Non-principal Branches of Lambert W. A Tale of 2 Circles

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Abstract. The Lambert W function is a multivalued function whose principal branch has been studied in detail. Non-principal branches, however, have been much less studied. Here, asymptotic series expansions for the non-principal branches are obtained, and their properties, including accuracy and convergence are studied. The expansions are investigated by mapping circles around singular points in the domain of the function into the range of the function using the new expansions. Different expansions apply for large circles around the origin and for small circles. Although the expansions are derived as asymptotic expansions, some surprising convergence properties are observed.

Keywords: Multivalued functions · Asymptotic expansions · Special functions · Convergence tests

1 Introduction

The Lambert W function owes its current status[1] in no small part to computer algebra systems. Because W allowed algebra systems to return closed-form solutions to problems from all branches of science, computer users, whether mathematicians or non-specialists discovered W in ways that a conventional literature search could not. One difficulty for users has been that Lambert W is multivalued, like arctangent or logarithm, but with an important difference. The branches of the elementary multivalued functions are trivially related, for example the branches of arctangent differ by π; similarly, the branches of logarithm differ by $2\pi i$. There are no simple relations, however, between the branches of W, and each branch must be labelled separately and studied separately.

1.1 Definitions

The branches of the Lambert W function are denoted $W_k(z)$, where k is the branch index. Each branch obeys [1]

$$W_k(z)e^{W_k(z)} = z \; , \tag{1}$$

[1] Citations of [1] as of July 2023: Google scholar 7283; Scopus 4588.

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and the different branches are distinguished by the definition

$$W_k(z) \to \ln_k z \text{ for } |z| \to \infty \ . \tag{2}$$

Here, $\ln_k z$ denotes the kth branch of logarithm [2], i.e. $\ln_k z = \ln z + 2\pi i$, with $\ln z$ as defined in [3]. The way in which condition (2) defines the branches of W is also illustrated in Fig. 1.

The principal branch $W_0(z)$ takes real values for $z \geq -e^{-1}$ and has been extensively studied. For example, the function $T(z) = \ -W_0(-z)$ is the exponential generating function for labelled rooted trees [4]; the convex analysis of W_0 was developed in [5]; it was shown in [6] that W_0 is a Bernstein function, and a Stieltjes function, and its derivative is completely monotonic; a model of chemical kinetics in the human eye uses $W_0(x)$ in [8]. Numerous papers have proposed numerical schemes for bounding or evaluating $W_0(x)$ for $x \in \mathbb{R}$, a recent example being [7].

In contrast, *non*-principal branches $k \neq 0$ have been less studied. They do have, nonetheless, some applications. The branch $W_{-1}(z)$ takes real values for $-e^{-1} \leq z < 0$. The real-valued function $W_{-1}(-\exp(-1 - \frac{1}{2}z^2))$ was used in [9] to obtain a new derivation of Stirling's approximation to $n!$ and Vinogradov has presented applications in statistics both for $W_{-1}(x)$ [10] and $W_0(x)$ [11].

1.2 Expansions

In [12], de Bruijn obtained an asymptotic expansion for $W_0(x)$ when $x \to \infty$; this was extended to the complex plane in [1]. Having obtained an expansion for large x, [1] continued by stating

> 'A similar but purely real-valued series is useful for the branch $W_{-1}(x)$ for $x < 0$. We can get a real-valued asymptotic formula from the above by using $\log(-x)$ in place of Log(z) and $\log(-\log(-x))$ in place of log(Log(z)). [...] This series is not useful for complex x because the branch cuts of the series do not correspond to those of W.'

We improve upon this point by proposing new, explicit series for all non-principal branches $k \neq 0$, and testing them numerically.

An important difference between W_0 and all other branches is behaviour at the origin. W_0 is analytic at the origin [13], and its Taylor expansion is known explicitly [13]; in contrast, all other branches are singular at the origin. Our interest here is to study asymptotic expansions both for $|z| \to \infty$ and, for non-principal branches, the neglected case $|z| \to 0$.

1.3 Branch Structure

To focus our discussion, we consider the plots shown in Fig. 1. The top set of axes show values of z in the domain of $W(z)$. The bottom set show values of W_k, where the branch indicator k is important; that is, the bottom axes show

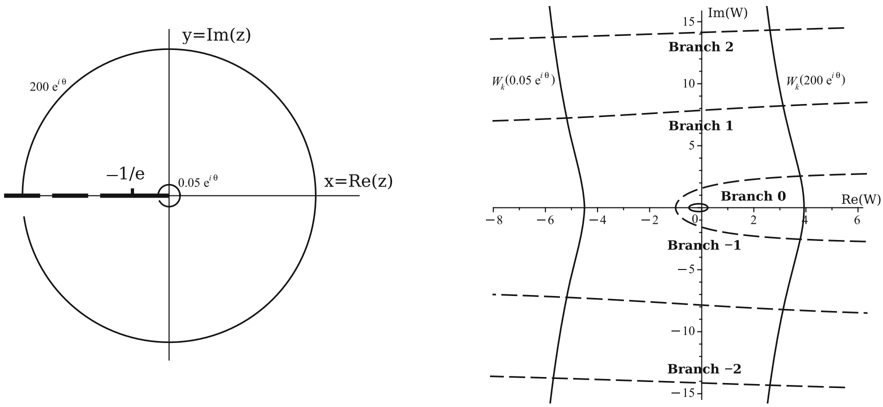

Fig. 1. The domains (left axes) and ranges (right axes) of the branches of the Lambert W function. The branches of W collectively fill all of the complex plane, although any one branch occupies only a disjoint strip of the plane. Each branch has a domain consisting of the entire complex plane, although the branch cuts differ according to the branch. The continuous curves in the range are constructed piecewise by mapping the circles successively with the different branches.

the ranges[2]. Although only one set of axes is used to show the domain, this is a simplification which avoids multiple figures.

There are actually several different domains, coinciding with the different branches of W. In contrast to more familiar multi-valued functions, such as $\ln z$, the different branches $W_k(z)$ do not share a single common domain. Specifically, the singular points and the branch cuts of $W_k(z)$ vary from branch to branch. In Fig. 1, the different branch cuts for different branches are compressed onto the negative real axis (of the top set of axes) using dashed and solid lines. For the principal branch W_0, the branch cut consists only of the dashed portion of the axis, i.e. $x \le -1/e$, and the solid segment is not a branch cut; the point $x = -1/e$ is the singular point. For the branches $k = \pm 1$, there are two branch cuts, both the dashed line and the solid line; they meet at $x = -1/e$. It is best to think of the cuts as distinct, even though they share a singular point and extend along the same axis. The distinction is that the dashed line for $k = -1$ maps to the boundary between W_0 and W_{-1}, with the boundary belonging to W_{-1}, while the solid line maps to the boundary between W_1 and W_{-1}, with the boundary belonging to W_{-1}. Similarly, the dashed line for $k = 1$ maps to the boundary between W_0 and W_1, but now the boundary belongs to W_0. In contrast to the dashed-line cuts, the solid-line cut maps to the boundary between W_1 and W_{-1}, with the boundary belonging to W_{-1}.

The origin is a second singular point for W_1 and W_{-1}. For all other branches, i.e. $k \ge 2$ and $k \le -2$, the two cuts merge into a single cut extending along the

[2] Note the plural. We regard each branch of W_k as a separate function with its own domain and range [14].

whole of the negative real axis, with the point $z = -1/e$ no longer being a singular point, and only the origin being singular. Two circles, both alike in dignity[3], are plotted in the domain; they are described by the equation $z = re^{i\theta}$ with $r = 200$ and $r = 0.05$ and $-\pi < \theta \leq \pi$. The circles are drawn so that one end of each circle touches the branch cut, while the other end stops short of the cut. This plotting convention reflects that the θ interval is closed on the top of the cut, when $\theta = \pi$.

The bottom axes in Fig. 1 show the ranges of the branches W_k. The branch boundaries are shown as black dashed lines. The curves plotted are the results from applying successively $W_{-2}, W_{-1}, W_0, W_1, W_2$ to the two circles shown in the top set of axes. The continuous curve in the positive-real half-plane corresponds to the large circle, while the small circle maps into two curves: the small closed curve around the origin and the continuous curve in the negative real half-plane.

1.4 Asymptotic Expansions

We briefly summarize Poincaré's theory of asymptotic expansions [15, Ch.1]. We begin with an example.

$$\begin{aligned} g(x) &= \int_0^\infty \frac{e^{-xt}\,\mathrm{d}t}{1+t} = \int_0^\infty e^{-xt}(1 - t + t^2 - t^3 + \ldots)\,\mathrm{d}t \\ &= \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \ldots \end{aligned} \tag{3}$$

The series in $1/x$ does not converge for any x, but if we substitute $x = 10$ into the equation, we obtain (evaluating the integral using Maple)

$$\int_0^\infty \frac{e^{-10t}\,\mathrm{d}t}{1+t} = 0.0915633\ldots = 0.1 - 0.01 + 0.002 - 0.0006 + 0.00024 - \ldots \tag{4}$$

Adding the first 4 terms, we obtain the approximation 0.0914, which approximates the integral with an error 0.00016. Our sum omitted the 5th term, and we note that its value, 0.00024, bounds the observed error. It is typical of asymptotic series that the error is bounded by the first omitted term in the sum.

The theory of asymptotic expansions generalizes the functions x^{-k} used in the example, with a sequence of gauge, or scale, functions $\{\phi_n(x)\}$ obeying the condition $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to \infty$. The series formed from these functions,

$$g(x) = \sum_{n=1}^{N} a_n \phi_n(x) \ , \tag{5}$$

has the property that it becomes more accurate as $x \to \infty$. Typically, the error is bounded by the omitted term $\phi_{N+1}(x)$. For an asymptotic expansion, the

[3] This whimsical Shakespearian reference emphasises the mathematical point that previous investigations have concentrated on the large circle and neglected the equally important small circle.

limit $N \to \infty$ is of less interest than the limit $x \to \infty$, and will not exist for a non-convergent expansion. This paper uses scale functions $\phi_n(z) = 1/\ln^n(z)$. In order for the functions to decrease with n, we require that $|\ln z| > 1$, which in turn requires $|z| > e$ or $|z| < e^{-1}$. Then they form an asymptotic sequence both in the limit $|z| \to \infty$ and $|z| \to 0$.

1.5 Outline

In Sect. 2, we revisit the derivation of the expansion of W given in [1] for large arguments, replacing the imprecise notation Log with the precise notation $\ln_k z$ defined above. We then use graphical methods to add to earlier treatments by demonstrating the accuracy of the approximations for the different branches. Although not all asymptotic expansions are convergent series, the expansions given here are convergent for some arguments. We show this convergence, but do not analyse the regions in detail.

In Sect. 3, the main motivation for this paper is taken up: the expansions for non-principal branches of W around the origin. We show that the key idea is to define a shifted logarithm which matches the asymptotic behaviour at the origin. Again we also consider convergence, and we uncover an unexpected result that several series, although based on different starting assumptions, none the less converge to correct values. The rates of convergence, however, are different, with the series based on shifted logarithms being best.

2 de Bruijn Series for Large *z*

Since the branches of W are defined so that $W_k(z)$ asymptotically approaches $\ln_k z$, we consider $W_k(z) = \ln_k z + v(z)$, and assume $v = o(\ln_k z)$. Then (1) gives

$$(\ln_k z + v(z))\ e^{\ln_k z + v} = (\ln_k z + v(z))\ ze^v = z\ .$$

To leading order, $e^{-v} = \ln_k z$, and assuming that v lies in the principal branch of logarithm, the approximation is (note the different branches of logarithm)

$$W_k(z) = \ln_k z - \ln_0(\ln_k z) + u(z)\ . \tag{6}$$

Neglecting temporarily the $u(z)$ term, we compare in Fig. 2 the one-term and two-term approximations to W. The line thickening shows where the approximations think the branch boundaries are. The term $\ln_k z$ alone is a significant over-estimate, and the branch boundaries are not close, but two terms, although under-estimating, are encouragingly closer. Our main interest, however, is the behaviour after including $u(z)$. Substituting (6) into (1) and introducing

$$\sigma = \frac{1}{\ln_k z}\ , \qquad \text{and} \qquad \tau = \frac{\ln(\ln_k z)}{\ln_k z}\ , \tag{7}$$

we can show that u obeys (more details of this demonstration are given below)

$$1 - \tau + \sigma u - e^{-u} = 0\ . \tag{8}$$

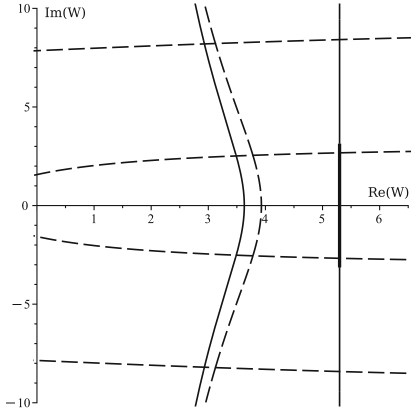

Fig. 2. A comparison between the exact value of W and the one-term and two-term approximations in (6). The dashed curve is the exact value. The straight line to the right is the one-term approximation; the central portion has been thickened to show where the approximation thinks the principal branch is. The solid curve to the left is the two-term approximation.

Equation (8) was solved for u by Comtet [16] as a series in σ:

$$u = \sum_{n=1}^{N} c_n \frac{(-\sigma)^n}{n!} \ , \tag{9}$$

$$c_n = \sum_{m=1}^{n} (-1)^{n-m} \begin{bmatrix} n \\ n-m+1 \end{bmatrix} \frac{\sigma^{-m}\tau^m}{m!} \ , \tag{10}$$

where $\left[{n \atop n-m+1}\right]$ is a Stirling Cycle number [17, p. 259], and we have written the series going to N terms, for later reference. The form of the expansion appears to be unchanged from the principal branch, but this is because the branch information is hidden in the variables σ and τ. The derivation of the expansion is for an asymptotic series, as defined in Sect. 1.4. Such series are not necessarily convergent[4], but in [19], the series (6) together with (9) was studied for $x \in \mathbb{R}$ and the series was shown to converge for $x > e$. The question naturally arises of where the series for principal and non-principal branches converge for $z \in \mathbb{C}$.

Since we are dealing with the accuracy and convergence of series on multiple domains of z and for multiple branches of W, we wish to avoid analyzing each branch separately and being tempted to present multiple repetitious plots of results. We thus use the plot shown in Fig. 3 to summarize our findings. The plot accumulates maps of the large circle shown above in Fig. 1 under successive branches W_k; these plots are compared with maps made by the corresponding series approximation (9) using 2 terms of the summation. The contours corre-

[4] Indeed, some authors define an asymptotic series as one that does not converge [18].

spond to circles of radii $r = 50, 10, 5, 3, 1, e^{-1}$. In each case the dashed curve is W and the solid curve is the series approximation.

In Fig. 3, we focus first on the approximation for the principal branch, indicated by the red curves. We see that for $r > 3$, the accuracy is acceptable, and improves for larger r, as expected. Since we are considering an asymptotic approximation, we fix the number of terms in the summation to 2, and consider changes with r. We note in particular that the exact and approximate curves for $r = 50$ are practically indistinguishable to the human eye. We can also investigate the convergence of the series. For $r > 10$ we can take more terms of the summation and observe improved accuracy (data not shown), indicating the series is convergent for larger r values (as well as asymptotic). For smaller values of r, the series loses accuracy, and in parallel fails to converge, the extraneous curves swamping the figure. Therefore, for $r < 3$ we plot only the values of W_0 and remove the distraction of the failed approximations.

Both the W curves and the approximations are smooth across the branch boundaries. This reflects the properties that

$$W_k(-x) = \lim_{y \uparrow 0} W_{k+1}(-x + iy)\ , \qquad \text{for } x < -1/e\ ,\ \text{and} \tag{11}$$

$$\ln_k(-x) = \lim_{y \uparrow 0} \ln_{k+1}(-x + iy)\ , \qquad \text{for } x < 0\ . \tag{12}$$

This does not ensure that the boundaries between the branches of W and of the approximations agree, although they approach each other with improved accuracy.

For branches $k \neq 0$, we observe something that is unexpected, namely, that the approximations show evidence of remaining accurate for all values of r down to $r = e^{-1}$. Indeed, the series appear convergent. This is difficult to justify graphically, but can be checked by extended summation for values where graphical evidence is weakest. In Table 1 we calculate approximations to $W_{-1}(-1/e) = -1$ and $W_{-1}(-0.4)$ using increasing numbers of terms in the sum. Adding up large numbers of terms in a sum can require additional intermediate precision for accuracy. For the table, Maple's default 10-digit accuracy had to be increased to 30 decimal digits for sums of more than 50 terms. The numerical results indicate convergence, but do not constitute a proof.

3 de Bruijn Series for Small z

A new feature associated with the analysis around the origin is the disappearance from the asymptotic analysis of the principal branch. Figure 4 shows a plot of values of W_k computed on a circle of radius $r = \frac{1}{20}$ and centred at the origin. The principal branch, shown in red, is the small closed curve around the origin, while all other branches form the continuous curve on the far left. It is important to note a difference between W_0 and W_{-1}. The real values of W_0 occur in the middle of its range, or to put it another way, the real values of W_0 do not coincide with the branch boundaries. In contrast, the real values of W_{-1} occur on one

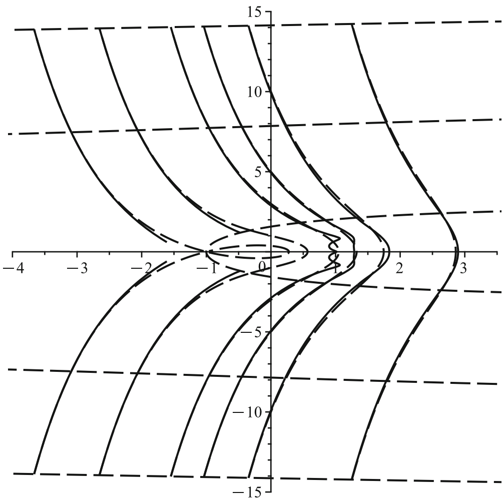

Fig. 3. A systematic test of expansion (9), using two terms of the summation. Each continuous curve is a concatenation of mappings of the same large circle using successively the various branches of W and of its approximations. The dashed curves are the exact values of W_k while the solid curves are the approximations. The contours correspond to circles of radii, from right to left, $r = 50, 10, 5, 3, 1, e^{-1}$. The approximations to the principal branch for $r < 3$ are so bad that they distract from the plots and have been omitted. For non-principal branches, all approximations are plotted.

of its branch boundaries. We want this difference to be reflected, if possible, in the asymptotic forms we use. As in the previous section, the leading asymptotic term is logarithm, and the problem is to match the branches of the logarithm term to W_{-1}, and more generally to all W_k for $k \neq 0$. Two possible asymptotic approximations are shown in Fig. 4 as the vertical lines to the right of the curve showing the values of W. The right-most line is the approximation $\ln_k z$ which was already used for the previous section. Since $W_{-1}(-0.01) = -6.473$, i.e. purely real, but $\ln(-0.01) = -4.605 + \pi i$ and $\ln_{-1}(-0.01) = -4.605 - \pi i$, it is clear that the approximations that worked well in the previous section, do not work here. For this reason, we introduce what we call a 'shifted log' by the definition

$$L_k(z) = \ln_k z - \operatorname{sgn}(k) i\pi \ , \quad \text{for } k \neq 0 \ . \tag{13}$$

We see that for this function $L_{-1}(-0.01) = -4.605$, and so is purely real where W_{-1} is real. This function is plotted in Fig. 4 as the straight line in between the other two contours. Notice that $W_{-1}(-e^{-1}) = -1$, and $L_{-1}(-e^{-1}) = -1$ also. Of course, $W_{-1}(z)$ is not differentiable at $z = -e^{-1}$, but $L_{-1}(z)$ is differentiable, showing that more terms in the series will be needed for numerical accuracy.

Table 1. Numerical tests of convergence for the expansion (9). The row $N = \infty$ refers to the value of W that the series is trying to reach. The series appears convergent, although painfully slowly.

N	value for $x = -e^{-1}$	value for $x = -0.4$
∞	-1	$-0.9441 - 0.4073\,i$
40	$-1.1568 - 0.1565\,i$	$-0.9665 - 0.3495\,i$
70	$-1.1190 - 0.1188\,i$	$-0.9259 - 0.3800\,i$
100	$-1.0997 - 0.0996\,i$	$-0.9232 - 0.4055\,i$
160	$-1.0789 - 0.0788\,i$	$-0.9448 - 0.4183\,i$

Having matched the leading-order behaviour of W_k using the shifted logarithm, we repeat the approach used above of substituting into $We^W = z$.

$$(L_k(z) + v(z)) \exp(L_k(z) + v(z)) = (L_k(z) + v(z))\,(-z) \exp(v(z)) = z$$
$$v(z) = -\ln(-L_k(z)) + u(z)\ .$$

It might seem that u will follow a pattern like $\ln(\ln(-L_k))$, but this is not so.

$$(L_k(z) - \ln(-L_k(z)) + u)) \exp(L_k(z) - \ln(-L_k(z)) + u))$$
$$= (L_k(z) - \ln(-L_k(z)) + u)\ \frac{-z}{-L_k(z)} \exp(u) = z\ .$$

Rearranging gives

$$1 - \frac{\ln(-L_k(z))}{L_k(z)} + \frac{u}{L_k(z)} - e^{-u} = 0\ . \tag{14}$$

Thus, if we redefine σ, τ by

$$\sigma = \frac{1}{L_k(z)} \quad \text{and} \quad \tau = \frac{\ln(-L_k(z))}{L_k(z)}\ , \tag{15}$$

we can return to (8) and (9).

It is remarkable that the fundamental relation (8), originally derived for the principal branch, has now reappeared twice: once for any branch ($|z| \gg 1$) and now for $|z| \ll e^{-1}$. Since (13) was chosen so that it is purely real where W_{-1} is real, we first compare plots for $-e^{-1} \le x < 0$. Figure 5 compares $W_{-1}(x)$ with two approximations, sum 9 for $N = 0$ and for $N = 3$. They are most accurate near $x = 0$ as expected.

Figure 6 shows a comparison in the complex plane for branches from $k = -2$ to $k = 2$. The contours are maps of small circles of radii $r = 0.25, 0.15, 0.05$. The series approximation was limited to $N = 1$ in order to obtain a visible separation of the exact and approximate contours. Recall that smaller values of r correspond to contours further to the left.

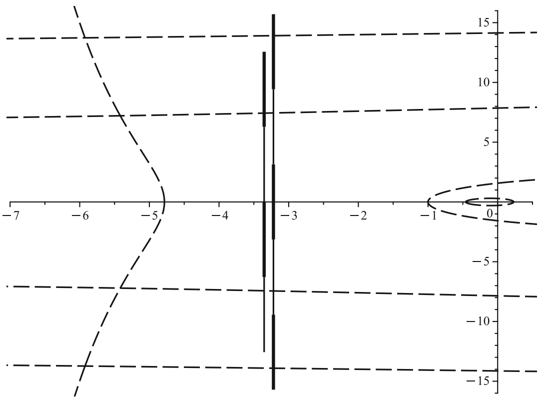

Fig. 4. A comparison of possible asymptotic approximations to W_k for small circles around the origin. The dashed curve shows $W_k(z)$ for $k \neq 0$. The two vertical lines show the two candidates: $\ln_k z$ is the right-most line and was used for large circles; the new shifted logarithm is the left line. The lines are sectioned into thick and thin segments. These show the branches of the approximations. The branches of $\ln_k z$ are seen to be not aligned with the boundaries of W, shown by the horizontal dashed lines. In contrast, the branches of the shifted logarithm are closer to the boundaries of the branches of W. Note that $W_{-1}(x)$ and the shifted logarithm are both purely real (although not equal, alas) for the same range of arguments, namely real and in the interval $[-e^{-1}, 0)$. For completeness, the map of the principal branch is also shown (around the origin), to emphasize that it does not participate in the asymptotic behaviour.

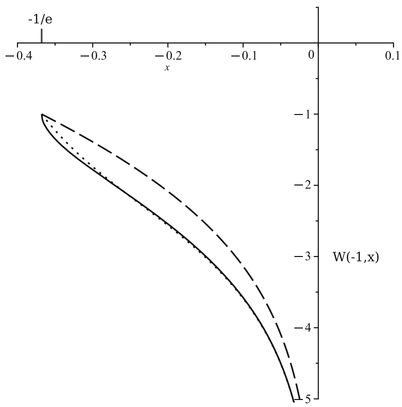

Fig. 5. Plots of $W_{-1}(x)$ and approximations based on (9) together with (15). The solid line shows W_{-1}; the dashed line shows (9) for $N = 0$; the dotted line shows $N = 3$.

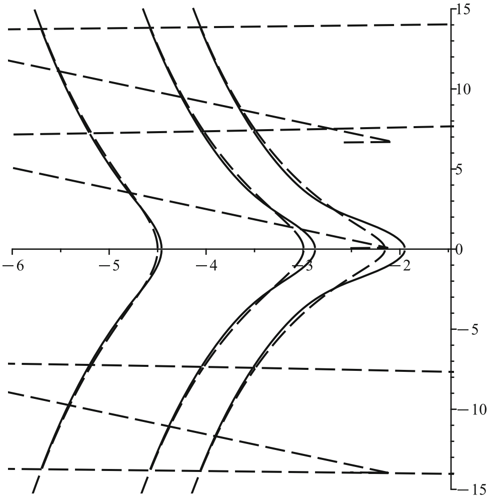

Fig. 6. Comparison of W_k, $k \neq 0$ and (9) using (15). The series uses $N = 1$ in order to separate the function and the approximation. The boundary between $k = -1$ and $k = 1$ is the negative real axis both for the function and for the approximation.

4 A Surprising Convergence

The approximation (7) used for $|z| \gg 1$ was discarded for $|z| \ll -e^{-1}$ because the branch boundaries were not aligned with the function near negative infinity. One could expect therefore that its accuracy would be bad, or wrong, or it would possibly return values for branches not requested. It is therefore surprising that in spite of starting from dismal estimates, the approximation manages to achieve results of reasonable accuracy. In Table 2, a comparison is made between series (9) based on (15) with the rejected series based on (7). Out of curiosity, we have tabulated the competing approximations when summed to one-term, two-terms and four-terms. The preferred series always performs better, but the other series also achieves good accuracy. As stated several times, (15) has the advantage of returning real values when W_{-1} is real, so we stick to our preferred series and do not pursue further discussion of this point.

5 A Further Variation

We briefly comment on a variation on the above series which can lead to more accurate estimates. We introduce a parameter during the derivation of the fundamental relation. During the derivation of (6), we considered the equation $\ln_k z + v = e^{-v}$, and argued that v is of smaller asymptotic order than $\ln_k z$. We thus neglected it on the left side of the equation and solved $\ln_k z = e^{-v}$ for v. We can note, however, that a constant is also of lower asymptotic order than

Table 2. Comparison of series (9) combined with (7) and then with (15). The various approximations are printed in adjacent columns for easy comparison. The errors reported in the last two columns report the errors in the 4-term summations.

x	k	W_k	$\ln_k x$	$L_k(x)$	Eq. (7) $N=0$	Eq. (15) $N=0$
-0.1	-1	-3.58	$-2.30-\pi i$	-2.30	$-3.66-0.94i$	-3.15
-0.01	-1	-6.47	$-4.61-\pi i$	-4.61	$-6.32-0.60i$	-6.13
-0.1	-2	$-4.45-7.31i$	$-2.30-3\pi i$	$-2.30-2\pi i$	$-4.58-7.61i$	$-4.20-7.50i$
-0.01	-2	$-6.90-7.08i$	$-4.61-3\pi i$	$-4.61-2\pi i$	$-6.96-7.40i$	$-6.66-7.22i$

x	k	W_k	Eq. (7) $N=2$	Eq. (15) $N=2$	Error (7)	Error (15)
-0.1	-1	-3.577	$-3.405-0.127i$	-3.591	0.213	0.013
-0.01	-1	-6.473	$-6.416+0.035i$	-6.481	0.066	0.008
-0.1	-2	$-4.449-7.307i$	$-4.448-7.314i$	$-4.442-7.305i$	0.0074	0.0071
-0.01	-2	$-6.896-7.081i$	$-6.891-7.086i$	$-6.894-7.079i$	0.0069	0.0039

$\ln_k z$, and instead of neglecting v, estimate the v on the left by a constant p: thus $\ln_k z + p = e^{-v}$. We now have the approximation

$$W_{k,dB}(z,p) = \ln_k(z) - \ln(p + \ln_k(z)) + u \ .$$

Substituting in $We^W = z$ leads now to the equation

$$(\ln_k z - \ln(p + \ln_k z) + u)\,\frac{1}{p + \ln_k z} = e^{-u} \ . \tag{16}$$

A simple manipulation allows us to convert this equation into yet another manifestation of the fundamental relation (8).

$$(\ln_k z + p - p - \ln(p + \ln_k z) + u)\,\frac{1}{p + \ln_k z} = 1 - \frac{p + \ln(p + \ln_k z)}{\ln_k z + p} + \frac{u}{p + \ln_k z} \ .$$

Thus, remarkably, we have

$$1 - \tau + \sigma u - e^{-u} = 0 \ , \text{ and } \sigma = \frac{1}{p + \ln_k z} \ , \tau = \frac{p + \ln(p + \ln_k z)}{\ln_k z + p} \ . \tag{17}$$

The contours in Fig. 3 would correspond to $p = 0$. The effect of p is greatest in the principal branch, where the approximation for the circle of radius $r = 3$ improves between the two figures, and for $r \le 1$, the approximations for $p = 1$ are good enough to be plotted (but still not good). The approximations for non-principal branches are little changed by the parameter.

6 Concluding Remarks

It was pointed out in Fig. 1 that the singular point $z_c = -e^{-1}$ is the place where different branch cuts meet. The point's singular nature is reflected in the drop in the accuracy of the various series seen above. It is interesting to extend the summation of the series to large numbers of terms so as to reach z_c, but it is not practical. The three branches $k = 0$ and $k = \pm 1$ share an expansion in the variable $\sqrt{2(ez+1)}$ [1], and for obtaining numerical values when z is in the neighbourhood of z_c, that expansion is much more convenient.

By concentrating the discussion on plots of the ranges of W_k, we have been able to condense the information more efficiently that by presenting results in the domains of the functions. We think this is a fruitful way to discuss multi-valued functions. Contrast Fig. 1 with the usual treatment in reference books of functions such as logarithm or arctangent. The books always present plots of the branch cuts in the domain, but never the ranges. The need to understand ranges is heightened by the fact that the ranges of W_k are not trivially related to each other, in contrast to the way in which $\ln_1 z$ is only $2\pi i$ different from $\ln_0 z$.

This paper has not attempted to supply formal proofs of the convergence properties of the series studied here. The aim has been to establish the correct forms of the expansions, and to demonstrate numerically their properties. Some of the surprising observations made here remain open problems, and invite both more detailed numerical investigation and formal analytical work.

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