Stability and Zero-Hopf Bifurcation Analysis of the Lorenz–Stenflo System Using Symbolic Methods

Bo Huang[1], Xiaoliang Li[2], Wei Niu[3,4](✉), and Shaofen Xie[5]

[1] LMIB – School of Mathematical Sciences, Beihang University, Beijing 100191, China
bohuang0407@buaa.edu.cn

[2] School of Business, Guangzhou College of Technology and Business, Guangzhou 510850, China

[3] Ecole Centrale de Pékin, Beihang University, Beijing 100191, China
wei.niu@buaa.edu.cn

[4] Beihang Hangzhou Innovation Institute Yuhang, Hangzhou 310051, China

[5] Academy of Mathematics and Systems Science, The Chinese Academy of Sciences, Beijing 100190, China
xieshaofen@amss.ac.cn

Abstract. This paper deals with the stability and zero-Hopf bifurcation of the Lorenz–Stenflo system by using methods of symbolic computation. Stability conditions on the parameters of the system are derived by using methods of solving semi-algebraic systems. Using the method of algorithmic averaging, we provide sufficient conditions for the existence of one limit cycle bifurcating from a zero-Hopf equilibrium of the Lorenz–Stenflo system. Some examples are presented to verify the established results.

Keywords: Averaging method · Limit cycle · Symbolic computation · Stability · Zero-Hopf bifurcation

1 Introduction and Main Results

In 1963, Edward Lorenz introduced a simplified mathematical chaotic model for atmospheric convection [1]. The chaotic model is a system of three ordinary differential equations now known as the Lorenz system. Since then, the research on dynamical behaviors of the Lorenz system and its generalizations has attracted great interest of scholars from various fields; the essence of chaos, characteristics of the chaotic system, bifurcations, and routes to chaos have been extensively studied (see [2–5] for instance).

The work was partially supported by National Natural Science Foundation of China (No. 12101032 and No. 12131004), Beijing Natural Science Foundation (No. 1212005), Philosophy and Social Science Foundation of Guangdong (No. GD21CLJ01), Social Development Science and Technology Project of Dongguan (No. 20211800900692).

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
F. Boulier et al. (Eds.): CASC 2023, LNCS 14139, pp. 183–198, 2023.
https://doi.org/10.1007/978-3-031-41724-5_10

Hyperchaos, as a dynamic behavior, is far more complex and has a greater potential than chaos in some non-traditional engineering and technological applications. It is well known that the minimal number of dimensions in which continuous-time hyperchaos can occur is 4; therefore, 4D autonomous differential systems are of main interest for research and applications of hyperchaos, especially 4D Lorenz-type hyperchaotic systems. In 1996, Stenflo [6] derived a system to describe the evolution of finite amplitude acoustic gravity waves in a rotating atmosphere. The Lorenz–Stenflo system is described by

$$\begin{aligned} \dot{x} &= a(y-x)+dw, \\ \dot{y} &= cx-y-xz, \\ \dot{z} &= -bz+xy, \\ \dot{w} &= -x-aw, \end{aligned} \tag{1}$$

where a, b, c, and d are real parameters; a, c, and d are the Prandtl, the Rayleigh, and the rotation numbers, respectively, and b is a geometric parameter. This system is rather simple and reduces to the classical Lorenz system when the parameter associated with the flow rotation, d, is set to zero. System (1) is chaotic as $a = 1$, $b = 0.7$, $c = 25$, and $d = 1.5$. Figure 1 shows the phase portraits of the system in 3D spaces.

This paper focuses on symbolic and algebraic analysis of stability and zero-Hopf bifurcation for the Lorenz–Stenflo system (1). We remark that, in the past few decades, symbolic methods have been explored extensively in terms of the qualitative analysis of dynamical systems (see [7–14] and the references therein). It should be mentioned that the zero-Hopf bifurcation of a generalized Lorenz–Stenflo system was already studied by Chen and Liang in [15]. However, the authors did not notice that the Lorenz–Stenflo system itself can exhibit a zero-Hopf bifurcation. The main goal of this paper is to fill this gap. Moreover, we study the zero-Hopf bifurcation of the Lorenz–Stenflo system in a parametric way by using symbolic methods. We recall that a (complete) zero-Hopf equilibrium of a 4D differential system is an isolated equilibrium point p_0 such that the Jacobian matrix of the system at p_0 has a double zero and a pair of purely imaginary eigenvalues. There are many studies of zero-Hopf bifurcations in 3D differential systems (see [16–21] and the references therein). The zero-Hopf bifurcations of hyperchaotic Lorenz systems can be found in [5,22]. Actually, there are very few results on the n-dimensional zero-Hopf bifurcation with $n > 3$. Our objective here is to study how many limit cycles can bifurcate from a zero-Hopf equilibrium of system (1) by using the averaging method. Unlike the usual analysis of zero-Hopf bifurcation, by means of symbolic computation, we would like to compute a partition of the parametric space of the involved parameters such that, inside every open cell of the partition, the system can have the maximum number of limit cycles that bifurcate from a zero-Hopf equilibrium.

On the number of equilibria of the Lorenz–Stenflo system, we recall from [6] that system (1) can have three equilibria, including the origin $E_0 = (x = 0, y = 0, z = 0, w = 0)$ and the two equilibria

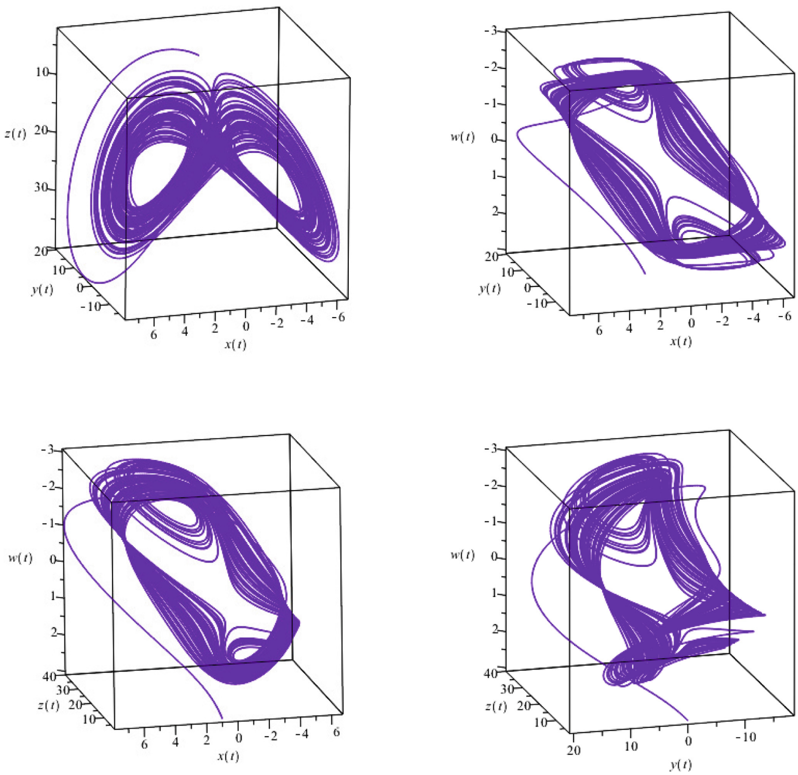


Fig. 1. The phase portraits of system (1) in different 3D projection spaces: $a = 1$, $b = 0.7$, $c = 25$, $d = 1.5$

$$E_{\pm} = \left(x = -aw,\ y = -\frac{a^2+d}{a}w,\ z = \frac{a^2+d}{b}w^2,\ w = \pm\sqrt{\frac{b(a^2c-a^2-d)}{a^2(a^2+d)}} \right)$$

if $\frac{b(a^2c-a^2-d)}{a^2(a^2+d)} > 0$. Otherwise, the origin is the unique equilibrium of the system. In fact, the above results can be easily verified by computing the Gröbner basis of the polynomial set $\{\dot{x}, \dot{y}, \dot{z}, \dot{w}\}$ with respect to the lexicographic term ordering determined by $x \succ y \succ z \succ w$.

The first goal of this paper is to study conditions on the parameters under which the Lorenz–Stenflo system (1) has a prescribed number of stable equilibrium points. Our result on this question is the following, and its proof can be found in Sect. 3.

Proposition 1. *The Lorenz–Stenflo system* (1) *can not have three stable equilibrium points; it has two stable equilibrium points if* $[a = 1]$ *and one of the following two conditions*

$$\begin{aligned} \mathcal{C}_1 &= [T_1 < 0,\ 0 < T_2,\ 0 < T_3,\ T_4 < 0,\ T_5 < 0,\ 0 < T_6], \\ \mathcal{C}_2 &= [0 < T_1,\ T_2 < 0,\ 0 < T_3,\ 0 < T_4,\ T_5 < 0,\ 0 < T_6] \end{aligned} \tag{2}$$

holds; and it has one stable equilibrium point if $[a = 1]$ *and one of the following five conditions*

$$
\begin{aligned}
\mathcal{C}_3 &= [0 < T_1,\, 0 < T_2,\, 0 < T_3,\, T_5 < 0,\, 0 < T_6],\\
\mathcal{C}_4 &= [0 < T_1,\, 0 < T_2,\, T_3 < 0,\, T_4 < 0,\, T_6 < 0],\\
\mathcal{C}_5 &= [0 < T_1,\, 0 < T_2,\, 0 < T_3,\, 0 < T_4,\, 0 < T_5,\, 0 < T_6],\\
\mathcal{C}_6 &= [0 < T_1,\, 0 < T_2,\, 0 < T_3,\, T_4 < 0,\, 0 < T_5,\, 0 < T_6],\\
\mathcal{C}_7 &= [0 < T_1,\, 0 < T_2,\, 0 < T_3,\, T_4 < 0,\, 0 < T_5,\, T_6 < 0]
\end{aligned}
\tag{3}
$$

holds. The explicit expressions of T_i *are the following:*

$$
\begin{aligned}
&T_1 = b, \quad T_2 = d - c + 1, \quad T_3 = d + 1,\\
&T_4 = bc - cd + d^2 + 2\,d + 1,\\
&T_5 = -bcd + bd^2 - 3\,bc - 2\,bd - 3\,b - 12\,d - 12,\\
&T_6 = b^2c + 2\,b^2d - bcd + bd^2 + 2\,b^2 + 10\,bd + 9\,b + 6\,d + 6.
\end{aligned}
$$

Remark 1. We remark that the condition $[a = 1]$ is used to facilitate the computation of the resulting semi-algebraic system (see Sect. 3) since the algebraic analysis usually involves heavy computation; see [8,9].

Example 1. Let

$$(a, b, c, d) = \left(1, \frac{1}{4}, -56, -29\right) \in \mathcal{C}_4.$$

Then the Lorenz–Stenflo system (1) becomes

$$
\begin{aligned}
&\dot{x} = y - x - 29w, \quad \dot{y} = -56x - y - xz,\\
&\dot{z} = -\frac{1}{4}z + xy, \quad \dot{w} = -x - w.
\end{aligned}
\tag{4}
$$

Its three equilibria are: $p_1 = (0, 0, 0, 0)$, $p_2 = (\frac{1}{2}, -14, -28, -\frac{1}{2})$ and $p_3 = (-\frac{1}{2}, 14, -28, \frac{1}{2})$. System (4) has only one stable equilibrium point p_1; see Fig. 2 (a); (b).

Example 2. Let

$$(a, b, c, d) = \left(1, \frac{1}{4}, \frac{55}{32}, -\frac{27}{64}\right) \in \mathcal{C}_2.$$

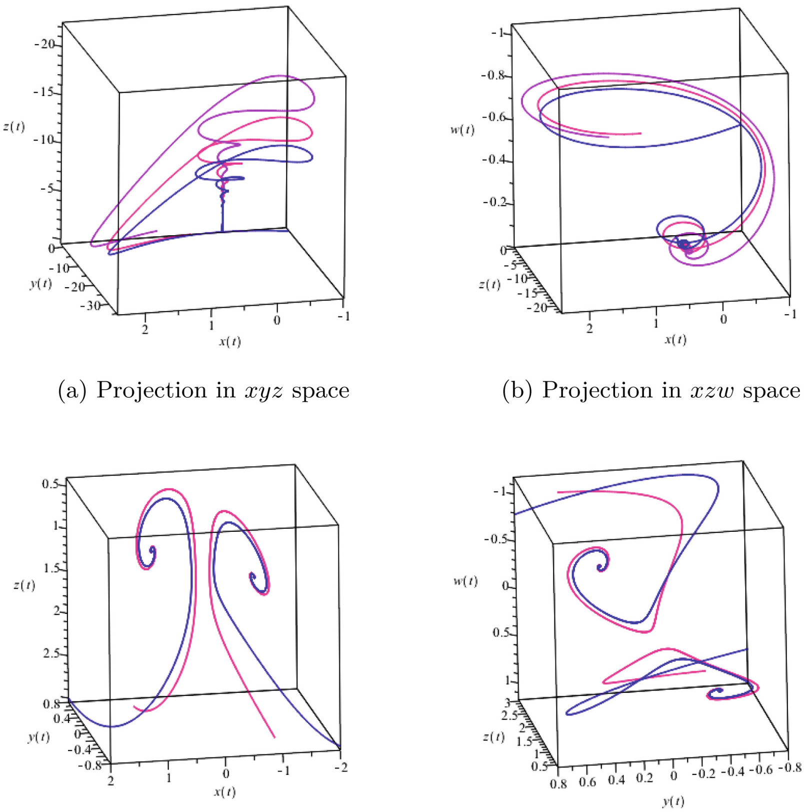
Then the Lorenz–Stenflo system (1) becomes

$$
\begin{aligned}
&\dot{x} = y - x - \frac{27}{64}w, \quad \dot{y} = \frac{55}{32}x - y - xz,\\
&\dot{z} = -\frac{1}{4}z + xy, \quad \dot{w} = -x - w.
\end{aligned}
\tag{5}
$$

Its three equilibria are:

$$p_1 = (0, 0, 0, 0), p_2 = (\frac{1}{74}\sqrt{2701}, \frac{1}{128}\sqrt{2701}, \frac{73}{64}, -\frac{1}{74}\sqrt{2701})$$

and $p_3 = (-\frac{1}{74}\sqrt{2701}, -\frac{1}{128}\sqrt{2701}, \frac{73}{64}, \frac{1}{74}\sqrt{2701})$. System (5) has two stable equilibria p_2 and p_3; see Fig. 2 (c); (d).



(a) Projection in xyz space

(b) Projection in xzw space

(c) Projection in xyz space

(d) Projection in yzw space

Fig. 2. Numerical simulations of local stability of the Lorenz–Stenflo system for the choice of parameter values given in Examples 1 and 2

Our second goal of this paper is to investigate the bifurcation of periodic solutions at the (complete) zero-Hopf equilibrium (that is, an isolated equilibrium with double zero eigenvalues and a pair of purely imaginary eigenvalues) of system (1). In the following, we characterize the periodic orbits bifurcating from the zero-Hopf equilibrium $E_0 = (0, 0, 0, 0)$ of system (1). The main techniques are based on the first order averaging method and some algebraic methods, such as the Gröbner basis [23] and real root classifications [24]. The techniques used here for studying the zero-Hopf bifurcation can be applied in theory to other high dimensional polynomial differential systems.

In the next proposition, we characterize when the equilibrium point $E_0 = (0, 0, 0, 0)$ is a zero-Hopf equilibrium.

Proposition 2. *The origin E_0 is a zero-Hopf equilibrium of the Lorenz–Stenflo system* (1) *if the conditions $2a + 1 = 0$, $b = 0$, $3c - 4 > 0$ and $12d - 1 > 0$ hold.*

We consider the vector (a, b, c, d) given by

$$
\begin{aligned}
a &= -\frac{1}{2} + \varepsilon a_1, \quad b = \varepsilon b_1, \\
c &= \frac{4}{3}(\beta^2 + 1) + \varepsilon c_1, \quad d = \frac{1}{3}\beta^2 + \frac{1}{12} + \varepsilon d_1,
\end{aligned} \tag{6}
$$

where $\varepsilon \neq 0$ is a sufficiently small parameter, the constants $\beta \neq 0$, a_1, b_1, c_1, and d_1 are all real parameters. The next result gives sufficient conditions for the bifurcation of a limit cycle from the origin when it is a zero-Hopf equilibrium.

Theorem 1. *For the vector given by* (6) *and* $|\varepsilon| > 0$ *sufficiently small, system* (1) *has, up to the first order averaging method, at most 1 limit cycle bifurcates from the origin, and this number can be reached if one of the following two conditions holds:*

$$
\begin{aligned}
\mathcal{C}_8 &= [b_1 < 0,\, 8\,\beta^2 a_1 - 4\,a_1 + 3\,c_1 - 12\,d_1 < 0] \wedge \bar{\mathcal{C}}, \\
\mathcal{C}_9 &= [0 < b_1,\, 0 < 8\,\beta^2 a_1 - 4\,a_1 + 3\,c_1 - 12\,d_1] \wedge \bar{\mathcal{C}},
\end{aligned} \tag{7}
$$

where $\bar{\mathcal{C}} = [\beta \neq 0, b_1 \neq 0, a_1 \neq 0, 4\beta^2 + 1 \neq 0, 8\,\beta^2 a_1 - 4\,a_1 + 3\,c_1 - 12\,d_1 \neq 0]$. *Moreover, the only limit cycle that exists (under the condition* $\mathcal{C}_8$ *or* $\mathcal{C}_9$*) is unstable.*

Theorem 1 shows that the Lorenz–Stenflo system (1) can have exactly 1 limit cycles bifurcating from the origin if the condition in (7) holds. In the following, we provide a concrete example of system (1) to verify this established result.

Corollary 1. *Consider the special family of the Lorenz–Stenflo system*

$$
\begin{aligned}
\dot{x} &= \left(\varepsilon + \frac{1}{2}\right)(x - y) + \left(\varepsilon + \frac{5}{12}\right)w, \\
\dot{y} &= \left(\varepsilon + \frac{8}{3}\right)x - xz - y, \\
\dot{z} &= xy + \varepsilon z, \\
\dot{w} &= -x + \left(\varepsilon + \frac{1}{2}\right)w.
\end{aligned} \tag{8}
$$

This system has exactly 1 limit cycle $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon))$ *bifurcating from the origin by using the first order averaging method, namely,*

$$
\begin{aligned}
x(t,\varepsilon) &= \frac{5}{12}\varepsilon\left(\bar{X}_3 - \bar{R}\cos t\right) + \mathcal{O}(\varepsilon^2), \\
y(t,\varepsilon) &= \frac{9}{5}\varepsilon\left(2\bar{X}_3 - \bar{R}\cos t - \bar{R}\sin t\right) + \mathcal{O}(\varepsilon^2), \\
z(t,\varepsilon) &= \varepsilon\bar{X}_4 + \mathcal{O}(\varepsilon^2), \\
w(t,\varepsilon) &= \frac{1}{6}\varepsilon\left(5\bar{X}_3 - \bar{R}\cos t + 2\bar{R}\sin t\right) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

where $(\bar{R}, \bar{X}_3, \bar{X}_4)$ *is a real solution of a semi-algebraic system (see Sect. 5). Moreover, the limit cycle is unstable.*

The rest of this paper is organized as follows. In Sect. 2, we recall the averaging method that we shall use for proving the main results. Section 3 is devoted to prove Proposition 1. The proofs of Proposition 2 and Theorem 1 are given in Sect. 4, and the proof of Corollary 1 is presented in Sect. 5. The paper is concluded with a few remarks.

2 Preliminary Results

The averaging method is one of the best analytical methods to study limit cycles of differential systems in the presence of a small parameter ε. The first order averaging method introduced here was developed in [25]. Recently, this theory was extended to an arbitrary order in ε for arbitrary dimensional differential systems, see [26]. More discussions on the averaging method, including some applications, can be found in [27,28].

We deal with differential systems in the form

$$\dot{\mathbf{x}} = \varepsilon F(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon), \tag{9}$$

with $\mathbf{x} \in D \subset \mathbb{R}^n$, D a bounded domain, and $t \geq 0$. Moreover we assume that $F(t, \mathbf{x})$ and $R(t, \mathbf{x}, \varepsilon)$ are T-periodic in t.

The averaged system associated to system (9) is defined by

$$\dot{\mathbf{y}} = \varepsilon f^0(\mathbf{y}), \tag{10}$$

where

$$f^0(\mathbf{y}) = \frac{1}{T} \int_0^T F(s, \mathbf{y}) ds. \tag{11}$$

The next theorem says under what conditions the equilibrium points of the averaged system (10) provide T-periodic orbits for system (9).

Theorem 2. *We consider system (9) and assume that the functions F, R, $D_{\mathbf{x}}F$, $D^2_{\mathbf{x}}F$ and $D_{\mathbf{x}}R$ are continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times D$, with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Moreover, we suppose that F and R are T-periodic in t, with T independent of ε.*

(i) *If $p \in D$ is an equilibrium point of the averaged system (10) such that*

$$\det(D_{\mathbf{x}} f^0(p)) \neq 0 \tag{12}$$

then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\mathbf{x}(t, \varepsilon)$ of system (9) such that $\mathbf{x}(0, \varepsilon) \to p$ as $\varepsilon \to 0$.

(ii) *If the equilibrium point $\mathbf{y} = p$ of the averaged system (10) has all its eigenvalues with negative real part then, for $|\varepsilon| > 0$ sufficiently small, the corresponding periodic solution $\mathbf{x}(t, \varepsilon)$ of system (9) is asymptotically stable and, if one of the eigenvalues has positive real part $\mathbf{x}(t, \varepsilon)$, it is unstable.*

The proof of Theorem 2 can be found in [25,28]. It follows from Lemma 1 of [26] that the expression of the limit cycle associated to the zero $\mathbf{y}^*$ of $f^0(\mathbf{y})$ can be described by

$$x(t, \mathbf{y}^*, \varepsilon) = \mathbf{y}^* + \mathcal{O}(\varepsilon). \tag{13}$$

The averaging method allows to find periodic solutions for periodic non-autonomous differential systems (see (9)). However, here we are interested in using it for studying the periodic solutions bifurcating from a zero-Hopf equilibrium point of the autonomous differential system (1). The steps for doing that are the following.

(i) First we must identify the conditions for which the system has a zero-Hopf equilibrium (see Proposition 2).
(ii) We write the linear part of the resulting system (plugging in the conditions obtained in **(i)**) at the origin in its real Jordan normal form by linear change of variables $(x, y, z, w) \mapsto (U, V, W, Z)$.
(iii) We scale the variables by setting $(U, V, W, Z) = (\varepsilon X_1, \varepsilon X_2, \varepsilon X_3, \varepsilon X_4)$, because the zero-Hopf bifurcation and the averaging method needs such a small parameter ε, and write the differential system in the form $\left(\frac{dR}{dt}, \frac{d\theta}{dt}, \frac{dX_3}{dt}, \frac{dX_4}{dt}\right)$ where $(X_1, X_2, X_3, X_4) = (R\cos\theta, R\sin\theta, X_3, X_4)$.
(iv) We take the angular variable θ as the new independent variable of the differential system. Obtaining a 3-dimensional periodic non-autonomous system $\frac{dR}{d\theta} = \cdots$, $\frac{dX_3}{d\theta} = \cdots$, $\frac{dX_4}{d\theta} = \cdots$ in the variable θ. In this way the differential system is written into the normal form of the averaging method for studying the periodic solutions.
(v) Going back through the change of variables we get the periodic solutions bifurcating from the zero-Hopf equilibrium of system (1).

Remark 2. A symbolic Maple program for the realization of certain steps on zero-Hopf bifurcation analysis of polynomial differential systems is developed in [29]. The program can be used for computing the higher-order averaged functions of nonlinear differential systems. The source code of the Maple program is available at https://github.com/Bo-Math/zero-Hopf. More details on the outline of the program, including some applications, can be found in [29].

3 Stability Conditions of the Lorenz–Stenflo System

The goal of this section is to prove Proposition 1. Let $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ be the equilibrium point of the Lorenz–Stenflo system (1). Namely, we have the algebraic system

$$\Psi = \{a(\bar{y} - \bar{x}) + d\bar{w} = 0,\ c\bar{x} - \bar{y} - \bar{x}\bar{z} = 0,\ -b\bar{z} + \bar{x}\bar{y} = 0,\ -\bar{x} - a\bar{w} = 0\}. \tag{14}$$

The Jacobian matrix of the Lorenz–Stenflo system evaluated at $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ is given by

$$\begin{pmatrix} -a & a & 0 & d \\ -\bar{z} + c & -1 & -\bar{x} & 0 \\ \bar{y} & \bar{x} & -b & 0 \\ -1 & 0 & 0 & -a \end{pmatrix},$$

and the characteristic polynomial of this matrix can be written as

$$P(\lambda) = t_0\lambda^4 + t_1\lambda^3 + t_2\lambda^2 + t_3\lambda + t_4,$$

where

$$\begin{aligned}
&t_0 = 1, \quad t_1 = 2\,a + b + 1,\\
&t_2 = a^2 + 2\,ab - ac + a\bar{z} + \bar{x}^2 + 2\,a + b + d,\\
&t_3 = a^2b - a^2c + a^2\bar{z} - abc + ab\bar{z} + 2\,a\bar{x}^2 + a\bar{x}\bar{y} + a^2 + 2\,ab + bd + d,\\
&t_4 = -a^2bc + a^2b\bar{z} + a^2\bar{x}^2 + a^2\bar{x}\bar{y} + a^2b + d\bar{x}^2 + bd.
\end{aligned}$$

By Routh–Hurwitz's stability criterion (e.g., [30]), $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ is a stable equilibrium point if the following algebraic system is satisfied

$$\begin{aligned}
D_1 &= t_1 = 2\,a + b + 1 > 0,\\
D_2 &= \det\begin{pmatrix} t_1 & t_0 \\ t_3 & t_2 \end{pmatrix} = 2\,a^3 + 4\,a^2b - a^2c + a^2\bar{z} + 2\,ab^2 - a\bar{x}\bar{y} + b\bar{x}^2 + 4\,a^2 + 4\,ab\\
&\quad - ac + 2\,ad + a\bar{z} + b^2 + \bar{x}^2 + 2\,a + b > 0,\\
D_3 &= \det\begin{pmatrix} t_1 & t_0 & 0 \\ t_3 & t_2 & t_1 \\ 0 & t_4 & t_3 \end{pmatrix} = -7\,a^3bc + 2\,abd - 4\,a^2bc - 2\,a^3b^2c + 8\,a^2bd - ab^2c\\
&\quad - 3\,a^4bc + a^3bc^2 - a^2b^3c - a^2\bar{x}^2\bar{y}^2 + 2\,ad^2 + a^3c^2 + 2\,a^3 + \big(a^4 + a^3b + a^3\\
&\quad + a^2b\big)\bar{z}^2 + \big(2\,a^5 + 3\,a^4b - 2\,a^4c + 2\,a^3b^2 - 2\,a^3bc + a^2b^3 + 5\,a^4 + 7\,a^3b\\
&\quad - 2\,a^3c + 2\,a^3d + 3\,a^2b^2 - 2\,a^2bc + 3\,a^2bd + ab^3 + 3\,a^3 + 4\,a^2b + a^2d + ab^2\\
&\quad + abd + ad\big)\bar{z} + (2\,ab + 2\,a)\,\bar{x}^4 + \big(4\,a^3b - 2\,a^3c + 4\,a^2b^2 - a^2bc - ab^2c\\
&\quad + 4\,a^3 + 8\,a^2b - 3\,a^2c + 4\,ab^2 - abc - 4\,abd + 4\,a^2 + 4\,ab - 4\,ad\big)\bar{x}^2 + 4\,a^4\\
&\quad + 2\,a^5 - abcd + \big(-a^2b + a^2\big)\,\bar{x}\bar{y}\bar{z} - 3\,a^2bcd - 2\,a^5c + a^4c^2 + 2\,a^5b + 4\,a^4b^2\\
&\quad + 2\,ab^3 + 4\,a^2d + 2\,ab^2 + 2\,ad + 8\,a^4b - 5\,a^4c + 10\,a^3b^2 - 3\,a^3c + 4\,a^3d\\
&\quad + 8\,a^2b^2 + 10\,a^3b + 4\,a^2b + 4\,a^2b^3 - ab^3c + 2\,ab^3d - a^2cd + 2\,ab^2d + 2\,abd^2\\
&\quad + \big(-2a^4 - a^3b + a^2b^2 + a^2bc - a^3 - a^2c + 2a^2d + ab^2 - abd + a^2 + ab\\
&\quad - ad\big)\bar{x}\bar{y} + 4\,a^3bd - 2\,a^3cd - 3\,a^2b^2c + 4\,a^2b^2d + a^2bc^2 + 2\,a^3b^3 + \big(-2\,a^2\\
&\quad + ab + a\big)\bar{x}^3\bar{y} + \big(2\,a^3 + a^2b + ab^2 + 3\,a^2 + ab\big)\,\bar{x}^2\bar{z} - acd > 0,\\
D_4 &= t_4 = -a^2bc + a^2b\bar{z} + a^2\bar{x}^2 + a^2\bar{x}\bar{y} + a^2b + d\bar{x}^2 + bd > 0.
\end{aligned} \tag{15}$$

Combining (14) and (15), we see that the Lorenz–Stenflo system has a prescribed number (say k) of stable equilibrium points if the following semi-algebraic system has k distinct real solutions:

$$\begin{cases} a(\bar{y} - \bar{x}) + d\bar{w} = 0,\ c\bar{x} - \bar{y} - \bar{x}\bar{z} = 0,\ -b\bar{z} + \bar{x}\bar{y} = 0,\ -\bar{x} - a\bar{w} = 0, \\ D_1 > 0,\ D_2 > 0,\ D_3 > 0,\ D_4 > 0, \end{cases} \tag{16}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$, and $\bar{w}$ are the variables. The above semi-algebraic system may be solved by the method of discriminant varieties of Lazard and Rouillier [31] (implemented as a Maple package by Moroz and Rouillier), or the method of Yang and Xia [24] for real solution classification (implemented as a Maple package DISCOVERER by Xia [32]; see also the recent improvements in [33] as well as the Maple package RegularChains[SemiAlgebraicSetTools]). However, in the presence of several parameters, the Yang–Xia method may be more efficient than that of Lazard–Rouillier, see [8].

Note that system (16) contains four free parameters a, b, c, d, and the total degree of the involved polynomials is 4, which makes the computation very difficult. In order to obtain simple sufficient conditions for system (16) to have a prescribed number of stable equilibrium points, the computation is done under the constraint $[a = 1]$. By using DISCOVERER or RegularChains, we obtain that system (16) has exactly two distinct real solutions with respect to the variables $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ if the condition $\mathcal{C}_1$ or $\mathcal{C}_2$ in (2) holds, and it has only one real solution if one of the conditions in (3) holds; system (16) can not have three distinct real solutions. This ends the proof of Proposition 1.

4 Zero-Hopf Bifurcation of the Lorenz–Stenflo System

This section is devoted to the proofs of Proposition 2 and Theorem 1.

Proof (**Proof of Proposition 2).** The characteristic polynomial of the linear part of the Lorenz–Stenflo system at the origin E_0 is

$$\begin{aligned} p(\lambda) = \lambda^4 + (2\,a + b + 1)\,\lambda^3 + \left(a^2 + 2\,ba - ac + 2\,a + b + d\right)\lambda^2 \\ + \left(a^2 b - a^2 c - abc + a^2 + 2\,ba + db + d\right)\lambda - a^2 bc + a^2 b + db. \end{aligned} \tag{17}$$

Imposing that $p(\lambda) = \lambda^2(\lambda^2 + \beta^2)$ with $\beta \neq 0$, we obtain $a = -\frac{1}{2}$, $b = 0$, $3c - 4 = 12d - 1 = 4\beta^2 > 0$. This completes the proof.

Proof (**Proof of Theorem 1).** Consider the vector defined by (6), then the Lorenz–Stenflo system becomes

$$\begin{aligned} \dot{x} &= \left(-\frac{1}{2} + \varepsilon a_1\right)(y - x) + \left(\frac{1}{3}\beta^2 + \frac{1}{12} + \varepsilon d_1\right) w, \\ \dot{y} &= \left(\frac{4}{3}(\beta^2 + 1) + \varepsilon c_1\right) x - y - xz, \\ \dot{z} &= -\varepsilon b_1 z + xy, \\ \dot{w} &= -x - \left(-\frac{1}{2} + \varepsilon a_1\right) w. \end{aligned} \tag{18}$$

We need to write the linear part of system (18) at the origin in its real Jordan normal form

$$\begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

when $\varepsilon = 0$. For doing that, we perform the linear change of variables $(x, y, z, w) \mapsto (U, V, W, Z)$ given by

$$
\begin{aligned}
x &= -\frac{\left(4\beta^2+1\right)U}{12\beta^2} + \frac{\left(4\beta^2+1\right)W}{12\beta^2},\\
y &= -\frac{\left(4\beta^2+1\right)U}{9\beta^2} - \frac{\left(4\beta^2+1\right)V}{9\beta} + \frac{\left(\beta^2+1\right)\left(4\beta^2+1\right)W}{9\beta^2},\\
z &= \beta Z,\\
w &= -\frac{U}{6\beta^2} + \frac{V}{3\beta} + \frac{\left(4\beta^2+1\right)W}{6\beta^2}.
\end{aligned}
\tag{20}
$$

In these new variables (U, V, W, Z), system (18) becomes a new system which can be written as $(\dot{U}, \dot{V}, \dot{W}, \dot{Z})$. By computing the second order Taylor expansion of expressions in this new system, with respect to ε, about the point $\varepsilon = 0$, we obtain

$$
\begin{aligned}
\dot{U} &= -\beta V + \frac{1}{4\beta}(UZ - WZ) + \varepsilon F_{1,1}(U, V, W, Z),\\
\dot{V} &= \beta U + \frac{1}{2}(ZW - ZU) + \varepsilon F_{1,2}(U, V, W, Z),\\
\dot{W} &= \frac{1}{4\beta}(ZU - ZW) + \varepsilon F_{1,3}(U, V, W, Z),\\
\dot{Z} &= \frac{\left(4\,\beta^2+1\right)^2}{108\beta^5}\Big(U^2 + (\beta^2+1)W^2 - (\beta^2+2)UW + \beta(UV - VW)\Big) - \varepsilon b_1 Z,
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
F_{1,1} &= \frac{1}{3\beta\ (4\,\beta^2+1)}\left(16\,\beta^4 a_1 + 8\,\beta^2 a_1 - 12\,\beta^2 d_1 + a_1 - 6\,d_1\right)V - \frac{1}{12\beta^2}\left(16\,\beta^4 a_1\right.\\
&\quad \left.+ 20\,\beta^2 a_1 + 24\,\beta^2 d_1 + 4\,a_1 - 3\,c_1 + 12\,d_1\right)W + \frac{1}{12\beta^2\ (4\,\beta^2+1)}\left(16\,\beta^4 a_1\right.\\
&\quad \left.+ 20\,\beta^2 a_1 - 12\,c_1\beta^2 + 24\,\beta^2 d_1 + 4\,a_1 - 3\,c_1 + 12\,d_1\right)U,\\
F_{1,2} &= -\frac{1}{4\,\beta^2+1}\left(4\,\beta^2 a_1 + a_1 - 2\,d_1\right)V - \frac{c_1 - 2\,d_1}{2\beta}W + \frac{4\,\beta^2 c_1 + c_1 - 2\,d_1}{2\beta\ (4\,\beta^2+1)}U,\\
F_{1,3} &= \frac{1}{3\beta\ (4\,\beta^2+1)}\left(4\,\beta^2 a_1 + a_1 - 6\,d_1\right)V - \frac{1}{12\beta^2}\left(16\,\beta^2 a_1 + 4\,a_1 - 3\,c_1\right.\\
&\quad \left.+ 12\,d_1\right)W + \frac{1}{12\beta^2\ (4\,\beta^2+1)}\left(16\,\beta^2 a_1 - 12\,c_1\beta^2 + 4\,a_1 - 3\,c_1 + 12\,d_1\right)U.
\end{aligned}
$$

After doing step **(iii)** and step **(iv)** (see Sect. 2), we write the differential system (21) into the normal form of the averaging method. By computing the first order averaged functions $f^0(\mathbf{y})$ in (11) (where $\mathbf{y} = (R, X_3, X_4)$), we obtain $f^0(\mathbf{y}) = (f_{1,1}(\mathbf{y}), f_{1,3}(\mathbf{y}), f_{1,4}(\mathbf{y}))$, where

$$
\begin{aligned}
f_{1,1}(\mathbf{y}) &= -\frac{R}{24\beta^3}\bar{f}_{1,1}(R, X_3, X_4),\\
f_{1,3}(\mathbf{y}) &= -\frac{1}{12\beta^3}\bar{f}_{1,3}(R, X_3, X_4),\\
f_{1,4}(\mathbf{y}) &= \frac{1}{216\beta^6}\bar{f}_{1,4}(R, X_3, X_4),
\end{aligned}
\tag{22}
$$

with

$$
\begin{aligned}
\bar{f}_{1,1}(R, X_3, X_4) &= 8\beta^2 a_1 - 3\beta X_4 - 4a_1 + 3c_1 - 12d_1,\\
\bar{f}_{1,3}(R, X_3, X_4) &= X_3\left(16\beta^2 a_1 + 3\beta X_4 + 4a_1 - 3c_1 + 12d_1\right),\\
\bar{f}_{1,4}(R, X_3, X_4) &= \left(16\beta^4 + 8\beta^2 + 1\right)R^2 + \left(32\beta^6 + 48\beta^4 + 18\beta^2 + 2\right)X_3^2\\
&\quad - 216\beta^5 X_4 b_1.
\end{aligned}
$$

It is obvious that system (22) can have at most one real solution with $R > 0$. Hence, system (18) can have at most one limit cycle bifurcating from the origin. Moreover, the determinant of the Jacobian of $(f_{1,1}, f_{1,3}, f_{1,4})$ is

$$
D_1(R, X_3, X_4) = \det\begin{pmatrix} \frac{\partial f_{1,1}}{\partial R} & \frac{\partial f_{1,1}}{\partial X_3} & \frac{\partial f_{1,1}}{\partial X_4}\\ \frac{\partial f_{1,3}}{\partial R} & \frac{\partial f_{1,3}}{\partial X_3} & \frac{\partial f_{1,3}}{\partial X_4}\\ \frac{\partial f_{1,4}}{\partial R} & \frac{\partial f_{1,4}}{\partial X_3} & \frac{\partial f_{1,4}}{\partial X_4} \end{pmatrix} = \frac{1}{10368\,\beta^{11}}\cdot \bar{D}_1(R, X_3, X_4),
$$

where

$$
\begin{aligned}
\bar{D}_1(R, X_3, X_4) &= -4608\,\beta^8 {a_1}^2 b_1 + 1152\,\beta^6 {a_1}^2 b_1 - 864\,\beta^6 a_1 b_1 c_1 + 3456\,\beta^6 a_1 b_1 d_1\\
&+ 576\,\beta^4 {a_1}^2 b_1 - 864\,\beta^4 a_1 b_1 c_1 + 3456\,\beta^4 a_1 b_1 d_1 + 324\,\beta^4 b_1 {c_1}^2 - 2592\,\beta^4 b_1 c_1 d_1\\
&+ 5184\,\beta^4 b_1 {d_1}^2 + \big(256\,\beta^6 a_1 + 192\,\beta^4 a_1 - 48\,\beta^4 c_1 + 192\,\beta^4 d_1 + 48\,\beta^2 a_1\\
&- 24\,\beta^2 c_1 + 96\beta^2 d_1 + 4a_1 - 3c_1 + 12d_1\big)R^2 + 324\beta^6 X_4^2 b_1 + \big(-256\beta^8 a_1\\
&- 256\beta^6 a_1 - 96\beta^6 c_1 + 384\beta^6 d_1 + 48\beta^4 a_1 - 144\beta^4 c_1 + 576\beta^4 d_1 + 56\beta^2 a_1\\
&- 54\beta^2 c_1 + 216\beta^2 d_1 + 8a_1 - 6c_1 + 24d_1\big)X_3^2 + (864\beta^7 a_1 b_1 + 864\beta^5 a_1 b_1\\
&- 648\beta^5 b_1 c_1 + 2592\beta^5 b_1 d_1)X_4 + \left(96\,\beta^7 + 144\,\beta^5 + 54\,\beta^3 + 6\,\beta\right)X_4 {X_3}^2\\
&+ (48\,\beta^5 + 24\,\beta^3 + 3\,\beta)X_4 R^2.
\end{aligned}
$$

It follows from Theorem 2 that system (18) can have one limit cycle bifurcating from the origin if the semi-algebraic system has exactly one real solution:

$$
\begin{cases} \bar{f}_{1,1}(R, X_3, X_4) = \bar{f}_{1,3}(R, X_3, X_4) = \bar{f}_{1,4}(R, X_3, X_4) = 0,\\ R > 0, \quad \bar{D}_1(R, X_3, X_4) \neq 0, \quad \beta \neq 0 \end{cases}
\tag{23}
$$

where R, X_3, and X_4 are the variables. Using DISCOVERER (or the package RegularChains[SemiAlgebraicSetTools] in Maple), we find that system (18) has exactly one real solution if and only if the one of the conditions $\mathcal{C}_8$ and $\mathcal{C}_9$ in (7) holds.

Remark that the stability conditions of the limit cycle may be derived by using the Routh–Hurwitz criterion to the characteristic polynomial of the Jacobian matrix of $(f_{1,1}, f_{1,3}, f_{1,4})$. In other words, more constraints on the principal diagonal minors of the Hurwitz matrix should be added to the algebraic system (23). By using similar techniques we can verify that the resulting semi-algebraic system has no real solution with respect to the variables R, X_3, X_4. Hence, we complete the proof of Theorem 1.

5 Zero-Hopf Bifurcation in a Special Lorenz–Stenflo System

Since the proof of Corollary 1 is very similar to that of Theorem 1, we omit some steps in order to avoid some long expressions.

The corresponding differential system $\left(\frac{dR}{dt}, \frac{d\theta}{dt}, \frac{dX_3}{dt}, \frac{dX_4}{dt}\right)$ (step **(iii)** in Sect. 2) associated to system (8) now becomes

$$\begin{aligned}
\frac{dR}{dt} &= \varepsilon\Big[\frac{1}{60}\,(-30\,R\cos\theta X_4 - 154\,R\cos\theta + 30\,X_3X_4 + 30\,X_3)\sin\theta \\
&\quad + \frac{1}{4}\,R\cos^2\theta X_4 - \frac{103}{60}\,R\cos^2\theta - \frac{1}{4}\,\cos\theta X_3X_4 \\
&\quad + \frac{7}{12}\,\cos\theta X_3 + \frac{7}{5}\,R\Big] + \mathcal{O}(\varepsilon^2),\\
\frac{d\theta}{dt} &= 1 + \varepsilon\Big[\frac{1}{60R}\big(-15R\cos\theta X_4 + 103R\cos\theta + 15X_3X_4 - 35X_3\big)\sin\theta \\
&\quad + \frac{1}{60R}\big(-30R\cos^2\theta X_4 - 154R\cos^2\theta + 30\cos\theta X_3X_4 + 30\cos\theta X_3 \\
&\quad + 172\,R\big)\Big] + \mathcal{O}(\varepsilon^2),\\
\frac{dX_3}{dt} &= \varepsilon\Big[-\frac{1}{4}X_3X_4 + \frac{1}{4}R\cos\theta X_4 + \frac{11}{12}X_3 - \frac{23}{60}R\cos\theta - \frac{11}{15}R\sin\theta\Big] + \mathcal{O}(\varepsilon^2),\\
\frac{dX_4}{dt} &= \varepsilon\Big[\frac{1}{108}\big(25\cos\theta R^2 - 25RX_3\big)\sin\theta + \frac{25}{108}R^2\cos^2\theta - \frac{25}{36}R\cos\theta X_3 \\
&\quad + \frac{25}{54}X_3^2 + X_4\Big] + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{24}$$

Hence, we have the normal form of averaging (step **(iv)** in Sect. 2)

$$\frac{dR}{d\theta} = \frac{dR/dt}{d\theta/dt}, \quad \frac{dX_3}{d\theta} = \frac{dX_3/dt}{d\theta/dt}, \quad \frac{dX_4}{d\theta} = \frac{dX_4/dt}{d\theta/dt}. \tag{25}$$

In order to find the limit cycles of system (8), we must study the real roots of the first order averaged functions

$$\begin{aligned}
f_{1,1}(R, X_3, X_4) &= \frac{1}{8}\,X_4R + \frac{13}{24}\,R,\\
f_{1,3}(R, X_3, X_4) &= -\frac{1}{4}\,X_4X_3 + \frac{11}{12}\,X_3,\\
f_{1,4}(R, X_3, X_4) &= \frac{25}{216}\,R^2 + \frac{25}{54}\,X_3^2 + X_4.
\end{aligned} \tag{26}$$

Moreover, the determinant of the Jacobian of $(f_{1,1}, f_{1,3}, f_{1,4})$ is

$$D_1(R, X_3, X_4) = -\frac{1}{32}\,X_4^2 - \frac{1}{48}\,X_4 + \frac{25}{864}\,X_3^2 X_4 + \frac{143}{288} + \frac{325}{2592}\,X_3^2 + \frac{25}{3456}\,R^2 X_4 - \frac{275}{10368}\,R^2. \tag{27}$$

Using the built in Maple command *RealRootIsolate* (with the option 'abserr'= $1/10^{10}$) to the semi-algebraic system

$$\begin{cases} f_{1,1}(R, X_3, X_4) = 0, \quad f_{1,3}(R, X_3, X_4) = 0, \quad f_{1,4}(R, X_3, X_4) = 0, \\ R > 0, \quad D_1(R, X_3, X_4) \neq 0, \end{cases} \tag{28}$$

we obtain a list of one real solution:

$$\Big[\bar{R} \approx 6.1185 \in \Big[\frac{6265}{1024}, \frac{50127}{8192}\Big], \quad \bar{X}_3 = 0, \quad \bar{X}_4 = -\frac{13}{3}\Big].$$

This verifies that system (8) has exactly one limit cycle bifurcating from the origin. Now we shall present the expression of the limit cycle. The limit cycles Λ of system (25) associated to system (8) and corresponding to the zero $(\bar{R}, \bar{X}_3, \bar{X}_4)$ given by (28) can be written as $\{(R(\theta, \varepsilon), X_3(\theta, \varepsilon), X_4(\theta, \varepsilon)), \theta \in [0, 2\pi]\}$, where from (13) we have

$$\Lambda := \begin{pmatrix} R(\theta, \varepsilon) \\ X_3(\theta, \varepsilon) \\ X_4(\theta, \varepsilon) \end{pmatrix} = \begin{pmatrix} \bar{R} \\ \bar{X}_3 \\ \bar{X}_4 \end{pmatrix} + \mathcal{O}(\varepsilon). \tag{29}$$

Moreover, the eigenvalues of the Jacobian matrix $\begin{pmatrix} \frac{\partial f_{1,1}}{\partial R} & \frac{\partial f_{1,1}}{\partial X_3} & \frac{\partial f_{1,1}}{\partial X_4} \\ \frac{\partial f_{1,3}}{\partial R} & \frac{\partial f_{1,3}}{\partial X_3} & \frac{\partial f_{1,3}}{\partial X_4} \\ \frac{\partial f_{1,4}}{\partial R} & \frac{\partial f_{1,4}}{\partial X_3} & \frac{\partial f_{1,4}}{\partial X_4} \end{pmatrix}$ at the point $(\bar{R}, \bar{X}_3, \bar{X}_4)$ are about $(-0.6546509493,\ 1.6546509493,\ 2)$. We have the corresponding limit cycles Λ is unstable.

Further, in system (24), the limit cycle Λ writes as

$$\begin{pmatrix} R(t, \varepsilon) \\ \theta(t, \varepsilon) \\ X_3(t, \varepsilon) \\ X_4(t, \varepsilon) \end{pmatrix} = \begin{pmatrix} \bar{R} \\ t \\ \bar{X}_3 \\ \bar{X}_4 \end{pmatrix} + \mathcal{O}(\varepsilon). \tag{30}$$

Finally, going back through the changes of variables, $(X_1, X_2, X_3, X_4) \mapsto (R\cos\theta, R\sin\theta, X_3, X_4)$, $(U, V, W, Z) \mapsto (\varepsilon X_1, \varepsilon X_2, \varepsilon X_3, \varepsilon X_4)$, and $(x, y, z, w) \mapsto (U, V, W, Z)$ given by (20), we have for system (8) the limit cycle:

$$\begin{aligned} x(t, \varepsilon) &= \frac{5}{12}\varepsilon\left(\bar{X}_3 - \bar{R}\cos t\right) + \mathcal{O}(\varepsilon^2), \\ y(t, \varepsilon) &= \frac{9}{5}\varepsilon\left(2\bar{X}_3 - \bar{R}\cos t - \bar{R}\sin t\right) + \mathcal{O}(\varepsilon^2), \\ z(t, \varepsilon) &= \varepsilon\bar{X}_4 + \mathcal{O}(\varepsilon^2), \\ w(t, \varepsilon) &= \frac{1}{6}\varepsilon\left(5\bar{X}_3 - \bar{R}\cos t + 2\bar{R}\sin t\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{31}$$

This completes the proof of Corollary 1.

6 Conclusions

In this paper, using symbolic computation, we analyzed the conditions on the parameters under which the Lorenz–Stenflo differential system has a prescribed number of (stable) equilibrium points. Sufficient conditions for the existence of one limit cycle bifurcating from the origin of the Lorenz–Stenflo system are derived by making use of the averaging method, as well as the methods of real solution classification. The special family of the Lorenz–Stenflo system (8) was provided as a concrete example to verify our established result. The algebraic analysis used in this paper is relatively general and can be applied to other n-dimensional differential systems. The zero-Hopf bifurcation of limit cycles from the equilibrium point (other than the origin) of the Lorenz–Stenflo system is also worthy of study. We leave this as a future problem.

References

1. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
2. Sparrow, C.: The Lorenz Equations: Bifurcation, Chaos, Strange Attractors; Applied Mathematical Sciences. Strange Attractors; Applied Mathematical Sciences. Springer, New York (1982). https://doi.org/10.1007/978-1-4612-5767-7
3. Robinson, C.: Nonsymmetric Lorenz attractors from a homoclinic bifurcation. SIAM J. Math. Anal. **32**, 119–141 (2000)
4. Yang, Q., Chen, G., Huang, K.: Chaotic attractors of the conjugate Lorenz-type system. Int. J. Bifurc. Chaos **17**, 3929–3949 (2007)
5. Montiel, L., Llibre, J., Stoica, C.: Zero-Hopf bifurcation in a hyperchaotic Lorenz system. Nonlinear Dyn. **75**, 561–566 (2014)
6. Stenflo, L.: Generalized Lorenz equations for acoustic-gravity waves in the atmosphere. Physica Scripta **53**, 83–84 (1996)
7. Wang, D., Xia, B.: Stability analysis of biological systems with real solution classification. In: Proceedings of ISSAC 2005, pp. 354–361. ACM Press, New York (2005)
8. Niu, W., Wang, D.: Algebraic approaches to stability analysis of biological systems. Math. Comput. Sci. **1**, 507–539 (2008)
9. Li, X., Mou, C., Niu, W., Wang, D.: Stability analysis for discrete biological models using algebraic methods. Math. Comput. Sci. **5**, 247–262 (2011)
10. Niu, W., Wang, D.: Algebraic analysis of stability and bifurcation of a self-assembling micelle system. Appl. Math. Comput. **219**, 108–121 (2012)
11. Chen, C., Corless, R., Maza, M., Yu, P., Zhang, Y.: An application of regular chain theory to the study of limit cycles. Int. J. Bifur. Chaos **23**, 1350154 (2013)
12. Boulier, F., Han, M., Lemaire, F., Romanovski, V.G.: Qualitative investigation of a gene model using computer algebra algorithms. Program. Comput. Softw. **41**(2), 105–111 (2015). https://doi.org/10.1134/S0361768815020048
13. Boulier, F., Lemaire, F.: Finding first integrals using normal forms modulo differential regular chains. In: Gerdt, V.P., Koepf, W., Seiler, W.M., Vorozhtsov, E.V. (eds.) CASC 2015. LNCS, vol. 9301, pp. 101–118. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-24021-3_8
14. Huang, B., Niu, W., Wang, D.: Symbolic computation for the qualitative theory of differential equations. Acta. Math. Sci. **42B**, 2478–2504 (2022)

15. Chen, Y., Liang, H.: Zero-zero-Hopf bifurcation and ultimate bound estimation of a generalized Lorenz-Stenflo hyperchaotic system. Math. Methods Appl. Sci. **40**, 3424–3432 (2017)
16. Llibre, J., Buzzi, C.A., da Silva, P.R.: 3-dimensional Hopf bifurcation via averaging theory. Disc. Contin. Dyn. Syst. **17**, 529–540 (2007)
17. Llibre, J., Makhlouf, A.: Zero-Hopf periodic orbits for a Rössler differential system. Int. J. Bifurc. Chaos **30**, 2050170 (2020)
18. Sang, B., Huang, B.: Zero-Hopf bifurcations of 3D quadratic Jerk system. Mathematics **8**, 1454 (2020)
19. Tian, Y., Huang, B.: Local stability and Hopf bifurcations analysis of the Muthuswamy-Chua-Ginoux system. Nonlinear Dyn. (2), 1–17 (2022). https://doi.org/10.1007/s11071-022-07409-3
20. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1993). https://doi.org/10.1007/978-1-4612-1140-2
21. Kuznetsov, Y.: Elements of Applied Bifurcation Theory. Springer, New York (2004)
22. Llibre, J., Candido, M.R.: Zero-Hopf bifurcations in a hyperchaotic Lorenz system II. Int. J. Nonlinear Sci. **25**, 3–26 (2018)
23. Buchberger, B.: Gröbner bases: an algorithmic method in polynomial ideal theory. In: Bose, N.K. (ed.) Multidimensional Systems Theory, pp. 184–232. Reidel, Dordrecht (1985)
24. Yang, L., Xia, B.: Real solution classifications of parametric semi-algebraic systems. In: Dolzmann A., Seidl A., Sturm T. (eds.) Algorithmic Algebra and Logic. Proceedings of the A3L, Norderstedt, Germany, pp. 281–289 (2005)
25. Buică, A., Llibre, J.: Averaging methods for finding periodic orbits via Brouwer degree. Bull. Sci. Math. **128**, 7–22 (2004)
26. Llibre, J., Novaes, D.D., Teixeira, M.A.: Higher order averaging theory for finding periodic solutions via Brouwer degree. Nonlinearity **27**, 563–583 (2014)
27. Sanders, J.A., Verhulst, F., Murdock, J.: Averaging Methods in Nonlinear Dynamical Systems, 2nd edn. Applied Mathematical Sciences Series Volume 59. Springer, New York (2007). https://doi.org/10.1007/978-0-387-48918-6
28. Llibre, J., Moeckel, R., Simó, C.: Central configuration, periodic orbits, and hamiltonian systems. In: Advanced Courses in Mathematics-CRM Barcelona Series. Birkhäuser, Basel, Switzerland (2015)
29. Huang, B.: Using symbolic computation to analyze zero-Hopf bifurcations of polynomial differential systems. In: Proceedings of ISSAC 2023, pp. 307–314. ACM Press, New York (2023). https://doi.org/10.1145/3597066.3597114
30. Lancaster, P., Tismenetsky, M.: The Theory of Matrices: With Applications. Academic Press, London (1985)
31. Lazard, D., Rouillier, F.: Solving parametric polynomial systems. J. Symb. Comput. **42**, 636–667 (2007)
32. Xia, B.: DISCOVERER: a tool for solving semi-algebraic systems. ACM Commun. Comput. Algebra **41**, 102–103 (2007)
33. Chen, C., Davenport, J.H., May, J.P., Moreno Maza, M., Xia, B., Xiao, R.: Triangular decomposition of semi-algebraic systems. J. Symb. Compt. **49**, 3–26 (2013)