On the Qualitative Analysis of the Equations of Motion of a Nonholonomic Mechanical System

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Abstract. The problem on the rotation of a dynamically asymmetric rigid body around a fixed point is considered. The body is fixed inside a spherical shell, which a ball and a disk adjoin to. The equations of motion of the mechanical system in the case of absence of external forces admit two additional first integrals and these are completely integrable. The nonintegrable case, when potential forces act upon the system, is also considered. The qualitative analysis of the equations of motion is done in the both cases: stationary sets are found and their Lyapunov stability is studied. A mechanical interpretation for the obtained solutions is given.

Keywords: Nonholonomic mechanical system · Qualitative analysis · Computer algebra

1 Introduction

The problem considered in this paper goes back to the Chaplygin work [1] of rolling a dynamically asymmetric balanced ball along a horizontal plane without slipping. The integrability of the system was revealed by Chaplygin with the help of its explicit reduction to quadratures. A sufficient number of works are devoted to the Chaplygin problem and its integrable generalizations (see, e.g., [2]). One of them is investigated in the paper. In [3] the generalization of system [2] is given. The motion of a dynamically asymmetric rigid body around fixed point O is considered (see Fig. 1). The body is rigidly enclosed in a spherical shell, the geometrical center of which coincides with the fixed point of the body. One ball and one disk adjoin to the spherical shell. It is supposed that slipping at a contact point of the ball with the shell is absent. The disk – nonholonomic hinge – concerns the external surface of the spherical shell. The centers of the balls and the axis of the disk are fixed in space. The study of dynamics of such systems is of interest, e.g., for robotics in the problems of the design and control of mobile spherical robots (see., e.g., [4]). The motion of the mechanical system is described by the differential equations [3]

F. Boulier et al. (Eds.): CASC 2023, LNCS 14139, pp. 213–232, 2023.
https://doi.org/10.1007/978-3-031-41724-5_12

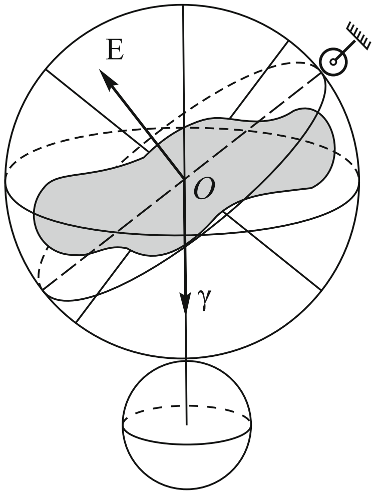

Fig. 1. The rigid body enclosed in a spherical shell, which a ball and a disk adjoin to.

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + R\boldsymbol{\gamma} \times \mathbf{N} + \mu\mathbf{E} + \mathbf{M_Q},\ D_1\dot{\boldsymbol{\omega}}_\mathbf{1} = D_1\boldsymbol{\omega}_\mathbf{1} \times \boldsymbol{\omega} + R_1\boldsymbol{\gamma} \times \mathbf{N},$$
$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega},\ \dot{\mathbf{E}} = \mathbf{E} \times \boldsymbol{\omega}, \quad (1)$$

and the equations of constraints

$$R\boldsymbol{\omega} \times \boldsymbol{\gamma} + R_1\boldsymbol{\omega}_\mathbf{1} \times \boldsymbol{\gamma} = 0,\ (\boldsymbol{\omega}, \mathbf{E}) = 0. \quad (2)$$

Here $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, R is the angular velocity of the body and the radius of the spherical shell, $\boldsymbol{\omega}_\mathbf{1} = (\omega_{1_1}, \omega_{1_2}, \omega_{1_3})$, R_1 is the angular velocity and the radius of the adjoint ball, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector of the axis connecting the fixed point with the center of the adjoint ball, $\mathbf{E} = (e_1, e_2, e_3)$ is the vector of the normal to the plane containing the fixed point and the axis of the disk, $\mathbf{I} = \mathrm{diag}(A, B, C)$ is the inertia tensor of the body, D_1 is the inertia tensor of the adjoint ball, $\mathbf{N}= (N_1, N_2,\ N_3)$, μ are indefinite factors related to the reactions of constraints (2), $\mathbf{M_Q}$ is the moment of external forces. One supposes that the position of the vectors $\mathbf{E}$ and $\boldsymbol{\gamma}$ with respect to each other is arbitrary.

By means of the equations of constraints (2) the differential Eqs. (1) are reduced to the form [3]:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + D\boldsymbol{\gamma} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma}) = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mu\mathbf{E} + \mathbf{M_Q},\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega},\ \dot{\mathbf{E}} = \mathbf{E} \times \boldsymbol{\omega}, \quad (3)$$

where $D = \frac{R^2}{R_1^2} D_1$.

The indefinite factor μ is found from the condition that the derivative of the 2nd relation (2) in virtue of differential Eqs. (3) is equal to zero.

If the body is subject to external forces, e.g., potential ones

$$\mathbf{M_Q} = \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}} + \mathbf{E} \times \frac{\partial U}{\partial \mathbf{E}},$$

where $U = U(\boldsymbol{\gamma}, \mathbf{E})$ is the potential energy of external forces, Eqs. (3) admit the following first integrals

$$2H = (\mathbf{I_Q}\,\boldsymbol{\omega}, \boldsymbol{\omega}) + 2U(\boldsymbol{\gamma}, \mathbf{E}) = 2\,h,\ V_1 = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1,\ V_2 = (\mathbf{E}, \mathbf{E}) = 1,$$
$$V_3 = (\boldsymbol{\gamma}, \mathbf{E}) = c_1,\ V_4 = (\boldsymbol{\omega}, \mathbf{E}) = 0 \qquad (4)$$

and are nonintegrable in the general case. Here $\mathbf{I_Q} = \mathbf{I} + D - D\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}$, $\boldsymbol{\gamma} \otimes \boldsymbol{\gamma} = [c_{ij}]$, $c_{11} = \gamma_1^2$, $c_{12} = \gamma_1\gamma_2, \ldots$

In the case of the absence of external forces ($U = 0$) and $(\mathbf{E} \times \boldsymbol{\gamma}) \neq 0$, Eqs. (3) have two additional first integrals

$$F_1 = (\mathbf{K}, \mathbf{E} \times \boldsymbol{\gamma}),\ F_2 = (\mathbf{K}, \mathbf{E} \times (\mathbf{E} \times \boldsymbol{\gamma})),$$

where $\mathbf{K} = \mathbf{I_Q}\,\boldsymbol{\omega} - (\mathbf{I_Q}\,\boldsymbol{\omega}, \mathbf{E})\mathbf{E}$, and then system (3) is completely integrable.

2 Problem Statement

The qualitative analysis of the problem under consideration was not conducted so far. In the present work, the qualitative analysis of the equations of motion (3) on the invariant set defined by the relation $V_4 = 0$ (4) is done. We find invariant sets of various dimension from the necessary conditions of extremum of the first integrals of the problem (or their combinations) and study their Lyapunov stability. The sets found in this way are called stationary ones. The stationary sets of zero dimension are known as stationary solutions, while the positive dimension sets are called stationary invariant manifolds (IMs).

We use the Routh–Lyapunov method [5] and some its generalizations [6] for the study of the problem. The computer analysis of the problem is mainly done symbolically. Computer algebra system (CAS) *Mathematica* and the software package [7] written in the language of this system are applied to solve computational problems. With the help of the package, the stability of the found solutions is investigated.

The paper is organized as follows. In Sect. 2 and 3, we describe finding stationary sets both in the case of absence of external forces and when potential forces act upon the mechanical system. Solutions obtained in these sections correspond to equilibria of the mechanical system. In Sect. 4, solutions corresponding to pendulum-type motions are presented. In Sect. 5, the stability of the found solutions is analyzed. In Sect. 6, we give some conclusions.

3 On Stationary Sets in the Case of Absence of External Forces

The equations of motion (3) in an explicit form on the invariant set $V_4 = 0$ when $U = 0$ are written as

$$\dot{\omega}_1 = -\frac{1}{\sigma_1}\Big[D((A-B)(B+D)\gamma_3\bar{\omega}_2 - (A-C)(C+D)\gamma_2\omega_3)\,\gamma_1\omega_1$$
$$+(B-C)((C+D)(B+D-D\gamma_2^2) - D(B+D)\gamma_3^2)\,\bar{\omega}_2\omega_3 + \mu\,[(C+D)$$
$$\times((B+D)e_1 + D\gamma_2(e_2\gamma_1 - e_1\gamma_2)) + D(B+D)\gamma_3(e_3\gamma_1 - e_1\gamma_3)]\Big],$$
$$\dot{\omega}_3 = -\frac{1}{\sigma_1}\Big[(A-B)((B+D)(A+D-D\gamma_1^2) - D(A+D)\,\gamma_2^2)\,\omega_1\bar{\omega}_2$$
$$-D((A-C)(A+D)\gamma_2\omega_1 - (B-C)(B+D)\gamma_1\bar{\omega}_2)\,\gamma_3\omega_3 + \mu\,[D(A+D)$$
$$\times\gamma_2\,(e_2\gamma_3 - e_3\gamma_2) + (B+D)(e_3(A+D-D\gamma_1^2) + De_1\gamma_1\gamma_3)]\Big],$$
$$\dot{\gamma}_1 = -\gamma_3\bar{\omega}_2 + \gamma_2\omega_3,\ \dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3,\ \dot{\gamma}_3 = -\gamma_2\omega_1 + \gamma_1\bar{\omega}_2,$$
$$\dot{e}_1 = -e_3\bar{\omega}_2 + e_2\omega_3,\ \dot{e}_2 = e_3\omega_1 - e_1\omega_3,\ \dot{e}_3 = -e_2\omega_1 + e_1\bar{\omega}_2, \qquad (5)$$

where $\bar{\omega}_2 = -\frac{e_1\omega_1 + e_3\omega_3}{e_2}$,

$$\mu = -\frac{1}{\sigma_2}\Big[(A-B)((B+D)(A+D)\,e_3 + D(B+D)\gamma_1(\gamma_3 e_1 - e_3\gamma_1)$$
$$+D(A+D)\gamma_2(\gamma_3 e_2 - e_3\gamma_2))\,\omega_1\bar{\omega}_2$$
$$-(A-C)(e_2(A+D)(C+D) + D(C+D)\gamma_1(e_1\gamma_2 - e_2\gamma_1)$$
$$+D(A+D)\gamma_3(e_3\gamma_2 - e_2\gamma_3))\,\omega_1\omega_3 + (B-C)((B+D)(C+D)e_1$$
$$+D(C+D)\gamma_2(e_2\gamma_1 - e_1\gamma_2) + D(B+D)\gamma_3(e_3\gamma_1 - e_1\gamma_3))\,\bar{\omega}_2\omega_3\Big],$$
$$\sigma_1 = D((B+D)(C+D)\,\gamma_1^2 + (A+D)(C+D)\,\gamma_2^2 + (A+D)(B+D)\,\gamma_3^2)$$
$$-(A+D)(B+D)(C+D),$$
$$\sigma_2 = (B+D)(C+D)\,e_1^2 + (A+D)(C+D)\,e_2^2 + (A+D)(B+D)\,e_3^2$$
$$-D[(C+D)(e_2\gamma_1 - e_1\gamma_2)^2 + (B+D)(e_3\gamma_1 - e_1\gamma_3)^2$$
$$+(A+D)(e_3\gamma_2 - e_2\gamma_3)^2],$$

Equations (5) admit the following first integrals:

$$2\,H = (A+D-D\gamma_1^2)\,\omega_1^2 + (B+D-D\gamma_2^2)\,\bar{\omega}_2^2 + (C+D-D\gamma_3^2)\,\omega_3^2$$
$$-2D(\gamma_1\gamma_2\omega_1\bar{\omega}_2 + \gamma_1\gamma_3\omega_1\omega_3 + \gamma_2\gamma_3\bar{\omega}_2\omega_3) = 2\,h,$$
$$V_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,\ V_2 = e_1^2 + e_2^2 + e_3^2 = 1,$$
$$V_3 = e_1\gamma_1 + e_2\gamma_2 + e_3\gamma_3 = c_1,$$
$$F_1 = -(A+D)(e_3\gamma_2 - e_2\gamma_3)\,\omega_1 + (B+D)(e_3\gamma_1 - e_1\gamma_3)\,\bar{\omega}_2$$
$$-(C+D)(e_2\gamma_1 - e_1\gamma_2)\,\omega_3 = c_2,$$
$$F_2 = [e_1\,(A+D-2D\gamma_1^2)(e_2\gamma_2 + e_3\gamma_3) - \gamma_1(A\,(e_2^2 + e_3^2)$$
$$-D((e_2^2 + e_3^2)(\gamma_1^2 - 1) + (e_3\gamma_2 - e_2\gamma_3)^2 + e_1^2(\gamma_2^2 + \gamma_3^2)))]\,\omega_1$$
$$+[e_2\,(B+D-2D\gamma_2^2)(e_1\gamma_1 + e_3\gamma_3) - \gamma_2(B\,(e_1^2 + e_3^2)$$
$$-D((e_1^2 + e_3^2)(\gamma_2^2 - 1) + (e_3\gamma_1 - e_1\gamma_3)^2 + e_2^2(\gamma_1^2 + \gamma_3^2)))]\,\bar{\omega}_2$$
$$+[e_3\,(e_1\gamma_1 + e_2\gamma_2)(C+D-2D\gamma_3^2) - \gamma_3(C\,(e_1^2 + e_2^2)$$
$$-D((e_2\gamma_1 - e_1\gamma_2)^2 + e_3^2(\gamma_1^2 + \gamma_2^2) + (e_1^2 + e_2^2)(\gamma_3^2 - 1)))]\,\omega_3 = c_3. \qquad (6)$$

Here F_1, F_2 are the additional integrals of the 3rd and 5th degrees, respectively.

As was remarked above, the stationary conditions for the first integrals of the problem (or their combinations) are used to obtain solutions of interest for us. In the problem under consideration, because of rather high degrees of the first integrals, another approach [8] turned out to be more effective for seeking the desired solutions: first, obtain the desired solutions from the equations of motion, and, then, find the conditions on the parameters of the problem under which these solutions satisfy the stationary equations for the first integrals.

Obviously, Eqs. (5) have the solution $\omega_1 = \omega_3 = 0$. These relations together with the integrals $V_1 = 1, V_2 = 1$ define the invariant manifold (IM) of codimension 4 for the equations of motion (5). It is easy to verify by direct calculation according to the IM definition. The equations of the IM are written as:

$$\omega_1 = \omega_3 = 0,\ e_1^2 + e_2^2 + e_3^2 = 1,\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{7}$$

With the help of maps on IM (7)

$$\omega_1 = \omega_3 = 0,\ \gamma_1 = \pm\sqrt{1 - \gamma_2^2 - \gamma_3^2},\ e_1 = \pm\sqrt{1 - e_2^2 - e_3^2}, \tag{8}$$

we find that the integral V_3 takes the form

$$e_2\gamma_2 + e_3\gamma_3 \pm \sqrt{1 - \gamma_2^2 - \gamma_3^2}\sqrt{1 - e_2^2 - e_3^2} = c_1$$

on this IM. Thus, IM (7) exists for any angles between the vectors $\mathbf{E}$ and $\boldsymbol{\gamma}$, i.e., it is the family of IMs.

The differential equations $\dot\gamma_2 = 0,\ \dot\gamma_3 = 0,\ \dot e_2 = 0,\ \dot e_3 = 0$ on IM (7) have the family of solutions:

$$\gamma_2 = \gamma_2^0 = \text{const},\ \gamma_3 = \gamma_3^0 = \text{const},\ e_2 = e_2^0 = \text{const},\ e_3 = e_3^0 = \text{const}. \tag{9}$$

The latter relations together with the IM equations determine four families of solutions for the equations of motion (5)

$$\begin{aligned}
&\omega_1 = \omega_3 = 0,\ e_1 = \pm\sqrt{1 - {e_2^0}^2 - {e_2^0}^2},\ e_2 = e_2^0,\ e_3 = e_3^0,\ \gamma_1 = \sqrt{1 - {\gamma_2^0}^2 - {\gamma_2^0}^2},\\
&\gamma_2 = \gamma_2^0,\ \gamma_3 = \gamma_3^0;\\
&\omega_1 = \omega_3 = 0,\ e_1 = \pm\sqrt{1 - {e_2^0}^2 - {e_2^0}^2},\ e_2 = e_2^0,\ e_3 = e_3^0,\ \gamma_1 = -\sqrt{1 - {\gamma_2^0}^2 - {\gamma_2^0}^2},\\
&\gamma_2 = \gamma_2^0,\ \gamma_3 = \gamma_3^0
\end{aligned} \tag{10}$$

that can be verified by substituting the solutions into these equations. Here $e_2^0, e_3^0,\ \gamma_2^0, \gamma_3^0$ are the parameters of the families. Evidently, the solutions belong to IM (7).

From a mechanical point of view, the elements of the families of solutions (10) correspond to equilibria of the mechanical system under study.

Using the stationary equations

$$\partial K_1/\partial\omega_1 = 0,\ \partial K_1/\partial\omega_3 = 0,\ \partial K_1/\partial\gamma_j = 0,\ \partial K_1/\partial e_j = 0\ (j = 1, 2, 3)$$

for the integral $2K_1 = 2\lambda_0 H - \lambda_1(V_1 - V_2)^2 - \lambda_2 F_1 F_2$ (λ_i = const), it is not difficult to show that this integral takes a stationary value both on IM (7) and solutions (10). For this purpose, it is sufficient to substitute expressions (8) (or (10)) into the above equations. These become identity.

Directly, from differential Eqs. (5), it is also easy to obtain the following their solutions:

$$\omega_1 = \omega_3 = 0,\ e_1 = \pm\gamma_1,\ e_2 = \pm\gamma_2,\ e_3 = \pm\gamma_3. \tag{11}$$

Relations (11) together with the integral $V_1 = 0$ define two IMs of codimension 6 of differential Eqs. (5) that is verified by direct computation according to the IM definition. The equations of these IMs have the form:

$$\omega_1 = \omega_3 = 0,\ e_1 \mp \gamma_1 = 0,\ e_2 \mp \gamma_2 = 0,\ e_3 \mp \gamma_3 = 0,\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{12}$$

On substituting expressions (12) into the stationary conditions for the integral

$$2K_2 = 2\lambda_0 H - \lambda_1 V_1 - \lambda_2 V_2 - 2\lambda_3 V_3 - 2\lambda_4 F_1 - 2\lambda_5 F_2\ (\lambda_i = \text{const})$$

we find the values $\lambda_2 = \lambda_1$, $\lambda_3 = \mp\lambda_1$ under which the integral K_2 assumes a stationary value on IMs (12).

The integrals K_1 and K_2 (under the corresponding values of λ_2, λ_3) are used for obtaining the sufficient conditions of stability of the above solutions.

The differential equations $\dot{\gamma}_2 = 0$, $\dot{\gamma}_3 = 0$ on each IMs (12) have the following family of solutions: $\gamma_2 = \gamma_2^0$ = const, $\gamma_3 = \gamma_3^0$ = const. Thus, geometrically, in space R^8, two-dimensional surface corresponds to each of IMs (12), each point of which is a fixed point of the phase space.

The integral V_3 takes the values ± 1 on IMs (12). Thus, IMs (12) correspond to the cases when the vectors $\mathbf{E}$ and $\boldsymbol{\gamma}$ are parallel or opposite in direction.

4 On Stationary Sets in the Case of the Presence of External Forces

Let the mechanical system under study be under the influence of external potential forces with the potential energy $U = (\mathbf{a}, \boldsymbol{\gamma}) + (\mathbf{b}, \mathbf{E})$, where $\mathbf{a} = (a_1, a_2, a_2)$, $\mathbf{b} = (b_1, b_2, b_2)$ are the indefinite factors. In this case, the equations of motion (3) on the invariant set $V_4 = 0$ are written as:

$$
\begin{aligned}
\dot{\omega}_1 &= -\frac{1}{\sigma_1}\Big[D((A-B)(B+D)\gamma_3\bar{\omega}_2 - (A-C)(C+D)\gamma_2\omega_3)\,\gamma_1\omega_1 \\
&\quad +(B-C)((C+D)(B+D-D\gamma_2^2) - D(B+D)\gamma_3^2)\,\bar{\omega}_2\omega_3 + \mu\,[(C+D) \\
&\quad \times((B+D)e_1 + D\gamma_2(e_2\gamma_1 - e_1\gamma_2)) + D(B+D)\gamma_3(e_3\gamma_1 - e_1\gamma_3)] \\
&\quad +((C+D)(B+D-D\gamma_2^2) - D(B+D)\gamma_3^2)M_{Q_1} + D(C+D)\gamma_1\gamma_2 M_{Q_2} \\
&\quad +D(B+D)\gamma_1\gamma_3 M_{Q_3}\Big], \\
\dot{\omega}_3 &= -\frac{1}{\sigma_1}\Big[(A-B)((B+D)(A+D-D\gamma_1^2) - D(A+D)\,\gamma_2^2)\,\omega_1\bar{\omega}_2 \\
&\quad -D((A-C)(A+D)\gamma_2\omega_1 - (B-C)(B+D)\gamma_1\bar{\omega}_2)\,\gamma_3\omega_3 + \mu\,[D(A+D) \\
&\quad \times\gamma_2\,(e_2\gamma_3 - e_3\gamma_2) + (B+D)(e_3(A+D-D\gamma_1^2) + De_1\gamma_1\gamma_3)] \\
&\quad +D\gamma_3((B+D)\gamma_1 M_{Q_1} + (A+D)\gamma_2 M_{Q_2}) + ((B+D)(A+D-D\gamma_1^2) \\
&\quad -D(A+D)\gamma_2^2)M_{Q_3}\Big], \\
\dot{\gamma}_1 &= -\gamma_3\bar{\omega}_2 + \gamma_2\omega_3,\ \dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3,\ \dot{\gamma}_3 = -\gamma_2\omega_1 + \gamma_1\bar{\omega}_2, \\
\dot{e}_1 &= -e_3\bar{\omega}_2 + e_2\omega_3,\ \dot{e}_2 = e_3\omega_1 - e_1\omega_3,\ \dot{e}_3 = -e_2\omega_1 + e_1\bar{\omega}_2,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\text{where } \mu = -\frac{1}{\sigma_2}\Big[(A-B)((B+D)(A+D)\,e_3 + D(B+D)\,\gamma_1 \\
&\quad \times(\gamma_3 e_1 - e_3\gamma_1) + D(A+D)\gamma_2(\gamma_3 e_2 - e_3\gamma_2))\,\omega_1\bar{\omega}_2 \\
&\quad -(A-C)(e_2(A+D)(C+D) + D(C+D)\gamma_1(e_1\gamma_2 - e_2\gamma_1) \\
&\quad +D(A+D)\gamma_3(e_3\gamma_2 - e_2\gamma_3))\,\omega_1\omega_3 + (B-C)((B+D)(C+D)\,e_1 \\
&\quad +D(C+D)\gamma_2(e_2\gamma_1 - e_1\gamma_2) + D(B+D)\gamma_3(e_3\gamma_1 - e_1\gamma_3))\,\bar{\omega}_2\omega_3 \\
&\quad -((C+D)((B+D)e_1 + D\gamma_2(e_2\gamma_1 - e_1\gamma_2)) \\
&\quad +D(B+D)\gamma_3(e_3\gamma_1 - e_1\gamma_3))M_{Q_1} - ((C+D)((A+D)e_2 \\
&\quad +D\gamma_1(e_1\gamma_2 - e_2\gamma_1)) + D(A+D)\gamma_3(e_3\gamma_2 - e_2\gamma_3))M_{Q_2} \\
&\quad -(D(A+D)\gamma_2(e_2\gamma_3 - e_3\gamma_2) + (B+D)((A+D)e_3 \\
&\quad +D\gamma_1(e_1\gamma_3 - e_3\gamma_1)))M_{Q_3}\Big]. \\
&M_{Q_1} = b_3e_2 - b_2e_3 + a_3\gamma_2 - a_2\gamma_3,\ M_{Q_2} = -b_3e_1 + b_1e_3 - a_3\gamma_1 + a_1\gamma_3, \\
&M_{Q_3} = b_2e_1 - b_1e_2 + a_2\gamma_1 - a_1\gamma_2.
\end{aligned}
$$

Here $\bar{\omega}_2, \sigma_1, \sigma_2$ have the same values as in Sect. 2.

The first integrals of Eqs. (13):

$$
\begin{aligned}
&2\,H = (A+D-D\gamma_1^2)\,\omega_1^2 + (B+D-D\gamma_2^2)\,\bar{\omega}_2^2 + (C+D-D\gamma_3^2)\,\omega_3^2 \\
&\quad -2D(\gamma_1\gamma_2\omega_1\bar{\omega}_2 + \gamma_1\gamma_3\omega_1\omega_3 + \gamma_2\gamma_3\bar{\omega}_2\omega_3) + a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \\
&\quad +b_1e_1 + b_2e_2 + b_3e_3 = 2\,h, \\
&V_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,\ V_2 = e_1^2 + e_2^2 + e_3^2 = 1, \\
&V_3 = e_1\gamma_1 + e_2\gamma_2 + e_3\gamma_3 = c_1.
\end{aligned} \tag{14}
$$

We shall seek solutions of differential Eqs. (13) of the following type:

$$\omega_1 = \omega_3 = 0,\ e_1 = e_1^0,\ e_2 = e_2^0,\ e_3 = e_3^0,\ \gamma_1 = \gamma_1^0,\ \gamma_2 = \gamma_2^0,\ \gamma_3 = \gamma_3^0, \quad (15)$$

where e_2^0, e_3^0, γ_2^0, γ_3^0 are some constants, and $e_1^0 = \pm\sqrt{1 - e_2^{0^2} - e_3^{0^2}}$, $\gamma_1^0 = \pm\sqrt{1 - \gamma_2^{0^2} - \gamma_3^{0^2}}$.

On substituting (15) into Eqs. (13) these take the form:

$$\begin{aligned}
&\bar{\mu}\,[(C+D)((B+D)\,e_1^0 + D\gamma_2^0(e_2^0\gamma_1^0 - e_1^0\gamma_2^0)) + D(B+D)\gamma_3^0(e_3^0\gamma_1^0 - e_1^0\gamma_3^0)]\\
&+((C+D)(B+D-D\gamma_2^{0^2}) - D(B+D)\gamma_3^{0^2})\bar{M}_{Q_1} + D(C+D)\gamma_1^0\gamma_2^0\bar{M}_{Q_2}\\
&+D(B+D)\gamma_1^0\gamma_3^0\bar{M}_{Q_3} = 0,\\
&\bar{\mu}\,[D(A+D)\gamma_2^0\,(e_2^0\gamma_3^0 - e_3^0\gamma_2^0) + (B+D)(e_3^0(A+D-D\gamma_1^{0^2}) + De_1^0\gamma_1^0\gamma_3^0)]\\
&+D\gamma_3^0((B+D)\gamma_1^0\bar{M}_{Q_1} + (A+D)\gamma_2^0\bar{M}_{Q_2}) + ((B+D)(A+D-D\gamma_1^{0^2})\\
&-D(A+D)\gamma_2^{0^2})\bar{M}_{Q_3} = 0.
\end{aligned} \quad (16)$$

$$\begin{aligned}
\text{Here } \bar{\mu} = &\frac{1}{\bar{\sigma}_2}\Big[[(C+D)((B+D)\,e_1^0 + D\gamma_2^0(e_2^0\gamma_1^0 - e_1^0\gamma_2^0)) + D(B+D)\\
&\times\gamma_3^0(e_3^0\gamma_1^0 - e_1^0\gamma_3^0)]\,\bar{M}_{Q_1} + [(C+D)((A+D)\,e_2^0 + D\gamma_1^0(e_1^0\gamma_2^0 - e_2^0\gamma_1^0))\\
&+D(A+D)\gamma_3^0(e_3^0\gamma_2^0 - e_2^0\gamma_3^0)]\,\bar{M}_{Q_2} + [D(A+D)\gamma_2^0(e_2^0\gamma_3^0 - e_3^0\gamma_2^0)\\
&+(B+D)((A+D)e_3^0 + D\gamma_1^0(e_1^0\gamma_3^0 - e_3^0\gamma_1^0))]\,\bar{M}_{Q_3}\Big],\\
\bar{\sigma}_2 = &(B+D)(C+D)\,e_1^{0^2} + (A+D)(C+D)\,e_2^{0^2} + (A+D)(B+D)\,e_3^{0^2}\\
&-D[(C+D)(e_2^0\gamma_1^0 - e_1^0\gamma_2^0)^2 + (B+D)(e_3^0\gamma_1^0 - e_1^0\gamma_3^0)^2\\
&+(A+D)(e_3^0\gamma_2^0 - e_2^0\gamma_3^0)^2],\ \bar{M}_{Q_1} = b_3e_2^0 - b_2e_3^0 + a_3\gamma_2^0 - a_2\gamma_3^0,\\
\bar{M}_{Q_2} = &-b_3e_1^0 + b_1e_3^0 - a_3\gamma_1^0 + a_1\gamma_3^0,\ \bar{M}_{Q_3} = b_2e_1^0 - b_1e_2^0 + a_2\gamma_1^0 - a_1\gamma_2^0.
\end{aligned}$$

Equations (16) are linear with respect to a_i, b_i $(i = 1, 2, 3)$. Considering them as unknowns, we find, e.g., b_2, b_3, as the expressions of a_1, a_2, a_3, b_1, e_i^0, γ_i^0:

$$\begin{aligned}
b_2 = &\frac{1}{e_1^0(e_1^{0^2} + e_2^{0^2} + e_3^{0^2})}\,(b_1e_2^0(e_1^{0^2} + e_2^{0^2} + e_3^{0^2}) + a_3(e_1^0e_3^0\gamma_2^0 - e_2^0e_3^0\gamma_1^0)\\
&-a_2((e_1^{0^2} + e_2^{0^2})\gamma_1^0 + e_1^0e_3^0\gamma_3^0) + a_1((e_1^{0^2} + e_2^{0^2})\gamma_2^0 + e_2^0e_3^0\gamma_3^0)),\\
b_3 = &\frac{1}{e_1^0(e_1^{0^2} + e_2^{0^2} + e_3^{0^2})}\,(b_1e_3^0(e_1^{0^2} + e_2^{0^2} + e_3^{0^2}) - a_3((e_1^{0^2} + e_3^{0^2})\gamma_1^0 + e_1^0e_2^0\gamma_2^0)\\
&+a_2e_2^0(e_1^0\gamma_3^0 - e_3^0\gamma_1^0) + a_1(e_2^0e_3^0\gamma_2^0 + (e_1^{0^2} + e_3^{0^2})\gamma_3^0)).
\end{aligned} \quad (17)$$

Assuming $e_3^0 = e_2^0$, $\gamma_3^0 = \gamma_2^0$ and $a_2 = a_3 = 0$, we obtain $\gamma_2^0 = -(b_1e_2^0 \pm b_2\sqrt{1 - 2e_2^{0^2}})/a_1$ from the 1st relation (17). The 2nd relation (17) under the above value of γ_2^0 takes the form $b_3 = b_2$. So, when $a_2 = a_3 = 0$, $b_3 = b_2$, we have 4 families of solutions of differential Eqs. (13):

$$\omega_1 = \omega_3 = 0,\ e_1 = -\sqrt{1-2e_2^{0\,2}},\ e_2 = e_3 = e_2^0,\ \gamma_1 = \mp\frac{\sqrt{a_1^2-2z_1^2}}{a_1},$$
$$\gamma_2 = -\frac{z_1}{a_1},\ \gamma_3 = -\frac{z_1}{a_1};$$
$$\omega_1 = \omega_3 = 0,\ e_1 = \sqrt{1-2e_2^{0\,2}},\ e_2 = e_3 = e_2^0,\ \gamma_1 = \pm\frac{\sqrt{a_1^2-2z_2^2}}{a_1},$$
$$\gamma_2 = -\frac{z_2}{a_1},\ \gamma_3 = -\frac{z_2}{a_1}. \tag{18}$$

Here $z_1 = b_1 e_2^0 + b_2\sqrt{1-2e_2^{0\,2}}$, $z_2 = b_1 e_2^0 - b_2\sqrt{1-2e_2^{0\,2}}$, and e_2^0 is the parameter of the families.

The integral V_3 takes the form $-(2e_2^0 z_1 \pm \sqrt{1-2e_2^{0\,2}}\sqrt{a_1^2-2z_1^2})/a_1 = c_1$ on the first two families of solutions (18), and on the last two families, it is $-(2e_2^0 z_2 \mp \sqrt{1-2e_2^{0\,2}}\ \sqrt{a_1^2-2z_2^2})/a_1 = c_1$. Thus, solutions (18) exist under any angles between vectors $\mathbf{E}$ and $\boldsymbol{\gamma}$.

From a mechanical point of view, the elements of the families of solutions (18) correspond to the equilibria of the mechanical system under study.

From the stationary conditions

$$\partial\Phi/\partial\omega_1 = 0,\ \partial\Phi/\partial\omega_3 = 0,\ \partial\Phi/\partial\gamma_j = 0,\ \partial\Phi/\partial e_j = 0\ (j = 1,2,3)$$

of the integral $2\Phi = 2\lambda_0 H - \lambda_1 V_1 - \lambda_2 V_2 - 2\lambda_3 V_3$ we find the constraints on λ_i, under which the first two families of solutions (18) satisfy these conditions:

$$\lambda_0 = -\frac{e_2^0\sqrt{a_1^2-2z_1^2} \pm \sqrt{1-2e_2^{0\,2}}\,z_1}{a_1^2 e_2^0},\ \lambda_2 = \frac{b_1 z_1 \mp b_2\sqrt{a_1^2-2z_1^2}}{a_1^2 e_2^0},\ \lambda_3 = \frac{z_1}{a_1 e_2^0}.$$

Having substituted the latter expressions into the integral Φ, we have:

$$2\Phi_{1,2} = \mp\frac{2(e_2^0\sqrt{a_1^2-2z_1^2} \pm \sqrt{1-2e_2^{0\,2}}\,z_1)}{a_1^2 e_2^0}\,H - V_1 - \frac{b_1 z_1 \mp b_2\sqrt{a_1^2-2z_1^2}}{a_1^2 e_2^0}\,V_2$$
$$-\frac{2z_1}{a_1 e_2^0}\,V_3. \tag{19}$$

By the same way, we find the integrals taking a stationary value on the elements of the last two families of solutions (18):

$$2\Phi_{3,4} = \pm\frac{2(e_2^0\sqrt{a_1^2-2z_2^2} \pm \sqrt{1-2e_2^{0\,2}}\,z_2)}{a_1^2 e_2^0}\,H - V_1 - \frac{b_1 z_2 \pm b_2\sqrt{a_1^2-2z_2^2}}{a_1^2 e_2^0}\,V_2$$
$$-\frac{2z_2}{a_1 e_2^0}\,V_3.$$

5 On Pendulum-Like Motions

In the problem under consideration, we could not obtain solutions corresponding to permanent rotations of the mechanical system. These motions are typical of rigid body dynamics. Basing on the analysis of the equations of motion (5) and (13), one can suppose that there are no such solutions. However, under the action of external potential forces the mechanical system can perform pendulum-like oscillations.

When $a_2 = a_3 = b_1 = 0$, the relations

$$\omega_3 = 0,\ \gamma_1 = \pm 1,\ \gamma_2 = \gamma_3 = e_1 = 0 \tag{20}$$

define two IMs of codimension 5 of the equations of motion (13).

The differential equations on these IMs are written as

$$\dot{\omega}_1 = \frac{b_3 e_2 - b_2 e_3}{A},\ \dot{e}_2 = e_3\omega_1,\ \dot{e}_3 = -e_2\omega_1$$

and describe the pendulum-like oscillations of the body with a fixed point relative to the axis Ox in the frame rigidly attached to the body.

The integral V_3 on IMs (20) is equal to zero identically that corresponds to the case of orthogonal vectors $\boldsymbol{\gamma}$, $\mathbf{E}$. The integral $\Psi = (V_1 - 1)V_3$ assumes a stationary value on IMs (20).

Let us consider another similar solution for equations (13). It is the IM of codimension 3:

$$\omega_1 = \gamma_3 = e_3 = 0. \tag{21}$$

This solution exists for $a_3 = b_3 = 0$.

The differential equations on IM (21)

$$\dot{\omega}_3 = \frac{b_2 e_1 - b_1 e_2 + a_2\gamma_1 - a_1\gamma_2}{C + D},$$
$$\dot{\gamma}_1 = \gamma_2\omega_3,\ \dot{\gamma}_2 = -\gamma_1\omega_3,\ \dot{e}_1 = e_2\omega_3,\ \dot{e}_2 = -e_1\omega_3$$

describe the pendulum-like oscillations of the body relative to the axis Oz. The motions exist under any angle between the vectors $\boldsymbol{\gamma}$, $\mathbf{E}$, because the integral V_3 on IM (21) takes the form: $e_1\gamma_1 + e_2\gamma_2 = c_1$. So, it is the family of IMs.

6 On the Stability of Stationary Sets

In this Section, we investigate the stability of the above found solutions on the base of the Lyapunov theorems on the stability of motion. To solve the problems, which often arise in the process of the analysis, the software package [7] written in *Mathematica* language is applied. In particular, the package gives a possibility to obtain the equations of the first approximation and their characteristic polynomial, using the equations of motion and the solution under study as input data, and then, to conduct the analysis of the polynomial roots, basing on the criteria of asymptotic stability of linear systems. When the problem of stability is solved by the Routh–Lyapunov method, the package, using the solution under study and the first integrals of the problem as input data, constructs a quadratic form and the conditions of its sign-definiteness in the form of the Sylvester inequalities. Their analysis is performed by means of *Mathematica* built-in functions, e.g., *Reduce, RegionPlot3D*.

6.1 The Case of Absence of External Forces

Let us investigate the stability of one of IMs (12), e.g.,

$$\omega_1 = \omega_3 = 0,\ e_1 - \gamma_1 = 0,\ e_2 - \gamma_2 = 0,\ e_3 - \gamma_3 = 0,\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

using the integral $2K_{2_1} = 2\lambda_0 H - \lambda_1(V_1 + V_2 - 2V_3) - 2\lambda_4 F_1 - 2\lambda_5 F_2$ for obtaining its sufficient conditions.

We use the maps

$$\omega_1 = 0,\ \omega_3 = 0,\ e_1 = \pm z,\ e_2 = \gamma_2,\ e_3 = \gamma_3,\ \gamma_1 = \pm z$$

on this IM. From now on, $z = \sqrt{1 - \gamma_2^2 - \gamma_3^2}$.

Introduce the deviations:

$$y_1 = \omega_1,\ y_2 = \omega_3,\ y_3 = e_1 - z,\ y_4 = e_2 - \gamma_2,\ y_5 = e_3 - \gamma_3,\ y_6 = \gamma_1 - z.$$

The 2nd variation of the integral K_{2_1} on the set defined by the first variations of the conditional integrals

$$\delta V_1 = \pm 2z\, y_6 = 0,\ \delta V_2 = 2(\gamma_2 y_4 + \gamma_3 y_5 \pm z\, y_3) = 0,$$
$$\delta V_3 = \gamma_2 y_4 + \gamma_3 y_5 \pm z\,(y_3 + y_6) = 0,$$

is written as:

$$2\delta^2 K_{2_1} = \alpha_{11} y_1^2 + \alpha_{12} y_1 y_2 + \alpha_{22} y_2^2 + \alpha_{33} y_3^2 + \alpha_{34} y_3 y_4 + \alpha_{24} y_2 y_4 + \alpha_{13} y_1 y_3 + \alpha_{23} y_2 y_3 + \alpha_{14} y_1 y_4 + \alpha_{44} y_4^2,$$

where

$$\alpha_{11} = \frac{((A-B)\,\gamma_2^2 + (B+D)(1-\gamma_3^2))\,\lambda_0}{2\gamma_2^2},\ \alpha_{12} = \pm\frac{(B+D)\,\gamma_3 z\lambda_0}{\gamma_2^2},$$
$$\alpha_{22} = \frac{((C+D)\,\gamma_2^2 + (B+D)\,\gamma_3^2)\,\lambda_0}{2\gamma_2^2},\ \alpha_{33} = \frac{(\gamma_2^2 - 1)\,\lambda_1}{2\gamma_3^2},\ \alpha_{34} = \mp\frac{\gamma_2\,\lambda_1 z}{\gamma_3^2},$$
$$\alpha_{24} = \Big(\frac{(C+D)\,\gamma_2}{\gamma_3} + \frac{(B+D)\,\gamma_3}{\gamma_2}\Big)\lambda_6 \mp (B-C)\,z\lambda_5,\ \alpha_{44} = -\frac{(\gamma_2^2+\gamma_3^2)\,\lambda_1}{2\gamma_3^2},$$
$$\alpha_{13} = \mp\frac{((A-B)\,\gamma_2^2 + B + D)\,z\lambda_5}{\gamma_2\gamma_3} - (A+D)\lambda_6,$$
$$\alpha_{23} = -\frac{1}{\gamma_2\gamma_3}((B+D)\,\gamma_3\lambda_5 \mp (C+D)\,\gamma_2 z\lambda_6) + (B-C)\,\gamma_2\lambda_5,$$
$$\alpha_{14} = -\frac{1}{\gamma_2\gamma_3}(((B+D) + (A-B)\,\gamma_2^2 \mp (B+D)\,\gamma_3 z\lambda_6)\,\gamma_2\lambda_5) - (A-B)\,\gamma_3\lambda_5.$$

The conditions of sign-definiteness of the quadratic form $\delta^2 K_{2_1}$

$$\begin{aligned}
&\Delta_1 = \frac{(\gamma_2^2-1)\,\lambda_1}{\gamma_3^2} > 0,\ \ \Delta_2 = \frac{\lambda_1^2}{\gamma_3^2} > 0,\\
&\Delta_3 = \frac{\lambda_1}{\gamma_2^2\gamma_3^2}[((C+D)\,\gamma_2^2 + (B+D)\,\gamma_3^2)\,\lambda_0\lambda_1 + ((C+D)^2\,\gamma_2^2 + ((B+D)^2\\
&\quad -(B-C)^2\,\gamma_2^2)\,\gamma_3^2)(\lambda_5^2+\lambda_6^2)] > 0,\\
&\Delta_4 = \frac{1}{\gamma_2^2\gamma_3^2}((C+D)(B+D+(A-B)\gamma_2^2) + (A-C)(B+D)\gamma_3^2)\\
&\quad \times[\lambda_0^2\lambda_1^2 + (B+C+2D+(A-B)\,\gamma_2^2 + (A-C)\,\gamma_3^2)\,\lambda_0\lambda_1(\lambda_5^2+\lambda_6^2)\\
&\quad +((C+D)(B+D+(A-B)\,\gamma_2^2)+(A-C)(B+D)\,\gamma_3^2)(\lambda_5^2+\lambda_6^2)^2] > 0. \qquad (22)
\end{aligned}$$

are sufficient for the stability of the IM under study.

The differential equations $\dot\gamma_2 = 0$, $\dot\gamma_3 = 0$ on IMs (12) have the family of solutions:

$$\gamma_2 = \gamma_2^0 = \text{const},\ \gamma_3 = \gamma_3^0 = \text{const}. \qquad (23)$$

Thus, each of IMs (12) can be considered as a family of IMs, where γ_2^0, γ_3^0 are the parameters of the family.

Let $\gamma_3^0 = \gamma_2^0$ and $\lambda_5 = \lambda_6 = \lambda_1$. Taking into consideration (23) and the above constraints, inequalities (22) take the form:

$$\begin{aligned}
&\frac{(\gamma_2^{0^2}-1)\,\lambda_1}{\gamma_2^{0^2}} > 0,\ \ \frac{\lambda_1^2}{\gamma_2^{0^2}} > 0,\\
&\frac{\lambda_1^2}{\gamma_2^{0^2}}((B+C+2D)\lambda_0 + 2((B+D)^2 + (C+D)^2 - (B-C)^2\gamma_2^{0^2})\,\lambda_1) > 0,\\
&\frac{\lambda_1^2}{\gamma_2^{0^4}}((B+D)(C+D) + ((A-D)(B+C) + 2(AD-BC))\,\gamma_2^{0^2})\\
&\times(\lambda_0^2 + 2(B+C+2D-(B+C-2\,A)\,\gamma_2^{0^2})\lambda_0\lambda_1 + 4((B+D)(C+D)\\
&+((A-D)(B+C)+2(AD-BC))\,\gamma_2^{0^2})\lambda_1^2) > 0.
\end{aligned}$$

With the help of the built-in function *Reduce*, we find the conditions of compatibility of the latter inequalities:

$$\begin{aligned}
&A > B > C > 0 \text{ and } A < B+C,\ D > 0 \text{ and}\\
&\Big[\Big(\Big(\lambda_0 > 0 \text{ and } \Big(\sigma_1 < \lambda_1 < \sigma_2 - \frac{\sigma_3}{4} \text{ or } \sigma_2 + \frac{\sigma_3}{4} < \lambda_1 < 0\Big) \text{ and}\\
&\quad \Big(-1 < \gamma_2^0 < -\frac{1}{\sqrt{2}} \text{ or } \frac{1}{\sqrt{2}} < \gamma_2^0 < 1\Big)\Big) \text{ or}\\
&\Big(\lambda_0 > 0 \text{ and } \sigma_2 + \frac{\sigma_3}{4} < \lambda_1 < 0 \text{ and } \Big(-\frac{1}{\sqrt{2}} \le \gamma_2^0 < 0 \text{ or } 0 < \gamma_2^0 \le \frac{1}{\sqrt{2}}\Big)\Big)\Big].
\end{aligned}$$

Here

$$\sigma_1 = \frac{(B + C + 2D)\lambda_0}{2((B - C)^2\gamma_2^{0^2} - (B^2 + C^2 + 2BD + 2D(C + D)))},$$

$$\sigma_2 = \frac{((B + C + 2D - (B + C - 2\,A)\gamma_2^{0^2})\lambda_0}{4((2BC + (B + C)D - A(B + C + 2D))\gamma_2^{0^2} - (B + D)(C + D))},$$

$$\sigma_3 = \frac{\sqrt{(B - C)^2 - 2(B - C)^2\gamma_2^{0^2} + (B + C - 2\,A)^2\gamma_2^{0^4}}\lambda_0}{(B + D)(C + D) + (A(B + C + 2D) - 2BC - (B + C)D)\gamma_2^{0^2}}.$$

The constraints on the parameter γ_2^0 give the sufficient conditions of stability for the elements of the family of IMs. The constraints imposed on the parameters λ_0, λ_1 isolate a subfamily of the family of the integrals K_{2_1}, which allows one to obtain these sufficient conditions. The analysis of stability of the 2nd IM of IMs (12) is done analogously.

Let us investigate the stability of IM (7), using the integral $2K_1 = 2\lambda_0 H - \lambda_1(V_1 - V_2)^2 - \lambda_2 F_1 F_2$ for obtaining sufficient conditions. The analysis is done in the map $\omega_1 = 0$, $\omega_3 = 0$, $\gamma_1 = -z_1$, $e_1 = -z_2$ on this IM. From now on, $z_1 = \sqrt{1 - \gamma_2^2 - \gamma_3^2}$, $z_2 = \sqrt{1 - e_2^2 - e_3^2}$.

In order to reduce the amount of computations we restrict our consideration by the case when the following restrictions are imposed on the geometry of mass of the mechanical system: $A = 3\,C/2,\ B = 2\,C,\ D = C/2$.

Introduce the deviations from the unperturbed solution:

$$y_1 = \omega_1,\ y_2 = \omega_2,\ y_3 = \gamma_1 + z_1,\ y_4 = e_1 + z_2.$$

The 2nd variation of the integral K_1 in the deviations on the set

$$\delta V_1 = -2z_1 y_3 = 0,\ \delta V_2 = -2z_2 y_4 = 0$$

has the form: $2\delta^2 K_1 = \beta_{11}y_1^2 + \beta_{12}y_1 y_2 + \beta_{22}y_2^2$, where $\beta_{11}, \beta_{12}, \beta_{22}$ are the expressions of $C, \gamma_2, \gamma_3, e_2, e_3$. These are bulky enough and presented entirely in Appendix.

Taking into consideration that $\gamma_2 = \gamma_2^0 = \text{const}$, $\gamma_3 = \gamma_3^0 = \text{const}$, $e_2 = e_2^0 = \text{const}$, $e_3 = e_3^0 = \text{const}$ (9) on IM (7), and introducing the restrictions on the parameters $\gamma_3^0 = \gamma_2^0$, $e_3^0 = e_2^0$, we write the conditions of positive definiteness of the quadratic form $2\delta^2 K_1$ (the Sylvester inequalities) as follows:

$$\begin{aligned}
\Delta_1 &= 2[\sqrt{1-2e_2^{0^2}}(\gamma_2^{0^2}+e_2^{0^2}(1-4\gamma_2^{0^2}))\\
&\quad -2e_2^0\gamma_2^0(1-2e_2^{0^2})\sqrt{1-2\gamma_2^{0^2}}\,]\,z+1>0,\\
\Delta_2 &= -\frac{1}{e_2^{0^2}}\Big(8\gamma_2^{0^2}+e_2^{0^2}(6-32\gamma_2^{0^2})-15-16e_2^0\sqrt{1-2e_2^{0^2}}\gamma_2^0\sqrt{1-2\gamma_2^{0^2}}\\
&\quad +2\Big(2e_2^0\gamma_2^0(1-2e_2^{0^2})(15-14e_2^{0^2}-16\gamma_2^{0^2}(1-4e_2^{0^2}))\sqrt{1-2\gamma_2^{0^2}}+\sqrt{1-2e_2^{0^2}}\\
&\quad \times(3e_2^{0^2}(2e_2^{0^2}-5)-(120e_2^{0^4}-106e_2^{0^2}+15)\,\gamma_2^{0^2}\\
&\quad +8(32e_2^{0^4}-16e_2^{0^2}+1)\,\gamma_2^{0^4})\Big)z+\Big(\gamma_2^{0^4}(15-8\gamma_2^{0^2})^2\\
&\quad +4e_2^0\sqrt{1-2e_2^{0^2}}\gamma_2^0\sqrt{1-2\gamma_2^{0^2}}(15-14e_2^{0^2}-16(1-4e_2^{0^2})\,\gamma_2^{0^2})\\
&\quad \times(3e_2^{0^2}(2e_2^{0^2}-5)-(120e_2^{0^4}-106e_2^{0^2}+15)\,\gamma_2^{0^2}+8(32e_2^{0^4}-16e_2^{0^2}+1)\,\gamma_2^{0^4})\\
&\quad +e_2^{0^2}(9e_2^{0^2}(5-2e_2^{0^2})^2-2(1504e_2^{0^6}-4508e_2^{0^4}+3420e_2^{0^2}-675)\,\gamma_2^{0^2}\\
&\quad +4(8736e_2^{0^6}-17264e_2^{0^4}+9761e_2^{0^2}-1785)\gamma_2^{0^4}-32(3840e_2^{0^6}-5312e_2^{0^4}\\
&\quad +2300e_2^{0^2}-325)\,\gamma_2^{0^6}-4096(1-4e_2^{0^2})^2(1-2e_2^{0^2})\,\gamma_2^{0^8})\Big)z^2\Big)>0.
\end{aligned} \tag{24}$$

Here $z = C\lambda_2$, $\lambda_0 = 1$.

The system of inequalities (24) has been solved graphically. The built-in function *RegionPlot3D* is used. The region, in which the inequalities have common values, is shown in Fig. 2 (dark region). Thus, when the values of the parameters z, e_2^0, γ_2^0 lie in this region, the IM under study is stable.

6.2 The Case of the Presence of External Forces

In this Subsection, we analyze the stability of the elements of the families of solutions (18). Let us investigate one of the first two families, e.g.,

$$\begin{aligned}
&\omega_1=\omega_3=0,\ e_1=-\sqrt{1-2e_2^{0^2}},\ e_2=e_3=e_2^0,\ \gamma_1=-\frac{\sqrt{a_1^2-2z^2}}{a_1},\\
&\gamma_2=-\frac{z}{a_1},\ \gamma_3=-\frac{z}{a_1},
\end{aligned} \tag{25}$$

where $z = b_1e_2^0 + b_2\sqrt{1-2e_2^{0^2}}$.

The integral

$$2\Phi_1=-\frac{2(e_2^0\sqrt{a_1^2-2z^2}+\sqrt{1-2e_2^{0^2}}\,z)}{a_1^2e_2^0}\,H-V_1-\frac{b_1z-b_2\sqrt{a_1^2-2z^2}}{a_1^2e_2^0}\,V_2-\frac{2z}{a_1e_2^0}\,V_3$$

is used for obtaining the sufficient conditions.

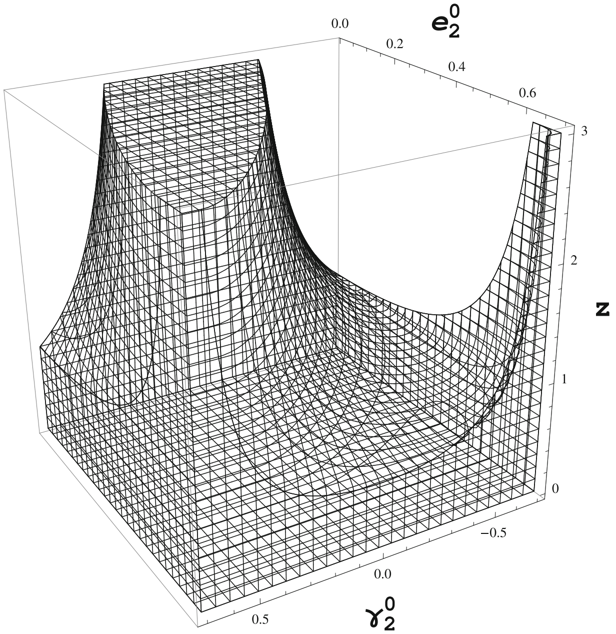

Fig. 2. The region of stability of the IM for $\gamma_2^0 \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}], e_2^0 \in (0, \frac{1}{\sqrt{2}}]$, $z \in (0, 3]$

In the deviations

$$y_1 = e_1 + \sqrt{1 - 2e_2^{0^2}},\ y_2 = e_2 - e_2^0,\ y_3 = e_3 - e_2^0,\ y_4 = \gamma_1 + \frac{\sqrt{a_1^2 - 2z^2}}{a_1},$$
$$y_5 = \gamma_2 + \frac{z}{a_1},\ y_6 = \gamma_3 + \frac{z}{a_1},\ y_7 = \omega_1,\ y_8 = \omega_2$$

on the linear manifold

$$\delta H = b_1 y_1 + b_2(y_2 + y_3) + a_1 y_4 = 0,\ \delta V_1 = -\frac{2}{a_1}\Big(z(y_5 + y_6) + \sqrt{a_1^2 - 2z^2}\, y_4\Big) = 0,$$
$$\delta V_2 = 2(e_2^0\,(y_2 + y_3) - \sqrt{1 - 2e_2^{0^2}} y_1) = 0,$$
$$\delta V_3 = e_2^0(y_5 + y_6) - \sqrt{1 - 2e_2^{0^2}} y_4 - \frac{1}{a_1}\Big(z(y_2 + y_3) + \sqrt{a_1^2 - 2z^2}\, y_1\Big) = 0$$

the 2nd variation of the integral Φ_1 has the form: $\delta^2\Phi_1 = Q_1 + Q_2$, where

$$Q_1 = \frac{1}{2a_1^2 e_2^{0^3}}\Big((3b_2 e_2^0\sqrt{1 - 2e_2^{0^2}} - b_1(1 - 4e_2^{0^2}))\, z + b_2(1 - e_2^{0^2})\sqrt{a_1^2 - 2z^2}$$
$$- a_1^2 e_2^0\Big)\, y_1^2 + \frac{1}{a_1^2 e_2^{0^2}}\Big(\sqrt{1 - 2e_2^{0^2}}(b_1 z - b_2\sqrt{a_1^2 - 2z^2}) - \sqrt{a_1^2 - 2z^2}\, z\Big)\, y_1 y_2$$
$$+ \frac{1}{a_1^2 e_2^0}\Big(b_2\sqrt{a_1^2 - 2z^2} - b_1 z\Big) y_2^2 + \frac{1}{a_1 e_2^{0^2}}\Big(e_2^0\sqrt{a_1^2 - 2z^2} - \sqrt{1 - 2e_2^{0^2}}\, z\Big)\, y_1 y_6$$

$$+\frac{2z}{a_1 e_2^0}\, y_2 y_6 - y_6^2,$$

$$Q_2 = -\frac{B+C+2D}{2a_1^2 e_2^0}\Big(\sqrt{1-2e_2^{0^2}}\, z + e_2^0\sqrt{a_1^2-2z^2}\Big)\, y_8^2$$

$$+\frac{(B+D)\sqrt{1-2e_2^{0^2}}}{a_1^2 e_2^{0^2}}\Big(\sqrt{1-2e_2^{0^2}}\, z + e_2^0\sqrt{a_1^2-2z^2}\Big)\, y_7 y_8$$

$$-\frac{1}{2a_1^4 e_2^{0^3}}\Big(a_1^2[(Ae_2^{0^2} + B(1-2e_2^{0^2}))(\sqrt{1-2e_2^{0^2}}\, z + e_2^0\sqrt{a_1^2-2z^2})$$

$$+D\sqrt{1-2e_2^{0^2}}((1-4e_2^{0^2})\, z + e_2^0\sqrt{1-2e_2^{0^2}}\sqrt{a_1^2-2z^2})]$$

$$-D\,[(1-8e_2^{0^2})(b_1^3 e_2^{0^3}\sqrt{1-2e_2^{0^2}} + b_2^3(1-2e_2^{0^2})^2 + 3b_1 b_2 e_2^0(1-2e_2^{0^2})\, z)$$

$$+e_2^0(3-8e_2^{0^2})\sqrt{a_1^2-2z^2}\, z^2]\Big)\, y_7^2.$$

The analysis of sign-definiteness of the quadratic forms Q_1 and Q_2 was done for the case when $b_1 = 0$ and $A = 3\,C/2$, $B = 2\,C$, $D = C/2$. Under these restrictions on the parameters, the conditions of negative definiteness of the quadratic forms Q_1 and Q_2 are respectively written as:

$$\varDelta_1 = -1 < 0,\ \ \varDelta_2 = -\frac{1}{a_1^2 e_2^{0^2}}\left(b_2(b_2(1-2e_2^{0^2}) + e_2^0\sqrt{a_1^2-2b_2^2(1-2e_2^{0^2})})\right) > 0,$$

$$\varDelta_3 = \frac{b_2}{a_1^4 e_2^{0^5}}\Big(2b_2 e_2^0\,(a_1^2(1-3e_2^{0^2}) - b_2^2\,(16e_2^{0^4} - 14e_2^{0^2} + 3))$$

$$+\sqrt{a_1^2-2b_2^2(1-2e_2^{0^2})}\,(a_1^2 e_2^{0^2} + b_2^2(16e_2^{0^4} - 10e_2^{0^2} + 1))\Big) < 0 \tag{26}$$

and

$$\varDelta_1 = -\frac{2\,C}{a_1^2 e_2^0}\Big(b_2\,(1-2e_2^{0^2}) + e_2^0\sqrt{a_1^2-2b_2^2(1-2e_2^{0^2})}\Big) < 0,$$

$$\varDelta_2 = \frac{C^2}{a_1^6 e_2^{0^4}}\Big(3a_1^4 e_2^{0^2}\,(5-2e_2^{0^2}) - 8b_2^4\,(1-2e_2^{0^2})^2(32e_2^{0^4} - 16e_2^{0^2} + 1)$$

$$+a_1^2 b_2^2\,(15 - 4e_2^{0^2}\,(60e_2^{0^4} - 83e_2^{0^2} + 34)) - 2b_2 e_2^0\,(1-2e_2^{0^2})$$

$$\times(a_1^2\,(14e_2^{0^2}-15) + 16b_2^2\,(8e_2^{0^4}-6e_2^{0^2}+1))\,\sqrt{a_1^2-2b_2^2(1-2e_2^{0^2})}\Big) > 0. \tag{27}$$

Taking into consideration the conditions for solutions (25) to be real

$$a_1 \neq 0 \text{ and } \left(e_2^0 = \pm\frac{1}{\sqrt{2}} \text{ or } \left(-\frac{1}{\sqrt{2}} < e_2^0 < \frac{1}{\sqrt{2}} \text{ and } -\sigma_1 \le b_2 \le \sigma_1\right)\right) \tag{28}$$

under the above restrictions on the parameters b_1, A, B, D, inequalities (26) and (27) are compatible when the following conditions

$$a_1 \neq 0,\ C > 0 \text{ and } \left(\left(b_2 < 0,\ \sigma_2 < e_2^0 \leq \frac{1}{\sqrt{2}}\right) \text{ or } \left(b_2 > 0,\ -\frac{1}{\sqrt{2}} \leq e_2^0 < -\sigma_2\right)\right) \tag{29}$$

hold.

Here $\sigma_1 = \sqrt{\dfrac{a_1^2}{2(1-2e_2^{0^2})}}$, $\sigma_2 = \sqrt{\dfrac{b_2^2}{a_1^2+2b_2^2}}$.

The latter conditions are sufficient for the stability of the elements of the family of solutions under study. Let us compare them with necessary ones which we shall obtain, using the Lyapunov theorem on stability in linear approximation [9].

The equations of the 1st approximation in the case considered are written as:

$$\begin{aligned}
\dot{y}_1 &= 2e_2^0 y_8 - \sqrt{z_1}\, y_7,\ \dot{y}_2 = e_2^0 y_7 + \sqrt{z_1}\, y_8,\ \dot{y}_3 = \left(e_2^0 - \frac{1}{e_2^0}\right) y_7 + \sqrt{z_1}\, y_8,\\
\dot{y}_4 &= \frac{b_2}{a_1}\left(\frac{z_1}{e_2^0}\, y_7 - 2\sqrt{z_1}\, y_8\right),\ \dot{y}_5 = \frac{1}{a_1}\left(\sqrt{a_1^2 - 2b_2^2 z_1}\, y_8 - b_2\sqrt{z_1}\, y_7\right),\\
\dot{y}_6 &= \frac{1}{a_1}\left(\frac{\sqrt{a_1^2 - 2b_2^2 z_1}\,(e_2^0\, y_8 - \sqrt{z_1}\, y_7)}{e_2^0} + b_2\sqrt{z_1}\, y_7\right),\\
\dot{y}_7 &= \frac{1}{z_2}\left(16a_1^2 b_2 e_2^{0^2}(y_3 - y_2) + 2a_1^2 e_2^0\sqrt{z_1}\,(5a_1 y_5 - 2b_2 y_1 - 3a_1 y_6)\right),\\
\dot{y}_8 &= \frac{1}{z_2}\left(2a_1^2[b_2\,(4e_2^{0^2} - 5)\, y_1 + 5a_1 y_5 + a_1 e_2^{0^2}(3y_6 - 7y_5)]\right.\\
&\quad + 10a_1^2 b_2 e_2^0\sqrt{z_1}\,(y_3 - y_2) + 2b_2^2 z_1\,(4e_2^{0^2} - 1)(a_1\,(y_5 + y_6) - 2b_2 y_1)\\
&\quad \left. - 4b_2 e_2^0 z_1\sqrt{a_1^2 - 2b_2^2 z_1}\,(a_1\,(y_5 + y_6) - 2b_2 y_1)\right).
\end{aligned} \tag{30}$$

Here $z_1 = 1 - 2e_2^{0^2}$, $z_2 = C(3a_1^2(2e_2^{0^2} - 5) - 8b_2 z_1\,(b_2\,(4e_2^{0^2} - 1) - 2e_2^0\sqrt{a1^2 - 2b_2^2 z_1}))$.

The characteristic equation of system (30) has the form:

$$\lambda^4\,(\lambda^4 + \alpha_1\lambda^2 + \alpha_2) = 0, \tag{31}$$

where

$$\begin{aligned}
\alpha_1 = \frac{4\,C}{z_2^2}&\left(a_1^4 e_2^0\,[2b_2\,(251e_2^{0^2} - 122e_2^{0^4} - 137) + 3(10e_2^{0^4} - 33e_2^{0^2} + 20)\right.\\
&\sqrt{a_1^2 - 2b_2^2 z_1}] + 8b_2^4 z_1^2\,[(64e_2^{0^4} - 24e_2^{0^2} + 1)\sqrt{a_1^2 - 2b_2^2 z_1} - 2b_2 e_2^0\,(64e_2^{0^4} - 40e_2^{0^2}\\
&+5)] - a_1^2 b_2^2 z_1\,[(432e_2^{0^4} - 518e_2^{0^2} + 47)\sqrt{a_1^2 - 2b_2^2 z_1} - 2b_2 e_2^0\,(560e_2^{0^4} - 706e_2^{0^2}\\
&\left.+173)]\right),
\end{aligned}$$

$$\alpha_2 = \frac{1}{z_2^2}\Big((8a_1^4 b_2^2 (240e_2^{0^6} - 408e_2^{0^4} + 206e_2^{0^2} - 19) + 12a_1^6 (4e_2^{0^2} (e_2^{0^2} - 3)$$
$$+5) - 64a_1^2 b_2^4 (64e_2^{0^6} - 80e_2^{0^4} + 24e_2^{0^2} - 1)\, z_1) - 8a_1^2 b_2 e_2^0 \sqrt{a_1^2 - 2b_2^2 z_1}$$
$$\times (a_1^2 (56e_2^{0^4} - 110e_2^{0^2} + 53) - 8b_2^2 (32e_2^{0^4} - 32e_2^{0^2} + 5)\, z_1)\Big).$$

The roots of the bipolynomial in the round brackets are purely imaginary when the conditions

$$\alpha_1 > 0,\ \alpha_2 > 0,\ \alpha_1^2 - 4\alpha_2 > 0$$

hold.

Taking into consideration (28), the latter inequalities are hold under the following constraints imposed on the parameters C, a_1, b_2, e_2^0:

$$C > 0 \text{ and } \Big[a_1 < 0 \text{ and } \Big(\Big(\Big(b_2 < \frac{3a_1}{\sqrt{2}} \text{ or } \frac{3a_1}{\sqrt{2}} < b_2 < \frac{a_1}{\sqrt{2}}\Big) \text{ and }$$
$$\frac{\rho_1}{2} \le e_2^0 \le \frac{1}{\sqrt{2}}\Big) \text{ or } \Big(b_2 = \frac{3a_1}{\sqrt{2}} \text{ and } \frac{\rho_1}{2} \le e_2^0 < \frac{1}{\sqrt{2}}\Big) \text{ or } \Big(b_2 = \frac{a_1}{\sqrt{2}} \text{ and }$$
$$-\frac{\rho_1}{2} < e_2^0 \le \frac{1}{\sqrt{2}}\Big) \text{ or} \Big(\frac{a_1}{\sqrt{2}} < b_2 < 0 \text{ and } \rho_2 < e_2^0 \le \frac{1}{\sqrt{2}}\Big)\Big)\Big] \text{ or}$$
$$C > 0 \text{ and } \Big[a_1 > 0 \text{ and } \Big(\Big(0 < b_2 < \frac{a_1}{\sqrt{2}} \text{ and } -\frac{1}{\sqrt{2}} \le e_2^0 < -\rho_2\Big) \text{ or}$$
$$\Big(\Big(\frac{a_1}{\sqrt{2}} < b_2 < \frac{3a_1}{\sqrt{2}} \text{ or } b_2 > \frac{3a_1}{\sqrt{2}}\Big) \text{ and } -\frac{1}{\sqrt{2}} \le e_2^0 \le -\frac{\rho_1}{2}\Big) \text{ or}$$
$$\Big(b_2 = \frac{a_1}{\sqrt{2}} \text{ and } -\frac{1}{\sqrt{2}} \le e_2^0 < -\frac{\rho_1}{2}\Big) \text{ or } \Big(b_2 = \frac{3a_1}{\sqrt{2}} \text{ and }$$
$$-\frac{1}{\sqrt{2}} < e_2^0 \le -\frac{\rho_1}{2}\Big)\Big)\Big]. \tag{32}$$

Here $\rho_1 = \sqrt{\dfrac{2b_2^2 - a_1^2}{b_2^2}},\ \rho_2 = \sqrt{\dfrac{a_1^2 + b_2^2 - \sqrt{b_2^2 (2a_1^2 + 5b_2^2)}}{2a_1^2 + 4b_2^2}}.$

The analysis of zero roots of characteristic Eq. (31) was done by the technique applied in [10]. The analysis shown that the characteristic equation has zero roots with simple elementary divisors. Whence it follows, the elements of the family of solutions under study are stable in linear approximation when conditions (32) hold. Comparing them with (29), we conclude that the sufficient conditions are close to necessary ones. The analogous result has been obtained for the 2nd family of solutions. Instability was proved for the rest of the families of solutions.

7 Conclusion

The qualitative analysis of the differential equations describing the motion of the nonholonomic mechanical system has been done. The solutions of these equations, which correspond to the equilibria and pendulum-like motions of the mechanical system, have been found. The Lyapunov stability of the solutions has been investigated. In some cases, the obtained sufficient conditions were compared with necessary ones. The analysis was done nearly entirely in symbolic form. Computational difficulties were in the main caused by the problem of bulky expressions: the differential equations are rather bulky, and the first integrals of these equations are the polynomials of the 2nd–5th degrees. Computer algebra system *Mathematica* was applied to solve computational problems. The results presented in this work show the efficiency of the approach used for the analysis of the problem as well as computational tools.

Appendix

$$
\begin{aligned}
\beta_{11} = {} & (4e_2^2)^{-1} C \Big[((e_3^2-1)(\gamma_2^2-5) - e_2^2(1-\gamma_2^2+z_1^2) + 2e_2\gamma_2 z_1 z_2)\lambda_0 \\
& +C\,[(5e_3^5\gamma_2(3\gamma_3^2 - z_1^2) + e_2e_3^4\gamma_3(43\gamma_2^2+20\gamma_3^2-25) + e_2\gamma_3(e_2^2-5) \\
& \times(5-3\gamma_2^2-\gamma_3^2+e_2^2(4\gamma_2^2+\gamma_3^2-2)) + e_2e_3^2\gamma_3(50-58\gamma_2^2-25\gamma_3^2 \\
& +e_2^2(59\gamma_2^2+17\gamma_3^2-27)) + e_3\gamma_2(e_2^2(65-37\gamma_2^2-46\gamma_3^2)+5(\gamma_2^2+3\gamma_3^2-5) \\
& +e_2^4(36\gamma_2^2+23\gamma_3^2-28)) + e_3^3\gamma_2(e_2^2(37\gamma_2^2+55\gamma_3^2-33)-5(2\gamma_2^2+7\gamma_3^2-6)))\,z_1 \\
& +(e_2^4\gamma_2\gamma_3(4(1-\gamma_2^2)-3\gamma_3^2)+e_2^2\gamma_2\gamma_3(21\gamma_2^2+16\gamma_3^2-25+e_3^2(53-57\gamma_2^2 \\
& -45\gamma_3^2)) - 5\gamma_2\gamma_3(e_3^2-1)(5-\gamma_2^2-\gamma_3^2+e_3^2(3\gamma_2^2+4\gamma_3^2-3)) - e_2^3e_3(10+36\gamma_2^4 \\
& -18\gamma_3^2+7\gamma_3^4+\gamma_2^2(41\gamma_3^2-46)) + e_2e_3(25-60\gamma_2^2+19\gamma_2^4+(44\gamma_2^2-45)\gamma_3^2 \\
& +15\gamma_3^4 - e_3^2(10-29\gamma_2^2+19\gamma_2^4+(53\gamma_2^2-35)\gamma_3^2+20\gamma_3^4)))\,z_2]\,\lambda_2\Big],
\end{aligned}
$$

$$
\begin{aligned}
\beta_{22} = {} & (4e_2^2)^{-1} C \Big[(3e_2^2+5e_3^2-(e_3\gamma_2-e_2\gamma_3)^2)\lambda_0 + C\,[(3e_2^5\gamma_3(4\gamma_2^2+\gamma_3^2-1) \\
& +e_2^4e_3\gamma_2(15-12\gamma_2^2+19\gamma_3^2) - 5e_3^3\gamma_2(5-\gamma_2^2-3\gamma_3^2+e_3^2(\gamma_2^2+4\gamma_3^2-1)) \\
& +e_2^2e_3\gamma_2(9\gamma_2^2-13\gamma_3^2-21+e_3^2(24-19\gamma_2^2+5\gamma_3^2)) + e_2e_3^2\gamma_3(5+11\gamma_2^2-15\gamma_3^2 \\
& +e_3^2(15-21\gamma_2^2+20\gamma_3^2)) + e_2^3\gamma_3(3(3-3\gamma_2^2-\gamma_3^2)+e_3^2(8-3\gamma_2^2+21\gamma_3^2)))\,z_1 \\
& +(3e_2^4\gamma_2\gamma_3(3-4\gamma_2^2-3\gamma_3^2)+e_2^2\gamma_2\gamma_3(e_3^2(9\gamma_2^2-15\gamma_3^2-14)+3(\gamma_2^2+\gamma_3^2-3)) \\
& +5e_3^2\gamma_2\gamma_3(5-\gamma_2^2-\gamma_3^2+e_3^2(3\gamma_2^2+4\gamma_3^2-3)) + e_2^3e_3(6+12\gamma_2^4+5\gamma_3^2-11\gamma_3^4 \\
& -\gamma_2^2(21+13\gamma_3^2)) + e_2e_3(5\gamma_3^2(\gamma_3^2-3)+\gamma_2^2(15+2\gamma_3^2)-3\gamma_2^4+e_3^2(13\gamma_2^4 \\
& +\gamma_2^2(11\gamma_3^2-23)-5(\gamma_3^2+4\gamma_3^4-2))))\,z_2]\,\lambda_2\Big],
\end{aligned}
$$

$$
\begin{aligned}
\beta_{12} = {} & (4e_2^2)^{-1}C\Big[2(e_2(e_2\gamma_3-e_3\gamma_2)z_1+(e_3(\gamma_2^2-5)-e_2\gamma_2\gamma_3)z_2)\lambda_0 \\
& +C\,[2e_2^4e_3\gamma_2\gamma_3(14\gamma_2^2+9\gamma_3^2-15)+10e_3\gamma_2\gamma_3(e_3^2-1)(5-\gamma_2^2-\gamma_3^2 \\
& +e_3^2(3\gamma_2^2+4\gamma_3^2-3)) + 2e_2^5(3+12\gamma_2^4+\gamma_3^2(\gamma_3^2-4)+\gamma_2^2(11\gamma_3^2-15)) \\
& +2e_2^2e_3\gamma_2\gamma_3(29-20\gamma_2^2-7\gamma_3^2+e_3^2(39\gamma_2^2+23\gamma_3^2-40)) + e_2^3(\gamma_3^2(10+3\gamma_3^2) \\
& -24\gamma_2^4-15+\gamma_2^2(51-13\gamma_3^2)+e_3^2(62\gamma_2^4+2\gamma_3^2(2\gamma_3^2-19)+2\gamma_2^2(31\gamma_3^2-46) \\
& +26)) + e_2(3\gamma_2^2(\gamma_2^2-5)+(15-2\gamma_2^2)\gamma_3^2-5\gamma_3^4+4e_3^4(8\gamma_2^2-5)(\gamma_2^2+2\gamma_3^2-1) \\
& +e_3^2(\gamma_2^2(98-53\gamma_3^2)+5(\gamma_3^2(2\gamma_3^2+7)-7)-35\gamma_2^4)) + 2(2e_2^3e_3\gamma_3(7\gamma_2^2+\gamma_3^2-4) \\
& +e_2^4\gamma_2(12\gamma_2^2+5\gamma_3^2-9)+e_2e_3\gamma_3(10-13\gamma_2^2+4e_3^2(8\gamma_2^2-5)+5\gamma_3^2) \\
& +5e_3^3\gamma_2(5-\gamma_2^2-3\gamma_3^2+e_3^2(\gamma_2^2+4\gamma_3^2-1))+e_2^2\gamma_2(15-6\gamma_2^2+2\gamma_3^2 \\
& +e_3^2(25\gamma_2^2+13\gamma_3^2-26)))\,z_1z_2]\,\lambda_2\Big].
\end{aligned}
$$

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