Aggregation Functions and Extent Structure Preservation in Formal Concept Analysis

Carlos Bejines, Domingo López-Rodríguez,
and Manuel Ojeda-Hernández(✉)

Universidad de Málaga, Andalucía Tech, Málaga, Spain
`manuojeda@uma.es`

Abstract. Formal Concept Analysis (FCA) is a mathematical framework for analysing data tables that capture the relationship between objects and attributes. The concept lattice derived from such a table is a representation of the implicit knowledge about this relationship, where each concept corresponds to a bicluster of objects and attributes. FCA has been widely used for knowledge acquisition and representation, conceptual data analysis, information retrieval and other applications. In this paper, we use an extension of the classical FCA to deal with fuzzy formal contexts, where the relationship between objects and attributes is modelled by truth values indicating the degree to which an object possesses a property or attribute. Fuzzy Formal Concept Analysis (FFCA) allows us to capture vague or imprecise information and handle uncertainty or ambiguity in data analysis. Our purpose is to use aggregation functions in order to manipulate and explore fuzzy formal concepts in different ways depending on the desired properties or criteria. In this work, we will focus on the structure of the extents of the concept lattice. We define the aggregation of fuzzy extents point-wise and study how it affects its structure. We characterise the aggregation functions that preserve the fuzzy extent structure and show that they depend on the number of objects in the context. Our results contribute to a better understanding of how aggregation functions can be used to manipulate and explore fuzzy formal concepts.

Keywords: Aggregation Function · Formal Concept Analysis · Fuzzy Sets

1 Introduction

Aggregation functions have become a significant area of research in Fuzzy Set Theory and its applications. The need to combine information, typically expressed as numerical values, into a single output for decision-making has generated interest in studying functions that enable such aggregation. Aggregation functions are now widely discussed in various conferences, and a biennial

congress, AGOP, is devoted to them. For further information on this topic, refer to [3,5].

Recently, there has been a considerable focus on developing a framework that concentrates on preserving the properties of fuzzy algebraic structures under aggregation functions. This framework is actively being developed and discussed, and more details can be found in [2,7,10,11].

There are some approaches to FCA that use aggregation functions but they are used in the classical setting, that is, they consider different measures on the concept lattice and aggregate these measures to a single number by using the operator [9,12]. In our approach, we focus on the concept lattice. In particular, the infimum and the supremum of the concept lattice are defined in terms of suprema and the infima of the powerset lattice [4,6]. The key point is that the supremum and the infimum are aggregation operators, and the question is whether changing these operations by another pair of aggregation operators defines a new concept lattice. Directly from the classical theory of FCA we know that we cannot interchange suprema with infima, therefore this claim will not be satisfied by some aggregation operators.

Another implication of this study would be in the study of algorithms for the computation of the concept lattice. Some of the most well-known algorithms, such as FastCbO [8] or InClose [1], use the intersection of extents to recursively find all the extents corresponding to a formal context. Since the fuzzy intersection operation is, as we will show later, a particular case of aggregation of fuzzy sets, we can expect that other aggregation functions (that preserve the extent structure) may help in accelerating the process by incorporating them into these algorithms.

The main section of this paper shows some of the properties that an aggregation operator must satisfy to be an internal operation in the set of extents, intents or formal concepts. Surprisingly, there are not many aggregation operators that preserve extents or intents. Even though this is only the first step in this line, experimentation hints that only the infimum and the projections on the components preserve these sets. In these preliminary steps, we will consider aggregation functions on the unit interval [0,1] and the Gödel t-norm, which is exactly the infimum.

The remainder of this work is structured as follows: in Sect. 2, we present the preliminary ideas about aggregation functions and fuzzy FCA that will help in the understanding of the results, that will be detailed in Sect. 3. Finally, in Sect. 4, the final conclusions and future research lines are commented.

2   Preliminaries

In this section, we outline some of the concepts which will be necessary to follow the paper. This work is set in the fuzzy framework so some concepts on fuzzy structures and methods are presented.

A complete residuated lattice $\mathbb{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a structure such that $(L, \wedge, \vee, 0, 1)$ is a complete lattice where 0 is the bottom element and 1 is

the top one, $(L, \otimes, 1)$ is a commutative monoid, that is, $\otimes$ is an associative and commutative binary operation and 1 is the identity element; and $(\otimes, \rightarrow)$ is an adjoint pair, that is

$$x \otimes y \leq z \text{ if and only if } y \leq x \rightarrow z.$$

In this particular paper, the role of $\mathbb{L}$ will be played by the unit interval $[0, 1]$.

One of the main concepts used in this work is that of aggregation function. Here is a short description of what they are and their most important types.

Definition 1. *([5]) Let $A : [0, 1]^n \longrightarrow [0, 1]$ be a function. We say that A is an aggregation function if:*

(A1) $A(0, ..., 0) = 0$ *and* $A(1, ..., 1) = 1$. *   (Boundary conditions)*
(A2) $A(x_1, ..., x_n) \leq A(y_1, ..., y_n)$ *whenever $x_i \leq y_i$ for each $1 \leq i \leq n$. (Monotonicity)*

Definition 2. *Given an arbitrary set S, an aggregation function $A : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ and two fuzzy subsets $X, Y : S \longrightarrow [0, 1]$, the fuzzy set $A(X, Y) : S \longrightarrow [0, 1]$ defined point-wise by*

$$A(X, Y)(t) := A(X(t), Y(t))$$

is the aggregation of X and Y using A.

We give now some brief preliminaries on Fuzzy FCA. A fuzzy formal context is a tuple $\mathbb{K} = (G, M, I)$ where G and M are the sets of objects and attributes, respectively and $I : G \times M \rightarrow [0, 1]$ is a fuzzy relation. The degree $I(g, m)$ is understood as the degree to which the object g has the attribute m. The concept-forming operators $\uparrow$ and $\downarrow$ are defined as follows, for a pair of fuzzy sets $X \in [0, 1]^G, Y \in [0, 1]^M$,

$$X^\uparrow(m) = \bigwedge_{g \in G} (X(g) \rightarrow I(g, m)) \qquad Y^\downarrow(g) = \bigwedge_{m \in M} (Y(m) \rightarrow I(g, m))$$

As in the classical case, a fuzzy formal concept is a pair $\langle X, Y \rangle \in [0, 1]^G \times [0, 1]^M$ such that $X^\uparrow = Y$ and $Y^\downarrow = X$. The set of fuzzy formal concepts, denoted by $\mathbb{B}(\mathbb{K})$ is a complete lattice with the following infima and suprema, let $\{\langle X_i, Y_i \rangle\}_{i \in I} \subseteq \mathbb{B}(\mathbb{K})$, then

$$\bigwedge \{\langle X_i, Y_i \rangle\}_{i \in I} = \left\langle \bigcap_{i \in I} X_i, \left(\bigcup_{i \in I} Y_i \right)^{\downarrow\uparrow} \right\rangle,$$

$$\bigvee \{\langle X_i, Y_i \rangle\}_{i \in I} = \left\langle \left(\bigcup_{i \in I} X_i \right)^{\uparrow\downarrow}, \bigcap_{i \in I} Y_i \right\rangle.$$

This hints at the preliminary idea of this work, the infimum operator is an aggregation function and the supremum operator is known as its dual aggregation function. Thus we wonder, assuming A is an aggregation function with dual A^*: can we ensure $\langle A(X_i), A^*(Y_i)^{\downarrow\uparrow} \rangle$ is a formal concept?

Table 1. A formal context (left) and an aggregation function (right) described as a table on the chain $\{0, 0.25, 0.5, 0.75, 1\}$.

$\mathbb{K}$	m_1	m_2	m_3
g_1	0.25	0.5	0
g_2	0.75	0.25	0
g_3	0	1	0.75

A	0	0.25	0.5	0.75	1
0	0	0	0	0	0.5
0.25	0	0	0.25	0.5	1
0.5	0	0.5	1	1	1
0.75	0	0.75	1	1	1
1	0.5	1	1	1	1

Formal context   Aggregation function

Example 1. Let L be the unit interval $[0, 1]$ and let us consider the formal context $\mathbb{K}$ and an aggregation function A whose restriction to $\{0, 0.25, 0.5, 0.75, 1\}$ are shown in Table 1. The formal concepts of the context above are the following.

$$\langle \{g_1, g_2, g_3\}, \qquad \{{}^{0.25}/m_2\} \rangle \tag{1}$$

$$\langle \{g_1, {}^{0.25}/g_2, g_3\}, \qquad \{{}^{0.5}/m_2\} \rangle \tag{2}$$

$$\langle \{{}^{0.5}/g_1, {}^{0.25}/g_2, g_3\}, \qquad \{m_2\})\rangle \tag{3}$$

$$\langle \{g_3\}, \qquad \{m_2, {}^{0.75}/m_3\} \rangle \tag{4}$$

$$\langle \{{}^{0.75}/g_3\}, \qquad \{m_2, m_3\} \rangle \tag{5}$$

$$\langle \{g_1, g_2\}, \qquad \{{}^{0.25}/m_1, {}^{0.25}/m_2\} \rangle \tag{6}$$

$$\langle \{g_1, {}^{0.25}/g_2\}, \qquad \{{}^{0.25}/m_1, {}^{0.5}/m_2\} \rangle \tag{7}$$

$$\langle \{{}^{0.5}/g_1, {}^{0.25}/g_2\}, \qquad \{{}^{0.25}/m_1, m_2\} \rangle \tag{8}$$

$$\langle \{{}^{0.25}/g_1, g_2\}, \qquad \{{}^{0.75}/m_1, {}^{0.25}/m_2\} \rangle \tag{9}$$

$$\langle \{{}^{0.25}/g_1, {}^{0.75}/g_2\}, \qquad \{m_1, {}^{0.25}/m_2\} \rangle \tag{10}$$

$$\langle \{{}^{0.25}/g_1, {}^{0.25}/g_2\}, \qquad \{m_1, m_2\} \rangle \tag{11}$$

$$\langle \varnothing, \qquad \{m_1, m_2, m_3\} \rangle \tag{12}$$

Consider for example concepts (7) and (8) above and the aggregation function A.

$$C_7 = \langle \{g_1, {}^{0.25}/g_2\}, \{{}^{0.25}/m_1, {}^{0.5}/m_2\} \rangle,$$
$$C_8 = \langle \{{}^{0.5}/g_1, {}^{0.25}/g_2\}, \{{}^{0.25}/m_1, m_2\} \rangle.$$

We show the aggregation of the two previous extents using A, according to Definition 2:

$$A\left(\{g_1, {}^{0.25}/g_2\}, \{{}^{0.5}/g_1, {}^{0.25}/g_2\}\right) = \left\{{}^{A(1,0.5)}/g_1, {}^{A(0.25,0.25)}/g_2\right\} =$$
$$= \{{}^{1}/g_1, {}^{0}/g_2\} = \{g_1\}.$$

An analogous procedure can be performed to aggregate the two intents. We should remember that we can only aggregate fuzzy sets over the same universe. Therefore, in general, we cannot aggregate extents and intents together.

As an extreme case, we can consider the top $\top$ and the bottom $\bot$ of the concept lattice:

$$\top = \left\langle \{g_1, g_2, g_3\}, \{^{0.25}/m_2\} \right\rangle,$$
$$\bot = \left\langle \{\varnothing\}, \{m_1, m_2, m_3\} \right\rangle,$$

and examine the aggregation of these concepts (defining the aggregation by parts –extents and intents–):

$$
\begin{aligned}
A(\top, \bot) &= A\left(\left\langle \{g_1, g_2, g_3\}, \{^{0.25}/m_2\} \right\rangle, \left\langle \{\varnothing\}, \{m_1, m_2, m_3\} \right\rangle \right) = \\
&= \left\langle A\left(\{g_1, g_2, g_3\}, \varnothing \right), A\left(\{^{0.25}/m_2\}, \{m_1, m_2, m_3\} \right) \right\rangle = \\
&= \left\langle \left\{ ^{A(1,0)}/g_1, {}^{A(1,0)}/g_2, {}^{A(1,0)}/g_3 \right\}, \left\{ ^{A(0,1)}/m_1, {}^{A(0.25,1)}/m_2, {}^{A(0,1)}/m_3 \right\} \right\rangle = \\
&= \left\langle \left\{ ^{0.5}/g_1, {}^{0.5}/g_2, {}^{0.5}/g_3 \right\}, \left\{ ^{0.25}/m_1, {}^{1}/m_2, {}^{0.25}/m_3 \right\} \right\rangle = \\
&= \left\langle \left\{ ^{0.5}/g_1, {}^{0.5}/g_2, {}^{0.5}/g_3 \right\}, \left\{ ^{0.25}/m_1, m_2, {}^{0.25}/m_3 \right\} \right\rangle.
\end{aligned}
$$

Therefore, $A(\top, \bot)$ is not a fuzzy formal concept.

The last example shows that $\langle A(X_i), A^*(Y_i)^{\downarrow\uparrow} \rangle$ is not a fuzzy formal concept in general. Nowadays, knowing the properties on aggregation functions which preserve fuzzy formal concepts is an open problem.

3 Aggregation in FCA: First Results

In this section, we wonder what conditions endow to an aggregation function in order to be closed on the set of fuzzy formal concepts. For a general aggregation function, it may be easy to find a formal context such that the aggregation of two given extents is not an extent.

Example 2. We continue our discussion using the same context and aggregation function as in Example 1. We got that the aggregation of the extents of C_1 and C_2 was the set $\{g_1\}$. We can check that it is not an extent, since

$$\{g_1\}^{\uparrow\downarrow} = \{^{0.25}/m_1, {}^{0.5}/m_2\}^{\downarrow} = \{g_1, {}^{0.25}/g_2\},$$

and therefore it is not a closed set of objects. The same reasoning (the computations are straightforward) can be done to deduce that the aggregation of the intents of C_1 and C_2 is not an intent of the formal context.

One can also easily check that the aggregation of the two concepts $\top$ and $\bot$ is not a concept, since:

$$\{^{0.5}/g_1, {}^{0.5}/g_2, {}^{0.5}/g_3\}^{\uparrow\downarrow} = \{g_1, g_2, g_3\},$$
$$\{^{0.25}/m_1, m_2, {}^{0.25}/m_3\}^{\downarrow\uparrow} = \{m_1, m_2, m_3\}.$$

Thus, A does not preserve the algebraic structures of extent or intent. Consequently, neither of concept.

Our aim in this work is to present a preliminary study of the conditions that allow us to determine if an aggregation function will preserve those structures.

Definition 3. *Let* $\mathbb{K} = (G, M, I)$ *be a formal context and denote by* $\mathrm{Ext}(\mathbb{K})$, $\mathrm{Int}(\mathbb{K})$ *and* $\underline{\mathbb{B}}(\mathbb{K})$, *the sets of extents, intents and formal concepts of* $\mathbb{K}$, *respectively. An aggregation function* A *is said to be:*

– *extent-consistent* with $\mathbb{K}$ if $A(X_1, X_2) \in \mathrm{Ext}(\mathbb{K})$ for all $X_1, X_2 \in \mathrm{Ext}(\mathbb{K})$.
– *intent-consistent* with $\mathbb{K}$ if $A(Y_1, Y_2) \in \mathrm{Int}(\mathbb{K})$ for all $Y_1, Y_2 \in \mathrm{Int}(\mathbb{K})$.
– *consistent* with $\mathbb{K}$ if $A(C_1, C_2) \in \underline{\mathbb{B}}(\mathbb{K})$ for all concepts C_1 and C_2 of $\mathbb{K}$.

Example 3. Continuing with the same situation as in Example 1, we can see that the aggregation function A is neither extent-consistent, intent-consistent, nor consistent at all.

However, there are aggregation functions that will always preserve the algebraic structure, e.g., the minimum operator or the projections. Let us consider the aggregation functions A_m, π_1 and π_2 given by:

$$A_m(x, y) := x \wedge y,$$
$$\pi_1(x, y) := x,$$
$$\pi_2(x, y) := y.$$

Proposition 1. *Let* $\mathbb{K} = (G, M, I)$ *be a formal context. Then:*

1. A_m, π_1 *and* π_2 *are extent- and intent-consistent with* $\mathbb{K}$.
2. π_1 *and* π_2 *are consistent with* $\mathbb{K}$.

Proof. 1. A_m is extent-consistent and intent-consistent since it is used in the definition of the intersection of fuzzy sets: let $S \in \{G, M\}$, then, given two extents or intents $X, Y \in [0,1]^S$, we have that $(X \cap Y)(s) := X(s) \wedge Y(s) = A_m(X(s), Y(s))$ for all $s \in S$. Since the set of extents and the set of intents are closed under intersections, then $A_m(X, Y) = A \cap Y$ is also an extent or an intent, respectively. It is evident that the two projections are extent- and intent-consistent since they always return one of their inputs.
2. The projections applied to concepts return one of their inputs, as mentioned before, and therefore, projections are consistent with $\mathbb{K}$.

Notice that A_m may not be consistent with some formal context $\mathbb{K}$, as shown in the next example.

Example 4. Following our running example, let us compute:

$$A_m(\top, \bot) = A_m\left(\langle \{g_1, g_2, g_3\}, \{^{0.25}/m_2\}\rangle, \langle \{\varnothing\}, \{m_1, m_2, m_3\}\rangle\right) =$$
$$= \langle A_m(\{g_1, g_2, g_3\}, \varnothing), A_m(\{^{0.25}/m_2\}, \{m_1, m_2, m_3\})\rangle =$$
$$= \langle \{g_1, g_2, g_3\} \cap \varnothing, \{^{0.25}/m_2\} \cap \{m_1, m_2, m_3\}\rangle =$$
$$= \langle \varnothing, \{^{0.25}/m_2\}\rangle.$$

which is not a concept of the formal context, as it is easy to check.

Let us now inspect a simple formal context, as given in the next table, for $a, b \in [0, 1]$:

	m_1	m_2
g_1	a	b

Let us suppose $a < b$. Then, the set of formal concepts (using, as we mentioned before, the Gödel logic structure) is:

$$\langle \{g_1\}, \{^a\!/m_1, {}^b\!/m_2\} \rangle,$$
$$\langle \{^b\!/g_1\}, \{^a\!/m_1, m_2\} \rangle,$$
$$\langle \{^a\!/g_1\}, \{m_1, m_2\} \rangle.$$

Note that if A is an aggregation function such that $A(a, b) \notin \{a, b, 1\}$, then A is not extent-consistent. By duality, it cannot be intent-consistent. This motivates the following:

Conjecture. If there exists $a, b \in [0, 1]$ such that $A(a, b) \notin \{a, b\}$ then there exists a formal context $\mathbb{K} = (G, M, I)$ such that A is not extent-consistent (dually, intent-consistent) with $\mathbb{K}$.

Notice that we have removed the possibility $A(a, b) = 1$ in this conjecture. Also, observe that A_m, π_1 and π_2 satisfy $A(x, y) \in \{x, y\}$ for all $x, y \in [0, 1]$. The proof of this conjecture would imply that the aggregation functions that are extent-consistent or intent-consistent belong to the average class. As a main result, fixed a formal context $\mathbb{K}$, we wish to find a total classification or identification of the aggregation functions which preserve the algebraic structure of extents and intents, that is, finding the ones which are extent-consistent with $\mathbb{K}$ and intent-consistent with $\mathbb{K}$.

4   Conclusions

In this work we have explored the properties an aggregation function must satisfy in order to preserve the structure of extents, intents or formal concepts in the setting of Fuzzy Formal Concept Analysis. We have discarded some preliminary hypotheses via a series of illustrative examples and some conjectures have arisen from experimentation. Even in the first steps in this line, we have found some interesting results.

As a prospect of near future work, we intend to study thoroughly this problem in order to prove or refute the conjectures presented. Experiments suggest that there are distinct situations depending on the size of the formal context. These results will also have an impact from the practical standpoint: the consistent aggregation functions could be incorporated into algorithms for concept lattice construction in order to reduce the computational cost of exploring and computing the set of extents, intents and concepts.

References

1. Andrews, S.: Making use of empty intersections to improve the performance of CbO-type algorithms. In: Bertet, K., Borchmann, D., Cellier, P., Ferré, S. (eds.) ICFCA 2017. LNCS (LNAI), vol. 10308, pp. 56–71. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-59271-8_4

2. Bejines, C., Ardanza-Trevijano, S., Chasco, M., Elorza, J.: Aggregation of indistinguishability operators. Fuzzy Sets Syst. **446**, 53–67 (2022)

3. Beliakov, G., Pradera, A., Calvo, T., et al.: Aggregation Functions: A guide for practitioners, vol. 221. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-73721-6

4. Bělohlávek, R.: Fuzzy Relational Systems. Springer, Heidelberg (2002). https://doi.org/10.1007/978-1-4615-0633-1

5. Calvo, T., Kolesárová, A., Komorníková, M., Mesiar, R.: Aggregation Operators: Properties, Classes and Construction Methods, pp. 3–104. Physica-Verlag, Heidelberg (2002). http://dl.acm.org/citation.cfm?id=774556.774559

6. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundation. Springer, Heidelberg (1999). https://doi.org/10.1007/978-3-642-59830-2

7. Jana, C., Pal, M., Wang, J.: A robust aggregation operator for multi-criteria decision-making method with bipolar fuzzy soft environment. Iran. J. Fuzzy Syst. **16**(6), 1–16 (2019)

8. Krajča, P., Outrata, J., Vychodil, V.: Advances in algorithms based on cbo. In: Kryszkiewicz, M., Obiedkov, S.A. (eds.) Proceedings of the 7th International Conference on Concept Lattices and Their Applications, Sevilla, Spain, 19–21 October 2010, CEUR Workshop Proceedings, vol. 672, pp. 325–337. CEUR-WS.org (2010)

9. Kuznetsov, S.O., Makhalova, T.: On interestingness measures of formal concepts. Inf. Sci. **442**, 202–219 (2018)

10. Pedraza, T., Ramos-Canós, J., Rodríguez-López, J.: Aggregation of weak fuzzy norms. Symmetry **13**(10), 1908 (2021)

11. Pedraza, T., Rodríguez-López, J., Valero, Ó.: Aggregation of fuzzy quasi-metrics. Inf. Sci. **581**, 362–389 (2021)

12. Poelmans, J., Kuznetsov, S.O., Ignatov, D.I., Dedene, G.: Formal concept analysis in knowledge processing: a survey on models and techniques. Expert Syst. Appl. **40**(16), 6601–6623 (2013). https://doi.org/10.1016/j.eswa.2013.05.007