N_α- Separation Axioms in Topological Spaces

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1 Introduction

A topological space (TS) is a fairly broad concept. More specificity is frequently desirable. Some studies in TS and their extensions in nonclassical TS are shown by many mathematicians [1–13]. One method is to define topological spaces with more constrained attributes using the separation axioms. In general, it is not true that a sequence in a topological space has only one limit. However, using the separation axioms, a type of space may be established in which the limit, if it exists at all, is unique. In 2015, N_a- open sets were initially examined by N.A Dawood and N.M Ali, see [14]; by using these sets, we study some classes of N_α- separation axioms and N_α- Ti separation for each $i = 0, 1, 2$, and look into some of their characteristics. Separation axioms have also been generalized to other generic topological spaces such as ordered topological spaces [15]. Ibrahim [16] presented and explored the features of a strong variant of α-open sets termed α_γ-open via operation in 2013. Khalaf and Ibrahim [17] extended their investigation of the features of operations defined on the collection of α-open sets introduced by Ibrahim [16], defining and discussing numerous properties of α_γ-regular, α-β-compact, and α_γ-connected spaces, as well as α-(γ, β)-continuous functions.

In this chapter, we use N_α- open sets in topological spaces to create new types of N_α-separation axioms and study some of their properties. There are also some definitions and theorems offered. Here in this work, all spaces *X* and *Y* are topological spaces, also the closure (interior resp.) of a subset *A* of *X* is denoted by *cl* (*A*) (*int* (*A*) resp.)

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2 Fundamental Concepts

We will cover some fundamental principles that will be useful in our work.

Definition 1 [15]
Assume that X is a topological space (TS), a set A is named N_a- open set ($N_\alpha - OS$) if for some α-open set $B \neq \varnothing$ satisfies $cl\ (B) \subseteq A$. Also, its complement is named a N_a- closed set ($N_\alpha - CS$). The collection of all N_α- open sets is referred as $N_\alpha O(X)$, and its complement by $N_\alpha C(X)$.

Remark 1 [15]
A set A is a ($N_\alpha - CS$) if for some α-closed set $\varnothing \neq B \neq X$ satisfies $\subseteq int\ (B)$.

Remark 2 [15]
(i) X and $\varnothing$ are N_α- open sets ($N_\alpha - OS$ s) in any (TS) X.
(ii) Any clopen set is ($N_\alpha - OS$).
(iii) Any set in discrete space is ($N_\alpha - OS$).

Theorem 1 [15]
Let X_1, X_2 be topological spaces (TSs). Then A_1 and A_2 are ($N_\alpha - OS$ s) in X_1, X_2 resp., if and only if $A_1 \times A_2$ is ($N_\alpha - OS$) in $X_1 \times X_2$.

Proposition 1 [15]
Let X be (TS). Then:

(i) Finite union of ($N_\alpha - OS$ s) is ($N_\alpha - OS$) also.
(ii) Finite intersection of ($N_\alpha - CS$ s) is ($N_\alpha - CS$) also.

Definition 2 [15]
The union of all N_a- open set of X contained in A is named N_a- interior of A and is denoted $N_\alpha{}^{int}(A)$, and the intersection of all ($N_\alpha - CS$) containing A is called N_α-closure of A, refereed by $N_\alpha{}^{cl}(A)$.

Definition 3 [18]
We say A is generalized N_α – closed set ($gN_\alpha - CS$) of a space X, if $N_\alpha{}^{cl}(A) \subseteq B$ whenever $A \subseteq B$ and B is ($N_\alpha - OS$).

The complement of ($gN_\alpha - CS$) is generalized N_α- open set ($gN_\alpha - OS$) in X.

Theorem 2 [18]
(i) If A is ($N_\alpha - CS$) in X, then it is ($gN_\alpha - CS$).
(ii) If A is ($N_\alpha - OS$) in X, then it is ($gN_\alpha - OS$).

Proposition 2 [15]
Suppose that (Y, t_y) is a subspace of a (TS) X with $A \subseteq Y \subseteq X$. Then:

(i) If $A \in N_a O(X)$, then $A \in N_\alpha O(Y)$.
(ii) If $A \in N_a O(Y)$, then $A \in N_\alpha O(X)$, where Y is clopen set in X

Definition 4 [19]
Let X_1, X_2 be (TSs) where, $f : X_1 \to X_2$ is a mapping, then f is named:

(i) $N_\alpha(N_{\alpha*}$- continues) resp. if f^{-1} (A) is $(N_\alpha - OS)$ in X_1 for each A open set $((N_\alpha - OS))$in X_2 resp.
(ii) $N_\alpha(N_{\alpha*}$- open) mapping if $f(A)$ is $(N_\alpha - OS)$ in X_2 for each open set $((N_\alpha - OS))$ A in X_1 resp.
(iii) gN_α-continuous ($gN_{\alpha*}$ -continuous) resp. mapping if for each open set $((gN_\alpha - OS))$ set A in Y respectively then $f^{-1}(A)$ is $(gN_\alpha - OS)$ in X.

3 Some Characteristics of N_α-Separation N_α- Axioms

In this section, we study ${N_\alpha}^{Ti}$- space X for each $i = 0, 1, 2$ and we discuss some of these spaces' characteristics and remarks. We will prove certain theorems in the following cases when X is a finite space.

Definition 5
Assume that X is a (TS). We say X is a ${N_\alpha}^{TO}$-space if for any $x \neq y$ in X, there exists $(N_\alpha - OS)$ A containing one of them but not other.

Theorem 3
Let X be a (TS). Then X is ${N_\alpha}^{To}$-space if and only if ${N_\alpha}^{Cl\{x\}} \neq {N_\alpha}^{Cl\{y\}}$.

Proof Let ${N_\alpha}^{Cl\{x\}} \neq {N_\alpha}^{Cl\{y\}}$, $\forall x \neq y$ in X. This implies ${N_\alpha}^{Cl\{x\}} \nsubseteq {N_\alpha}^{Cl\{y\}}$ or ${N_\alpha}^{Cl\{y\}}$ $\nsubseteq {N_\alpha}^{Cl\{x\}}$. Suppose ${N_\alpha}^{Cl\{x\}} \nsubseteq {N_\alpha}^{Cl\{y\}}$, hence $X \notin {N_\alpha}^{Cl\{y\}}$, thus $x \in ({N_\alpha}^{Cl\{y\}})^c$, which is $(N_\alpha - OS)$ and $y \notin ({N_\alpha}^{Cl\{y\}})^c$. Thus, X is ${N_\alpha}^{To}$-space, assume X is ${N_\alpha}^{To}$-space; hence, for each $x \neq y$ in X, there exists $(N_\alpha - OS)$ G such that $x \in G, y \notin G$ or $y \in G$, $x \notin G$. Hence, G^c is $(Na - CS)$. $x \notin G^c$, $y \in G^c y$; hence, $x \notin {N_\alpha}^{Cl\{y\}}$, $x \in {N_\alpha}^{Cl\{x\}}$; this means $x \notin {N_\alpha}^{Cl\{y\}}$. Thus, ${N_\alpha}^{Cl\{x\}} \neq {N_\alpha}^{\{y\}}$.

Definition 6
Let X be a (TS). Then X is named ${N_\alpha}^{T1}$-space if each pair of distinct points x and y of X, there exist two N_α- open sets A, B containing x and y, respectively, such that $y \notin A, x \notin B$.

Proposition 3
Let X be a (TS). Then X is ${N_\alpha}^{T1}$ -space if and only if $\{x\}$ is $(N_\alpha - CS)$ $\forall x \in X$.

Proof Assume that X is ${N_\alpha}^{T1}$-space, to show that each $\{x\}$ is $(N_\alpha - CS)$, this means we must show that $X/\{x\}$ is $(N_\alpha - OS)$ for each singleton set $\{x\}$ in X.

Let $y \in X/\{x\}$, then $y \neq x$ in X, since X is ${N_\alpha}^{T1}$ space, then there exists $(N_\alpha - OS)$ G with $y \in G$ and $x \notin G$. This implies that $y \in G \subseteq \frac{X}{\{x\}}$; this implies $X/\{x\}$ is $(N_\alpha - OS)$. Hence, $\{x\}$ is $(N_\alpha - CS)$.

Conversely: Let $\{x\}$ be $(N_\alpha - CS)$, $\forall$ $x \in X$, to prove X is ${N_\alpha}^{T1}$ -space. Let $x \neq y$ in X, hence $\{x\}$, $\{y\}$ are $(N_\alpha - CSs)$ hence $\{x\}^c$, $\{y\}^c$ are $(N_\alpha - OSs)$ and $y \in \{x\}^c$, $x \notin \{x\}^c$, $x \in \{y\}^c$, $y \notin \{y\}^c$. Therefore, X is Na^{T1}-space.

Definition 7
Let X be a (TS). Then X is named ${N_\alpha}^{T2}$-space if for any two distinct points x, y in X there exists two $(N_\alpha - OSs)$ X satisfy $x \in A_1$, $y \in A_2$ and $A_1 \cap A_2 = \varnothing$

Proposition 4
If X is Na^{T2}- space, then $A = \{(x, y) : x = y, x, y \in X\}$ is $(Na - CS)$.

Proof Assume that X is ${N_\alpha}^{T2}$- space, to prove A is $(N_\alpha - CS)$, let $(x, y) \in A^c \subseteq X \times X/A$, this mean x and y are two distinct points in X, where X is ${N_\alpha}^{T1}$-space then for some A_1, $A_2 \in N_\alpha O(X)$ satisfy $x \in A_1$, $y \in A_2$ and A_1, A_2 are disjoint sets, hence $(x, y) \in A_1 \times A_2 \subseteq A^c$, but $A_1 \times A_2 \in N_\alpha O(X \times X)$ (see Theorem 1), hence A^c is N_α- open set, thus A is $(N_\alpha - CS)$.

Proposition 5
If f, $g : x \to y$ are $N_{\alpha *}$- continuous and Y is ${N_\alpha}^{T2}$ space, then the set $A = \{x : x \in X f(x) = g(x)\}$ is $(N_\alpha - CS)$.

Proof If $x \notin A$, then $x \in A^c$ this mean that $f(x) \neq g(x)$ in Y, since Y is ${N_\alpha}^{T2}$-space, then there exist B_1, $B_2 \in N_\alpha O(Y)$ such that $f(x) \in B_1$, $g(x) \in B_2$ and $B_1 \cap B_2 = \varnothing$, but $f^{-1}(B_1)$, $g^{-1}(B_2) \in N_\alpha O(X)$ since f, g are $N_{\alpha *}$-continuous, hence $x \in f^{-1}(B_1)$, $x \in g^{-1}(B_2)$ hence $x \in f^{-1}(B_1) \cap g^{-1}(B_2)$, let $B = f^{-1}(B_1) \cap g^{-1}(B_2)$, where B is $(N_\alpha - OS)$. Now we shall prove $B \subseteq A^c$, i.e $B \cap A = \varnothing$. Suppose that $B \cap A \neq \varnothing$ this mean $y \in B \cap A$; thus, $y \in A$, $y \in B$. Hence, $y \in f^{-1}(B_1)$, $y \in g^{-1}(B_2)$, hence $f(y) \in B_1$, $g(y) \in B_2$, $y \in A$. Thus, $f(y) = g(y)$, since $y \in A$, hence $B_1 \cap B_2 \neq \varnothing$, which is a contradiction, thus $B \subseteq A^c$, thus $A^c \in N_\alpha O(X)$,, hence $A \in N_\alpha C(x)$.

Proposition 6
If X and Y are ${N_\alpha}^{Ti}$- space, then $X \times Y$ is ${N_\alpha}^{Ti}$- space $\forall i = 0, 1, 2$

Proof Assume that X and Y are ${N_\alpha}^{Ti}$- space. Put $i = 0$ and take $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$, then for any two distinct points x_1 and x_2 in X, there exists $A_1 \in N_\alpha O(X)$ such that $x_1 \in A_1$, $x_2 \notin A_1$ *or* $x_1 \notin A_1$, $x_2 \in A_1$, also $y_1 \neq y_{2,}$ then there exists $A_2 \in N_\alpha O(Y)$ such that $y_1 \in A_2$, $y_2 \notin A_2$ or $y_1 \notin A_2$, $y_2 \in A_2$ then $(x_1, y_1) \in A_1 \times A_2$ $(x_2, y_2) \notin A_1 \times A_2$ *or* $(x_1, y_1) \notin A_1 \times A_2 (x_2, y_2) \in A_1 \times A_2$ but $A_1 \times A_2$ is $(N_\alpha - OS)$ in $X \times Y$ (see Theorem 1). Hence $X \times Y\, {N_\alpha}^{Ti}$- space. Similarly, we can prove other states for $i = 1, 2$.

Proposition 7
If X is ${N_\alpha}^{Ti}$, then it is ${N_\alpha}^{Ti-1}$ –space, where $i = 2, 1$.

Proof The proof is consider from Definitions 5, 6, and 7.

Theorem 4
The inverse image of ${N_\alpha}^{Ti}$ -space under injective $N_{\alpha *}$- continuous mapping is also ${N_\alpha}^{Ti}$ space, where $i = 0, 1, 2$

We shall prove only when $i = 2$ and the other cases are similarly.

Proof Let $f : X \to Y$ be injective, $N_{\alpha *}$- continuous mapping and $x_1 \neq x_2$ in X, since f is injective then $y_1 = f(x_1) \neq f(x_2) = y_2$ in Y where Y is ${N_\alpha}^{T2}$ then there exist two disjoint N_α- open set A_1, A_2 in Y satisfy $y_1 \in A_1$, $y_2 \in A_2$, since f is $N_{\alpha *}$- continuous

Diagram 1 The relationship between $N_\alpha{}^{Ti}$-spaces and $gN_\alpha{}^{Ti}$ spaces

$N_\alpha T2$	→	$N_\alpha T1$	→	$N_\alpha T0$
↓				↓
$gN_\alpha T2$	→	$gN_\alpha T1$	→	$gN_\alpha T0$

than $f^{-1}(A_1), f^{-1}(A_2)$ are $(N_\alpha - OSs)$ in X such that $x_1 \in f^{-1}(A_1)$, $x_2 \in f^{-1}(A_2)$ and $f^{-1}(A_1) \cap f^{-1}(A_2) = \varnothing$. Therefore, X is $N_\alpha{}^{T2}$-space.

Theorem 5
If $f : X \to Y$ is injective N_α- continuous and Y is T_2 space, then X is $N_\alpha{}^{T2}$ – space.

Proof Similar to the proof of Theorem 4.

Definition 8
Let X be a (TS). Then X is called $gN_\alpha{}^{Ti}$-space, where $i = 0, 1, 2$ if:

- (i) $i = 0$ if for any $x \neq y$ in X, there exists $(gN_\alpha - OS)$ A containing one of them but not other.
- (ii) $i = 1$ if for any $x \neq y$ in X, there exist two $(gN_\alpha - OSs)$ A, B containing x and y, respectively, satisfy $y \notin A$, $x \notin B$.
- (iii) $i = 2$ if for each pair of distinct point x, y in X there exist disjoint $(gN_\alpha - OSs)$ A, B such that $x \in A$, $y \in B$.

Proposition 8
Every $N_\alpha{}^{Ti}$- space is $gN_\alpha{}^{Ti}$ space.

Proof The proof is in hand, from Theorem 2 where every N_α- open set is gN_α-open set.

By Propositions 7 and 8 we have the following Diagram 1

Theorem 6
If $f : X \to Y$ is injective gN_α- continuous and Y is T_2 – space than X is $gN_\alpha - T_2$ space.

Proof Assume that $x \neq y$ in X, since f is injective, thus $f(x) \neq f(y)$ in Y where Y is T_2 space, then there exists disjoint open sets A, B satisfy $f(x) \in A, f(y) \in B$ and $A \cap B = \varnothing$, since f is gN_α- continuous, then $f^{-1}(A), f^{-1}(B)$ *are* $(gN_\alpha - OSs)$ in X see (Definition 4(iii)) where $x \in f^{-1}(A)$, $y \in f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \varnothing$. Hence, X is $gN_\alpha - T_2$ space.

4 Conclusion and Future Work

We use N_α-open sets in topological spaces to generate new sorts of N_α-separation axioms and investigate some of their features in this research. Some theorems are also provided. In future work, we will discuss in nonclassical (TS) such as neutrosophic/fuzzy/soft topological spaces.

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