A Note on the Number of (Maximal) Antichains in the Lattice of Set Partitions

Dmitry I. Ignatov(✉)

School of Data Analysis and Artificial Intelligence, Laboratory for Models and Methods of Computational Pragmatics, National Research University Higher School of Economics, Moscow, Russia
dignatov@hse.ru
http://www.hse.ru

Abstract. Set partitions and partition lattices are well-known objects in combinatorics and play an important role as a search space in many applied problems including ensemble clustering. Searching for antichains in such lattices is similar to that of in Boolean lattices. Counting the number of antichains in Boolean lattices is known as the Dedekind problem. In spite of the known asymptotic for the latter problem, the behaviour of the number of antichains in partition lattices has been paid less attention. In this short paper, we show how to obtain a few first numbers of antichains and maximal antichains in the partition lattices with the help of concept lattices and provide the reader with some related heuristic bounds. The results of our computational experiments confirm the known values and are also recorded in the Online Encyclopaedia of Integer Sequences (see https://oeis.org/A358041).

Keywords: Formal Concept Analysis · partition lattice · maximal antichains · concept lattices · enumerative combinatorics

1 Introduction

Partitions and their lattices are among the basic combinatorial structures [1] and have various applications, for example, blocks of a partition of objects are known as clusters in data analysis [2,3], while in social network analysis the partition blocks of graph vertices (actors) are known as social communities [4,5]. As the Boolean lattice of an n-element set, the lattice of all partitions of this set plays a fundamental role as an ordered search space when we need to find a partition with certain properties, e.g. when we search for partitions with a concrete number of blocks with no two specific elements in one block (cf. constrained clustering [6] or granular computing [7]) or generate functional dependencies over a relational database (cf. partition pattern structures [8]). In Formal Concept Analysis, there are also interesting attempts to employ the idea of independence for data analysis via partitions of objects w.r.t. their attributes where a special variant of Galois connection appears [9–11].

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In combinatorics, special attention is paid to the number of antichains in Boolean lattices, that is to the number of all possible families of mutually incomparable sets. This problem is known as the Dedekind problem [12] and its asymptotic is well studied [13,14]. However, an analogous problem for antichains of partitions has been paid less attention. For example, we know a few values for the number of antichains in the partition lattice for n up to 5[1]. Another interesting question, for which there is the famous Sperner theorem, is about the size of the maximum antichain in the Boolean lattice (actually, its width) [15]. R.L. Graham overviewed the results on maximum antichains of the partition lattice [16]. Also, the number of maximal antichains (w.r.t. their extensibility) in the Boolean lattice [17], a sibling of the Dedekind problem, was algorithmically attacked and we know these numbers up to $n = 7$[2] [18]. The lattices of maximal antichains for event sets play an important role in parallel programming [19].

In this paper, we not only confirm the results on the number of antichains in the partition lattice, but also share our recent results on the number of maximal antichains in the partition lattice up to $n = 5$, show recent progress for $n = 6$, and provide some useful bounds for this number. All these results were obtained with the help of concept lattices isomorphic to the partition lattice and parallel versions of classic algorithms designed for that purpose.

2 Basic Definitions

Formal Concept Analysis is an applied branch of modern lattice theory aimed at data analysis, knowledge representation and processing with the help of (formal) concepts and their hierarchies. Here we reproduce basic definitions from [1,20] and our related tutorial [21].

First, we recall several notions related to lattices and partitions.

Definition 1. *A partition of a nonempty set A is a set of its nonempty subsets $\sigma = \{B \mid B \subseteq A\}$ such that $\bigcup\limits_{B\in\sigma} B = A$ and $B \cap C = \emptyset$ for all $B, C \in \sigma$. Every element of σ is called* block.

Definition 2. *A poset $\mathbf{L} = (L, \leq)$ is a **lattice**, if for any two elements a and b in L the supremum $a \vee b$ and the infimum $a \wedge b$ always exist. $\mathbf{L}$ is called a **complete lattice**, if the supremum $\bigvee X$ and the infimum $\bigwedge X$ exist for any subset A of L. For every complete lattice $\mathbf{L}$ there exists its largest element, $\bigvee L$, called the **unit element** of the lattice, denoted by $\mathbf{1}_L$. Dually, the smallest element $\mathbf{0}_L$ is called the **zero element**.*

Definition 3. *A partition lattice of set A is an ordered set $(Part(A), \vee, \wedge)$ where $Part(A)$ is a set of all possible partitions of A and for all partitions σ and ρ supremum and infimum are defined as follows:*

$$\sigma \vee \rho = \left\{ \bigcup conn_{\sigma,\rho}(B) \mid \forall B \in \sigma \right\},$$

[1] https://oeis.org/A302250.
[2] https://oeis.org/A326358.

$$\sigma \wedge \rho = \{B \cap C \mid \exists B \in \sigma, \exists C \in \rho : B \cap C \neq \emptyset\}, \textit{where}$$

$conn_{\sigma,\rho}(B)$ is the connected component to which B belongs to in the bipartite graph (σ, ρ, E) such that $(B, C) \in E$ iff $C \cap B \neq \emptyset$.

Definition 4. *Let A be a set and let $\rho, \sigma \in Part(A)$. The partition ρ is finer than the partition σ if every block B of σ is a union of blocks of ρ, that is $\rho \leq \sigma$.*

Equivalently one can use the traditional connection between supremum, infimum and partial order in the lattice: $\rho \leq \sigma$ iff $\rho \vee \sigma = \sigma$ ($\rho \wedge \sigma = \rho$).

Definition 5. *A* ***formal context*** *$\mathbb{K} = (G, M, I)$ consists of two sets G and M and a relation I between G and M. The elements of G are called the* ***objects*** *and the elements of M are called the* ***attributes*** *of the context. The notation gIm or $(g, m) \in I$ means that the object g has attribute m.*

Definition 6. *For $A \subseteq G$, let*

$$A' := \{m \in M \mid (g, m) \in I \textit{ for all } g \in A\}$$

and, for $B \subseteq M$, let

$$B' := \{g \in G \mid (g, m) \in I \textit{ for all } m \in B\}.$$

These operators are called ***derivation operators*** *or* ***concept-forming operators*** *for $\mathbb{K} = (G, M, I)$.*

Let (G, M, I) be a context, one can prove that operators

$$(\cdot)'' \colon 2^G \to 2^G, \ (\cdot)'' \colon 2^M \to 2^M$$

are closure operators (i.e. idempotent, extensive, and monotone).

Definition 7. *A* ***formal concept*** *of a formal context $\mathbb{K} = (G, M, I)$ is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. The sets A and B are called the extent and the intent of the formal concept (A, B), respectively. The* ***subconcept-superconcept relation*** *is given by $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$ ($B_2 \subseteq B_1$).*

This definition implies that every formal concept has two constituent parts, namely, its extent and intent.

Definition 8. *The set of all formal concepts of a context $\mathbb{K}$ together with the order relation $\leq$ forms a complete lattice, called the* ***concept lattice*** *of $\mathbb{K}$ and denoted by $\mathfrak{B}(\mathbb{K})$.*

Definition 9. *For every two formal concepts (A_1, B_1) and (A_2, B_2) of a certain formal context their* ***greatest common subconcept*** *is defined as follows:*

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)'').$$

The ***least common superconcept*** *of (A_1, B_1) and (A_2, B_2) is given as*

$$(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)'', B_1 \cap B_2).$$

We say supremum instead of “least common superconcept”, and instead of “greatest common subconcept” we use the term infimum.

In Fig. 1, one can see the context whose concept lattice is isomorphic to the partition lattice of a four-element set and the line (or Hasse) diagram of its concept lattice.

	1|234	12|34	2|134	123|4	23|14	13|24	3|124
(1, 2)		×		×			×
(1, 3)			×	×		×	
(1, 4)			×		×		×
(2, 3)	×			×	×		
(2, 4)	×					×	×
(3, 4)	×	×	×				

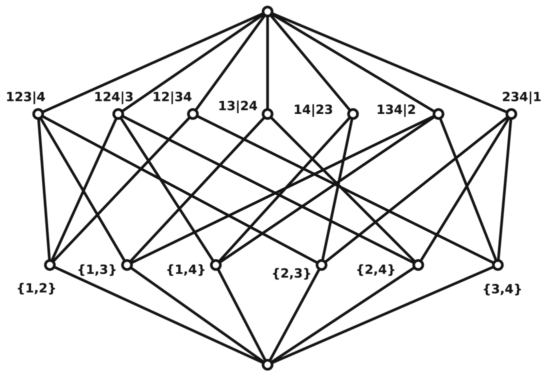

Fig. 1. The formal context (left) and the line diagram of the concept lattice (right) which is isomorphic to $\mathcal{P}_4$.

Theorem 1 *(Ganter & Wille [20]). For a given partially ordered set* $\mathfrak{P} = (P, \leq)$ *the concept lattice of the formal context* $\mathbb{K} = (J(P), M(P), \leq)$ *is isomorphic to the Dedekind–MacNeille completion of* $\mathfrak{P}$*, where* $J(P)$ *and* $M(P)$ *are sets of join-irreducible and meet-irreducible elements of* $\mathfrak{P}$*, respectively.*

A join-irreducible[3] lattice element cannot be represented as the supremum of strictly smaller elements; dually, for meet-irreducible elements. If $(P, \leq)$ is a lattice, then $\mathbb{K} = (J(P), M(P), \leq)$ is called its **standard context**.

Theorem 2 *(Bocharov et al. [2]). For a given partition lattice* $\mathfrak{L} = (Part(A), \vee, \wedge)$ *there exist a formal context* $\mathbb{K} = (P_2, A_2, I)$*, where* $P_2 = \{\{a, b\} \mid a, b \in A \text{ and } a \neq b\}$, $A_2 = \{\sigma \mid \sigma \in Part(A) \text{ and } |\sigma| = 2\}$ *and* $\{a, b\} I \sigma$ *when* a *and* b *belong to the same block of* σ*. The concept lattice* $\underline{\mathfrak{B}}(P_2, A_2, I)$ *is isomorphic to the initial lattice* $(Part(A), \vee, \wedge)$*.*

There is a natural bijection between elements of $\mathfrak{L} = (Part(A), \vee, \wedge)$ and formal concepts of $\underline{\mathfrak{B}}(P_2, A_2, I)$. Every $(A, B) \in \underline{\mathfrak{B}}(P_2, A_2, I)$ corresponds to $\sigma = \bigwedge B$ and every pair $\{i, j\}$ from A is in one of σ blocks, where $\sigma \in Part(A)$. Every $(A, B) \in \underline{\mathfrak{B}}(J(\mathfrak{L}), M(\mathfrak{L}), \leq)$ corresponds to $\sigma = \bigwedge B = \bigvee A$.

[3] join- and meet-irreducible elements are also called supremum- and infimum-irreducible elements, respectively.

3 Problem Statement

Let us denote the partition lattice of set $[n] = \{1, \ldots, n\}$ by $\mathcal{P}_n = (Part([n]), \leq)$, where $Part([n])$ is the set of all partitions of $[n]$.

Two related problems, which we are going to consider are as follows.

Problem 1 (#ACP). *Count the number of antichains of* $\mathcal{P}_n = (Part([n]), \leq)$ *for a given* $n \in \mathbb{N}$.

Problem 2 (#MaxACP). *Count the number of maximal antichains of* $\mathcal{P}_n = (Part([n]), \leq)$ *for a given* $n \in \mathbb{N}$.

4 Proposed Approach

Our approach to computing maximal antichains of the considered lattice is a direct consequence of the Dedekind-MacNeille completion and the basic theorem of FCA. The first one allows building the minimal extension of a partial order such that this extension forms a lattice. From the second theorem, we know that every complete lattice can be represented by a formal context built on the supremum- and infimum-irreducible elements of the lattice.

When Klaus Reuter was studying jump numbers of partial orders $(P, \leq)$, he found their connection with the number of maximal antichains and reported about it as follows [22]: "Originally we have discovered a connection of the concept lattice of $(P, P, \not\geq)$ to the jump number of P. Later on, we learned from Wille that this lattice is isomorphic to the lattice of maximal antichains of P. Thus with speaking about $MA(P)$ it is now quite hidden that we have gained most of our results by knowledge of Formal Concept Analysis." Here, $MA(P)$ denotes the set of maximal antichains of $(P, \leq)$.

An order ideal $\downarrow X$ of $X \subseteq P$ is a set $\{y \in P \mid \exists x \in X : y \leq x\}$, while $\uparrow X$ denotes the order filter generated by X (dually defined).

The lattice of maximal antichains of P, $(MA(P), \leq)$ is defined by $A_1 \leq A_2$ iff $\downarrow A_1 \subseteq \downarrow A_2$ for $A_1, A_2 \in MA(P)$.

It is known that two fundamental lattices related to orders, the distributive lattice of order ideals and the lattice of the Dedekind-MacNeille completion can be naturally described by FCA means [22]: $\underline{\mathfrak{B}}(P, P, \leq)$ represents the Dedekind-MacNeille completion (completion by cuts) of $(P, \leq)$, while $\underline{\mathfrak{B}}(P, P, \not\geq)$ represents the lattice of order ideals of $(P, \leq)$ (which is isomorphic to the lattice of all antichains of $(P, \leq)$).

The observation made by Wille makes it possible to fit the lattice of maximal antichains in this framework: $\underline{\mathfrak{B}}(P, P, \not>)$ represents the lattice of maximal antichains of $(P, \leq)$.

Proposition 1 ([22], Proposition 2.1). $(MA(P), \leq)$ *is isomorphic to* $\underline{\mathfrak{B}}(P, P, \not>)$.

Corollary 1. #MaxACP *(Problem 2) is equivalent to determining the number of formal concepts of* $\underline{\mathfrak{B}}(Part([n]), Part([n]), \not>)$.

So, our approach has two steps:

– 1. Generate the formal context $\mathbb{K} = (Part([n]), Part([n]), \ngtr)$ for a given n.
– 2. Count the cardinality of $\mathbf{L}_n = \underline{\mathfrak{B}}(Part([n]), Part([n]), \ngtr)$.

$\mathbb{K}(\mathcal{P}_3)$	$1\mid 23$	$12\mid 3$	$2\mid 13$
(1, 2)		×	
(1, 3)			×
(2, 3)	×		

$\ngeq$	0	4	2	1	7
0		×	×	×	×
4			×	×	×
2		×		×	×
1		×	×		×
7					

$\ngtr$	0	4	2	1	7
0	×	×	×	×	×
4		×	×	×	×
2		×	×	×	×
1		×	×	×	×
7					×

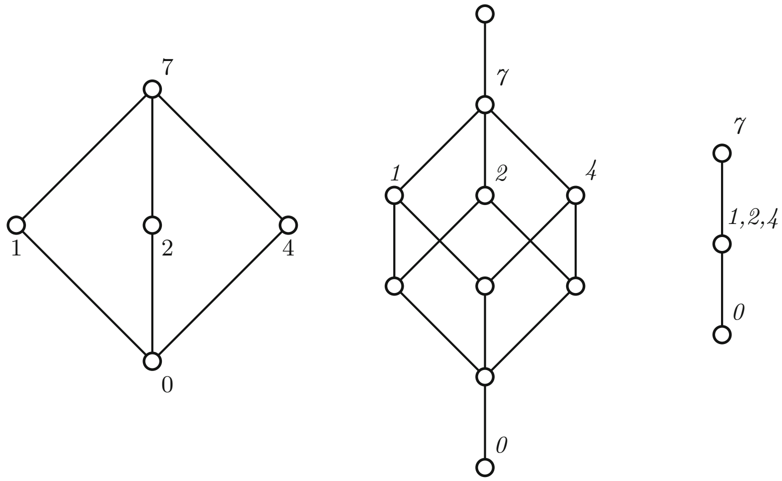

Fig. 2. The formal contexts $\mathbb{K}(\mathcal{P}_3) = (J(\mathcal{P}_3), M(\mathcal{P}_3), \leq)$ (left), $(Part([3]), Part([3]), \ngeq)$ (centre), and $(Part([3]), Part([3]), \ngtr)$ (right) along with the line diagrams of their concept lattices [20] (bottom line), respectively.

The line diagram of $\underline{\mathfrak{B}}(Part([3]), Part([3]), \ngeq)$, which is isomorphic to the lattice of maximal antichains $(MA(\mathcal{P}_3), \leq)$, and its formal context are given in Fig. 2, the right column. The context for the lattice isomorphic to the lattice of ideals of $\mathcal{P}_3$ is in the centre, while the original context for the lattice isomorphic to $\mathcal{P}_3$ is shown on the right. The nodes of $\underline{\mathfrak{B}}(J(\mathcal{P}_3), M(\mathcal{P}_3), \leq)$ are labelled with integers, whose binary codes correspond to concept extents. For example, label 4 encodes the extent of concept $((2,3), 1|23)$ since $4_{10} = 100_2$. The orders $\ngeq$ and $\ngtr$ are taken with respect to hierarchical order on concepts of $\underline{\mathfrak{B}}(J(\mathcal{P}_3), M(\mathcal{P}_3), \leq$). The labels of the two remaining lattices are given with reduced attribute labelling.

Note that some rows and columns of the third context can be removed without affecting the lattice structure. For example, duplicated rows 2 and 4.

Columns and rows obtained as an intersection of other columns and rows, respectively, can also be removed without affecting the concept lattice structure. This procedure is called reducing the context [20]. Thus, for moderately large n we use the so-called standard contexts of concept lattices, $\mathbb{K}(\mathbf{L}) = (J(\mathbf{L}), M(\mathbf{L}), \leq)$, where $\mathbf{L} = (L, \leq)$ is a finite lattice, and $J(\mathbf{L})$ and $M(\mathbf{L})$ are join- and meet-irreducible elements of $\mathbf{L}$ [20].

The first step is trivial, while for the second step, we have plenty of algorithms both in FCA [23] and Frequent Closed Itemset mining [24] communities. However, having in mind the combinatorial nature of the problem, and the almost doubly-exponential growth of the sequence, we cannot use a fast algorithm which relies on recursion or (execution tree will be humongous) sophisticated structures like FP-trees due to memory constraints. We rather need a parallelisable solution which does not require the memory size of $O(|L|)$ and can be easily resumed, for example, after the break of computation for monthly routine maintenance. So, we set our eye on Ganter's Next Closure algorithm [25,26], which does not refer to the list of generated concepts and uses little storage space.

Since the extent of a concept uniquely defines its intent, to obtain the set of all formal concepts, it is enough to find closures either of subsets of objects or subsets of attributes.

We assume that there is a linear order ($<$) on G. The algorithm starts by examining the set consisting of the object maximal with respect to $<$ ($max(G)$) and finishes when the canonically generated closure is equal to G. Let A be a currently examined subset of G. The generation of A'' is considered canonical if $A'' \setminus A$ does not contain $g < max(A)$. If the generation of A'' is canonical (and A'' is not equal to G), the next set to be examined is obtained from A'' as follows:

$$A'' \cup \{g\} \setminus \{h | h \in A'' \text{ and } g < h\}, \text{ where } g = max(\{h | h \in G \setminus A''\}).$$

Otherwise, the set examined at the next step is obtained from A in a similar way, but the added object must be less (w.r.t. $<$) than the maximal object in A:

$$A'' \cup \{g\} \setminus \{h | h \in A \text{ and } g < h\}, \text{ where } g = max(\{h | h \in G \setminus A \text{ and } h < max(A)\}).$$

The pseudocode of NextClosure is given in Algorithm 1.

The NextClosure algorithm is enumerative and produces the set of all concepts in time $O(|G|^2|M||L|)$ and also has polynomial delay $O(|G|^2|M|)$. For our counting purposes, Step 5 of the algorithm should be replaced with $|L| := |L| + 1$, while Step 12 should return $|L|$.

Our modification of the algorithm features parallel computing, saving of intermediate results as pairs (A'', $|L|$), and representation of sets as binary vectors with integers as well as usage of bit operations on them.

Algorithm 1. NextClosure

Input: $\mathbb{K} = (G, M, I)$ is a context
Output: L is the concept set

```
L := ∅, A := ∅, g := max(G)
while A ≠ G do
    A := A'' ∪ {g} \ {h|h ∈ A and g < h}
    if {h|h ∈ A and g ≤ h} = ∅ then
        L := L ∪ {(A'', A')}
        g := max({h|h ∈ G \ A''})
        A := A''
    else
        g := max({h|h ∈ G \ A and h < g})
    end if
end while
return L
```

5 Results and Recent Progress

The results for #ACP problem were published in OEIS by John Machacek on Apr 04 2018. We have validated them with the used approach. While our results on #MaxACP were obtained by Oct 29 2022. They are summarised for n up to 5 in Table 1.

Table 1. The confirmed (the first row) and the obtained (the last row) results

n	1	2	3	4	5
#ACP, OEIS A302250	2	3	10	347	79814832
#MaxACP, OEIS A358041	1	2	3	32	14094

All the contexts and codes are available on GitHub: https://github.com/dimachine/SetPartAnti. We used IPython for its ease of implementation and speeded it up with Cython and multiprocess(ing) libraries. To compute all the known values for #MaxACP it took about 357 ms, while similar experiments for #ACP took 26 min 44 s on a laptop with 2.9 GHz 6-core processor, Intel Core i9.

To compute #MaxACP for $n = 6$, we used Intel Core i9-12900KS with 24 threads (at maximum capacity) and 3.4 GHz of base processor frequency. Sixty branches of computation have been completed with 250201481250 maximal antichains, while twelve branches are still in progress (see Fig. 3) with the preliminary sum 1320200000000 obtained during more than one month of computations.

As for the lower and upper bounds and asymptotic analysis on the number of (maximal) antichains of set partitions, it is more complex than that of set subsets.

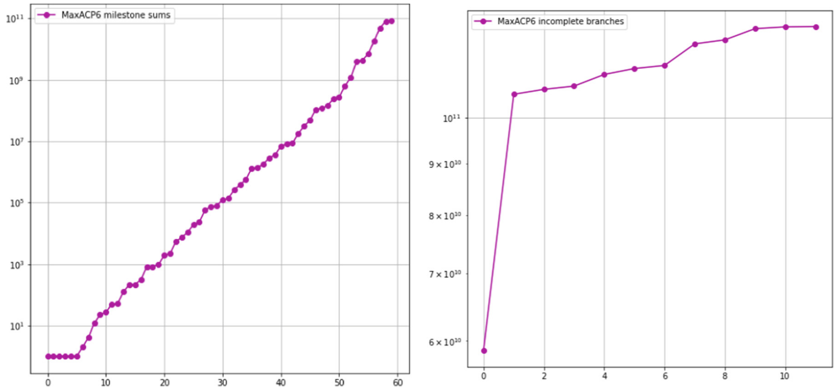

Fig. 3. Completed (left) and incomplete branches (right) for our current $ma(\mathcal{P}_6)$ computation

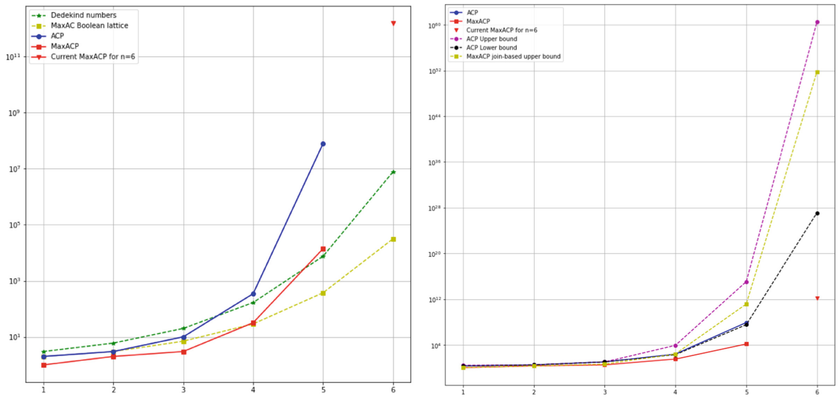

Fig. 4. Comparison with D_n and $ma(\mathcal{B}_n)$ (left) and with the lower and upper bounds of $acp(\mathcal{P}_n)$ (right) for our current $macp(\mathcal{P}_6)$ computation

The size of the level sets of the partition lattice is given by Stirling numbers of the second kind, while the sizes of the level sets of the Boolean lattice are given by binomial coefficients. The lower bound and the asymptotic for the Boolean lattice are based on the size of its largest level set(s) (maximal antichain), so we could use similar logic for the partition lattice. However, the maximal value of the Stirling number of the second kind, $\max\limits_{k \leq n} \left\{ {n \atop k} \right\}$, is not always equal to the size of the maximum antichain in $\mathcal{P}_n$ and the connection between these numbers is non-linear with unknown constants [27][4].

Thus, from [27], we know that

[4] The question on the equality was posed by G.C. Rota [16].

$$d(\mathcal{P}_n) = \max_{k \le n} \left\{ {n \atop k} \right\} \Theta(n^a (\ln n)^{-a-1/4}), \text{ where}$$

$d(\mathcal{P}_n)$ is is the size of the maximal antichain in $\mathcal{P}_n$ and $a = \frac{2 - e \ln 2}{4} \approx 0.02895765$.

Luckily, according to [16], it was first shown that a maximal antichain has at most $\max\limits_{k \le n} \left\{ {n \atop k} \right\}$ elements for $n \le 20$, while later it was obtained that the discrepancy arises when $n \ge 3.4 \cdot 10^6$ [28,29]. Thus, a simple lower bound for #ACP problem is given by $2^{\max\limits_{k \le n} \left\{ {n \atop k} \right\}} \le 2^{d(\mathcal{P}_n)}$, and can be further improved by considering not only the partition lattice level for $\max\limits_{k \le n} \left\{ {n \atop k} \right\}$.

Proposition 2. *$acp(\mathcal{P}_n) \ge \sum\limits_{k=1}^{n} 2^{\left\{ {n \atop k} \right\}} - n + 1$ for $n \ge 1$.*

Proof. Each partition lattice level contains partitions in k blocks for a given $1 \le k \le n$. These partitions form a maximal antichain and each of its subsets forms an antichain. The number of unique antichains by each level is given by $2^{\left\{ {n \atop k} \right\}} - 1$ since the empty set should be counted only once.

For the upper bounds, we can use knowledge of FCA, where the largest number of concepts of a context with n objects and m attributes is given by $2^{\min(n,m)}$. Since the Bell numbers B_n count the size of the Partition lattice on n elements, the number of objects (and attributes) in $\mathbb{K} = (Part([n]), Part([n]), \not>)$ and $\mathbb{K} = (Part([n]), Part([n]), \not\geq)$ is given by B_n. Thus, the trivial upper bound is given by 2^{B_n} but it is equivalent to the powerset of all partitions. We can notice the $\mathbf{0}$ and $\mathbf{1}$ of the set partition lattice are represented by empty column and empty row in the context inducing the order ideals lattice, while for the context inducing the lattice of maximal antichains, they are represented by full row and column, respectively. This implies slightly better upper bounds $2^{B_n - 1} + 2$ with $n > 1$ (although, it is still valid for $n = 1$ giving $3 > acp(1) = 2^{B_1} = 2$) for #ACP and $2^{B_n - 1}$ for #MaxACP.

Remark 1. Since we deal with lattices, which are partial orders (reflexive, antisymmetric, and transitive), their incidence relations can be represented with formal contexts with identical sets of objects and attributes where each object-attribute pair on the main diagonal belongs to the incidence relation (the main diagonal is full) while all the pairs below the diagonal do not.

Proposition 3. *Let $\mathbf{L} = (L, \le)$ be a finite lattice, then $|\underline{\mathfrak{B}}(L, L, \not\geq)| \le 2^{|L|-2} + 2$.*

Proof. 1) Let $|L| = 1$, then $|\underline{\mathfrak{B}}(L, L, \not\geq)| = 2$ which is less than $2\frac{1}{2}$. 2) Let $|L| = 2$, then $|\underline{\mathfrak{B}}(L, L, \not\geq)| = 3$ which is equal to the right-hand side of the inequality.
3) For $|L| \ge 3$, let us consider the subcontext $(L \setminus \mathbf{1}, L \setminus \mathbf{0}, \not\geq)$. Recalling the structure of the incidence table for a partial order with all empty pairs below the main diagonal, we obtain that one of the context objects, $\mathbf{0}$, and one of its attributes, $\mathbf{1}$, are represented by a full row and a full column, respectively, while

the main diagonal is full and the pairs above the main diagonal belong to $\ngeq$. It is so, since for every pair (a, b) above the main diagonal of the original context $|(L, L, \leq)|$ only one of the cases fulfils 1) $a < b$ or 2) $a \nless b$ (which implies $a \ngeq b$, i.e. a and b incomparable). Either case implies $a \ngeq b$.

At the same time, the first subdiagonal is empty since $\ngeq$ is antireflexive. It implies that the number of concepts $|\mathfrak{B}(L \setminus \mathbf{1}, L \setminus \mathbf{0}, \ngeq)| \leq 2^{\min(|L\setminus\mathbf{1}|-1, |L\setminus\mathbf{0}|-1)} = 2^{|L|-2}$. Going back to the original context, we obtain two more concepts for the deleted object $\mathbf{1}$, $(\mathbf{1}'', \mathbf{1}') = (L, \emptyset)$ and for the deleted attribute $\mathbf{0}$, $(\mathbf{0}', \mathbf{0}'') = (\emptyset, L)$, respectively.

Unfortunately, even these slightly better upper bounds are overly high, but at least we can do better by providing an upper bound for $macp(n)$, which can be also estimated via the sizes of the standard context for $MA(\mathcal{P}_n)$. Thus, for $MA(\mathcal{P}_n)$ the upper bound is as follows:

$$2^{\min(|J(MA(\mathcal{P}_6))|, |M(MA(\mathcal{P}_6))|)} = 2^{\min(172,188)} \approx 5.986 \cdot 10^{51} \ .$$

Table 2. The sizes of standard context for $MA(\mathcal{P}_n)$ compared to Bell numbers for n up 7

n	1	2	3	4	5	6	7
Bell numbers	1	2	5	15	52	203	877
$J(MA(\mathcal{P}_n))$	0	1	2	8	37	172	814
$M(MA(\mathcal{P}_n))$	0	1	2	9	42	188	856

The size of the standard context for the lattice of antichains on partitions for a fixed n is given by Bell numbers both for join- and meet-irreducible elements (see Table 2).

Since we know $macp(\mathcal{P}_n) \leq acp(\mathcal{P}_n)$, we can try to further sharpen this inequality by discarding some of those antichains that are not maximal.

Proposition 4. *$macp(\mathcal{P}_n) \leq acp(\mathcal{P}_n) - \sum\limits_{k=1}^{n} 2^{\left\{ n \atop k \right\}} + 2n - 1$ for $n \geq 1$.*

Proof. We subtract from $acp(\mathcal{P}_n)$ the number of all non-maximal antichains obtained by each level of the partition lattice, which gives us a decrement $2^{\left\{ n \atop k \right\}} - 2$ for each k (the empty set is counted only once).

Let us use $\Delta(n)$ for $acp(n) - macp(n)$ and $D_l(n)$ for the decrement by levels $\sum\limits_{k=1}^{n} 2^{\left\{ n \atop k \right\}} - 2n + 1$. In Table 3, it is shown that for the first three values $\Delta(n)$ and $D_l(n)$ coincide, but later the antichains different from the level antichain's subsets appear.

Proposition 4 gives us a tool to establish an improved upper bound for $macp(\mathcal{P}_n)$.

Table 3. The signed relative error $\frac{\Delta(n)-D_l(n)}{\Delta(n)}$

n	1	2	3	4	5
$\Delta(n)$	1	1	7	315	79800738
$D_l(n)$	1	1	7	189	33588219
Relative error	0	0	0	0.4	≈ 0.5791

Proposition 5. $macp(\mathcal{P}_n) \leq 2^{B_n-2} - \sum\limits_{k=1}^{n} 2^{\left\{ n \atop k \right\}} + 2n + 1$ *for* $n \geq 1$.

Proof. We directly plug in $2^{B_n-2}+2$ in the previous inequality. Note that for $n=1$, $macp(\mathcal{P}_n) = 1 < 2^{1-2} - 2^{\left\{ 1 \atop 1 \right\}} + 2 + 1 = 1\frac{1}{2}$.

Since $\binom{n}{k} \leq \left\{ n \atop k \right\}$, we could expect that the Dedekind numbers D_n and the number of maximal antichains of the Boolean lattice, $ma(\mathcal{B}_n)$, are good candidates for heuristic lower bounds. As we can see from Fig. 4, they become lower than their counterparts for the set partition lattice already at $n=4$.

It is known that $B_n < \left(\frac{0.792n}{\ln(n+1)}\right)^n$ for all positive integers n [30]. So, $\log B_n$ is bounded by a superlinear function in n[5]. Thus, we can try a linear approximation for the logarithms of the number of maximal antichains, $macp(\mathcal{P}_6)$, and that of antichains, $acp(\mathcal{P}_6)$, respectively, by a tangential line passing through the line segments $[\log macp(\mathcal{P}_4), \log macp(\mathcal{P}_5)]$ and $[\log acp(\mathcal{P}_4), \log acp(\mathcal{P}_5)]$, respectively. Let us consider the natural logarithm, ln. Thus, these heuristic lower bounds are as follows:

$$e^{\ln^2 acp(5)/\ln acp(4)} \approx 1.25 \cdot 10^{26} \text{ and } e^{\ln^2 macp(5)/\ln macp(4)} \approx 273562462667.8.$$

The latter heuristic lower bound is already about 5.74 times smaller than the currently precomputed estimate of $macp(6)$, i.e. 1570401481250.

6 Conclusion

We hope that this paper will stimulate the interest of the conceptual structures community in computational combinatorics, both from algorithmic and theoretic points of view. Recent progress in computing such numbers as the Dedekind number for $n=9$ due to high-performance computing and FCA-based algorithms can be relevant here [31].

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[5] We use log when the logarithm base is not specified.

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