On the Existence of Fuzzy Contractual Allocations, Fuzzy Core and Perfect Competition in an Exchange Economy

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Abstract. The fuzzy core is well-known in theoretical economics, it is widely applied to model the conditions of perfect competition. In contrast, the original author's concept of fuzzy contractual allocation as a specific element of the fuzzy core is not so widely known in the literature, but it also represents a (refined) model of perfect competition. This motivates the study of its validity: the existence of fuzzily contractual allocations in an economic model; it also implies the existence (non-emptiness) of the fuzzy core and develops an approach from [15]. The proof is based on two well-known theorems: Michael's theorem on the existence of a continuous selector for a point-to-set mapping and Brouwer's fixed point theorem. In literature, only the non-emptiness of the fuzzy core was proven under essentially stronger assumptions—typically, it applies replicated economies and Edgeworth equilibria.

Keywords: Fuzzy core · Fuzzy contractual allocation · Edgeworth equilibria · Perfect competition · Existence theorems

1 Introduction

In modern economic theory, the idea of perfect competition is implemented in many ways. Among others one can find the famous Aumann [3] approach based on a model with a non-atomic set of economic agents, non-standard economies according to Brown–Robinson [5] and, of course, the asymptotic Debreu–Scarf Theorem [6], as well as other results, including the contractual approach developed by the author. The history of the idea of perfect competition goes back to Edgeworth and his well-known conjecture [7] that the core (contract curve) shrinks into equilibrium. The proof of this conjecture, based on the idea of replication of economic agents, was proposed in [6]. Later it turned out that the limit

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allocations from the core of the replicated economy, named by Aliprantis as Edgeworth equilibria, are elements of the fuzzy core of the economy—a concept introduced in [2]. Edgeworth equilibrium is an attainable allocation whose r-fold repetition belongs to the core of the r-fold replica of the original economy, for any positive integer r.

Due to Debreu–Scarf theorem on the limit coincidence of equilibria and the core for the replicated economy, the fuzzy core was started to be also applied to state the existence of competitive equilibrium. As a result, now the fuzzy core is widely used in theoretical economics, e.g. see [1,8,9]. One can see also [4] as one of the latest results on the existence of fussy core (under essentially stronger assumptions than in [15]). The original author's concept of fuzzy contractual allocation [10,12–14] is not so widely known in the literature, but it also represents an effective model of perfect competition in its simplest form. The idea of fuzzy contractual allocation is that, in the current contractual situation, agents can break contracts *asymmetrically*, without coordination with other individuals and without transferring information about their intentions, i.e. acting in a secret manner. Further, individuals can try to find a new contract, such that this contractual interaction—a break and a signing of a new contract—is beneficial to each of its participants. If this happens, we are talking about fuzzy contractual domination. Allocations that are not dominated in this sense are called fuzzy contractual. They have the highest level of stability and, as it follows from the analysis, every fuzzy contractual allocation belongs to the fuzzy core and presents competitive equilibrium. This allows us to state that it is a model of perfect competition.

Thus, both notions—fuzzy core and fuzzy contractual allocation—play key roles in modern economic theory, and the conditions under which they exist have a high theoretical meaning. The paper examines this problem and states the existence of fuzzy contractual allocations for an economy under very weak conditions[1]. This also implies the non-emptiness of the fuzzy core. Our proof is based on two well-known theorems, they are Michael's theorem on the existence of a continuous selector for a point-to-set mapping and Brouwer's fixed point theorem. A direct proof of the existence of fuzzy contractual allocations is a new result, while the non-emptiness of the fuzzy core is well known (under stronger assumptions). In [15] I suggested the direct proof of fuzzy core non-emptiness, which is efficient and shortest one among others; it also was stimulating our modern study, which develops our approach. As a result, I have produced new results that can be incorporated in proving the existence of Walrasian equilibrium in economies, even with infinite-dimensional commodity spaces, e.g. see [11].

[1] A convex model with a compact set $\mathcal{A}(X)$ of feasible allocations and preferences that are continuously extendable to a neighborhood of $\mathcal{A}(X)$.

2 An Economic Model, Fuzzy Core and Contractual Approach

I consider a typical exchange economy in which L denotes the (finite-dimensional) *space of commodities*. Let $\mathcal{I} = \{1, \ldots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i \subset L$, an initial endowment $\mathbf{e}_i \in L$, and a preference relation described by a point-to-set mapping $\mathcal{P}_i : X \Rightarrow X_i$ where $X = \prod_{j \in \mathcal{I}} X_j$ and $\mathcal{P}_i(x)$ denotes the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i relative to allocation $x \in X$. It is also can be applied the notation $y_i \succ_i x_i$ which is equivalent to $y_i \in \mathcal{P}_i(x)$ (to simplify notations; preferences can indirectly depend on other agents consumption $x_j \in X_j$ $j \in \mathcal{I}$, $j \neq i$). So, the pure exchange model may be represented as a triplet

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

Let us denote by $\mathbf{e} = (\mathbf{e}_i)_{i \in \mathcal{I}}$ the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i \in \mathcal{I}} X_i$ and let

$$\mathcal{A}(X) = \{\, x \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i \,\}$$

be the set of all *feasible allocations*. Now let us recall some definitions.

A pair (x, p) is said to be a *quasi-equilibrium* of $\mathcal{E}$ if $x \in \mathcal{A}(X)$ and there exists a linear functional $p \neq 0$ onto L such that

$$\langle p, \mathcal{P}_i(x) \rangle \geq p x_i = p \mathbf{e}_i, \quad \forall i \in \mathcal{I}.$$

A quasi-equilibrium such that $x_i' \in \mathcal{P}_i(x)$ actually implies $p x_i' > p x_i$ is a *Walrasian or competitive equilibrium.*

An allocation $x \in \mathcal{A}(X)$ is said to be dominated (blocked) by a nonempty coalition $S \subseteq \mathcal{I}$ if there exists $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} \mathbf{e}_i$ and $y_i^S \in \mathcal{P}_i(x)$ $\forall i \in S$.

The *core* of $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that are blocked by no (nonempty) coalition.

Everywhere below we assume that model $\mathcal{E}$ satisfies the following assumption.

(A) *For each $i \in \mathcal{I}$, $X_i \subset L$ is a convex closed subset, $\mathbf{e}_i \in X_i$ and, for every $x = (x_j)_{j \in \mathcal{I}} \in \mathcal{A}(X)$:*

$$\mathcal{P}_i(x) = \mathrm{co}[\mathcal{P}_i(x) \cup \{x_i\}] \setminus \{x_i\}$$

is a convex set.

Notice that due to (**A**) preferences may be satiated, i.e., $\mathcal{P}_i(x) = \emptyset$ is possible for some agent i and $x \in X$. However, if $\mathcal{P}_i(x) \neq \emptyset$, then preference is *locally non-satiated* at the point x.

For the existence of objects under study, we apply the following (weak) preference continuity assumption.

(**C**) *For each* $i \in \mathcal{I}$ *there is a point-to-set mapping* $\hat{\mathcal{P}}_i : X \Rightarrow L$ *such that for every* $x \in \mathcal{A}(X)$ *the image* $\hat{\mathcal{P}}_i(x)$ *is convex, open in* L*, implements*

$$\mathcal{P}_i(x) = \hat{\mathcal{P}}_i(x) \cap X_i$$

and for every $y_i \in \hat{\mathcal{P}}_i(x)$ *the set*

$$\hat{\mathcal{P}}_i^{-1}(y_i) = \{z \in X \mid y_i \in \hat{\mathcal{P}}_i(z)\}$$

is open one in X.

Remark 1. Notice that our modern assumptions (**A**) and (**C**) are a bit stronger of applied in [15]. In (**A**) I assume in addition that $\alpha y_i + (1-\alpha)x_i \in \mathcal{P}_i(x)$ for every $y_i \in \mathcal{P}_i(x)$ and $\alpha \in [1, 0)$. For (**C**) now I assumed also that images $\mathcal{P}_i(x)$, $x \in \mathcal{A}(X)$ can be extended to a neighbourhood of X_i, $i \in \mathcal{I}$.

Also, below without loss of generality to simplify notations, I will assume that X_i is convex and has *full dimension*, i.e. $\operatorname{int} X_i \neq \emptyset\ \forall i \in \mathcal{I}$.

In the framework of model $\mathcal{E}$, a formal mechanism of contracting and recontracting can be introduced. This mechanism reflects the idea that any group of agents can find and realize some (permissible) within-the-group exchanges of commodities referred to as contracts. The mechanism defines the rules of contracting.

2.1 Fuzzy Core and Fuzzy Contractual Allocations

The concept of the fuzzy core is fruitfully working in the theory of economic equilibrium. I recall that any vector

$$t = (t_1, \ldots, t_n) \neq 0, \quad 0 \leq t_i \leq 1, \quad \forall i \in \mathcal{I}$$

maybe identified with a fuzzy coalition, where the real number t_i is interpreted as the measure of agent i participation in the coalition. A coalition t is said to dominate (block) an allocation $x \in \mathcal{A}(X)$ if there exists $y^t \in \prod_{\mathcal{I}} X_i$ such that

$$\sum_{i \in \mathcal{I}} t_i y_i^t = \sum_{i \in \mathcal{I}} t_i \mathbf{e}_i \iff \sum_{i \in \mathcal{I}} t_i (y_i^t - \mathbf{e}_i) = 0 \tag{1}$$

and

$$y_i^t \succ_i x_i, \quad \forall i \in \operatorname{supp}(t) = \{i \in \mathcal{I} \mid t_i > 0\}. \tag{2}$$

The set of all feasible allocations which cannot be dominated by fuzzy coalitions is called the *fuzzy core* of the economy $\mathcal{E}$ and is denoted by $\mathcal{C}^f(\mathcal{E})$.

We begin with a study of the specific properties of the fuzzy core allocations. The elements of fuzzy core are defined via conditions (1), (2) which for non-satiated preferences, i.e., when $\mathcal{P}_i(x) \neq \emptyset$, $\forall i \in \mathcal{I}$, the domination may be equivalently rewritten in the form[2]

$$0 \notin \sum_{i \in \mathcal{I}} t_i (\mathcal{P}_i(x) - \mathbf{e}_i).$$

[2] Admitting some inaccuracy in formulas here and below, we identify a vector with a one-element set containing it.

Thus $x \in \mathcal{C}^f(\mathcal{E})$ is now equivalent to[3]

$$0 \notin \mathrm{co}[\underset{\mathcal{I}}{\cup}(\mathcal{P}_i(x) - \mathbf{e}_i)], \tag{3}$$

that after applying separation theorem allows us to conclude that the elements of the fuzzy core are quasi-equilibria. Below we propose other useful in applications characterizations of fuzzy core points presented in "geometrical" terms (introduced in [10]). To this end, let us consider the sets

$$\Upsilon_i(x) = \mathrm{co}(\mathcal{P}_i(x) \cup \{\mathbf{e}_i\}), \quad i \in \mathcal{I}.$$

Due to the convexity of $\mathcal{P}_i(x)$, for $\mathcal{P}_i(x) \neq \emptyset$, conclude

$$\mathrm{co}(\mathcal{P}_i(x) \cup \{\mathbf{e}_i\}) = \underset{0 \le \lambda \le 1}{\cup}[\lambda \mathcal{P}_i(x) + (1-\lambda)\mathbf{e}_i] = \underset{0 \le \lambda \le 1}{\cup} \lambda(\mathcal{P}_i(x) - \mathbf{e}_i) + \mathbf{e}_i, \quad i \in \mathcal{I}.$$

This implies that the condition $z + \mathbf{e} \in \prod_{\mathcal{I}} \Upsilon_i(x)$, where $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$, is equivalent to the existence of $0 \le \lambda_i \le 1$ and $[y_i \in \mathcal{P}_i(x) \neq \emptyset$ and $y_i = \mathbf{e}_i$, if $\mathcal{P}_i(x) = \emptyset]$, $i \in \mathcal{I}$ such that

$$z = (\lambda_1(y_1 - \mathbf{e}_1), \ldots, \lambda_n(y_n - \mathbf{e}_n)).$$

Hence, due to (1), (2)

$$x \in \mathcal{C}^f(\mathcal{E}) \iff \nexists z \in L^{\mathcal{I}},\, z \neq 0: \quad z + \mathbf{e} \in \prod_{\mathcal{I}} \Upsilon_i(x) \quad \& \quad \sum_{i \in \mathcal{I}} z_i = 0 \iff$$

$$\prod_{\mathcal{I}} \Upsilon_i(x) \bigcap \mathcal{A}(L^{\mathcal{I}}) = \{\mathbf{e}\}, \tag{4}$$

where $\mathcal{A}(L^{\mathcal{I}})$ is a subspace defined by the balance constraints of a pure exchange economy:

$$\mathcal{A}(L^{\mathcal{I}}) = \{(z_1, \ldots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}.$$

Notice that characterization (4) is also valid for satiated preferences. In doing so, we have proven the following

Proposition 1. *An allocation $x \in \mathcal{A}(X)$ is the element of fuzzy core if and only if relation (4) is true.*

The direct and effective proof of fuzzy core non-emptiness is based on relation (4) and I suggested it earlier in [15]. In the case of a *2-agent economy*, Fig. 1 presents a graphic illustration of conducted analysis in the Edgeworth's box for a 2-goods economy. In this case, an allocation $x = (x_1, x_2)$ lying in the fuzzy core is equivalent to the convex hulls of $\mathcal{P}_1(x_1) \cup \{\mathbf{e}_1\}$ and of $[\bar{\mathbf{e}} - \mathcal{P}_2(\bar{\mathbf{e}} - x_1)] \cup \{\mathbf{e}_1\}$, $\bar{\mathbf{e}} = \mathbf{e}_1 + \mathbf{e}_2$ having only one point, $\mathbf{e}_1$, in common.

[3] Clearly, for a dominating fuzzy coalition t one may always think that $\sum_{i \in \mathcal{I}} t_i = 1$.

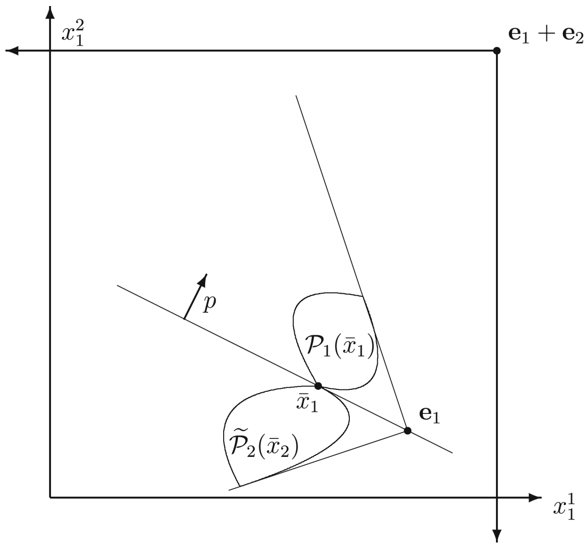

Fig. 1. *Fuzzy core*

One more important notion, which probably still is not good enough qualified in theoretical economy, is the notion of fuzzy contractual allocation. First I recall briefly the conceptual apparatus of the theory of barter contracts, see [10,12,13].

Any vector $v = (v_i)_{i\in\mathcal{I}} \in L^{\mathcal{I}}$ satisfying $\sum_{i\in\mathcal{I}} v_i = 0$ is called a barter (exchange) *contract*. Such barter contracts are used in pure exchange economies, as well as in the consumption sector in the economy with production. In what follows, we assume that any barter agreement is valid. With every finite collection V of (permissible) contracts, it can be associated allocation $x(V) = \mathbf{e} + \sum_{v\in V} v$, where the vector $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n) \in X$ is an initial endowments allocation. If $\mathbf{e} + \sum_{v\in U} v \in X \ \forall U \subseteq V$, i.e., if any part of the contracts is broken one can get anyway a feasible allocation, then we call V a *web* of contracts. The consideration of webs of contracts allows us to study a huge massive of contractual interactions including different possibilities of contracts breaking (one of them is a fuzzy contractual interaction) for details see [10,12,13].

Let V be a web of contracts. For every $v \in V$ we consider and put into correspondence an n-dimension vector

$$t^v = (t_1^v, t_2^v, \dots, t_n^v), \quad 0 \le t_i^v \le 1, \quad \forall i \in \mathcal{I},$$

and let

$$v^t = (t_1^v v_1, t_2^v v_2, \dots, t_n^v v_n)$$

be the vector of commodity bundles formed from contract $v = (v_i)_{i\in\mathcal{I}}$ when all agents "break" individual bundles (fragments) of this contract in shares

$(1-t_i^v)_{i\in\mathcal{I}}$. Denote $T(V) = T = \{t^v \mid v \in V\}$ and introduce

$$V^T = \{v^t \mid v \in V,\ t^v \in T\}, \quad \Delta(V^T) = \sum_{v^t \in V^T} v^t. \tag{5}$$

Definition 1. *An allocation $x \in \mathcal{A}(X)$ is called fuzzy contractual if there exists a web V such that $x = x(V)$ and for every $T(V)$ there is no barter contract $w = (w_1, \dots, w_n) \in L^{\mathcal{I}}$, $\sum_{i\in\mathcal{I}} w_i = 0$, such that for*

$$\xi_i = \xi_i(T, V, w) = \mathbf{e}_i + \Delta_i(V^T) + w_i, \quad i \in \mathcal{I} \tag{6}$$

one has

$$\xi_i \succ_i x_i \quad \forall i:\ \xi_i \neq x_i. \tag{7}$$

So, for this kind of allocation the negation of domination means that the implementing web of contracts is *stable* relative to asymmetric *partial breakings of contracts* with or without concluding a new contract.

In economic terms, this notion can be explained in the following way. During recontracting agents may make mistakes, coordination among coalition members may work imperfectly, information can be hiden and so on. As a result, an agent i can (erroneously) think that after the partial breaking of current contracts he/she will have a commodity bundle $x_i^T = \mathbf{e}_i + \Delta_i(V^T)$ and that commodities from x_i^T may be mutually beneficial exchanged so that to dominate the current allocation $x = (x_i)_{\mathcal{I}}$. If allocation $x(V)$ is not fuzzy contractual, then the last possibility may (potentially) destroy agreements and allocation will be changed. Thus fuzzy contractual allocations are protected from this kind of agreement destructions. Notice, that agents also allow only break contracts and do not conclude a new one.

We continue from a preliminary result describing mathematical properties of fuzzy contractual allocations, that is of interest in its own right.

Proposition 2. *An allocation $x \in \mathcal{A}(X)$ is fuzzy contractual if and only if*[4]

$$\mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] = \emptyset \quad \forall i \in \mathcal{I} \tag{8}$$

and

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(x) + \mathrm{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}] \bigcap \mathcal{A}(L^{\mathcal{I}}) = \{\mathbf{e}\}. \tag{9}$$

Notice that in this proposition $\mathcal{P}_i(x) = \emptyset$ is possible for some $i \in \mathcal{I}$: by definition $\emptyset + A = \emptyset$ for any $A \subseteq L$. Condition (8) indicates that a partial break of contracts without signing of a new one cannot be beneficial. The requirement (9) denies the existence of a dominating coalition after the partial asymmetric break of the contract $v = (x - \mathbf{e})$.

Figure 2 illustrates Proposition 2 result in the Edgeworth's box. Here $\widetilde{\mathcal{P}}_2(x_2) = \bar{\mathbf{e}} - \mathcal{P}_2(\bar{\mathbf{e}} - x_1)$, $\bar{\mathbf{e}} = \mathbf{e}_1 + \mathbf{e}_2$ and one can see that preferred bundles are extended along linear segment with endpoints x_1, $\mathbf{e}_1$.

[4] *A linear* segment with ends $a, b \in L$ is the set $[a, b] = \mathrm{co}\{a, b\} = \{\lambda a + (1-\lambda) b \mid 0 \le \lambda \le 1\}$.

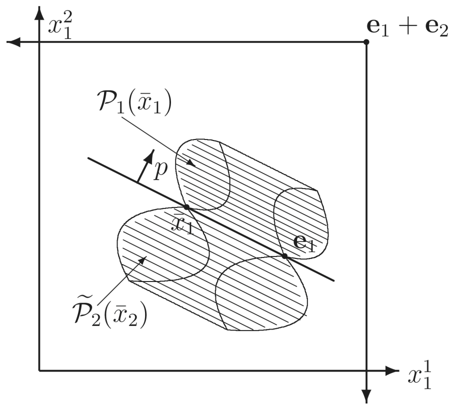

Fig. 2. *Fuzzy contractual allocations*

Proof of Proposition 2. Let x be a fuzzy contractual allocation implemented by a web V, i.e., $x = x(V)$ for some web V, satisfying Definition 1. Then (8) is clearly true one. Suppose that (9) is false and therefore does exist $y = (y_i)_\mathcal{I} \neq \mathbf{e}$ which belongs to the left part of equality (9). Consider coalition $S = \{i \in \mathcal{I} \mid y_i \neq \mathbf{e}_i\}$. Notice $\mathcal{P}_i(x) \neq \emptyset$, $i \in S$ and find $z_i \in \mathcal{P}_i(x)$, $i \in S$ such that $y_i = z_i + \lambda_i(\mathbf{e}_i - x_i)$, for some real $0 \leq \lambda_i \leq 1$, $i \in S$ and $y_i = \mathbf{e}_i$, $i \notin S$. Determine $w_i = y_i - \mathbf{e}_i$, $i \in \mathcal{I}$. Since $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ then $\sum_\mathcal{I} w_i = 0$ and therefore $w = (w_i)_{i \in \mathcal{I}}$ is a contract with $\text{supp}\,(w) = S \neq \emptyset$. One can write

$$z_i = y_i - \mathbf{e}_i + \lambda_i(x_i - \mathbf{e}_i) + \mathbf{e}_i = w_i + \lambda_i \sum_{v \in V} v_i + \mathbf{e}_i, \quad i \in S.$$

Now for all $v \in V$ put $t_i = t_i^v = \lambda_i$, $i \in S$, and $t_i = 1$, $i \notin S$ and apply $T(V) = \{t^v\}_{v \in V}$ for allocation $x = x(V)$. We have $x^T = \mathbf{e} + \Delta(V^T)$, whereby construction $x_i^T = \mathbf{e}_i + t_i(x_i - \mathbf{e}_i)$, $\forall i \in \mathcal{I}$. Therefore, by construction

$$\xi_i = w_i + x_i^T = z_i \in \mathcal{P}_i(x), \quad \forall i \in S = \{j \in \mathcal{I} \mid x_i \neq \xi_i\},$$

that contradicts (7).

Show that if a contractual allocation x satisfies (8) and (9) then it is fuzzy contractual relative to the web $V = \{x - \mathbf{e}\}$. Assume contrary and find $T = \{t\}$ and a contract $w = (w_i)_\mathcal{I}$, $\text{supp}\,(w) = S \neq \emptyset$, such that

$$w_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \in \mathcal{P}_i(x), \ \forall i \in S \iff z_i = w_i + \mathbf{e}_i \in \mathcal{P}_i(x) + t_i(\mathbf{e}_i - x_i), \ \forall i \in S.$$

Let us determine $z_i = \mathbf{e}_i$ for $i \notin S$. Now due to contract's definition conclude $\sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i$ that implies the allocation $z \neq \mathbf{e}$ belongs to the left part of (9) and this is a contradiction. ■

Notice that as soon as for every feasible allocation $x = (x_i)_\mathcal{I}$ we have

$$\mathbf{e}_i \in \Upsilon_i(x) \subset (\mathcal{P}_i(x) + \text{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}, \quad \forall i \in \mathcal{I},$$

then due to Propositions 1, 2 every fuzzy contractual allocation belongs to the fuzzy core of economy. However, in general, the property of an allocation to be fuzzy contractual is still a bit stronger than being an element of fuzzy core. The following result clarifies the relationships between two fuzzy notions.

Lemma 1. *Let* $x \in \mathcal{A}(X)$ *and* $\mathcal{P}_i(x) \neq \emptyset$ *for all* $i \in \mathcal{I}$. *Then* $x \in \mathcal{C}^f(\mathcal{E})$ *implies*:

$$\prod_{i \in \mathcal{I}} (\mathcal{P}_i(x) + \mathrm{co}\{0, \mathbf{e}_i - x_i\}) \bigcap \mathcal{A}(L^{\mathcal{I}}) = \emptyset. \tag{10}$$

The comparing of formulas (10) and (9) makes clearer the difference between fuzzy core allocation and fuzzy contractual one. One can see that this difference is not too big that allows us in appropriate circumstances to interpret allocations from fuzzy core as fuzzy contractual ones.[5] Moreover, the fact that every element of fuzzy core is a quasi-equilibrium (this is why fuzzy core is so popular in existence theory) can be also easily derived from formula (10). In fact, separating sets in (10) by a (non-zero) linear functional $\pi = (p_1, \ldots, p_n) \in L^{\mathcal{I}}$ one can conclude:

(*i*) $p_i = p_j = p \neq 0$ for each $i, j \in \mathcal{I}$; this is so because π is bounded on $\mathcal{A}(L^{\mathcal{I}}) = \{(z_1, \ldots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}$. So, one can take p as a price vector.

(*ii*) Due to construction and in view of preferences are locally non-satiated at the point $x \in \mathcal{A}(X)$ the points x_i and $\mathbf{e}_i$ belong to the closure of $\mathcal{P}_i(x)$+co$\{0, \mathbf{e}_i - x_i\}$. Therefore, via separating property we have

$$\sum_{j \neq i} p\mathbf{e}_j + px_i \geq \sum_{\mathcal{I}} p\mathbf{e}_j \;\Rightarrow\; px_i \geq p\mathbf{e}_i \;\; \forall i \in \mathcal{I},$$

that is possible only if $px_i = p\mathbf{e}_i$ $\forall i \in \mathcal{I}$. So, we obtain budget constraints for consumption bundles.

(*iii*) By separation property for each i we also have

$$\langle p, \mathcal{P}_i(x) + \mathrm{co}\{0, \mathbf{e}_i - x_i\}\rangle \geq p\mathbf{e}_i,$$

that by (*ii*) implies $\langle p, \mathcal{P}_i(x)\rangle \geq px_i = p\mathbf{e}_i$. So we proved that p is (quasi)equilibrium prices for allocation $x = (x_i)_{i \in \mathcal{I}}$.

As a result, one can see that if an economic model is such that every quasi-equilibrium is equilibrium, then every fuzzy core allocation is fuzzy contactual one and therefore two fuzzy concepts are equivalent each other. Conditions delivering this fact are well known in literature; for example, it is the case when an economy is irreducible. Moreover there is also a nice possibility to describe fuzzy contractual allocation as an equilibrium with nonstandard prices, see [14].

[5] Earlier in literature allocations from fuzzy core were interpreted only as Edgeworth's equilibria and served as a technical tool more than an economic concept.

3 Existence Theorems

Theorem 1. *Let in an exchange economy* $\mathcal{A}(X)$ *be bounded and assumptions* (**A**), (**C**) *hold. Then fuzzy contractual allocations do exist.*

Corollary 1. *In Theorem 1 conditions fuzzy core is non-empty, i.e.* $\mathcal{C}^f(\mathcal{E}) \neq \emptyset$.

The existence of contractual core, fuzzy and fuzzy-contractual allocations can be established by applying Brouwer (or Kakutani) fixed point theorem and Michael's [16] continuous selector theorems. The proof of Theorem 1 is presented below; we use characterization described in Proposition 2.

4 Proofs

For the further analysis we need auxiliary lemmas. Let $NS \subset \mathcal{A}(X)$ be an area of all lower unstable contractual allocations, i.e. $x \in \mathcal{A}(X)$ for which (8) is false; also let $\Omega \subset \mathcal{A}(X)$ be a subset consisting the points $x \in \mathcal{A}(X)$ for which (9) is false. Below I study some properties of these sets.

Lemma 2. *If economy* $\mathcal{E}$ *obeys* (**A**) *and* (**C**), *then* NS *and* Ω *are open in* $\mathcal{A}(X)$.

Proof. An area of all lower stable contractual allocations is specified as

$$LS = \{x = (x_i)_{i \in \mathcal{I}} \in \mathcal{A}(X) \mid \mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] = \emptyset \ \forall i \in \mathcal{I}\}$$

and now I consider its supplement $NS = \mathcal{A}(X) \setminus LS$, this is the set of all allocations for which there is an agent interested in a partial break of current contract $v = x - \mathbf{e}$. Suppose $\mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] \neq \emptyset$ for some $i \in \mathcal{I}$. It means there is $y_i \in \hat{\mathcal{P}}_i(x) \cap [x_i, \mathbf{e}_i]$, $y_i \neq x_i$. Since $\hat{\mathcal{P}}_i(x)$ is assumed to be an open one, there is a *finite* set $A \subset \hat{\mathcal{P}}_i(x)$ such that

$$y_i \in \operatorname{int}(\operatorname{co}A) \quad \Rightarrow \quad x \in \Theta = \bigcap_{a \in A} \hat{\mathcal{P}}_i^{-1}(a),$$

where $\Theta \subset X$ is open in X. Due to $\hat{\mathcal{P}}_i(z)$, $z \in \mathcal{A}(X)$ are also assumed to be convex ones we conclude $y_i \in \operatorname{co}A \subset \hat{\mathcal{P}}_i(z) \ \forall z \in \Theta$. Now if $\varepsilon > 0$ is so that

$$z_i \in L, \ ||z_i - y_i|| < \varepsilon \ \Rightarrow \ z_i \in \operatorname{co}A \ \ \& \ \ x' \in \mathcal{A}(X), \ ||x' - x|| < \varepsilon \ \Rightarrow \ x' \in \Theta,$$

then for these allocations one can conclude

$$[x'_i, \mathbf{e}_i] \cap \operatorname{co}A \neq \emptyset \quad \Rightarrow \quad [x'_i, \mathbf{e}_i] \cap \mathcal{P}_i(x') \neq \emptyset \ \ \forall x' \in \mathcal{A}(X) : \ ||x' - x|| < \varepsilon.$$

As a result we conclude NS is the neighbourhood of every its point and, therefore, is an open subset of $\mathcal{A}(X)$.

Next I consider $\Omega \subset \mathcal{A}(X)$. The reasoning is similar to that presented above: for every $x \in \Omega$ one can find $t = (t_i)_{i\in\mathcal{I}} \in [0,1]^{\mathcal{I}}$ and a contract $w = (w_i)_{i\in\mathcal{I}}$, $\sum_{i\in\mathcal{I}} w_i = 0$, such that

$$y_i = w_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \in \mathcal{P}_i(x), \quad \forall i:\ y_i \neq x_i.$$

For these $i \in \mathcal{I}$ there are finite $A_i \subset L$ such that

$$y_i \in \text{int}(\text{co}A_i) \subset \hat{\mathcal{P}}_i(z), \quad \forall z \in \bigcap_{i:y_i\neq x_i} \hat{\mathcal{P}}_i^{-1}(A_i) \cap \mathcal{A}(X) \subset \Omega.$$

It implies there is $\varepsilon > 0$ such that

$$z = (z_i)_{i\in\mathcal{I}} \in \mathcal{A}(X),\ ||z - x|| < \varepsilon \quad \Rightarrow \quad y_i = w_i + t_i(z_i - \mathbf{e}_i) + \mathbf{e}_i \in \tilde{\mathcal{P}}_i(z), \quad \forall i:\ y_i \neq x_i.$$

As a result $\Omega \subset \mathcal{A}(X)$ is a neighbourhood of every its point and therefore it is an open subset in $\mathcal{A}(X)$, as we wanted to prove. ■

Let us study other properties of these allocations from Ω. Assuming $x \in \Omega$ we consider the set of contracts $\varphi(x)$ that fuzzily block this allocation:

$$\varphi(x) = \{(v_i, t_i)_{\mathcal{I}} \in (L \times [0,1])^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0, v \neq 0 : \forall i \notin \text{supp}\,(v), t_i = 1 \quad \&$$

$$\forall i \in \mathcal{I} \quad v_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i = g_i(x), \quad \forall i \in \text{supp}\,(v), \quad g_i(x) \succ_i x_i\}. \tag{11}$$

The following lemma presents crucial properties of the point-to-set mapping $\varphi(\cdot)$. First I recall the definition of lower hemicontinuous[6] point-to-set mapping.

Definition 2. *Let Y, Z be topological spaces. A point-to-set mapping $\psi : Y \Rightarrow Z$ is called lower hemicontinuous (l.h.c.) iff*

$$\psi^{-1}(V) = \{y \in Y \mid \psi(y) \cap V \neq \emptyset\}$$

is open for every open $V \subset Z$. For a metric spaces Y, Z a l.h.c. mapping can be equivalently characterized as follows:

For every $y \in Y$, $z \in \psi(y) \subset Z$ and every sequence $y_m \to y$ there is a subsequence $y_{m_k} \in Y$ and a sequence $z_k \in \psi(y_{m_k})$, $m, k \in \mathbb{N}$ such that $z_k \to z$ for $k \to \infty$.

Lemma 3. *If $x \in \Omega$ the set $\varphi(x)$ is convex and non-empty. Moreover, the mapping $\varphi : \Omega \Rightarrow (L \times [0,1])^{\mathcal{I}}$ is lower hemicontinuous one.*

Proof. I first state the convexity of $\varphi(x)$. Let $(w', t'), (w'', t'') \in \varphi(x)$ and $\alpha \in (0,1)$. Then, from the convexity of preferences (**A**), having in mind $t'_i = 1$ for $i \notin \text{supp}\,(w')$ and, similarly, $t''_i = 1$ for $i \notin \text{supp}\,(w'')$ we have:

$$\forall i \in \text{supp}\,(w') \cup \text{supp}\,(w'')$$

[6] According to the modern views, the term semi-continuous mapping is specifically applied for a function—point-to-point map—and hemicontinuous for a correspondence.

$$\alpha(w'_i + t'_i(x_i - \mathbf{e}_i) + \mathbf{e}_i) + (1-\alpha)(w''_i + t''_i(x_i - \mathbf{e}_i) + \mathbf{e}_i) \succ_i x_i.$$

Thus, with respect to $t = \alpha t' + (1-\alpha)t''$, for any $\alpha \in [0,1]$ the contract

$$w = \alpha w' + (1-\alpha)w'' : \quad (w,t) \in \varphi(x).$$

Next, we show that due to (**C**) point-to-set mapping $\varphi(\cdot)$, defined in (11) is lower hemicontinuous.

Indeed, let $(v,t) \in \varphi(x)$ be fixed. Now according to (11), for $i \in \mathcal{I}$ such that $v_i \neq 0$ we have

$$g_i(x) = v_i + t_i(x_i - \mathbf{e}_i) + \mathbf{e}_i \in \mathcal{P}_i(x).$$

Clearly, without loss of generality it is enough to study the case $g_i(x) \in \operatorname{int} \mathcal{P}_i(x)$. Now let $A_i \subset \operatorname{int} \mathcal{P}_i(x)$ be a finite subset such that $g_i(x) \in \operatorname{int}(\mathrm{co}A_i)$, i.e. $\mathrm{co}A_i$ is a neighborhood of $g_i(x)$. We specify

$$V_i = \bigcap_{a \in A_i} \mathcal{P}_i^{-1}(a).$$

Due to (**C**) and (**A**) this is an open neighborhood of $x \in \mathcal{A}(X)$ such that $g_i(x) \in \mathrm{co}A_i \subset \mathcal{P}_i(y)$ for every $y \in V_i$. So, if $x^m \in \mathcal{A}(X)$, $x^m \to x$ for the natural $m \to \infty$, then for some $k \in \mathbb{N}$ we have: $\forall m \geq k$ $\forall i \in \mathrm{supp}\,(v)$

$$g_i(x^m) = v_i + t_i(x_i^m - \mathbf{e}_i) + \mathbf{e}_i \in \mathrm{co}A_i \subset \mathcal{P}_i(x^m) \quad \& \quad g_i(x^m) \to g_i(x).$$

As a result, via (11) one concludes $(v^m, t^m) = (v,t) \in \varphi(x^m)$ for all $m \in \mathbb{N}$ big enough. This proves, by definition, $\varphi(\cdot)$ is lower hemicontinuous in $x \in \Omega \subset \mathcal{A}(X)$. ■

In the proof of Lemma 4 below I apply the following Michael theorem (see [16] p. 368, Th 3.1‴, (c)) on the existence of a continuous selector in its simplified finite-dimensional presentation.[7]

Theorem 2 (Michael, 1956). *Let Y and Z be subsets of finite-dimensional linear spaces. Then every l.h.c. point-to-set mapping $\psi : Y \rightrightarrows Z$ having nonempty convex images $\psi(y) \subset Z$ $\forall y \in Y$ has a continuous selector.*

Lemma 4. *There is a continuous function $h : \Omega \to \mathcal{A}(X)$ such that for some continuous $\xi_i : \Omega \to X_i$, $\gamma_i : \Omega \to [0,1]$ such that $\xi_i(x) \in \mathcal{P}_i(x) \cup \{x_i\}$ one has*

$$h_i(x) = \xi_i(x) + \gamma_i(x)(\mathbf{e}_i - x_i), \quad \forall x \in \Omega, \ i \in \mathcal{I}$$

and, moreover, for any $x \in \Omega$ there exists $i \in \mathcal{I}$ such that $h_i(x) = \xi_i(x) \succ_i x_i$, i.e. $\xi_i(x) \in \mathcal{P}_i(x)$ and $\gamma_i(x) = 0$.

[7] Note that in original paper item (c) has a typo for the range of $\phi : X \to \mathcal{K}(Y)$. Author denoted $\mathcal{K}(Y)$ as a set of all convex subsets of Y, but speak and prove the result for a narrower class of sets $\mathcal{D}(Y) \subset \mathcal{K}(Y)$, see p. 372. Here I present a less general result, to avoid a cumbersome specification of $\mathcal{D}(Y)$.

Proof. According to assumptions and Lemma 3, the correspondence $\varphi(\cdot)$ specified in (11) obeys all requirements of Michael's theorem on the existence of continuous selector: a lower hemicontinuous correspondence having domain $\Omega \subset \mathcal{A}(X)$, and with convex non-empty images. Thus, there is a continuous mapping satisfying

$$(v,t)(\cdot) : \Omega \to (L \times [0,1])^{\mathcal{I}} \text{ such that } (v(x), t(x)) \in \varphi(x) \;\; \forall x \in \Omega.$$

By definition, we have $\sum_{i \in \mathcal{I}} v_i(x) = 0$ and, $t_i(x) = 1$, $g_i(x) = x_i$ for $i \notin \operatorname{supp}(v(x))$ and

$$\forall i \in \operatorname{supp}(v), \quad v_i(x) + t_i(x)(x_i - \mathbf{e}_i) + \mathbf{e}_i = g_i(x) \in \mathcal{P}(x).$$

Therefore, for $f_i(x) = v_i(x) + \mathbf{e}_i = g_i(x) + t_i(x)(\mathbf{e}_i - x_i)$, $i \in \mathcal{I}$ we obtain $\sum_{i \in \mathcal{I}} f_i(x) = \sum_{i \in \mathcal{I}} \mathbf{e}_i$. Thus, we find a continuous mapping $f : \Omega \to \mathcal{A}(L^{\mathcal{I}})$ such that

$$\forall x \in \Omega \;\; \mathbf{e} \neq f(x) \in \prod_{i \in \mathcal{I}} [(\mathcal{P}_i(x_i) + \operatorname{co}\{0, \mathbf{e}_i - x_i\}) \cup \{\mathbf{e}_i\}] \bigcap \mathcal{A}(L^{\mathcal{I}}).$$

Now we define $t^{min}(x) = \min_{j \in \mathcal{I}}(t_j(x))$ and specify

$$h_i(x) = \frac{x_i + g_i(x) + (t_i(x) - t^{min}(x))(\mathbf{e}_i - x_i)}{2}, \quad x \in \Omega, \; i \in \mathcal{I}. \tag{12}$$

So, as

$$\sum_{i \in \mathcal{I}} (g_i(x) + t_i(x)(\mathbf{e}_i - x_i)) = \sum_{i \in \mathcal{I}} \mathbf{e}_i, \quad \sum_{i \in \mathcal{I}} (\mathbf{e}_i - x_i) = 0$$

we conclude

$$\sum_{i \in \mathcal{I}} h_i(x) = \sum_{i \in \mathcal{I}} \mathbf{e}_i.$$

Moreover, for $i \in \mathcal{I}$ such that $t_i(x) = t^{min}(x)$ we have $h_i(x) = \frac{x_i + g_i(x)}{2}$ that due to $x_i \in \operatorname{cl} \mathcal{P}_i(x)$ and $g_i(x) \in \mathcal{P}_i(x)$ gives $h_i(x) \in \mathcal{P}_i(x)$. Now we need to show only that $h_i(x) \in X_i \; \forall i \in X_i$.

For $g_i(x) = x_i$ one can put $\frac{(t_i(x) - t^{min}(x))}{2} = \alpha_i \in [0,1]$ and by (12) conclude

$$h_i(x) = (1 - \alpha_i)x_i + \alpha_i \mathbf{e}_i \in X_i.$$

For $g_i(x) \in \mathcal{P}_i(x) \subset X_i$ via (12) for $\beta_i = t_i(x) - t^{min}(x)$ we have $(1-\beta_i)x_i + \beta_i \mathbf{e}_i \in X_i$ and therefore

$$h_i(x) = \frac{g_i(x)}{2} + \frac{(1 - \beta_i)x_i + \beta_i \mathbf{e}_i}{2} \in X_i.$$

This proves the map specified in (12) has range $\mathcal{A}(X)$. Now putting

$$\xi_i(x) = \frac{x_i + g_i(x)}{2}, \quad \gamma_i(x) = \frac{t_i(x) - t^{min}(x)}{2}, \quad i \in \mathcal{I},$$

we can redefine the map $h : \Omega \to \mathcal{A}(X)$; it obeys all requirements of Lemma 4. ■

Proof of Theorem 1. Recall that

$$LS = \{x = (x_i)_{i \in \mathcal{I}} \in \mathcal{A}(X) \mid \mathcal{P}_i(x) \cap [x_i, \mathbf{e}_i] = \emptyset \ \forall i \in \mathcal{I}\}$$

is an area of all lower stable contractual allocations and we consider its supplement $NS = \mathcal{A}(X) \backslash LS$, this is the set of all allocations for which there is an agent interested in a partial break of current contract $v = x - \mathbf{e}$. For $x \in NS$ condition (8) is false. Also we specified $\Omega \subset \mathcal{A}(X)$ as a subset consisting the points $x \in \mathcal{A}(X)$ for which (9) is false. Now let us suppose that

$$\Omega \cup NS = \mathcal{A}(X)$$

and show that it is impossible. According to the assumptions and Lemma 2, NS and Ω are an open subsets of $\mathcal{A}(X)$.

We specify $q : NS \to \mathcal{A}(X)$ by formula

$$q(x) = \frac{x + \mathbf{e}}{2}$$

and "glue" this mapping with $h(\cdot)$ defined in Lemma 4, setting

$$f(x) = \alpha(x)q(x) + \beta(x)h(x), \quad x \in \mathcal{A}(X),$$

where $\alpha : \mathcal{A}(X) \to [0,1]$, $\beta : \mathcal{A}(X) \to [0,1]$ are continuous functions, such that $\alpha(x) = 1$ for $x \in NS \setminus \Omega$, $\beta(x) = 1$ for $x \in \Omega \setminus NS$, and $\alpha(x) + \beta(x) = 1$ $\forall x \in \mathcal{A}(X)$. For example they can be specified as

$$\alpha(x) = \frac{\rho(x, LS)}{\rho(x, LS) + \rho(x, \mathcal{A}(X) \setminus \Omega)}, \quad \beta(x) = \frac{\rho(x, \mathcal{A}(X) \setminus \Omega)}{\rho(x, LS) + \rho(x, \mathcal{A}(X) \setminus \Omega)}, \quad x \in \mathcal{A}(X),$$

where $\rho(x, S)$ is a distance from the point x to the set $S \subset \mathcal{A}(X)$.[8] Obviously, for $\Omega \cup NS = \mathcal{A}(X)$ the mapping $f : \mathcal{A}(X) \to \mathcal{A}(X)$ is continuous and, by Brouwer's theorem, it must have a fixed point $\bar{x} = f(\bar{x})$. However, where is it?

Suppose $\bar{x} \in NS \setminus \Omega$. Then $f(\bar{x}) = q(x) = \frac{x+\mathbf{e}}{2} \neq \bar{x}$, since otherwise $\bar{x} = \mathbf{e} \notin NS$.

Suppose $\bar{x} \in \Omega \setminus NS$. Then $f(\bar{x}) = h(\bar{x}) = (h_j(\bar{x}))_{j \in \mathcal{I}}$ and by Lemma 4 there is $i \in \mathcal{I}$ such that $h_i(\bar{x}) \in \mathcal{P}_i(\bar{x})$ that is impossible by (**A**).

Suppose $\bar{x} \in NS \cap \Omega$. Now $f(\bar{x}) = \alpha(\bar{x})q(\bar{x}) + \beta(\bar{x})h(\bar{x})$. Clearly $\alpha(\bar{x}) > 0$ and $\beta(\bar{x}) > 0$, since the contrary is impossible. Recall that we also have $h_i(\bar{x}) = \xi_i(\bar{x}) + \gamma_i(\bar{x})(\mathbf{e}_i - \bar{x}_i)$ $\forall i \in \mathcal{I}$. Now for $i_0 \in \mathcal{I}$, which is interested in a partial breaking of the contract $\bar{x} - \mathbf{e}$, at the fixed point we have

$$\bar{x}_{i_0} = \alpha \left[\bar{x}_{i_0} + \frac{1}{2}(\mathbf{e}_{i_0} - \bar{x}_{i_0})\right] + \beta[\xi_{i_0}(\bar{x}) + \gamma_{i_0}(\bar{x})(\mathbf{e}_{i_0} - \bar{x}_{i_0})]$$

$$= \bar{x}_{i_0} + \alpha\frac{1}{2}(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \beta[\xi_{i_0}(\bar{x}) - \bar{x}_{i_0} + \gamma_{i_0}(\bar{x})(\mathbf{e}_{i_0} - \bar{x}_{i_0})] \quad \Rightarrow$$

$$\left[\frac{1}{2}\alpha + \beta\gamma_{i_0}(\bar{x})\right](\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \beta(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) = 0. \tag{13}$$

[8] It is standardly defined as $\rho(x, S) = \inf_{y \in S} \rho(x, y)$.

Clearly $\xi_{i_0}(\bar{x}) = \bar{x}_{i_0}$ is impossible, otherwise (13) implies $\bar{x}_{i_0} = \mathbf{e}_{i_0}$. Therefore $\xi_{i_0}(\bar{x}) \in \mathcal{P}_{i_0}(\bar{x})$. Also at a fixed point $\bar{x} \in NS$ for some $\lambda \in (0, 1]$ we have

$$\lambda \left[\frac{1}{2}\alpha + \beta\gamma_{i_0}(\bar{x}) \right] = \mu > 0, \quad \mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \bar{x}_{i_0} \in \mathcal{P}_{i_0}(\bar{x}).$$

At the same time, due to $(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) \in \mathcal{P}_{i_0}(\bar{x}) - \bar{x}_{i_0}$ and (**A**) we conclude

$$\lambda\beta(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) \in \mathcal{P}_{i_0}(\bar{x}) - \bar{x}_{i_0} \quad \Rightarrow \quad \exists \eta_{i_0}(\bar{x}) \in \mathcal{P}_{i_0}(\bar{x}) : \ \lambda\beta(\xi_{i_0}(\bar{x}) - \bar{x}_{i_0}) = \eta_{i_0}(\bar{x}) - \bar{x}_{i_0}.$$

Now, due to (13) and (**A**) we have $\mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \eta_{i_0}(\bar{x}) - \bar{x}_{i_0} = 0 \Rightarrow$

$$\frac{\mu(\mathbf{e}_{i_0} - \bar{x}_{i_0}) + \bar{x}_{i_0}}{2} + \frac{\eta_{i_0}(\bar{x})}{2} = \bar{x}_{i_0} \quad \Rightarrow \quad \bar{x}_{i_0} \in \mathrm{co}\mathcal{P}_{i_0}(\bar{x}) = \mathcal{P}_{i_0}(\bar{x}),$$

which is impossible.

Thus, the assumption $\Omega \cup NS = \mathcal{A}(X)$ implies the existence of a continuous mapping $f : \mathcal{A}(X) \to \mathcal{A}(X)$ with no fixed point in $\mathcal{A}(X)$. This contradicts Brouwer's theorem. So, the assumption that there are no fuzzy contractual allocations lead us to a contradiction and it proves the theorem. ■

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