Effective Algorithm for Computing Noetherian Operators of Positive Dimensional Ideals

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Abstract. An effective algorithm for computing Noetherian operators of positive dimensional ideals is introduced. It is shown that an algorithm for computing Noetherian operators of zero dimensional ideals, that was previously published by the authors [https://doi.org/10.1007/s00200-022-00570-7], can be generalized to that of positive dimensional ideals. The key ingredients of the generalization are the prime decomposition of a radical ideal and a maximal independent set. The results of comparison between the resulting algorithm with another existing one are also given.

Keywords: Noetherian operator · Partial differential operator · Primary ideal · Positive dimensional ideal

1 Introduction

This is the continuation of the authors' paper [16] that introduces an algorithm for computing Noetherian operators of zero dimensional ideals.

In the 1930s, W. Gröbner addressed the problem of characterizing ideal membership with differential conditions [11]. Later in the 1960s, L. Ehrenspreis and V. P. Palamodov obtained a complete description of primary ideals and modules in terms of differential operators [7,8,21]. At the core of the results, one has the notion of Noetherian operators to describe a primary module (and ideal).

Recently several authors, including the authors of the present paper, have studied the Noetherian operators in the context of symbolic computation. In [3–6], Y. Cid-Riz, J. Chen et al. give algorithms for computing Noetherian operators and the Macaulay2 implementation. They use the Hilbert schemes and Macaulay dual spaces for studying and computing them. In [16], the authors propose a different algorithm for computing Noetherian operators of zero dimensional ideals. The theory of holonomic D-modules and local cohomology play key roles in this

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approach. Notably, as the authors' algorithm [16] is constructed by mainly linear algebra techniques, the algorithm is much faster than the algorithms presented by Y. Cid-Riz, J. Chen et al. in computational speed.

In this paper, by adopting the framework proposed in [16], we consider a method for computing Noetherian differential operators of a positive dimensional primary ideal. We show that the use of the maximally independent set allows us to reduce the computation of Noetherian operators of positive dimensional primary ideals to that of zero dimensional cases. Accordingly, as the resulting algorithm of computing Noetherian operators of positive dimensional primary ideals consists mainly of linear algebra computation, it is also effective.

This paper is organized as follows. In Sect. 2, following [16], we recall results of Noetherian operators of zero dimensional primary ideals. In Sect. 3, we review some mathematical basics that are utilized in our main results. Section 4 consists of three subsections. In Sect. 4.1, we describe an algorithm for computing Noetherian operators of positive dimensional ideals. In Sect. 4.2 we give results of benchmark tests. In Sect. 4.3, we introduce a concept of Noetherian representations and we present an algorithm for computing Noetherian representations as an application of our approach.

2 Noetherian Operators of Zero Dimensional Ideals

Here we recall the algorithm for computing Noetherian operators of zero dimensional ideals that is published in [16].

Through this paper, we use the notation X as the abbreviation of n variables $x_1, x_2, \ldots, x_n$, K as a subfield of the field $\mathbb{C}$ of complex numbers and $\mathbb{Q}$ as the field of rational numbers. The set of natural numbers $\mathbb{N}$ includes zero. For $f_1, \ldots, f_r \in K[X] = K[x_1, \ldots, x_n]$, let $\langle f_1, \ldots, f_r \rangle$ denote the ideal in $K[X]$ generated by $f_1, \ldots, f_r$ and $\sqrt{\langle f_1, \ldots, f_r \rangle}$ denote the radical of the ideal $\langle f_1, \ldots, f_r \rangle$. If an ideal $I \subset K[X]$ is primary and $\sqrt{I} = \mathfrak{p}$, then we say that I is $\mathfrak{p}$-primary.

Let $D = K[X][\partial]$ denote the ring of partial differential operators with coefficients in $K[X]$ where $\partial = \{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$, $\partial_{x_i} = \frac{\partial}{\partial x_i}$ with relations $x_i x_j = x_j x_i$, $\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}$, $\partial_{x_j} x_i = x_i \partial_{x_j}$ $(i \neq j)$, $\partial_{x_i} x_i = x_i \partial_{x_i} + 1$ $(1 \leq i, j \leq n)$, i.e. $D = \{\sum_{\beta \in \mathbb{N}^n} c_\beta \partial^\beta | c_\beta \in K[X]\}$ where $\partial^\beta = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\cdots\partial_{x_n}^{\beta_n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$. For $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$, $|\beta| := \sum_{i=1}^n \beta_i$. The set of all terms of ∂ is denoted by $\mathrm{Term}(\partial)$ and that of X is denoted by $\mathrm{Term}(X)$.

Let us fix a term order $\succ$ on $\mathrm{Term}(\partial)$. For a given partial differential operator of the form

$$\psi = c_\alpha \partial^\alpha + \sum_{\partial^\alpha \succ \partial^\beta} c_\beta \partial^\beta \quad (c_\alpha, c_\beta \in K[X]),$$

we call ∂^α the *head term*, c_α the *head coefficient* and ∂^β the lower terms. We denote the head term by $\mathrm{ht}(\psi)$, the head coefficient by $\mathrm{hc}(\psi)$ and the set of lower terms of ψ as $\mathrm{LL}(\psi) = \left\{ \partial^\lambda \in \mathrm{Term}(\psi) \,\middle|\, \partial^\lambda \neq \mathrm{ht}(\psi) \right\}$. For a finite subset $\Psi \subset D$, $\mathrm{ht}(\Psi) = \{\mathrm{ht}(\psi) \,|\, \psi \in \Psi\}$, $\mathrm{LL}(\Psi) = \bigcup_{\psi \in \Psi} \mathrm{LL}(\psi)$.

For instance, let $\psi = x_1^3x_2^2\partial_{x_1}^3\partial_{x_2}^2\partial_{x_3} + x_3^2\partial_{x_1}^2\partial_{x_3} + x_1x_3\partial_{x_2}\partial_{x_3} + x_1^2x_2x_3$ be a partial differential operator in $\mathbb{Q}[x_1, x_2, x_3][\partial_{x_1}, \partial_{x_2}, \partial_{x_3}]$ and $\succ$ the graded lexicographic term order on $\mathrm{Term}(\{\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\})$ with $\partial_{x_1} \succ \partial_{x_2} \succ \partial_{x_3}$. Then, $\mathrm{ht}(\psi) = \partial_{x_1}^3\partial_{x_2}^2\partial_{x_3}$, $\mathrm{hc}(\psi) = x_1^3x_2^2$ and $\mathrm{LL}(\psi) = \{\partial_{x_1}^2\partial_{x_3}, \partial_{x_2}\partial_{x_3}, 1\}$.

For each $1 \le i \le n$, we write the standard unit vector as

$$\boldsymbol{e}_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots, 0).$$

The definition of Noetherian operators is the following.

Theorem 1 (Ehrenspreis-Palamodov [7,8,21]). *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal in $K[X]$ and proper. There exist partial differential operators $\psi_1, \psi_2, \ldots, \psi_\ell \in D$ with the following property. A polynomial $g \in K[X]$ lies in the ideal $\mathfrak{q}$ if and only if $\psi_1(g), \psi_2(g), \ldots, \psi_\ell(g) \in \mathfrak{p}$.*

Definition 1. *The partial differential operators $\psi_1, \psi_2, \ldots, \psi_\ell$ that satisfy Theorem 1 are called Noetherian operators of the primary ideal $\mathfrak{q}$.*

The core of the algorithm for computing Noetherian operators of zero dimensional ideals, that is introduced in [16], is the following theorem. Actually, this is the generalization of the result of L. Hörmander [14, Theorem 7.76 and pp. 235].

Theorem 2 ([16, Theorem 5]). *Let I be a zero-dimensional ideal generated by $f_1, \ldots, f_r$ in $K[X]$ and $\mathfrak{q}$ a primary component of a minimal primary decomposition of I with $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Let $\mathcal{N}_s(I)$ be the set of all partial differential operators $\varphi = \sum_{\beta \in \mathbb{N}^n, |\beta| < s} c'_\beta \partial^\beta \quad (c'_\beta \in K[X])$, such that $\varphi(f) \in \mathfrak{p}$ for all $f \in I$ where s is a natural number that satisfies $\mathfrak{p}^s \subset \mathfrak{q}$. Let $\mathrm{NT}_\mathfrak{q}$ be the set of all partial differential operators $\psi = \sum_{\beta \in \mathbb{N}^n, |\beta| < s} c_\beta \partial^\beta \quad (c_\beta \in K[X])$, such that the commutator $[\psi, x_i] = \psi x_i - x_i \psi \in \mathcal{N}_{s-1}(I)$ for $i = 1, 2, \ldots, n$ and $\psi(f_j) \in \mathfrak{p}$ for $j = 1, 2, \ldots, r$. Then,*

(i) $g \in K[X]$, $\psi(g) \in \mathfrak{p}$ for all $\psi \in \mathrm{NT}_\mathfrak{q} \iff g \in \mathfrak{q}$.
(ii) Further, one can choose $\psi_1, \psi_2, \ldots, \psi_\ell \in \mathrm{NT}_\mathfrak{q}$ such that

$$g \in K[X], \psi_k(g) \in \mathfrak{p} \text{ for } k = 1, 2, \ldots, \ell \iff g \in \mathfrak{q}.$$

In what follows, the notation $\mathrm{NT}_\mathfrak{q}$, that is introduced in Theorem 2, is utilized as the set of Noetherian operators of the primary ideal $\mathfrak{q}$.

Proposition 1 ([16, Proposition 1]). *Let $\mathfrak{q}$ be a zero dimensional primary ideal in $K[X]$ and $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Then, the set $\mathrm{NT}_\mathfrak{q}$, that is from Theorem 2, is a finite dimensional vector space over the field $K[X]/\mathfrak{p}$.*

Definition 2. *Let $\succ$ be a term order on $\mathrm{Term}(\partial)$, $\mathfrak{q}$ a zero dimensional primary ideal in $K[X]$ and $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Let $\mathrm{NB}_\mathfrak{q}$ be a basis of the vector space $\mathrm{NT}_\mathfrak{q}$ over the field $K[X]/\mathfrak{p}$ such that*

$$\text{for all } \psi \in \mathrm{NB}_\mathfrak{q}, \mathrm{hc}(\psi) = 1, \mathrm{ht}(\psi) \notin \mathrm{ht}(\mathrm{NB}_\mathfrak{q} \setminus \{\psi\}) \text{ and } \mathrm{ht}(\psi) \notin \mathrm{LL}(\mathrm{NB}_\mathfrak{q}).$$

Then, the basis $\mathrm{NB}_\mathfrak{q}$ is called a reduced basis of the vector space $\mathrm{NT}_\mathfrak{q}$ over $K[X]/\mathfrak{p}$ w.r.t. $\succ$.

The algorithm that is presented in [16] always outputs a reduced basis of the vector space if we input a zero dimensional primary ideal.

3 Mathematical Basics

Here we quickly review some mathematical basics of maximally independent sets, extensions of ideals and Noetherian operators.

3.1 Extension and Contraction

Definition 3. *Let I be a proper ideal in $K[X]$ and $U \subset X$. Then U is called an independent set modulo I if $K[U] \cap I = \{0\}$. Moreover, $U \subset X$ is called a maximal independent set (MIS) modulo I if it is an independent set modulo I and the cardinality of U is equal to the dimension of I.*

For a finite subset Y, the cardinality of Y is written by $|Y|$.

Definition 4. *Let I be an ideal in $K[X]$, $U \subset X$ and $Y = X \backslash U$. Then, the extension I^e of I to $K(U)[Y]$ is the ideal generated by the set I in the ring $K(U)[Y]$ where $K(U)$ is the field of rational functions with variables U. If J is an ideal in $K(U)[Y]$, then the contraction J^c of J to $K[X]$ is defined as $J \cap K[X]$.*

The following lemmas are fundamental in commutative algebra and computer algebra. See [2].

Lemma 1. *Let I be an ideal in $K[X]$. If $U \subset X$ is a MIS modulo I, then I^e is a zero dimensional ideal of $K(U)[X \backslash U]$.*

Lemma 2 ([2, Lemma 1.122, Lemma 8.97]).

(1) Let $\mathfrak{p}$ be a prime ideal in $K[X]$ and U a MIS modulo $\mathfrak{p}$ and $Y = X \backslash U$. Then $\mathfrak{p}^e$ is prime in $K(U)[Y]$ and $\mathfrak{p} = \mathfrak{p}^{ec} = (\mathfrak{p}^e)^c$.

(2) Let $\mathfrak{p}$ be a prime ideal in $K[X]$ and U a MIS modulo I and $Y = X \backslash U$. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of $K[X]$, then $\mathfrak{q}^e$ is $\mathfrak{p}^e$-primary in $K(U)[Y]$ and $\mathfrak{q} = \mathfrak{q}^{ec}$.

Let $\succ$ be a term order on $\mathrm{Term}(Y)$. For a polynomial $g \in K(U)[Y]$, we denote the head coefficient of g by $\mathrm{hc}(g)$. In the following three lemmas, we fix subsets $U \subset X$ and $Y = X \backslash U$.

Lemma 3 ([2, Lemma 8.91]). *Let $\succ$ be a term order on $\mathrm{Term}(Y)$. Suppose J is an ideal of $K(U)[Y]$, and G is a Gröbner basis w.r.t. $\succ$ of J such that $G \subset K[X]$. Let I be the ideal generated by G in $K[X]$, and set f as a least common multiple of $\{\mathrm{hc}(g) | g \in G\}$ (i.e. $f = LCM\{\mathrm{hc}(g) | g \in G\}$), where $\mathrm{hc}(g) \in K[U]$ is taken of g as an element of $K(U)[Y]$. Then, $J^c = I : f^\infty$.*

Lemma 4 ([2, Proposition 8.94]). *Let $\succ$ be a block term order on $\mathrm{Term}(X)$ with $Y \gg U$, and suppose I is an ideal of $K[X]$ and G is a Gröbner basis of I w.r.t. $\succ$. Set f as a least common multiple of $\{\mathrm{hc}(g) | g \in G\}$ (i.e. $f = LCM\{\mathrm{hc}(g) | g \in G\}$), where $\mathrm{hc}(g) \in K[U]$ is taken of g as an element of $K(U)[Y]$. Then, $I^{ec} = I : f^\infty$.*

Lemma 5 ([2, Lemma 8.95]). *Let $I = \langle f_1, \ldots, f_r \rangle \subset K[X]$. Suppose $q \in K[X]$ and $s \in \mathbb{N} \backslash \{0\}$ are such that $I : q^s = I : q^\infty$. Then, $I = \langle f_1, \ldots, f_r, q^s \rangle \cap (I : q^s)$.*

In [12,13], J. Hoffmann and V. Levandovskyy provided more information on the extension and contraction from both theoretical and algorithmic point of view.

3.2 Noetherian Operators of a Primary Ideal $\mathfrak{q}^e \subset K(U)[Y]$

Here we discuss the relations between Noetherian operators and local cohomology classes for extensions of ideals. This discussion is basically the same as Sect. 3.1 of [16]. See [16,18,19,23,24] for details.

Throughout this subsection, let I be an ideal in $K[X]$, U a MIS modulo I, $\mathfrak{q}$ a primary component of the minimal primary decomposition of I such that a MIS modulo $\mathfrak{q}$ is U, $\sqrt{\mathfrak{q}} = \mathfrak{p}$, $Y = X \backslash U$ and $|Y| = \ell$. Then, by Lemma 1, I^e, $\mathfrak{q}^e$ and $\mathfrak{p}^e$ are zero dimensional ideals in $K(U)[Y]$.

Let $H^\ell_{[Z]}(K(U)[Y])$ denote an algebraic local cohomology group, with support on $Z = \{a \in \overline{K(U)}^\ell | g(a) = 0, \forall g \in \mathfrak{p}^e\}$, defined as

$$H^\ell_{[Z]}(K(U)[Y]) = \lim_{k\to\infty} \mathrm{Ext}^\ell_{K(U)[Y]} \left(K(U)[Y]/(\mathfrak{p}^e)^k, K(U)[Y]\right)$$

where $\overline{K(U)}$ be an algebraic closure of the field $K(U)$ of rational functions.

Set $H_{\mathfrak{q}^e} = \{\psi \in H^\ell_{[Z]}(K(U)[Y]) | q\psi = 0, \forall q \in \mathfrak{q}^e\}$. Then, the following holds

$$\begin{aligned} H_{\mathfrak{q}^e} &\cong \mathcal{H}om_{K(U)[Y]} \left(K(U)[Y]/\mathfrak{q}^e, H^\ell_{[Z]}(K(U)[Y])\right) \\ &= \mathcal{H}om_{K(U)[Y]} \left(K(U)[Y]/I^e, H^\ell_{[Z]}(K(U)[Y])\right). \end{aligned}$$

Let $\mathcal{D}^e = K(U)[Y][\{\partial_y | y \in Y\}]$ denote the ring of partial differential operators with coefficients in $K(U)[Y]$. Then, since $K(U)[Y] \subset \mathcal{D}^e$, we also have

$$\begin{aligned} H_{\mathfrak{q}^e} &\cong \mathcal{H}om_{\mathcal{D}^e} \left(\mathcal{D}^e/\mathcal{D}^e\mathfrak{q}^e, H^\ell_{[Z]}(K(U)[Y])\right) \\ &= \mathcal{H}om_{\mathcal{D}^e} \left(\mathcal{D}^e/\mathcal{D}^e I^e, H^\ell_{[Z]}(K(U)[Y])\right). \end{aligned}$$

Noetherian operators are considered as follows.

Definition 5. *The set of $\mathcal{D}^e$-linear homomorphisms $\mathcal{H}om_{\mathcal{D}^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$ between the two left $\mathcal{D}^e$-modules are called the Noetherian space of $\mathfrak{q} \subset K[X]$ w.r.t. U where $M_{\mathfrak{q}^e} = \mathcal{D}^e/\mathcal{D}^e\mathfrak{q}^e$ and $M_{\mathfrak{p}^e} = \mathcal{D}^e/\mathcal{D}^e\mathfrak{p}^e$ are $\mathcal{D}^e$-modules.*

The Noetherian space has the structure of the right $K(U)[Y]/\mathfrak{p}^e$-module.

Example 1. Let us consider a primary ideal

$$\begin{aligned} \mathfrak{q} = \ & \langle x_1^4 - 3x_2x_1x_0^2 + 2x_3x_0^3, x_2x_1^3 - 2x_3x_1^2x_0 + x_2^2x_0^2, \\ & x_3x_1^3 - 2x_2^2x_1x_0 + x_2x_3x_0^2, x_2^2x_1^2 - 2x_2x_3x_1x_0 + x_3^2x_0^2, x_3^2x_1 - x_2^3\rangle \end{aligned}$$

in $\mathbb{Q}[x_0, x_1, x_2, x_3]$. Then, a MIS modulo $\mathfrak{q}$ is $\{x_2, x_3\}$. A Gröbner basis G of $\mathfrak{q}^e$ w.r.t. the lexicographic term order with $x_0 \succ x_1$ is $G = \{(x_3x_0 - x_2^4)^2, x_3^2x_1 - x_2^3\}$ in $\mathbb{Q}(x_2, x_3)[x_0, x_1]$. It is obvious that $\sqrt{\mathfrak{q}^e} = \langle x_3x_0 - x_2^4, x_3^2x_1 - x_2^3\rangle$. Hence, the Noetherian space of $\mathfrak{q} \subset \mathbb{Q}[x_0, x_1, x_2, x_3]$ w.r.t. $\{x_0, x_1\}$ is $\mathrm{Span}_R\left(1, \frac{\partial}{\partial x_0}\right)$ where $R = \mathbb{Q}(x_0, x_1)[x_2, x_3]/\langle x_3x_0 - x_2^4, x_3^2x_1 - x_2^3\rangle$.

Proposition 2. *Let $M_{I^e} = \mathcal{D}^e/\mathcal{D}^e I^e$. Then,*

$$\mathcal{H}om_{\mathcal{D}^e}\left(M_{I^e}, M_{\mathfrak{p}^e}\right) \cong \mathcal{H}om_{\mathcal{D}^e}\left(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e}\right).$$

The proposition above says that the primary ideal $\mathfrak{q}^e \subset K(U)[Y]$ can be determined by I^e and the prime ideal $\mathfrak{p}^e$.

4 Main Results

Here, first we generalize the algorithm for computing Noetherian operators of a zero dimensional ideal [16] to that of positive dimensional ideal. Second, we compare the resulting algorithm with another existing one [4]. Third, we discuss a Noetherian representation of an ideal as an application of the Noetherian operators.

4.1 Generalization

By utilizing a MIS modulo an ideal, we are able to generalize Theorem 2 to the following.

Lemma 6. *Let I be an ideal generated by $f_1, \ldots, f_r$ in $K[X]$, U a MIS modulo I, $\mathfrak{q}$ a primary component of the minimal primary decomposition of I such that the MIS modulo $\mathfrak{q}$ is U and $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Let $\mathcal{N}_s(I^e)$ be the set of all partial differential operators $\varphi = \sum_{\beta \in \mathbb{N}^\ell, |\beta| < s} c'_\beta \partial^\beta$ ($c'_\beta \in K(U)[Y]$), such that $\varphi(f) \in \mathfrak{p}^e$ for all $f \in I^e \subset K(U)[Y]$ where s is a natural number that satisfies $(\mathfrak{p}^e)^s \subset \mathfrak{q}^e$ in $K(U)[Y]$. Let $\mathrm{NT}_{\mathfrak{q}^e}$ be the set of all partial differential operators $\psi = \sum_{\beta \in \mathbb{N}^\ell, |\beta| < s} c_\beta \partial^\beta$ ($c_\beta \in K(U)[Y]$), such that the commutator $[\psi, y] = \psi y - y\psi \in \mathcal{N}_{s-1}(I^e)$ for each $y \in Y$ and $\psi(f_j) \in \mathfrak{p}^e$ for $j = 1, 2, \ldots, r$. Then,*

(i) $g \in K(U)[Y]$, $\psi(g) \in \mathfrak{p}^e$ for all $\psi \in \mathrm{NT}_{\mathfrak{q}^e} \iff g \in \mathfrak{q}^e$ in $K(U)[Y]$.
(ii) Further, one can choose $\psi_1, \psi_2, \ldots, \psi_t \in \mathrm{NT}_{\mathfrak{q}^e}$ such that

$$g \in K(U)[Y],\ \psi_k(g) \in \mathfrak{p}^e \text{ for } k = 1, 2, \ldots, t \iff g \in \mathfrak{q}^e.$$

Proof. As we describe in Sect. 3.2, I^e, $\mathfrak{q}^e$ and $\mathfrak{p}^e$ are zero dimensional ideals in $K(U)[Y]$ and Noetherian operators of the primary ideal $\mathfrak{q}^e \subset K(U)[Y]$ can be determined by I^e. Since it can be regarded as the same setting of Theorem 2, this lemma holds. □

By combining Proposition 1 and Lemma 6, we have the following corollary.

Corollary 1. *Using the same notation as in Lemma 6, then, the set $\mathrm{NT}_{\mathfrak{q}^e}$ is a finite dimensional vector space over the field $K(U)[Y]/\mathfrak{p}^e$.*

Definition 6. *Using the same notation as in Lemma 6, let $\succ$ be a term order on $\mathrm{Term}(\{\partial_y | y \in Y\})$. Let $\mathrm{NB}_{\mathfrak{q}^e}$ be a basis of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$ over the field $K(U)[Y]/\mathfrak{p}^e$ such that*

$$\text{for all } \psi \in \mathrm{NB}_{\mathfrak{q}^e}, \mathrm{hc}(\psi) = 1, \mathrm{ht}(\psi) \notin \mathrm{ht}(\mathrm{NB}_{\mathfrak{q}^e} \setminus \{\psi\}) \text{ and } \mathrm{ht}(\psi) \notin \mathrm{LL}(\mathrm{NB}_{\mathfrak{q}^e}).$$

Then, the basis is called a reduced basis $\mathrm{NB}_{\mathfrak{q}^e}$ of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$ over $K(U)[Y]/\mathfrak{p}^e$ w.r.t. $\succ$.

For $\psi \in K(U)[Y][\{\partial_y | y \in Y\}]$ (or $f \in K(U)[Y]$), we define $\mathrm{dlcm}(\psi)$ (or $\mathrm{dlcm}(f)$) as the least common multiple of all denominators of coefficients in $K(U)$ of ψ (or f). For instance, set $\psi = xy\partial_x^2\partial_y^2 + \frac{1}{u^2}x\partial_x\partial_y^2 + \frac{4}{w}\partial_y$ in $K(u,w)[x,y][\partial_x,\partial_y]$, then $\mathrm{dlcm}(\psi) = u^2 w$. Hence, $\mathrm{dlcm}(\psi)\cdot\psi$ is in $(K[u,w][x,y])[\partial_x,\partial_y]$.

Theorem 3. *Using the same notation as in Lemma 6, the following holds.*

(i) $g \in K[X]$, $\psi(g) \in \mathfrak{p} \subset K[X]$ *for all* $\psi \in \mathrm{NT}_{\mathfrak{q}^e} \cap K[X][\partial] \iff g \in \mathfrak{q}$ *in* $K[X]$.

(ii) One can choose $\psi_1, \psi_2, \ldots, \psi_\ell \in \mathrm{NT}_{\mathfrak{q}^e} \cap K[X][\partial]$ *such that*

$$g \in K[X],\ \psi_k(g) \in \mathfrak{p} \subset K[X] \text{ for } k = 1,2,\ldots,\ell \iff g \in \mathfrak{q} \subset K[X].$$

Proof. (i) ($\Rightarrow$) For $g \in K[X]$, assume that $\psi(g) \in \mathfrak{p} \subset K[X]$ for all $\psi \in \mathrm{NT}_{\mathfrak{q}^e} \cap K[X][\partial]$. As we have $\mathfrak{p} \subset \mathfrak{p}^e$, by Lemma 6, $g \in \mathfrak{q}^e$ in $K(U)[Y]$. Thus, by Lemma 2, $g \in \mathfrak{q}^e \cap K[X] = \mathfrak{q}^{ec} = \mathfrak{q}$.
($\Leftarrow$) For $g \in K[X]$, assume that $g \in \mathfrak{q}$ in $K[X]$. As we have $\mathfrak{q} \subset \mathfrak{q}^e$, thus by Lemma 6, for all $\psi \in \mathrm{NT}_{\mathfrak{q}^e} \cap K[X][\partial] \subset \mathrm{NT}_{\mathfrak{q}^e}$, $\psi(g) \in \mathfrak{p}^e$ in $K[X]$. By Lemma 2, $g \in \mathfrak{p}^e \cap K[X] = \mathfrak{p}^{ec} = \mathfrak{p}$.
(ii) Since Lemma 6 holds, there exist $\psi_1, \psi_2, \ldots, \psi_t \in \mathrm{NT}_{\mathfrak{q}^e}$ such that "$g \in K[X] \subset K(U)[Y],\ \psi_k(g) \in \mathfrak{p}^e$ for $k = 1,2,\ldots,t$ if and only if $g \in \mathfrak{q}^e$". Let us consider the finitely many partial differential operators

$$\mathrm{dlcm}(\psi_1)\psi_1, \mathrm{dlcm}(\psi_2)\psi_2, \ldots, \mathrm{dlcm}(\psi_t)\psi_t.$$

Note that $(\mathrm{dlcm}(\psi_k)\psi_k)(g) \in K[X][\partial]$ $(k = 1,2,\ldots,t)$, $(\mathrm{dlcm}(\psi_k)\psi_k)(g) \subset \mathfrak{p}^e \cap K[X][\partial] = \mathfrak{p}^{ec} = \mathfrak{p}$ and $g \in \mathfrak{q}^e \cap K[X] = \mathfrak{q}^{ec} = \mathfrak{q}$. As $K(U)$ is a field,

$$g \in K[X],\ (\mathrm{dlcm}(\psi_k))(g) \in \mathfrak{p} \text{ for } k = 1,2,\ldots,t \text{ if and only if } g \in \mathfrak{q}$$

holds. □

Let $\{\varphi_1, \ldots, \varphi_t\}$ be a basis of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$. Then, by the proof of Theorem 3, $\mathrm{dlcm}(\varphi_1)\varphi_1, \ldots, \mathrm{dlcm}(\varphi_t)\varphi_t$ become Noetherian operators of $\mathfrak{q} \subset K[X]$. Thus, we need an algorithm for computing a basis of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$ where $\mathfrak{q}^e$ is zero dimensional in $K(U)[Y]$. Since Lemma 6 is essentially the same as Theorem 2, we can naturally generalize the algorithm for computing Noetherian operators of zero dimensional ideals to that of positive dimensional ideals.

Before describing the main algorithm, we give the following lemma and corollaries for efficiency. Note that these facts follow from Lemma 6 because if $\psi \in \mathrm{NT}_{\mathfrak{q}^e}$, then the commutator $[\psi, y] \in \mathrm{NT}_{\mathfrak{q}^e}$ for each $y \in Y$.

Lemma 7. *Using the same notation as in Lemma 6, let $\succ$ be a term order on* $\mathrm{Term}(\{\partial_y | y \in Y\})$ *and* $|Y| = \ell$. *If* $\partial^\alpha \notin \mathrm{NT}_{\mathfrak{q}^e}$. *Then, for all* $\partial^\lambda \in \{\partial^{\alpha+\gamma} | \gamma \in \mathbb{N}^\ell\}$, $\partial^\lambda \notin \mathrm{ht}(\mathrm{NT}_{\mathfrak{q}^e})$.

Let M be a set of terms of $\mathrm{Term}(\{\partial_y | y \in Y\})$. We define the neighbors of M as $\mathrm{Neighbor}(M, Y) = \{\partial^\lambda \partial_y | \partial^\lambda \in M, y \in Y\}$. The following corollary that is the generalization of Corollary 1 of [16] is useful to compute possible candidates of head terms of $\mathrm{NT}_{\mathfrak{q}^e}$.

Corollary 2. *Using the same notation as in Lemma 6, let* $\succ$ *be a term order on* $\mathrm{Term}(\{\partial_y | y \in Y\})$ *and* $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{N}^\ell$. *Let* $\Lambda_{\mathfrak{q}}^{(\lambda)} = \{\partial^{\lambda'} \in \mathrm{ht}(\mathrm{NT}_{\mathfrak{q}^e}) | \partial^\lambda \succ \partial^{\lambda'}\}$. *If* $\partial^\lambda \in \mathrm{ht}(\mathrm{NT}_{\mathfrak{q}^e})$, *then for each* $1 \leq i \leq \ell$, $\partial^{\lambda - e_i}$ *is in* $\Lambda_{\mathfrak{q}}^{(\lambda)}$, *provided* $\lambda_i \geq 1$.

If $\partial^\lambda \in \mathrm{ht}(\mathrm{NT}_{\mathfrak{q}^e})$, then by Corollary 2, there is a possibility that an element of $\mathrm{Neighbor}(\{\partial^\lambda\}, Y)$ belongs to $\mathrm{ht}(\mathrm{NT}_{\mathfrak{q}^e})$. The following algorithm computes possible candidates of head terms of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$ w.r.t. a term order $\succ$ on $\mathrm{Term}(\{\partial_y | y \in Y\})$ where Y is a subset of X.

Sub-algorithm (Headcandidate)

Specification: Headcandidate$(Y, \partial^\tau, \succ, \Lambda, \mathrm{FL})$
Making new candidates for head terms.

Input: Y: set of variables in X $(|Y| = \ell)$, $\partial^\tau \in \mathrm{Term}(\{\partial_y | y \in Y\})$,
$\succ$: a term order on $\mathrm{Term}(\{\partial_y | y \in Y\})$, $\Lambda = \{\partial^\alpha \in \mathrm{ht}(\mathrm{NT}_{\mathfrak{q}^e}) | \partial^\tau \succ \partial^\alpha\}$
FL: set of $\mathrm{Term}(\{\partial_y | y \in Y\})$ such that $\forall \partial^\alpha \in \mathrm{FL}$, $\partial^\alpha \notin \Lambda$.
Output: CT: set of new candidates for head terms.

BEGIN
$\mathrm{CT} \leftarrow \emptyset$; $B \leftarrow \mathrm{Neighbor}(\{\partial^\tau\}, Y)$; $B \leftarrow B \backslash (B \cap \{\partial^{\alpha+\gamma} \mid \partial^\alpha \in \mathrm{FL}, \gamma \in \mathbb{N}^\ell\})$;
while $B \neq \emptyset$ **do**
  select $\partial^{\tau'} = \partial^{(\tau'_1, \tau'_2, \ldots, \tau'_\ell)}$ from B; $B \leftarrow B \backslash \{\partial^{\tau'}\}$;
  for each i **from** 1 **to** ℓ **do** $Flag \leftarrow 1$;
    if $\tau'_i \neq 0$ **then**
      if $\partial^{\tau' - e_i} \notin \Lambda$ **then** $Flag \leftarrow 0$; **break**; **end-if**
    end-if
  end-for
  if $Flag = 1$ **then** $\mathrm{CT} \leftarrow \mathrm{CT} \cup \{\partial^{\tau'}\}$; **end-if**
end-while
return CT;
END

The following corollary that is the generalization of Corollary 2 of [16] is utilized to compute the candidates of lower terms.

Corollary 3. *Using the same notations as in Corollary 2, let* $\Gamma_{\mathfrak{q}^e}$ *denote the set of lower terms in* $\mathrm{NT}_{\mathfrak{q}^e}$ *and* $\Gamma_{\mathfrak{q}}^{(\lambda)} = \left\{ \partial^{\lambda'} \in \Gamma_{\mathfrak{q}^e} \,\middle|\, \partial^\lambda \succ \partial^{\lambda'} \right\}$.
If $\partial^\lambda = \partial^{(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_\ell)} \in \Gamma_{\mathfrak{q}^e}$, *then for each* $i = 1, 2, \ldots, \ell$, $\partial^{\lambda - e_i}$ *is in* $\Gamma_{\mathfrak{q}}^{(\lambda)} \cup \Lambda_{\mathfrak{q}}^{(\lambda)}$, *provided* $\lambda_i \geq 1$.

The algorithm **Noether** decides head terms of a reduced basis $\mathrm{NB}_{\mathfrak{q}^e}$ of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$ from bottom to up w.r.t. a term order $\succ$ on $\mathrm{Term}(\{\partial_y | y \in Y\})$. The algorithm consists of three main blocks, computing candidates for head terms (**Headcandidate**), computing for candidate of lower terms and solving a system of linear equations. For each block, the algorithm makes use of several sets as intermediate data. We fix the meaning of the sets as follows.

- CT is a set of candidates of head terms w.r.t. $\prec$.
- CL is a set of candidates of lower terms for some $\partial^\lambda \in$ CT.
- FL is a set of terms that do not belong to $\mathrm{ht}(\mathrm{NB}_{\mathfrak{q}^e})$ w.r.t. $\prec$.

The Sub-algorithm "**DetermineP**" that is utilized in Algorithm 1, determines indeterminates c_τs that are coefficients of the partial differential operators ψ.

Remark 1. Let $I = \langle f_1, \ldots, f_r \rangle \subset K[X]$ and $\mathfrak{q}$ a primary component of the minimal primary decomposition of I such that the dimension of I is equal to that of $\mathfrak{q}$. Let U be a MIS modulo $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $Y = X \backslash U$. If a partial differential operator ψ is in the reduced basis $\mathrm{NB}_{\mathfrak{q}^e}$ of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$ w.r.t. a term order $\succ$, then ψ satisfies the following condition $(N^{(*)})$

$$(N^{(*)}) \qquad \text{“}\psi(f_i) \in \mathfrak{p}^e \text{ in } K(U)[Y] \text{ and } [\psi, y] \in \mathrm{Span}_{K(U)[Y]/\mathfrak{p}^e}(\mathrm{NB}_{\mathfrak{q}^e})\text{”}$$

where $1 \leq i \leq r$ and $y \in Y$.

It is clear that $1 \in \mathrm{Span}_{K(U)[Y]/\mathfrak{p}^e}(\mathrm{NB}_{\mathfrak{q}^e})$, and hence, by Corollary 2, $\{\partial_y | y \in Y\}$ becomes a set of candidates of the head terms.

Remark 2. It is reported that algorithms, published in [1,15,22], for computing a prime decomposition of the radical $\sqrt{I}$ are much faster than those for computing primary decomposition of a polynomial ideal I in $K[X]$. One can utilized the algorithms for computing a prime component of $\sqrt{I}$. In fact, the MIS modulo $\sqrt{I}$ can be also obtained as a by-product when we compute the prime component.

Algorithm 1 (Noether)

Specification: Noether$(\{f_1, f_2, \ldots, f_r\}, \mathfrak{p}, U, Y, \succ)$
Computing Noetherian operators for a primary component $\mathfrak{q}$ of the primary decomposition of $\langle f_1, f_2, \ldots, f_r \rangle$ where $\sqrt{\mathfrak{q}} = \mathfrak{p}$.
Input: $\{f_1, f_2, \ldots, f_r\} \subset K[X]$,
$\mathfrak{p}$: associate prime ideal of a primary component $\mathfrak{q}$ of the minimal primary decomposition of $\langle f_1, f_2, \ldots, f_r \rangle$ s.t. U is a MIS modulo $\mathfrak{p}$,
$U \subset X$: MIS modulo $\langle f_1, f_2, \ldots, f_r \rangle$,
$Y := X \backslash U$, $(|Y| = \ell)$, $\succ$: term order on $\mathrm{Term}(\{\partial_y | y \in Y\})$.
Output: NB: a (reduced) basis of the vector space $\mathrm{NT}_{\mathfrak{q}^e}$.
BEGIN
NB $\leftarrow \{1\}$; CT $\leftarrow \{\partial_y | y \in Y\}$; CL $\leftarrow \emptyset$; FL $\leftarrow \emptyset$; EE $\leftarrow \emptyset$;
  while CT $\neq \emptyset$ **do**
  $\partial^\lambda \leftarrow$ Take the smallest element in CT w.r.t. $\succ$; CT $\leftarrow$ CT $\backslash \{\partial^\lambda\}$;
  E $\leftarrow \{\partial^\gamma \in \mathrm{EE} \,|\, \partial^\lambda \succ \partial^\gamma\}$; EE $\leftarrow$ EE $\backslash$E;
  EL $\leftarrow \{\partial^{(\gamma_1,\ldots,\gamma_\ell)} \in \mathrm{E} | \partial^{(\gamma_1,\ldots,\gamma_\ell)-e_i} \in \mathrm{ht}(\mathrm{NB}) \cup \mathrm{LL}(\mathrm{NB}),$ provided $\gamma_i \geq 1\}$;
  CL $\leftarrow$ CL $\cup$ EL;
  $\psi \leftarrow \partial^\lambda + \sum_{\partial^\tau \in \mathrm{CL}} c_\tau \partial^\tau$;   /* ($c_\tau$s are indeterminates) */
  $\psi' \leftarrow$ **DetermineP**$(\{f_1, \ldots, f_r\}, \psi, \mathfrak{p}, \mathrm{NB}, \{c_\tau | \partial^\tau \in \mathrm{CL}\}, U, Y)$;
    if $\psi' \neq 0$ **then**
      NB $\leftarrow$ NB $\cup \{\psi'\}$;
      CT $\leftarrow$ **Headcandidate**$(Y, \partial^\lambda, \succ, \mathrm{ht}(\mathrm{NB}), \mathrm{FL}) \cup$ CT;

EE $\leftarrow$ (Neighbor(LL(ψ')) $\cup$ EE) $\setminus$ CL;
else
FL $\leftarrow$ FL $\cup\{\partial^\lambda\}$; CL $\leftarrow$ CL $\cup\{\partial^\lambda\}$;
end-if
end-while
return NB ;
END

Sub-algorithm (DetermineP)

Specification: DetermineP$(\{f_1, f_2, \ldots, f_r\}, \psi, \mathfrak{p}, \mathrm{NB}, \{c_\tau | \tau \in \mathrm{CL}\}, U, Y)$
Determining c_τs that are coefficients of the partial differential operator ψ.
Input: $\{f_1, f_2, \ldots, f_r\}, \psi, \mathfrak{p}, \mathrm{NB}, \{c_\tau | \tau \in \mathrm{CL}\}, U, Y$: described in Algorithm 1.
Output: ψ': if $\psi' = 0$, then ψ is not a Noetherian operator of $\mathfrak{q}_i$, otherwise ψ' is a Noetherian operator of $\mathfrak{q}_i$ where $\mathrm{ht}(\psi') = \mathrm{ht}(\psi)$.
BEGIN
$L \leftarrow \emptyset$; $C \leftarrow$ NB; $Y' \leftarrow Y$; $i \leftarrow 1$; $\{\varphi_1, \varphi_2, \ldots, \varphi_s\} \leftarrow$ NB; /* $|\,\mathrm{NB}\,| = s$ */
for each i **from** 1 **to** s **do**
$g \leftarrow$ Compute the normal form of $\psi(f_i)$ w.r.t. $\mathfrak{p}^e$ in $K(U)[Y]$;
if $g \neq 0$ **then**
$L \leftarrow L \cup \{g = 0\}$;
end-if
end-for
while $Y' \neq \emptyset$ **do**
Select y from Y'; $Y' \leftarrow Y' \setminus \{y\}$; $b_i \leftarrow [\psi, y]$; $C \leftarrow C \cup \{b_i\}$; $i \leftarrow i + 1$;
end-while
$\boldsymbol{v}= (\partial^{\alpha 1}\ \partial^{\alpha 2} \cdots\ \partial^{\alpha \ell}) \leftarrow$ Make a vector from $\mathrm{Term}(C) = \{\partial^{\alpha 1}, \cdots, \partial^{\alpha \ell}\}$;
$M \leftarrow$ Get the $\ell \times (s + |Y|)$ matrix that satisfies $(\varphi_1 \cdots \varphi_s\ b_1 \cdots b_{|Y|}) = \boldsymbol{v}M$;
$\left(\begin{array}{c|c} E_s & \cdots \\ \hline 0 & A \end{array}\right) \leftarrow$ Reduce M by elementary operations of matrix over $K(U)[Y]/\mathfrak{p}^e$;
$L \leftarrow L \cup \{a' = 0 \mid a'$ is an entry of the matrix $A\}$;
if the system of linear equations L has no solution over $K(U)[Y]/\mathfrak{p}^e$ **then**
return 0;
else
$\psi' \leftarrow$ Get the (unique) solution of L and substitute the solution into c_τs of ψ;
return $\mathrm{dlcm}(\psi')\psi'$;
end-if
END

In the sub-algorithm **DetermineP**, E_s is the identity matrix of size s. Then, it is known that A is the zero matrix if and only if $b_1, b_2, \ldots, b_{|Y|} \in \mathrm{Span}_{K(U)[Y]/\mathfrak{p}^e}(\mathrm{NB})$. Hence, the sub-algorithm checks the condition $(N^{(*)})$ (see Remark 1). Notice that the Sub-algorithm, consists of linear algebra techniques except for computing a normal form of $\psi(f_i)$ w.r.t. $\mathfrak{p}^e$ in $K(U)[X]$.

The correctness and termination follow from Theorem 3 and Corollary 1. As Algorithm 1 is essentially the same as the case of zero dimensional ideal, we omit the proof. We refer the readers to [16, Theorem 6] for details.

Example 2. Let $I = \langle f_1, f_2, f_3\rangle \subset \mathbb{Q}[x, y, z]$ where $f_1 = x^6z + 9x^4yz + x^4z + 27x^2y^2z + 6x^2yz + 27y^3z + 9y^2z$, $f_2 = x^6 + 6x^4y + 9x^2y^2 + z^2$, $f_3 = z^3$. Then, the prime decomposition of $\sqrt{I}$ is $\sqrt{I} = \langle x^2 + 3y, z\rangle \cap \langle x, z\rangle$.

Let us consider the first prime ideal $\mathfrak{p} = \langle x^2 + 3y, z\rangle$, then a MIS modulo $\mathfrak{p}$ is $\{y\}$. Let $\succ$ be the total degree lexicographic term order with $\partial_x \succ \partial_z$.
We execute **Noether**$(\{f_1, f_2, f_3\}, \mathfrak{p}^e, \{y\}, \{x, z\}, \succ)$ where $\mathfrak{p}^e$ is the extension of $\mathfrak{p}$ to $\mathbb{Q}(y)[x, z]$.

(0) Set NB $= \{1\}$, CT $= \{\partial_z, \partial_x\}$ and CL $=$ FL $=$ EE $= \emptyset$.

(1) Take the smallest element ∂_z in CT and update CT to $\{\partial_x\}$. Since CL $=$ EE $= \emptyset$, there does not exist possible candidates of the lower terms. Set $\psi = \partial_z$ and check the conditions $(N^{(*)})$ *i.e.* execute the sub-algorithm **DetermineP**, then
$\psi(f_1) = x^6 + 9x^4y + x^4 + 27x^2y^2 + 6x^2y + 27y^3 + 9y^2 \in \mathfrak{p}^e$,
$\psi(f_2) = z \in \mathfrak{p}^e$, $\psi(f_3) = 3z^2 \in \mathfrak{p}^e$,
$[\psi, x] = 0 \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$, $[\psi, z] = 1 \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$.
Hence, ψ satisfies the condition $(N^{(*)})$. Renew NB as $\{1, \partial_z\}$ and CT as
$$\{\partial_x\} \cup \textbf{Headcandidate}\Big(\{x, z\}, \partial_z, \succ, \mathrm{ht}(\mathrm{NB}), \emptyset\Big) = \{\partial_x, \partial_z^2, \partial_x\partial_z\}.$$

(2) Take the smallest element ∂_x in CT and update CT to $\{\partial_z^2, \partial_x\partial_z\}$. Since CL $=$ EE $= \emptyset$, there does not exist possible candidates of the lower terms. Set $\psi = \partial_x$ and check the conditions $(N^{(*)})$, then
$\psi(f_1) = 6x^5z + 36x^3yz + 4x^3z + 54xy^2z + 12xyz \in \mathfrak{p}^e$,
$\psi(f_2) = 6x^5 + 24x^3y + 18xy^2 \in \mathfrak{p}^e$, $\psi(f_3) = 0 \in \mathfrak{p}^e$,
$[\psi, x] = 1 \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$, $[\psi, z] = 0 \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$.
Hence, ψ satisfies the condition $(N^{(*)})$. Renew NB as $\{1, \partial_z, \partial_x\}$ and CT as
$$\{\partial_z^2, \partial_x\partial_z\} \cup \textbf{Headcandidate}\Big(\{x, z\}, \partial_x, \succ, \mathrm{ht}(\mathrm{NB}), \emptyset\Big) = \{\partial_z^2, \partial_x\partial_z, \partial_x^2\}.$$

(3) Take the smallest element ∂_z^2 in CT and update CT to $\{\partial_x\partial_z, \partial_x^2\}$. Since CL $=$ EE $= \emptyset$, there does not exist possible candidates of the lower terms. Set $\psi = \partial_z^2$ and check the conditions $(N^{(*)})$, then
$\psi(f_1) = 0 \in \mathfrak{p}^e$, $\psi(f_2) = 2 \notin \mathfrak{p}^e$, $\psi(f_3) = 2z \in \mathfrak{p}^e$.
Hence, ψ does not satisfy the condition $(N^{(*)})$. Update FL $= \{\partial_z^2\}$ and CL $= \{\partial_z^2\}$.

(4) Take the smallest element $\partial_x\partial_z$ in CT and update CT to $\{\partial_x^2\}$. Set $\psi = \partial_x\partial_z + c_{(0,2)}\partial_z^2$ where $c_{(0,2)}$ is an indeterminate. Then,
$\psi(f_1) = 6x^5 + 36x^3y + 4x^3 + 54xy^2 + 12xy \in \mathfrak{p}^e$,
$\psi(f_2) = 2c_{(0,2)}$, $\psi(f_3) = 2c_{(0,2)}z \in \mathfrak{p}^e$, $[\psi, x] = \partial_z \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$,
$[\psi, z] = \partial_x + 2c_{(0,2)}\partial_z \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$.
Hence, when $c_{(0,2)} \equiv 0 \pmod{\mathfrak{p}^e}$, then ψ satisfies the condition $(N^{(*)})$. Set $c_{(0,2)} = 0$, and renew NB as $\{1, \partial_z, \partial_x, \partial_x\partial_z\}$ and CT as
$$\{\partial_x^2\} \cup \textbf{Headcandidate}\Big(\{x, z\}, \partial_x\partial_z, \succ, \mathrm{ht}(\mathrm{NB}), \mathrm{FL}\Big) = \{\partial_x^2, \partial_x^2\partial_z\}.$$

(5) Take the smallest element ∂_x^2 in CT and update CT to $\{\partial_x^2\partial_z\}$. Set $\psi = \partial_x^2 + c_{(0,2)}\partial_z^2$ where $c_{(0,2)}$ is an indeterminate. Then,
$\psi(f_1) = 30x^4z + 108x^2yz + 12x^2z + 54y^2z + 12yz \in \mathfrak{p}^e$,
$\psi(f_2) = 30x^4 + 72x^2y + 18y^2 + 2c_{(0,2)}$, $\psi(f_3) = 2c_{(0,2)}z \in \mathfrak{p}^e$,
$[\psi, x] = 2\partial_x \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$, $[\psi, z] = 2c_{(0,2)}\partial_z \in \mathrm{Span}_{\mathbb{Q}(y)[x,z]/\mathfrak{p}^e}(\mathrm{NB})$.
Hence, when $c_{(0,2)} \equiv -36y^2 \pmod{\mathfrak{p}^e}$, then $\psi = \partial_x^2 - 36y^2\partial_z^2$ satisfies the condition $(N^{(*)})$. Set $c_{(0,2)} = -36y^2$, and renew NB as $\{1, \partial_z, \partial_x, \partial_x\partial_z, \partial_x^2 - 36y^2\partial_z^2\}$ and CT as

$$\{\partial_x^2\partial_z\} \cup \textbf{Headcandidate}\Big(\{x, z\}, \partial_x^2, \succ, \mathrm{ht}(\mathrm{NB}), \mathrm{FL}\Big) = \{\partial_x^2\partial_z, \partial_x^3\}.$$

Update $\mathrm{EE} = \{\partial_x\partial_z^2, \partial_z^3\}$.

(6) Take the smallest element $\partial_x^2\partial_z$ in CT and update CT to $\{\partial_x^3\}$. Since EE = E = EL, thus $\mathrm{CL} = \{\partial_x\partial_z^2, \partial_z^3, \partial_z^2\}$. Set

$$\psi = \partial_x^2\partial_z + c_{(1,2)}\partial_x\partial_z^2 + c_{(0,3)}\partial_z^3 + c_{(0,2)}\partial_z^2$$

where $c_{(1,2)}, c_{(0,3)}, c_{(0,2)}$ are indeterminates. Then,
$\psi(f_1) = 30x^4 + 108x^2y + 12x^2 + 54y^2 + 12y \notin \mathfrak{p}^e$,
$\psi(f_2) = 2c_{(0,2)}$, $\psi(f_3) = 6c_{(0,2)}z + 6c_{(0,3)}$.
Hence, ψ does not satisfy the condition $(N^{(*)})$. Update $\mathrm{FL} = \{\partial_x^2\partial_z, \partial_z^2\}$ and $\mathrm{CL} = \{\partial_x^2\partial_z, \partial_x\partial_z^2, \partial_z^3, \partial_z^2\}$.

(7) Take the smallest element ∂_x^3 in CT and update CT to $\emptyset$. Set

$$\psi = \partial_x^3 + c_{(2,1)}\partial_x^2\partial_z + c_{(1,2)}\partial_x\partial_z^2 + c_{(0,3)}\partial_z^3 + c_{(0,2)}\partial_z^2$$

where $c_{(2,1)}, c_{(1,2)}, c_{(0,3)}, c_{(0,2)}$ are indeterminates. Then,
$\psi(f_1) \equiv -24c_{(2,1)}y \pmod{\mathfrak{p}^e}$, $\psi(f_2) \equiv -216xy + 2c_{(0,2)} \pmod{\mathfrak{p}^e}$,
$\psi(f_3) \equiv 6c_{(0,3)} \pmod{\mathfrak{p}^e}$,
$[\psi, x] = 3\partial_x^2 + 2c_{(2,1)}\partial_x\partial_z + c_{(1,2)}\partial_z^2$,
$[\psi, z] = c_{(2,1)}\partial_x^2 + 2c_{(1,2)}\partial_x\partial_z + 3c_{(0,3)}\partial_z^2 + 2c_{(0,2)}\partial_z$.
Thus,

$$(1, \partial_z, \partial_x, \partial_x\partial_z, \partial_x^2 - 36y^2\partial_z^2, [\psi, x], [\psi, z]) = (1,\ \partial_z,\ \partial_x,\ \partial_x\partial_z,\ \partial_x^2,\ \partial_z^2)A$$

where

$$A = \begin{pmatrix} 1\,0\,0\,0 & 0 & 0 & 0 \\ 0\,1\,0\,0 & 0 & 0 & 2c_{(0,2)} \\ 0\,0\,1\,0 & 0 & 0 & 0 \\ 0\,0\,0\,1 & 0 & 2c_{(2,1)} & 2c_{(1,2)} \\ 0\,0\,0\,0 & 1 & 3 & c_{(2,1)} \\ 0\,0\,0\,0 & -36y^2 & c_{(1,2)} & 3c_{(0,3)} \end{pmatrix}.$$

By the Gaussian elimination method, we obtain

$$A \longrightarrow \left(\begin{array}{c|cc} 1\,0\,0\,0\,0 & 0 & 0 \\ 0\,1\,0\,0\,0 & 0 & 2c_{(0,2)} \\ 0\,0\,1\,0\,0 & 0 & 0 \\ 0\,0\,0\,1\,0 & 2c_{(2,1)} & 2c_{(1,2)} \\ 0\,0\,0\,0\,1 & 3 & c_{(2,1)} \\ \hline 0\,0\,0\,0\,0 & c_{(1,2)}+108y^2 & 3c_{(0,3)}+36y^2c_{(2,1)} \end{array}\right).$$

We have the following system of linear equations over $\mathbb{Q}(y)[x,y]/\mathfrak{p}^e$

$-24c_{(2,1)}y = 0, -216xy + 2c_{(0,2)} = 0, 6c_{(0,3)} = 0, c_{(1,2)} + 108y^2 = 0,$
$3c_{(0,3)} + 36y^2c_{(2,1)} = 0.$

Hence, we have the solution $\{c_{(2,1)} = 0, c_{(1,2)} = -108y^2, c_{(0,3)} = 0, c_{(0,2)} = 108xy\}$. Therefore, we obtain $\psi = \partial_x^3 - 108y^2\partial_x\partial_z^2 + 108xy\partial_z^2$. Renew NB as $\{1, \partial_z, \partial_x, \partial_x\partial_z, \partial_x^2 - 36y^2\partial_z^2, \partial_x^3 - 108y^2\partial_x\partial_z^2 + 108xy\partial_z^2\}$ and CT as

$$\textbf{Headcandidate}\Big(\{x,z\}, \partial_x^3, \succ, \mathrm{ht}(\mathrm{NB}), \mathrm{FL}\Big) = \{\partial_x^4\}.$$

Update EE $= \{\partial_x^2\partial_z^2, \partial_x\partial_z^3\}$.

(8) Take the smallest element ∂_x^4 in CT and update CT to $\emptyset$. Since EL $= \emptyset$, set

$$\psi = \partial_x^4 + c_{(2,1)}\partial_x^2\partial_z + c_{(1,2)}\partial_x\partial_z^2 + c_{(0,3)}\partial_z^3 + c_{(0,2)}\partial_z^2$$

where $c_{(2,1)}, c_{(1,2)}, c_{(0,3)}, c_{(0,2)}$ are indeterminates. Then,
$\psi(f_1) \equiv -24c_{(2,1)}y \pmod{\mathfrak{p}^e}$, $\psi(f_2) \equiv -936y + 2c_{(0,2)} \pmod{\mathfrak{p}^e}$,
$\psi(f_3) \equiv 6c_{(0,3)} \pmod{\mathfrak{p}^e}$.
Thus, we get $c_{(2,1)} = 0, c_{(0,2)} = 468y, c_{(0,3)} = 0$. Furthermore,
$[\psi, x] = 4\partial_x^3 + 2c_{(2,1)}\partial_x\partial_z + c_{(1,2)}\partial_z^2 = 4\partial_x^3 + c_{(1,2)}\partial_z^2$,
$[\psi, z] = c_{(2,1)}\partial_x^2 + 2c_{(1,2)}\partial_x\partial_z + 3c_{(0,3)}\partial_z^2 + 2c_{(0,2)}\partial_z = 2c_{(1,2)}\partial_x\partial_z + 2c_{(0,2)}\partial_z$.
Thus,

$$\begin{aligned}&(1, \partial_z, \partial_x, \partial_x\partial_z, \partial_x^2 - 36y^2\partial_z^2, \partial_x^3 - 108y^2\partial_x\partial_z^2 + 108xy\partial_z^2, [\psi,x], [\psi,z])\\ &= (1,\ \partial_z,\ \partial_x,\ \partial_x\partial_z,\ \partial_x^2,\ \partial_z^2,\ \partial_x^3,\ \partial_x\partial_z^2)B\end{aligned}$$

where

$$B = \begin{pmatrix} 1\,0\,0\,0 & 0 & 0 & 0 & 0 \\ 0\,1\,0\,0 & 0 & 0 & 0 & 2c_{(0,2)} \\ 0\,0\,1\,0 & 0 & 0 & 0 & 0 \\ 0\,0\,0\,1 & 0 & 0 & 0 & 2c_{(1,2)} \\ 0\,0\,0\,0 & 1 & 0 & 0 & 0 \\ 0\,0\,0\,0 & -36y^2 & 108xy & c_{(1,2)} & 0 \\ 0\,0\,0\,0 & 0 & 1 & 4 & 0 \\ 0\,0\,0\,0 & 0 & -108y^2 & 0 & 0 \end{pmatrix}.$$

By the Gaussian elimination method, we obtain

$$B \longrightarrow \left(\begin{array}{c|cc} 1\,0\,0\,0\,0\,0 & 0 & 0 \\ 0\,1\,0\,0\,0\,0 & 0 & 2c_{(0,2)} \\ 0\,0\,1\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,1\,0\,0 & 0 & 2c_{(1,2)} \\ 0\,0\,0\,0\,1\,0 & 0 & 0 \\ 0\,0\,0\,0\,0\,1 & 4 & 0 \\ \hline 0\,0\,0\,0\,0\,0 & c_{(1,2)} - 432xy & 0 \\ 0\,0\,0\,0\,0\,0 & 432y^2 & 0 \end{array}\right).$$

The system of linear equations $\{c_{(1,2)} - 432xy = 0, 432y^2 = 0\}$ does not have any solution. Thus, ψ does not satisfy the condition $(N^{(*)})$. Update $\mathrm{FL} = \{\partial_x^4, \partial_x^2\partial_z, \partial_z^2\}$.

Now, we stop computing because of $\mathrm{CT} = \emptyset$. Hence,

$$1, \partial_z, \partial_x, \partial_x\partial_z, \partial_x^2 - 36y^2\partial_z^2, \partial_x^3 - 108y^2\partial_x\partial_z^2 + 108xy\partial_z^2$$

are Noetherian operators of the primary component, whose radical is $\mathfrak{p}$, of I.

In Fig. 1, an element of ht(NB) is displayed as $\circ$ and an element of FL is displayed as $*$.

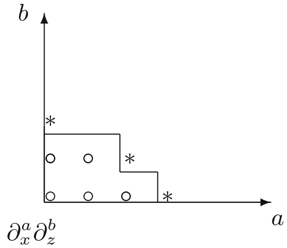

Fig. 1. Elements of ht(NB) and FL

Algorithm 1 is implemented in the computer algebra system Risa/Asir [17]. One can download the source codes from the following website:

https://www.rs.tus.ac.jp/~nabeshima/softwares.html.

When we input the second prime ideal $\langle x, z\rangle$ to the Risa/Asir implementation, then it outputs $1, \partial_x$ as the Noetherian operators.

4.2 Comparisons

In [4], the computer algebra system Macaulay2 [9] package NoetherianOperators, that implements another algorithm for computing Noetherian operators introduced in [3], is published. Let us compare an output of our Risa/Asir implementation with that of the Macaulay2 implementation.

Let $f = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + x^2z^2 + 5xy^4 + 2xyz^2 + xz^3 + y^5 + y^2z^2 \in \mathbb{Q}[x, y, z]$ and $J = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$. Then, J is a primary ideal with $\sqrt{J} = \langle x + y, z\rangle$, and $\{y\}$ is a MIS modulo J.

Macaulay2 implementation returns the following Noetherian operators of J if we input J.

$$\{1,\ \partial_z,\ \partial_x,\ \partial_x\partial_z,\ \partial_x^2,\ 3y\partial_x^2\partial_z + 2\partial_z^2,\ \partial_x^3,\\ (-162y^2-36)\partial_x^4 + (240y^3+540y)\partial_x^3\partial_z - 1620\partial_x^2\partial_z + (4860y^2+1080)\partial_x\partial_z^2\\ +4860y\partial_z^2\},$$

where $\partial_x := \frac{\partial}{\partial x}, \partial_z := \frac{\partial}{\partial z}$.

Our Risa/Asir implementation returns the following Noetherian operators of J if we input J.

$$\left\{1,\ \partial_z,\ \partial_x,\ \partial_x\partial_z,\ \partial_x^2,\ y\partial_x^2\partial_z + \frac{2}{3}\partial_z^2,\ \partial_x^3,\ y\partial_x^4 + 15y^2\partial_x^3\partial_z - 30y\partial_x\partial_z^2 - 30\partial_z^2\right\}.$$

As is evident from the outputs above, the output of our Risa/Asir implementation is simpler than that of Macaulay2. This is because Algorithm 1 returns a reduced basis of the finite dimensional vector space $\mathrm{NT}_{\mathfrak{q}^e}$ over $\mathbb{Q}(y)[x,z]/\langle x+y, z\rangle$. In contract, the output of Macaulay2 contains a redundant term $\partial_x^2\partial_z$. This is one of advantages of Algorithm 1.

Next, we give results of benchmark tests. All results in this paper have been computed on a PC with [OS: Ubuntu Linux, CPU: Intel(R) Core(TM) i9-7900X CPU @ 3.30 GHz, RAM: 128 GB]. The time is given in CPU-seconds. In Table 1, "$>10m$" means it takes more than 10 min.

Note that as the Macaulay2 implementation [4] allows only a primary ideal as the input, thus we use the following eight positive-dimensional primary ideals in $\mathbb{Q}[x,y,z]$ (or $\mathbb{Q}[x,y,z,w]$) for the comparisons. We use the total degree lexicographic term order with $\partial_x \succ \partial_y \succ \partial_z$ (or $\partial_x \succ \partial_y$, $\partial_x \succ \partial_z$).

1. $F_1 = \{x^8+4x^6y+6x^4y^2+4x^2y^3+y^4, z^4+2z^2+1\} \subset \mathbb{Q}[x,y,z]$, $\sqrt{\langle F_1\rangle} = \langle x^2+y, z^2+1\rangle$, and a MIS modulo $\sqrt{\langle F_1\rangle}$ is $\{y\}$.
2. $F_2 = \{3x^2+(y^2+z)^7, 7(y^2+z)^6x+10(y^2+z)^9, x^3+(y^2+z)^7x+(y^2+z)^{10}\} \subset \mathbb{Q}[x,y,z]$, $\sqrt{\langle F_2\rangle} = \langle x, y^2+z\rangle$, and a MIS modulo $\sqrt{\langle F_2\rangle}$ is $\{z\}$.
3. $F_3 = \{3(x+z^2+1)^2y+y^6, (x+z^2+1)^3+6(x+z^2+1)y^5+10y^9, (x+z^2+1)^3y+(x+z^2+1)y^6+y^{10}\} \subset \mathbb{Q}[x,y,z]$, $\sqrt{\langle F_3\rangle} = \langle y, x+z^2+1\rangle$, and a MIS modulo $\sqrt{\langle F_3\rangle}$ is $\{z\}$.
4. $F_4 = \{3(x+y)^2(z^2+w)+(z^2+w)^8+(z^2+w)^7, (x+y)^3+8(x+y)(z^2+w)^7+7(x+y)(z^2+w)^6+9(z^2+w)^8, (x+y)^3(z^2+w)+(x+y)(z^2+w)^8+(x+y)(z^2+w)^7+(z^2+w)^9\} \subset \mathbb{Q}[x,y,z,w]$, $\sqrt{\langle F_4\rangle} = \langle x+y, z^2+w\rangle$, and a MIS modulo $\sqrt{\langle F_4\rangle}$ is $\{y,z\}$.
5. $F_5 = \{3(x^2+z^2)^2+(y+z)^{11}, 11(y+z)^{10}(x^2+z^2)+19(y+z)^{18}+17(y+z)^{16}, ((x^2+z^2)^3+(x^2+z^2)(y+z)^{11}+(y+z)^{17}+(y+z)^{19})^2\} \subset \mathbb{Q}[x,y,z]$, $\sqrt{\langle F_5\rangle} = \langle x^2+z^2, y+z\rangle$, and a MIS modulo $\sqrt{\langle F_5\rangle}$ is $\{z\}$.
6. $F_6 = \{(3(x+w)^2+y^{10}+y^9)^2, ((10y^9+9y^8)(x+w)+13y^{12}+(z^2+w)^2)^2, y(z^2+w), (x+w)^3+(x+w)y^{11}+y^{17}+y^{19}\} \subset \mathbb{Q}[x,y,z,w]$, $\sqrt{\langle F_6\rangle} = \langle x+w, y, z^2+w\rangle$, and a MIS modulo $\sqrt{\langle F_6\rangle}$ is $\{w\}$.
7. $F_7 = \{4(x^2+z)^3+2(y+z)^5(x^2+z)+y^7, (5(y+z)^4(x^2+z)^2+7(y+z)^6(x^2+z)+12(y+z)^{11})^3, (x^2+z)^4+(y+z)^5(x^2+z)^2+(y+z)^7(x^2+z)+(y+z)^{12}\} \subset \mathbb{Q}[x,y,z]$, $\sqrt{\langle F_7\rangle} = \langle x^2+z, y+z\rangle$, and a MIS modulo $\sqrt{\langle F_7\rangle}$ is $\{z\}$.

8. $F_8 = \{3(2x^2+z)^2(y^2+2)+(y^2+2)^{13}+(y^2+2)^{12}+(y^2+2)^{11}, (2x^2+z)^3+13(2x^2+z)(y^2+2)^{12}+12(2x^2+z)(y^2+2)^{11}+11(2x^2+z)(y^2+2)^{10}+15(y^2+2)^{14}, ((2x^2+z)^3(y^2+2)+(y^2+2)^{15}+(2x^2+z)(y^2+2)^{12}+(2x^2+z)(y^2+2)^{13})^2\} \subset \mathbb{Q}[x,y,z]$, $\sqrt{\langle F_8\rangle} = \langle y^2+2, 2x^2+z\rangle$, and a MIS modulo $\sqrt{\langle F_8\rangle}$ is $\{z\}$.

In the benchmark tests, we use the Macaulay2 implementation with Strategy => "MacaulayMatrix" and our Risa/Asir implementation with computing an associate prime and a MIS, namely, the CPU time of "New implementation (Risa/Asir)", in Table 1, contains the sum of the computation times of $\sqrt{\langle F_i\rangle}$[1], a MIS modulo $\sqrt{\langle F_i\rangle}$ and Algorithm 1 for each $i \in \{1, 2, \dots, 8\}$.

Table 1. Comparisons of Noetherian operators

Problem	Macaulay2	New implementation (Risa/Asir) (Algorithm 1)
1	0.280	0.0156
2	11.389	0.1875
3	5.898	0.03125
4	27.816	0.0180
5	>10 m	0.8288
6	>10 m	1.172
7	>10 m	2.922
8	>10 m	4.875

As is evident from Table 1, our new implementation is much faster in comparison with Macaulay2 implementation because Algorithm 1 mainly consists of linear algebra techniques. This is one of the big advantages of the new algorithm.

4.3 Computing Noetherian Representations

Here we introduce an algorithm for computing a Noetherian representation that can be regarded as an alternative primary ideal decomposition of a polynomial ideal. As we described in Sect. 3 and Sect. 4.1, Noetherian operators encode primary components of a polynomial ideal. Thus, they can be utilized to characterize an ideal.

Definition 7. *Let I be an ideal in $K[X]$, $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$ a primary decomposition of I where $\mathfrak{q}_i$ is a primary ideal for $1 \le i \le t$. Let $\mathrm{NB}_i \subset K(U_i)[Y_i][\{\partial_y | y \in Y_i\}]$ be a basis of the vector space $\mathrm{NT}_{\mathfrak{q}_i{}^e}$ where U_i is a MIS modulo $\mathfrak{q}_i$ and $Y_i = X \backslash U_i$. Then,*

$$\{(\sqrt{\mathfrak{q}_1}, \mathrm{NB}_1, U_1), (\sqrt{\mathfrak{q}_2}, \mathrm{NB}_2, U_2), \dots, (\sqrt{\mathfrak{q}_t}, \mathrm{NB}_t, U_t)\}$$

is called a Noetherian representation of I and written as Noether(I).

[1] A function noro_pd.prime_dec [15], that computes a prime decomposition of a radical ideal, is available in a program file noro_pd.rr that is contained in the OpenXM package [20].

By combining an algorithm for computing a prime decomposition of $\sqrt{I}$ [1,15,22], Lemma 3, 4, 5 and Algorithm 1, we can construct an algorithm for computing Noether(I) without computing a primary decomposition of I. The following algorithm is based on Gianni-Trager-Zacharias algorithm [10] of computing a primary ideal decomposition.

Algorithm 2 (noetherian-rep)

Specification: noetherian-rep(F)
Computing Noetherian representation of $\langle F\rangle$.
Input: $F \subset K[X]$.
Output: $\mathrm{NR} = \{(\mathfrak{p}_1, \mathrm{NB}_1, U_1), \ldots, (\mathfrak{p}_t, \mathrm{NB}_t, U_t)\}$: Noetherian representation of $\langle F\rangle$.

BEGIN
$Flag \leftarrow 1$; $\mathrm{NR} \leftarrow \emptyset$;
while $Flag = 1$ **do**
  $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} \leftarrow \bigcap_{i=1}^{k} \mathfrak{p}_i$ is the minimal prime decomposition of $\sqrt{\langle F\rangle}$; $(*)$
  $\mathfrak{p}_{max} \leftarrow$ Select a maximal dimensional prime ideal $\mathfrak{p}_{max}$ from $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$;
  $U \leftarrow$ Compute a MIS modulo $\mathfrak{p}_{max}$; $Y \leftarrow X \backslash U$;
  $\succ_b \leftarrow$ Set a block term order with $U \gg Y$;
  $M \leftarrow \{\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} | \dim(\mathfrak{p}) = \dim(\mathfrak{p}_{max}), U$ is a MIS modulo $\mathfrak{p}\}$;
  $\succ \leftarrow$ Set a term order on $\mathrm{Term}(\{\partial_y | y \in Y\})$;
  while $M \neq \emptyset$ **do**
    $\mathfrak{p}_m \leftarrow$ Select $\mathfrak{p}$ form M ; $M \leftarrow M \backslash \{\mathfrak{p}_m\}$;
    $\mathrm{NB} \leftarrow$ **Noether**$(F, \mathfrak{p}_m, U, Y, \succ)$;
    $\mathrm{NR} \leftarrow \mathrm{NR} \cup \{(\mathfrak{p}_m, \mathrm{NB}, U)\}$;
  end-while
  if $Y \neq \emptyset$ **then**
    $G \leftarrow$ Compute a Gröbner basis of $\langle F\rangle$ w.r.t. $\succ_b$ in $K[U, Y] = K[X]$;
    $h \leftarrow \mathrm{LCM}\{\mathrm{hc}(g) | g \in G\}$ where G is regarded as a subset of $K[U][Y]$;
    if h is a constant **then**
      $Flag \leftarrow 0$;
    else
      $s \leftarrow$ Compute a natural number with $\langle H\rangle : h^\infty = \langle H\rangle : h^s$;
      $F \leftarrow \{F \cup \{h^s\}\}$;
    end-if
  else
    $Flag \leftarrow 0$;
  end-if
end-while
return NR;
END

As we mentioned in Remark 2, in general, an algorithm for computing a prime decomposition of the radical $\sqrt{I}$, at $(*)$, is much faster that that for computing primary decomposition of a polynomial ideal I in $K[X]$.

Theorem 4. *Algorithm 2 terminates and outputs correctly.*

Proof. By utilizing Lemma 5, we have $\langle F\rangle = \langle F \cup \{h_1^{s_1}\}\rangle \cap (\langle F\rangle : h_1^{s_1})$ where $h_1 = \mathrm{LCM}\{\mathrm{hc}(g)|g \in G \subset K[U_1][Y_1]\}$, G is a Gröbner basis of $\langle F\rangle$ w.r.t. a block term order with $U_1 \gg Y_1$ on Term(X) in $K[X]$, U_1 is a MIS modulo $\langle F\rangle$, $Y_1 = X\backslash U_1$ and s_1 is a natural number that satisfying $\langle F\rangle : h_1^{\infty} = \langle F\rangle : h_1^{s_1}$. In the second while-loop, a Noetherian representation of $\langle F\rangle : h_1^{s_1}$ is obtained because of Lemma 3 and 4. Renew $F_2 := F \cup \{h_1^{s_1}\}$. Again, by utilizing Lemma 6, we have $\langle F_2\rangle = \langle F_2 \cup \{h_2^{s_2}\}\rangle \cap (\langle F_2\rangle : h_2^{s_2})$ where $h_2 = \mathrm{LCM}\{\mathrm{hc}(g)|g \in G_2 \subset K[U_2][Y_2]\}$, G_2 is a Gröbner basis of $\langle F_2\rangle$ w.r.t. a block term order with $U_2 \gg Y_2$ on Term(X) in $K[X]$, U_2 is a MIS modulo $\langle F_2\rangle$, $Y_2 = X\backslash U_2$ and s_2 is a natural number satisfies $\langle F_2\rangle : h_2^{\infty} = \langle F_2\rangle : h_2^{s_2}$. In the second while-loop, a Noetherian representation of $\langle F_2\rangle : h_2^{s_2}$ is obtained by the same reason above. We repeat the same procedure until h_i becomes a constant ($i \in \mathbb{N}$). Then, the union NR of all triples is a Noetherian representation of the input ideal $\langle F\rangle$ because of $\langle F\rangle = (\cap_{i=2}^{t}(\langle F_i\rangle : h_i^{s_i})) \cap (\langle F\rangle : h_1^{s_1})$. As $K[X]$ is a Noetherian ring, the number t is finite. Thus, Algorithm 2 terminates and outputs correctly. □

We illustrate the algorithm with the following example.

Example 3. Let us consider the ideal I of Example 2, again. As we described in Example 2, we have $\sqrt{I} = \langle x^2 + 3y, z\rangle \cap \langle x, z\rangle$ as the prime decomposition of $\sqrt{I}$. Since $\{y\}$ is the MIS modulo $\langle x^2 + 3y, z\rangle$ and $\langle x, z\rangle$, thus $M = \{\langle x^2 + 3y, z\rangle, \langle x, z\rangle\}$. We have $\mathrm{NR} = \{(\langle x^2 + 3y, z\rangle, \mathrm{NB}, \{y\}), (\langle x, z\rangle, \{1, \partial_z\}, \{y\})\}$ in Example 2.

The reduced Gröbner basis G of I w.r.t. a block term order with $\{x, z\} \gg \{y\}$ is $G = \{z^3, (3y+1)x^4z + (18y^2+6y)x^2z + 27y^3z + 9y^2z, x^6 + 6yx^4 + 9y^2x^2 + z^2\}$ in $\mathbb{Q}[x, y, z]$. Then, $h = \mathrm{LCM}\{\mathrm{hc}(g)|g \in G \subset \mathbb{Q}[y][x, z]\} = 3y + 1$ in $\mathbb{Q}[y]$ and $\langle F\rangle : h^{\infty} = \langle F\rangle : h$. We set $F' = \{3y+1\} \cup \{f_1, f_2, f_3\}$. In this case, $\langle F'\rangle$ is zero dimensional, namely, the MIS modulo $\langle F'\rangle$ is the empty set.

The prime decomposition of $\sqrt{\langle F'\rangle}$ is

$$\sqrt{\langle F'\rangle} = \langle x, 3y+1, z\rangle \cap \langle x-1, 3y+1, z\rangle \cap \langle x+1, 3y+1, z\rangle.$$

Thus, for each prime ideal, Algorithm 1 outputs the reduced basis of the vector space as follows:

$$\begin{aligned}\mathrm{NZ} = \{&(\langle x, 3y+1, z\rangle, \{1, \partial_x, \partial_z, \partial_x\partial_z, \partial_z^2 - \partial_z^2, \partial_x^3 - 3\partial_x\partial_z^2\},\ \emptyset),\\ &(\langle x-1, 3y+1, z\rangle, \{1, \partial_x, \partial_z, \partial_x\partial_z, \partial_x^2 - 4\partial_z^2, \partial_x^3 - 12\partial_x\partial_z^2 - 36\partial_z^2\},\ \emptyset),\\ &(\langle x+1, 3y+1, z\rangle, \{1, \partial_x, \partial_z, \partial_x\partial_z, \partial_x^2 - 4\partial_z^2, \partial_x^3 - 12\partial_x\partial_z^2 + 36\partial_z^2\},\ \emptyset)\}.\end{aligned}$$

Therefore, Noether$(I) = \mathrm{NR} \cup \mathrm{NZ}$.

We remark that bases of the primary ideals that are associated to $(\langle x^2+3y, z\rangle$, NB $\emptyset)$, $(\langle x-1, 3y+1, z\rangle, \{1, \partial_x, \partial_z, \partial_x\partial_z, \partial_x^2 - 4\partial_z^2, \partial_x^3 - 12\partial_x\partial_z^2 - 36\partial_z^2\}\ \emptyset)$ and $(\langle x+1, 3y+1, z\rangle, \{1, \partial_x, \partial_z, \partial_x\partial_z, \partial_x^2 - 4\partial_z^2, \partial_x^3 - 12\partial_x\partial_z^2 + 36\partial_z^2\}\ \emptyset)$ are the following $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$. respectively.

$$\mathfrak{q}_1 = \{z^3, x^4z + 6x^2yz + 9y^2z, 9x^4y^2 + 54x^2y^3 - x^2z^2 + 81y^4 - 6yz^2, x^6 + 6x^4y + 9x^2y^2 + z^2\},$$
$$\mathfrak{q}_2 = \{3y+1, z^3, 4x^2-3xz^2-8x+4z^2+4, x^2z-2xz+z, 12x^3-32x^2+28x+z^2-8\},$$
$$\mathfrak{q}_3 = \{3y+1, z^3, 4x^2+3xz^2+8x+4z^2+4, x^2z+2xz+z, 12x^3+32x^2+28x-z^2+8\}.$$

Since we can check $\langle \mathfrak{q}_1 \rangle \subset \langle \mathfrak{q}_2 \rangle$ and $\langle \mathfrak{q}_1 \rangle \subset \langle \mathfrak{q}_3 \rangle$, thus $\mathfrak{q}_2$ and $\mathfrak{q}_3$ are redundant, namely, the following is also a Noetherian representation of $\langle F \rangle$:

$$\text{Noether}(I) = \text{NR} \cup \{(\langle x, 3y+1, z\rangle, \{1, \partial_x, \partial_z, \partial_x\partial_z, \partial_z^2 - \partial_z^2, \partial_x^3 - 3\partial_x\partial_z^2\},\ \emptyset)\}.$$

The Noetherian representation above corresponds to the minimal primary decomposition of I.

Since we adapt the Gianni-Trager-Zacharias algorithm [10] of computing a primary decomposition, there is a possibility that the output of Algorithm 2 contains redundant components, like the above. After obtaining the decomposition, it is possible to delete the redundant components by checking the inclusions.

In Sect. 6 of [16], an algorithm for computing generators of a zero dimensional primary ideal $\mathfrak{q}$ from a triple $(\mathfrak{p}, \text{NB}, \emptyset)$ is introduced where $\mathfrak{q}$ is $\mathfrak{p}$-primary and NB is a basis of the vector space $\text{NT}_{\mathfrak{q}}$ in $K[X][\partial]$. Even if $\mathfrak{q}$ is not zero dimensional, we can utilize the algorithm for computing generators of $\mathfrak{q}^e$ in $K(U)[Y]$ where U is a MIS of $\mathfrak{q}$ and $Y = X \backslash U$. As $\mathfrak{q}^{ec} = \mathfrak{q}$, generators of $\mathfrak{q}$ can be obtained by the algorithm that is published in [16]. Actually, in Example 3, $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$ were computed by the algorithm. Therefore, by combining Algorithm 3 and the algorithm for computing generators (and techniques of [15]), one can construct an algorithm for computing a minimal primary decomposition of a polynomial ideal $I \subset K[X]$ and the Noetherian representation Noether(I), simultaneously.

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