal{D}$. We define a function $\phi^{\mathcal{P}}$ from $\sigma^{\mathcal{P}}$ to $\sigma^{\mathcal{P}}$ by the following cases:*

1. *$\phi^{\mathcal{P}}(\sigma_i^{\mathcal{P}}) = \sigma_{i-1}^{\mathcal{P}}$ if $\sigma_i^{\mathcal{P}}$ is an **O** move;*
2. *$\phi^{\mathcal{P}}(\sigma_i^{\mathcal{P}}) = m_A$ if $\sigma_i^{\mathcal{P}}$ is a **P** move and S_n is the conclusion of a $\rightarrow^{\mathsf{L}}$ whose principal formula occurrence is A;*
3. *$\phi^{\mathcal{P}}(\sigma_i^{\mathcal{P}}) = \sigma_k^{\mathcal{P}}$ if $\sigma_i^{\mathcal{P}}$ is a **P** move and a defense, and $\sigma_k^{\mathcal{P}}$ is the last unanswered **O** challenge in $\sigma_{\leq i-1}^{\mathcal{P}}$.*

Lemma 4. *Let $\mathscr{D}$ be a* **S***-derivation derivation of* $\vdash F$*. If* $\mathcal{P}$ *is a path in* $\mathscr{D}$*, then* $\mathsf{p}^{\mathcal{P}} = \langle \sigma^{\mathcal{P}}, \phi^{\mathcal{P}} \rangle$ *is a play for F. Moreover, if* $\mathcal{P}$ *is a branch of* $\mathscr{D}$*, then* $\langle \sigma^{\mathcal{P}}, \phi^{\mathcal{P}} \rangle$ *is won by* **P**.

Proof. Suppose that the proposition holds for any path whose length is at most $k \geq 1$ and let $\mathcal{P} = \mathcal{P}', S$ by a path of length $k + 1$. We should check that $\sigma^{\mathcal{P}'} \cdot m^{\mathbf{O}} \cdot n^{\mathbf{P}}$ and $\phi^{\mathcal{P}}$ forms a play where m and n are the two moves associated to S. There are as many cases as those detailed in the Definition 12 of $\sigma^{\mathcal{P}}$. We only consider some of them. Let t be the last move of $\sigma^{\mathcal{P}'}$.

- If S is obtained by an$\rightarrow^{\mathsf{L}}$-rule whose principal formula is $A \rightarrow B$ and S is the left-hand premise of another $\rightarrow^{\mathsf{L}}$-rule whose principal formula is $C \rightarrow D$, then $m = \langle ?, \bullet \rangle$, $n = \langle ?, A \rangle$, $\phi^{\mathcal{P}}(m) = t$ and $\phi^{\mathcal{P}}(n) = m_{A \rightarrow B}$. The move t is the **P** move associated to the last element $\Sigma \vdash G$ of $\mathcal{P}'$. This latter sequent is obtained by a $\rightarrow^{\mathsf{L}}$. Thus, by construction t is $\langle ?, C \rangle$, and since C is atomic, and t is a justified move by induction hypothesis, then m is justified. The move $m_{A \rightarrow B}$ is an **O**-move that asserts $A \rightarrow B$, since $n = \langle ?, A \rangle$ and $m_{A \rightarrow B}$ is justified by hypothesis, then m is justified as well.
- If S is obtained by an $\rightarrow^{\mathsf{R}}$-rule whose principal formula is $A \rightarrow B$ and S is the premise of another $\rightarrow^{\mathsf{R}}$-rule whose principal formula is $G \rightarrow (A \rightarrow B)$, then $m = \langle ?, G \rangle$, $n = \langle !, A \rightarrow B \rangle$, $\phi^{\mathcal{P}}(m) = t$ and $\phi^{\mathcal{P}}(n) = m$. The move t is associated to the last sequent $\Gamma \vdash G \rightarrow (A \rightarrow B)$ of $\mathcal{P}'$. This latter sequent is obtained by a $\rightarrow^{\mathsf{R}}$ with principal formula $G \rightarrow (A \rightarrow C)$ thus $t = \langle !, G \rightarrow (A \rightarrow C) \rangle$ and since t is justified by induction hypothesis, then also m is. The fact that n is justified is immediate.
- If S is obtained by an Ax-rule whose principal formula is a and S is the premise of $\rightarrow^{\mathsf{R}}$, then the principal formula of this rule must be $B \rightarrow a$ for some B. In this case $m = \langle ?, B \rangle$ and $n = \langle !, a \rangle$. Remark that $t = \langle B \rightarrow a \rangle$, and since t is justified by induction hypothesis, then also m is. By definition of $\phi^{\mathcal{P}}$, we have that $\phi^{\mathcal{P}}(n) = m$ and thus also m is justified. We should check that $n = \langle !, a \rangle$ is a repetition. This easily follows by observing that S must be of the form $\Delta, a, B \vdash a$ for some Δ and by applying Proposition 5.

The fact that $\mathsf{p}^{\mathcal{P}}$ is won by **P** whenever $\mathcal{P}$ is a branch, follows from the fact that the last move of $\mathsf{p}^{\mathcal{P}}$ must be $\langle !, a \rangle$ for some atom a.

Lemma 5. *Let $\mathscr{D}$ be a proof of $\vdash F$ and $\mathcal{P}$ a path in $\mathscr{D}$. The following holds:*

1. *if $\mathscr{D}$ is a* ***ST****-derivation then $\langle \sigma^{\mathcal{P}}, \phi^{\mathcal{P}} \rangle$ is a* S*-play;*
2. *if $\mathscr{D}$ is a* ***LF****-derivation then $\langle \sigma^{\mathcal{P}}, \phi^{\mathcal{P}} \rangle$ is a* LF*-play.*

Proof. Both statement are proven by induction on $|\mathcal{P}|$. We only detail the interesting case of (2), i.e., when $\mathcal{P} = \mathcal{P}', S$ and the last **O**-move of $\sigma^{\mathcal{P}}$ is a defense asserting either a complex formula $A \rightarrow B$ or an atomic formula a. By the construction of $\sigma^{\mathcal{P}}$, S can only be a sequent $\Gamma \vdash G$ that is the right-hand premise of a $\rightarrow^{\mathsf{L}}$ with principal formula $C \rightarrow (A \rightarrow B)$ (resp $C \rightarrow a$). As a consequence, G must be an atomic formula b, and thus either $\Gamma \vdash b$ is obtained by another $\rightarrow^{\mathsf{L}}$-rule or by an Ax-rule. In the former case, since $\mathscr{D}$ is a **ST**-derivation, then $A \rightarrow B$ is the principal formula of $\Gamma \vdash b$, and by construction of $\sigma^{\mathcal{P}}$ its last move must be $\langle ?, A \rangle$ and must be justified by $\langle !, A \rightarrow B \rangle$ and we can conclude. In the latter case, since $\mathscr{D}$ is an **ST**-derivation, then $b = a$ and $\Gamma \vdash G$ is $\Delta, a \vdash a$ for some multiset Δ. Thus, the last move of $\sigma^{\mathcal{P}}$ must be $\langle !, a \rangle$.

Let $\mathcal{D}$ be a **S**-derivation of $\vdash F$. Let $\mathcal{T}^{\mathcal{D}}$ be the tree in which any branch is equal to a $\sigma^{\mathcal{B}}$ for a branch $\mathcal{B}$ of $\mathcal{D}$. Let $\phi^{\mathcal{D}}$ be the union of all $\phi^{\mathcal{B}}$ for a branch $\mathcal{B}$ of $\mathcal{D}$.

Theorem 6. *If $\mathcal{D}$ is a **S**-derivation of $\vdash F$, then $\mathcal{S}^{\mathcal{D}} = \langle \mathcal{T}^{\mathcal{D}}, \phi^{\mathcal{D}} \rangle$ is a winning strategy for F. Moreover, if $\mathcal{D}$ is a **LF**-derivation then $\mathcal{S}^{\mathcal{D}}$ is a Lorenzen-Felscher strategy and if $\mathcal{D}$ is a **ST**-derivation, then $\mathcal{S}^{\mathcal{D}}$ is a Stubborn strategy.*

Proof. Each branch of $\mathcal{S}^{\mathcal{D}}$ is a play won by **P** in virtue of Lemma 4. The other conditions in the definition of strategy follows easily by the construction of the sequences composing $\mathcal{S}^{\mathcal{D}}$. The fact that $\mathcal{S}^{\mathcal{D}}$ is a Lorenzen-Felscher (resp. Stubborn) strategy when $\mathcal{D}$ is a **LF**-derivation (resp. **ST**-derivation) follows from Lemma 5.

Corollary 1. *Strategies, Lorenzen-Felscher Strategies and Stubborn Strategies are sound and complete for $\mathsf{IL}^{\rightarrow}$.*

We conclude by establishing that there is a bijective correspondence between the classes of winning strategies and derivations that we have considered.

Theorem 7. *The following statements hold:*

1. *The set of **S**-derivations is in one-to-one correspondence with the set of winning strategies;*
2. *The set of **LF**-derivations is in one-to-one correspondence with the set of Lorenzen-Felscher winning strategies*
3. *The set of **ST**-derivations is in one-to-one correspondence with the set of Stubborn winning strategies.*

Proof. The procedure we have used to transform winning strategies into derivations (see Definition 11) and the one we have used to obtain the converse result (see Definitions 12 and 13) are one the inverse of the other. Thus, the result follows.

6 Conclusion and Future Work

We have defined different classes of Lorenzen-style dialogical plays for intuitionistic logic by restricting the way in which **P** can play during a game. We have shown that winning strategies for such games naturally corresponds to particular GKi derivations obtained by limiting the application of GKis-rules in proof search procedures.

The correspondence between Stubborn strategies and **ST**-derivation, as well as the result we used to prove that the latter are sound and complete with respect to $\mathsf{IL}^{\rightarrow}$ (Lemma 2), suggest the existence of a one-to-one correspondence between these strategies and Hyland-Ong Winning Innocent Strategies. In future work, we want to study this correspondence in order to use dialogical logic to define denotational semantics of the simply typed lambda calculus [18], for which Hyland-Ong game semantics is a fully abstract denotational semantics [11,19]. Moreover, the results in [1,2] would suggest a way to define a dialogical system for the constructive modal logic CK.

The semantics of formal argumentation systems are often specified through the help of concepts originated in dialogic logic (e.g. E-strategies see [24]). We think it would be interesting to study a more abstract version of our stubborn strategies in the context of formal argumentation.

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