The Form of Fuzzy Implication Functions Satisfying a Multiplicative Sincov's Equation

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Abstract. The analysis of the additional properties of fuzzy implication functions often leads to studying some functional equations. Among them, the solution of a multiplicative Sincov's type equation connected with the characterization of power-based implications was recently published. In this paper, we provide a counterexample of this previous result, and the corrected result is presented, describing all fuzzy implication functions that satisfy the equation. Finally, we illustrate the result with several examples.

Keywords: Fuzzy implication function · Functional equation · Sincov's equation

1 Introduction

In the last decades, one of the leading research lines on the topic of fuzzy implication functions is the study of the additional properties that these operators may fulfil [2]. This research line is boosted by two crucial features of fuzzy implication functions. First of all, the not-so-demanding axioms of their definition allow the existence of a plethora of families of these operators, each of these with their own expression, method of construction, and additional properties. Second, many of these additional properties are not exclusively connected with one of such families, but members of different families can fulfill them. Therefore, in some recent studies, an additional property is fixed, and the corresponding

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functional equation is solved, finding all those fuzzy implication functions fulfilling the property. As important examples, we can highlight the law of importation [9], the distributivity properties [5] or the invariance property [6], among many others.

One of the functional equations which has been studied recently is a multiplicative version of Sincov's functional equation given by

$$I(x,y)\cdot I(y,z)=I(x,z), \quad 1>x>y>z>0, \tag{1}$$

where I is a fuzzy implication function. The importance of this functional equation is twofold. Namely, it is used in [8] to characterize the family of power based implications, and moreover, the fulfilment of Eq. (1) in the domain $x\le y\le z$ is a necessary and sufficient condition for being a unidimensional T'-preorder, with T' the product t-norm (see [4]). In [3], a generalization of this functional equation by understanding the internal product as the product t-norm and changing it to a general arbitrary continuous Archimedean t-norm was deeply analysed. Focusing on the original Eq. (1), due to its importance, in [1], a characterization result of all fuzzy implication functions satisfying Eq. (1) was presented. Unfortunately, as it will be proved later in this paper, that characterization result is not entirely correct, and a revision is needed. This constitutes the main contribution of this paper. First, we will present an example of a fuzzy implication function satisfying Eq. (1) but not being a solution provided by the characterization theorem. After that, a new characterization result that provides the solution to our problem will be proved jointly with several examples that illustrate the result.

The structure of the paper is as follows. After the preliminaries, in Sect. 3, the characterization result given in [1] is recalled, and the counterexample proving that the result needs revision is presented. Then, in Sect. 4, the corrected result is proved along with some illustrative examples. Finally, the paper ends with some concluding remarks.

2 Preliminaries

In this section, we will recall the basic definitions and concepts of fuzzy implication functions that will be used throughout the paper. First, the definition of a fuzzy implication function is provided.

Definition 1 ([2, Definition 1.1.1.]). *A binary operation $I\colon [0,1]^2\to[0,1]$ is said to be a* fuzzy implication function *if it satisfies, for all $x,y,z\in[0,1]$:*

(I1) $I(x,z)\ge I(y,z)$, *when* $x\le y$,
(I2) $I(x,y)\le I(x,z)$, *when* $y\le z$,
(I3) $I(0,0)=I(1,1)=1$, *and* $I(1,0)=0$.

From the definition it is clear that $I(0,x)=I(x,1)=1$ for all $x\in[0,1]$. However, neither $I(x,0)$ nor $I(1,x)$ are determined for all $x\in(0,1)$. This flexibility allows the potential fulfilment of many additional properties from which we recall next to the ordering property, which will be used later.

Definition 2 **([2, Definition 1.3.1.]).** *We say that a fuzzy implication function I satisfies the* ordering property*, if:*

$$x \leq y \quad \Longleftrightarrow \quad I(x, y) = 1, \quad x, y \in [0, 1]. \tag{OP}$$

On the other hand, the monotonicities of the fuzzy implication function imply the following straightforward result.

Lemma 1. *Let $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication function. Define*

$$Q_I := \{(x, y) \in [0,1]^2 : x > y \text{ and } I(x, y) = 1\}.$$

Then it holds that $I(u, v) = 1$ for each $(u, v) \in R_I$, where

$$R_I := \bigcup \{[0, x] \times [y, 1] : (x, y) \in Q_I\}. \tag{2}$$

3 Previous Characterization Result and Counterexample

In [1], the characterization result that was presented for the fuzzy implication functions satisfying Eq. (1) stated as follows.

Theorem 1 **([1, Theorem 14]).** *Let I be a fuzzy implication function. If I solves Eq. (1), then there exist $y_0 \in [0,1]$ and a non-increasing function $f\colon (y_0, 1) \to (0, +\infty)$ such that I is given by:*

$$I(x, y) = \begin{cases} \frac{f(x)}{f(y)}, & \text{if } y \in (y_0, 1), x \in (y, 1), \\ 0, & \text{if } y \in [0, y_0), x \in (y, 1). \end{cases} \tag{3}$$

Conversely, for every point $y_0 \in [0,1]$ and for every function $f\colon (y_0, 1) \to (0, +\infty)$, every mapping $I\colon [0,1]^2 \to \mathbb{R}$ which on the set

$$\{(x, y) \in [0, 1] \mid y \in (y_0, 1), x \in (y, 1) \text{ or } y \in [0, y_0), x \in (y, 1)\}$$

is given by Eq. (3) is a solution to Eq. (1) postulated for all $x, y, z \in [0,1]$ such that $1 > x > y > z > y_0$.

As it has been already aforementioned, this theorem contains an error and not every solution can be described in such way. Indeed, let us define an example of a fuzzy implication function which satisfies Eq. (1) but it is not of the form given by Eq. (3):

Example 1. Let us define the following operator $I\colon [0,1]^2 \to [0,1]$ given by:

$$I(x, y) = \begin{cases} 1, & \text{if } (x \leq y) \text{ or } (x < \frac{1}{2}) \text{ or } (y > \frac{1}{2}), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Figure 1 gives the structure of I. It is easy to check that such a function is a fuzzy implication function. Now, on the one hand, by a simple computation, it can be proved that this implication fulfils Sincov's equation (1). However, it is clear that this implication cannot be written in the form of Eq. (3). Consequently, this results in a counterexample of Theorem 1, which needs a deep revision.

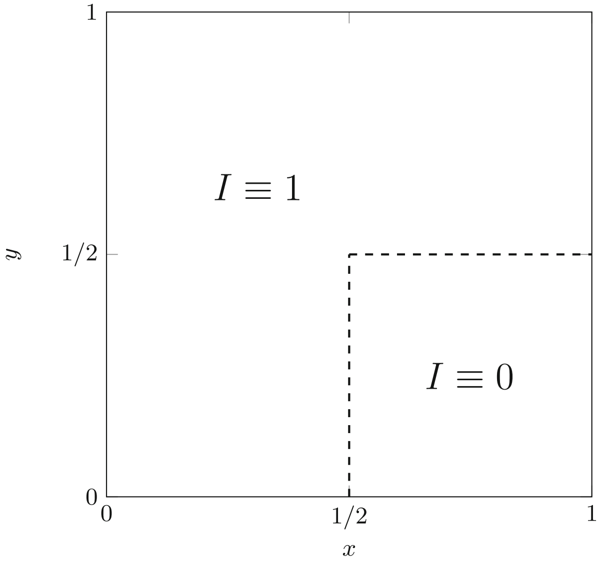

Fig. 1. Plot of the fuzzy implication function used in Example 1.

This example can be easily generalized. Indeed, there is nothing special about point $\frac{1}{2}$ used in the construction, and we can take any $y_0 \in (0,1)$. Moreover, the fuzzy implication function does not need to be constant on the two triangles with vertices:

– $(0,0), (y_0, y_0), (y_0, 0)$,
– $(y_0, y_0), (1, y_0), (1, 1)$.

Example 2. Let us consider $y_0 \in (0,1)$ and the non-increasing functions $f\colon (y_0, 1) \to (0, +\infty)$ and $g\colon (0, y_0) \to (0, +\infty)$. Then the binary operator $I\colon [0,1]^2 \to [0,1]$ given by

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{f(x)}{f(y)}, & \text{if } y \in (y_0, 1), x \in (y, 1), \\ \frac{g(x)}{g(y)}, & \text{if } x \in (0, y_0), y \in (0, x), \\ 0, & \text{otherwise}, \end{cases}$$

is a fuzzy implication function which is a solution of Eq. (1). In Fig. 2 the structure of I is depicted. However, again it is easy to check that it cannot be written in the form of Eq. (3).

4 New Characterization Result

The main goal of this section is to present a new characterization result that fixes Theorem 1.

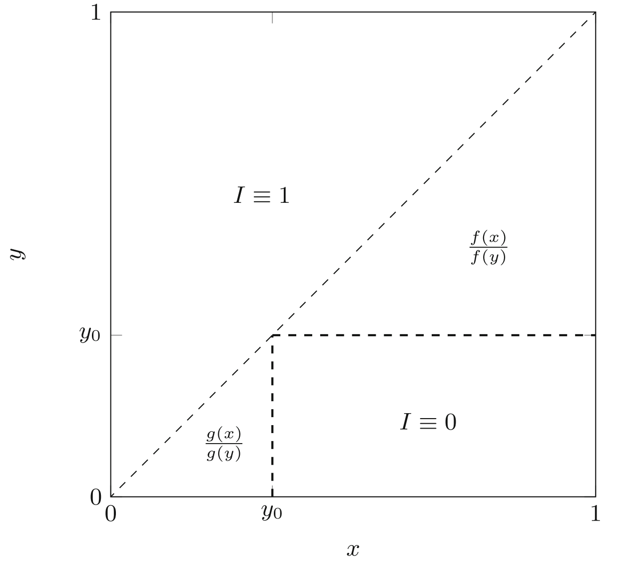

Fig. 2. Plot of the fuzzy implication function used in Example 2.

Assume that J is a nontrivial interval and denote $\Delta_J := \{(s,t) \in J^2 : s \leq t\}$. Gergely Kiss and Jens Schwaiger in [7] proved the following result.

Theorem 2 (see [7, **Theorem 3.11**]). *Let* $f \colon \Delta_J \to \mathbb{R}$ *be a solution of*

$$f(s,t) \cdot f(t,u) = f(s,u), \quad (s,t),(t,u) \in \Delta_J. \tag{5}$$

Then there exists a countable (possibly empty) family $\mathcal{S}$ *of pairwise disjoint nontrivial intervals* $I \subseteq J$ *and a function* $d \colon \bigcup_{I \in \mathcal{S}} I \to \mathbb{R} \backslash \{0\}$ *such that*

$$f(x,y) = \frac{d(y)}{d(x)}, \quad x,y \in I, \ I \in \mathcal{S}, \, x \leq y. \tag{6}$$

Moreover, fixed $I \in \mathcal{S}$*,* $x \in I$ *and* $y,z \in (J \backslash I)$ *such that* $z < x < y$*, then*

$$f(x,y) = f(z,x) = 0.$$

Further,

$$f(x,x) = \begin{cases} 1, & \text{if } x \in \bigcup_{I \in \mathcal{S}} I, \\ 1 \text{ or } 0, & \text{otherwise.} \end{cases}$$

Moreover, as it was observed in the last sentence of the proof of this theorem, it holds also that $f(x,y) = 0$ if $x,y \notin \bigcup_{I \in \mathcal{S}} I$ and $x < y$.

We will apply the above theorem to obtain a new characterization of the family of fuzzy implication functions fulfiling Eq. (1) amending the gap in Theorem 1, our earlier result.

Corollary 1. *Let $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication which satisfies Eq. (1). Then, there exists a countable (possibly empty) family $\mathcal{S}$ of pairwise disjoint nontrivial intervals $I \subseteq [0,1]$ and a non-increasing function $d\colon \bigcup_{I\in\mathcal{S}} I \to [0,+\infty)$ such that:*

(a) if $x, y \in I$ with $I \in \mathcal{S}$ and $x > y$, then $I(x,y) = d(x)/d(y)$,
(b) if $x > y$ and x, y do not belong to the same member of $\mathcal{S}$ (or one or both of them are outside the set $\bigcup \mathcal{S}$), then $I(x,y) = 0$,
(c) if $x < y$ and $x \in \operatorname{int} \bigcup_{I\in\mathcal{S}} I$, then $I(x,y) = 1$.

Conversely, for every countable family $\mathcal{S}$ of pairwise disjoint nontrivial intervals $I \subseteq [0,1]$ and every non-increasing function $d\colon \bigcup_{I\in\mathcal{S}} I \to [0,+\infty)$, the map $I\colon [0,1]^2 \to [0,1]$ described by (a), (b), (c) *satisfies Eq. (1). Moreover, if I is mixed monotone (in the sense of Definition 1) on the (possibly empty) set that is not covered by* (a), (b), (c), *then it is a fuzzy implication function.*

Proof. Assume that I is a fuzzy implication function that satisfies Eq. (1). We will follow the idea of [7, Remark 3.15]. Define $f\colon \Delta_{[0,1]} \to [0,1]$ as $f(x,y) = I(y,x)$ when $x < y$ and $f(x,x) = 1$ for $x \in [0,1]$. One can see that f is a solution of Eq. (5) on $\Delta_{[0,1]}$. By Theorem 2 there exists a family $\mathcal{S}$ of disjoint nontrivial intervals $I \subseteq J = [0,1]$ and a function $d\colon \bigcup_{I\in\mathcal{S}} I \to \mathbb{R}\setminus\{0\}$ such that Eq. (6) holds. Further, by the same theorem, $I(x,y) = f(y,x) = 0$ for $x > y$ in the following two cases: $(x, y \notin \bigcup_{I\in\mathcal{S}} I)$ or $(x \in I$ with $I \in \mathcal{S}$ and $y \notin I)$. Therefore, (b) holds. Since I attains only non-negative values, we can assume that $d\colon \bigcup_{I\in\mathcal{S}} I \to (0,+\infty)$. We thus have

$$I(x,y) = \frac{d(x)}{d(y)}, \quad x, y \in I,\ I \in \mathcal{S},\ x > y. \tag{7}$$

It is important to remember that $I(x,x)$ needs not to be equal to $f(x,x)$, thus we have excluded in Eq. (7) the possibility $x = y$. Thus, (a) holds. Moreover, without loss of generality, thanks to (I3), we may assume that $d(0) = 1$.

At this point, since map d is monotone, has at most countably many points of discontinuity. Moreover, d is bounded on every closed subinterval contained in $\bigcup_{I\in\mathcal{S}} I$. Therefore, the value $I(x,y)$ is arbitrarily close to 1 when $x, y \in [x_0, y_0] \subseteq I$ are such that $x > y$ and x approaches to y. Consequently, using Lemma 1, for every $x, y \in [0,1]$ such that $x < y$ and $x \in \operatorname{int} \bigcup_{I\in\mathcal{S}} I$ we have $I(x,y) = 1$, thus (c) holds. Further, since I is non-decreasing with respect to the second variable, then the map d is non-increasing on each interval from the family $\mathcal{S}$.

Conversely, any function $I\colon [0,1]^2 \to [0,1]$ described by (a), (b), (c), for any countable family $\mathcal{S}$ of pairwise disjoint nontrivial intervals $I \subseteq [0,1]$ and any non-increasing function $d\colon \bigcup_{I\in\mathcal{S}} I \to [0,+\infty)$, it is easy to check that Eq. (1) holds. We cannot ensure that $\bigcup_{I\in\mathcal{S}} I \supseteq [0,1]$ as it can be seen in Fig. 3. Thus we have to assume monotonicity for points $x, y \notin \bigcup_{I\in\mathcal{S}} I$, for I to be a fuzzy implication function. □

Note that each fuzzy implication function, continuous or not, such that $I(x,y) = 0$ for $x > y$ trivially solves Eq. (1). In this case the family $\mathcal{S}$ is empty.

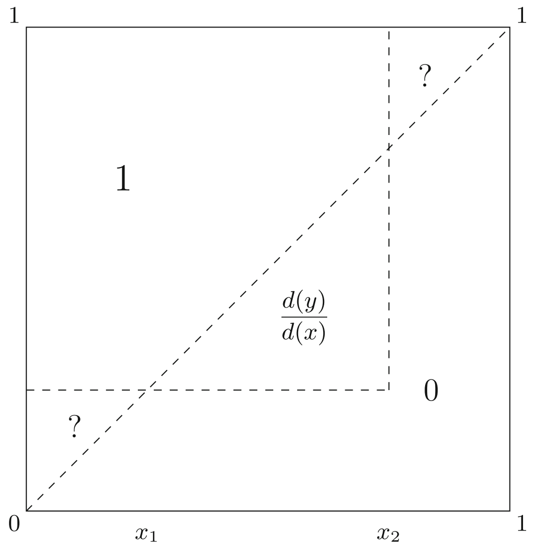

Fig. 3. In the above example $\mathcal{S} = \{(x_1, x_2)\}$, and d is strictly monotone on (x_1, x_2). The set Q_I is empty, I is discontinuous and not determined everywhere on $[0, 1]^2$ and **(OP)** may hold or not.

We can see exemplary application of Corollary 1 in Fig. 4. Such fuzzy implication function can be constant and equal to 1 on some intervals for $x > y$.

Next, we will deal with the continuous case.

Corollary 2. *Assume that* $I\colon [0,1]^2 \to [0,1]$ *is a continuous fuzzy implication function that satisfies Eq.* (1)*. Then there exists a continuous and non-increasing function* $d\colon [0,1] \to [0,1]$ *such that* $d(1) = 0$*,* $d(0) = 1$ *and* $d(x) > 0$ *for* $x < 1$ *and*

$$I(x, y) = \frac{d(x)}{d(y)}, \quad x, y \in [0, 1],\ x > y \tag{8}$$

with the convention $0/0 = 1$*. Moreover, in this case* I *satisfies* **(OP)** *if and only if* d *is decreasing.*

Conversely, for every continuous and non-increasing function $d\colon [0,1] \to [0,1]$ *such that* $d(1) = 0$*,* $d(0) = 1$ *and* $d(x) > 0$ *for* $x < 1$*, the map* $I\colon [0,1]^2 \to [0,1]$ *defined by* (8) *and* $I(x, y) = 1$ *for* $x \le y$ *satisfies Eq.* (1) *and is a fuzzy implication function. Moreover, it satisfies* **(OP)** *if and only if* d *is decreasing.*

Proof. Let us assume that I is a continuous fuzzy implication function that satisfies (1). From Corollary 1, we know that there exist a family $\mathcal{S}$ and a non-increasing function d such that (a), (b) and (c) hold. Continuity of d follows from continuity of I. Moreover, the continuity of I implies that either $\bigcup_{I \in \mathcal{S}} I = \varnothing$ or $\bigcup_{I \in \mathcal{S}} I \supseteq (0, 1)$ since otherwise I would have a $1 - 0$ discontinuity at each point $(x_0, x_0) \in [0, 1]^2$ such that $x_0 \in (0, 1) \setminus \bigcup_{I \in \mathcal{S}} I$.

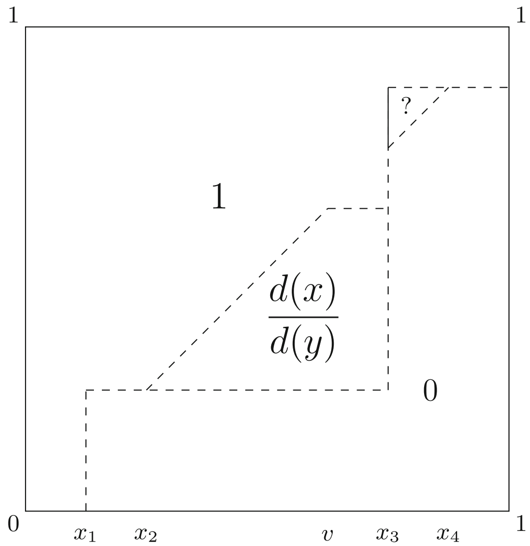

Fig. 4. In the above example $\mathcal{S} = \{(0, x_1), (x_2, x_3), (x_4, 1)\}$, $d(x) = 1$ on $(0, x_1) \cup (x_4, 1)$ and d is arbitrary monotone on (x_2, x_3) and constant on (v, x_3). Moreover, I is discontinuous and not determined everywhere on $[0, 1]^2$ and **(OP)** does not hold.

However, the case $\bigcup_{I\in\mathcal{S}} I = \varnothing$ also leads to a contradiction with continuity. Using (b) we can see that $I(x, y) = 0$ for $x > y$, but from (I3) we also know that $I(1, 1) = 1$. So if we take an arbitrary non-decreasing sequence $(x_n)_{\mathbb{N}}$, such that $x_n \in [0, 1]$ and $\lim_{n\to+\infty} x_n = 1$, then $\lim_{n\to+\infty} I(1, x_n) = 0 \neq 1 = I(1, 1)$. Thus such solutions do not exist.

Therefore, the only viable case is when $\bigcup_{I\in\mathcal{S}} I \supseteq (0, 1)$. In that case, function d has finite limits at 0 and 1, and thus it can be continuously extended to $[0, 1]$. From the proof of Corollary 1, we know that without loss of generality, $d(0) = 1$. Note that the last part of the condition (I3) implies that $[0, 1] \notin \mathcal{S}$. Therefore, if we want to extend the map d in such a way that it is defined at 1, then necessarily $d(1) = 0$. This, in turn, implies that because of the second part of (I3), one needs to adopt a convention that $0/0 = 1$. After this agreement, Eq. (7) covers the cases $I(1, 1)$ and $I(1, 0)$ as well. As a consequence, Eq. (7) holds for all $x, y \in [0, 1]$ such that $x > y$.

Conversely, if we consider map $I\colon [0, 1]^2 \to [0, 1]$ defined by Eq. (8) and $I(x, y) = 1$ for $x \leq y$, where $d\colon [0, 1] \to [0, 1]$ is an arbitrary continuous and non-increasing function such that $d(1) = 0$, $d(0) = 1$ and $d(x) > 0$ for $x < 1$, then it is easy to check that I satisfies Eq. (1) and is a fuzzy implication function.

Regarding **(OP)**, we know that if $\bigcup_{I\in\mathcal{S}} I \supseteq (0, 1)$, then $I(x, x) = 1$ for all $x \in [0, 1]$. From this, it also follows that $I(x, y) = 1$ for all $x, y \in [0, 1]$, such that $x \leq y$. If d is constant on a nontrivial subinterval $[v, u] \subseteq \bigcup_{I\in\mathcal{S}} I$, then by Eq. (7) we get $I(u, v) = 1$. Conversely, if $I(u, v) = 1$ for some $u > v$, then d is constant on $[v, u]$. Thus, I satisfies **(OP)** if and only if d is decreasing.

5 Conclusions

In this paper, we obtained a description of fuzzy implication functions that satisfy the multiplicative Sincov's functional equation described in Eq. (1). With this contribution, we corrected a gap in our earlier work [1], and we supplemented the research with some illustrative examples. The new characterization result is based on a new result of Kiss and Schwaiger, published in [7], that deals with the Sincov's equation. From these papers, this equation shows its importance in information science and economy. We believe further studies on the topic will bring new interesting results and further applications of the equation.

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References

1. Baczyński, M., Fechner, W., Massanet, S.: A functional equation stemming from a characterization of power-based implications. In: 2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), pp. 1–6 (2019)
2. Baczyński, M., Jayaram, B.: Fuzzy Implications, Studies in Fuzziness and Soft Computing, vol. 231. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-69082-5
3. Baczyński, M., Fechner, W., Massanet, S.: On a generalization of multiplicative Sincov's equation for fuzzy implication functions. Fuzzy Sets Syst. **451**, 196–205 (2022). Recent Trends in Aggregation - In Honour of Radko Mesiar's 70th Birthday
4. Boixader, D., Recasens, J.: Generation and characterization of fuzzy t-preorders. Appl. Artif. Intell. **29**(5), 514–530 (2015)
5. Dombi, J., Baczyński, M.: General characterization of implication's distributivity properties: the preference implication. IEEE Trans. Fuzzy Syst. **28**(11), 2982–2995 (2020)
6. Fernandez-Peralta, R., Massanet, S., Mir, A.: On strict T-power invariant implications: properties and intersections. Fuzzy Sets Syst. **423**, 1–28 (2021)
7. Kiss, G., Schwaiger, J.: Sincov's and other functional equations and negative interest rates. Aequat. Math. **97**, 629–637 (2023). https://doi.org/10.1007/s00010-022-00936-9. in Press
8. Massanet, S., Recasens, J., Torrens, J.: Some characterizations of T-power based implications. Fuzzy Sets Syst. **359**, 42–62 (2019)
9. Massanet, S., Torrens, J.: Characterization of fuzzy implication functions with a continuous natural negation satisfying the law of importation with a fixed t-norm. IEEE Trans. Fuzzy Syst. **25**(1), 100–113 (2017)