1 Complex Numbers and Complex Plane

Abstract

In this chapter we recall some concepts from basic courses in mathematical analysis of real-valued functions of one and several variables as well as from a course in linear algebra, namely complex numbers and their various forms, arithmetic operations on them (Sect. 1.1), and basic topological notions in the vector space $\mathbb{R}^2$ (Sect. 1.5). The novel notion of the stereographic projection in Sect. 1.2 provides a geometric interpretation of the extended complex plane. Complex-valued functions of a real variable and various curves in the complex plane are considered in more detail in Sects. 1.3 and 1.4, respectively.

1.1 Complex Numbers

A number is the basic concept of mathematics, which evolved throughout the history of humankind. The emergence and formation of this concept went hand in hand with the emergence and development of mathematics. Practical human activities, on the one hand, and internal needs of mathematics, on the other, determined the development of the concept of numbers.

The necessity of counting objects led to the emergence of the concept of the set of natural numbers ($\mathbb{N}$). Starting with natural numbers, the number system expanded in response to the need to describe quantities that could not be accommodated within the existing (previous) number system. As a result, sets of integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), and real numbers ($\mathbb{R}$) appeared in mathematics such that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Complex numbers arose from the need to find solutions of polynomial equations, for example, $x^2 + 1 = 0$. The first written mention of complex numbers as square roots of negative numbers can be found in Girolamo Cardano's book in 1545. For nearly two centuries, complex numbers remained mysterious, had a poor reputation and were generally not considered legitimate. The active use of complex numbers

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in calculations began with works of Leonard Euler (1707–1783), in particular, with his famous formula (1.9) introduced in 1748. The first systematic description of complex numbers, arithmetic operations on them and their geometric interpretation was conducted by Carl Gauss (1777–1855) in his memoir "Theoria residuorum biquadraticorum" (1828, 1832). The term "complex number" is due to C. Gauss in 1831. The 2000-year and engaging history of complex numbers is presented in [7].

Definition 1.1 The set $\mathbb{C}$ of complex numbers is the set of ordered pairs (x, y) of real numbers x and y, equipped with algebraic operations of addition and multiplication:

$$(x_1, y_1) + (x_2, y_2) \stackrel{def}{=} (x_1 + x_2,\ y_1 + y_2), \tag{1.1}$$

$$(x_1, y_1) \cdot (x_2, y_2) \stackrel{def}{=} (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2). \tag{1.2}$$

▲

It is clear that two complex numbers (x_1, y_1) and (x_2, y_2) are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. From the definition it follows that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0), \qquad (x_1, 0) \cdot (x_2, 0) = (x_1x_2, 0). \tag{1.3}$$

Thus, addition and multiplication on complex numbers of the form $(x, 0)$ coincide with the corresponding algebraic operations on real numbers. Therefore, we can identify each real number x with the complex number $(x, 0)$, i.e., $\mathbb{R} \ni x := (x, 0) \in \mathbb{C}$, and after this identification one can state that $\mathbb{R} \subset \mathbb{C}$. In addition, one can verify that for any real number a

$$a \cdot (x, y) = (a, 0) \cdot (x, y) = (ax, ay). \tag{1.4}$$

The complex number $(0, 1)$ is called the *imaginary unit* and is denoted by the Latin letter i. It is easy to check that

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1 \quad \text{and} \quad (0, y) = (0, 1) \cdot (y, 0) = iy.$$

Based on these notations, any complex number can be represented as

$$(x, y) = (x, 0) + (0, y) = x \cdot (1, 0) + y \cdot (0, 1) = x + iy,$$

which is called the *algebraic form* of a complex number. The algebraic form of a complex number is usually denoted by one letter $z := x + iy$. Moreover, the number x is called the *real part* of the complex number z and is denoted $\mathrm{Re}(z)$, while the number y is called the *imaginary part* of z and is denoted $\mathrm{Im}(z)$.

The *conjugate* of a complex number $z = x + iy$ is the complex number $\overline{z} := x - iy$. The *modulus* or *absolute value* of z is defined by

$$|z| := \sqrt{x^2 + y^2}.$$

Note that $|z| = |\overline{z}|$ and $z \cdot \overline{z} = x^2 + y^2 = |z|^2$.

Subtraction and division of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are defined as follows:

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2), \qquad \frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} \ (z_2 \neq 0). \tag{1.5}$$

The set of complex numbers with respect to the introduced operations forms a *field*, i.e., it is an *Abelian group*[1] with respect to addition with $0 = (0, 0)$ as the additive identity; the nonzero elements in $\mathbb{C}$ form an Abelian group with respect to multiplication with $1 = (1, 0)$ as the multiplicative identity; and multiplication distributes over addition.

Exercise 1.1 Prove that all these field properties are fulfilled.

From (1.3) it follows that the field of complex numbers includes the field of real numbers as a subfield. The reader is invited to make sure that all extensions of the field $\mathbb{R}$ obtained by joining the root of the equation $x^2 + 1 = 0$ to it are isomorphic to the field $\mathbb{C}$.

Based on Definition 1.1, (1.1) and (1.4) we can assert that the set of complex numbers is a *real vector space*,[2] or more precisely, the vector space $\mathbb{R}^2$. This makes the complex numbers a Cartesian plane (coordinate plane), called the *complex plane*. Clearly that the real numbers lie on the horizontal x-axis, called the *real axis*, and the y-axis is called the *imaginary axis* of the complex plane. This allows to give the geometric interpretation of complex numbers and arithmetic operations defined on $\mathbb{C}$ and, conversely, to express some geometric properties and constructions in terms of complex numbers. For instance, conjugation is the reflection symmetry with respect to the real axis; multiplication by -1 is the central symmetry about the origin.

[1] Recall that a *group* is a set of elements together with a binary operation on this set such that the following three requirements, known as group axioms, are satisfied: the binary operation is associative, there is a unique identity with respect to this operation, and every element of this set has an inverse with respect to this operation. In an Abelian group, the binary operation is additionally commutative.

[2] It is a set of objects called vectors, which may be added together and multiplied by real numbers (scalar multiplication). This set is an Abelian group under addition, and scalar multiplication has the following properties: $x(\mathbf{u} + \mathbf{v}) = x\mathbf{u} + x\mathbf{v}$, $(x + y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$, $(xy)\mathbf{v} = x(y\mathbf{v})$, and $1\mathbf{v} = \mathbf{v}$ for all $x, y \in \mathbb{R}$ and all vectors $\mathbf{u}$ and $\mathbf{v}$.

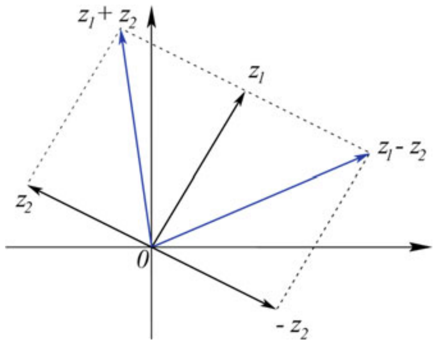

Fig. 1.1 Geometric interpretation of the sum and subtraction two complex numbers

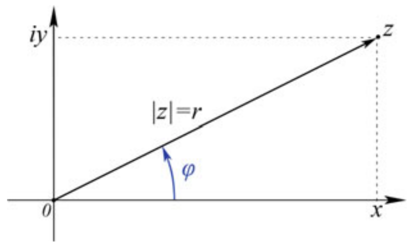

Fig. 1.2 The argument of z

Furthermore, sum and subtraction of two complex numbers coincides with the sum and subtraction of the corresponding vectors in $\mathbb{R}^2$ (Fig. 1.1). The absolute value of a complex number is the length of the corresponding vector (the usual Euclidean norm) in the vector space $\mathbb{R}^2$.

Other Forms of Writing Complex Numbers

It is known that the position of the point (x, y) is also determined by the pair (r, φ), where r is the distance of (x, y) to the origin and φ is the counterclockwise angle (measured in radians) between the positive x-axis and the ray from the origin through (x, y); the values r and φ are called the *polar coordinates* of (x, y), and

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{1.6}$$

Substituting these relations into the algebraic form of a complex number $z = x + iy$ yields the *trigonometric form:*

$$z = |z|\,(\cos\varphi + i\,\sin\varphi), \tag{1.7}$$

where the angle φ is called the *argument* of z (Fig. 1.2).

Note that the argument of each nonzero complex number z is defined ambiguously, and up to the term $2\pi k$ ($k \in \mathbb{Z}$); in addition, there exists a unique angle $\varphi_0 \in (-\pi, \pi]$ such that $\varphi = \varphi_0 + 2\pi k$. This angle φ_0 is called the *principal value* of the argument of z and is denoted $\arg(z)$. The set of all arguments of z is denoted by

$$\mathrm{Arg}(z) := \{\varphi_0 + 2\pi k : \ k \in \mathbb{Z}\}.$$

The argument of 0 is not defined. The principal value of the argument of z can be considered as a real-valued function defined on $\mathbb{C} \setminus \{0\}$ and it can be expressed from the formulas (1.6) in terms of the inverse trigonometric function arctan :

$$\arg(z) = \begin{cases} \arctan \frac{y}{x}, & \text{if } x > 0; \\ \pi + \arctan \frac{y}{x}, & \text{if } x < 0, \ \ y > 0; \\ \pi, & \text{if } x < 0, \ \ y = 0; \\ -\pi + \arctan \frac{y}{x}, & \text{if } x < 0, \ \ y < 0. \end{cases} \tag{1.8}$$

Example 1.1 It is easy to verify that

- the principal value of the argument of each positive number x $(y = 0)$ is zero, and the set of all arguments of x is $\mathrm{Arg}(x) = \{2\pi k \colon k \in \mathbb{Z}\}$;
- $\arg(1 - i) = -\frac{\pi}{4}$, and $\mathrm{Arg}(1 - i) = \{-\frac{\pi}{4} + 2\pi k \colon k \in \mathbb{Z}\}$;
- $\arg(-3) = \pi$, and $\mathrm{Arg}(-3) = \{\pi + 2\pi k \colon k \in \mathbb{Z}\}$.

Let us define the exponential function of an imaginary number $i\alpha$ by the following way:

$$e^{i\alpha} \stackrel{def}{=} \cos\alpha + i \sin\alpha \quad (\alpha \in \mathbb{R}), \tag{1.9}$$

which is known as Euler's formula (the proof is given in Example 5.4). From (1.9) it is clear that $|e^{i\alpha}| = 1$. In addition, it is easy to verify that

$$\begin{aligned} e^{i\alpha_1} \cdot e^{i\alpha_2} &= (\cos\alpha_1 + i \sin\alpha_1)(\cos\alpha_2 + i \sin\alpha_2) \\ &= \big(\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2\big) + i\big(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2\big) \\ &= \cos(\alpha_1 + \alpha_2) + i \sin(\alpha_1 + \alpha_2) \\ &= e^{i(\alpha_1 + \alpha_2)}. \end{aligned} \tag{1.10}$$

In (1.10) we used the addition formulas for sine and cosine. Similarly, it is proved that

$$\left(e^{i\alpha}\right)^n = e^{in\alpha}, \qquad \frac{e^{i\alpha_1}}{e^{i\alpha_2}} = e^{i(\alpha_1 - \alpha_2)}.$$

Using Euler's formula (1.9), we get from (1.7) the *exponential form* of a complex number: $z = |z|\, e^{i\varphi}$. This form well illustrates the essence of multiplication and division of complex numbers. If $z_1 = |z_1|\, e^{i\varphi_1}$ and $z_2 = |z_2|\, e^{i\varphi_2}$, then

$$z_1 \cdot z_2 = |z_1||z_2|\, e^{i(\varphi_1+\varphi_2)}, \qquad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\, e^{i(\varphi_1-\varphi_2)} \ \ (z_2 \neq 0).$$

Thus, when multiplying (respectively dividing) two complex numbers, their moduli are multiplied (resp. divided):

$$|z_1 \cdot z_2| = |z_1||z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

and arguments are added (resp. subtracted):

$$\varphi_1 + \varphi_2 \in \mathrm{Arg}(z_1 \cdot z_2), \qquad \varphi_1 - \varphi_2 \in \mathrm{Arg}(\frac{z_1}{z_2}).$$

Definition 1.2 A complex number z is called an n^{th} root of a complex number a, if $z^n = a$. Here, $n \in \mathbb{N}$ and $a \neq 0$. ▲

Let us derive a formula for finding n^{th} roots of a complex number $a = |a|\, e^{i\theta}$ ($\theta \in (-\pi, \pi)$). If $z = |z|\, e^{i\varphi}$ is an n^{th} root of a, then according to the definition

$$|z|^n\, e^{in\varphi} = |a|\, e^{i\theta} \iff \begin{cases} |z|^n = |a|, \\ n\varphi = \theta + 2\pi k, \quad k \in \mathbb{Z}, \end{cases}$$

whence

$$\begin{cases} |z| = \sqrt[n]{|a|}, \\ \varphi_k = \frac{\theta+2\pi k}{n}, \quad k \in \mathbb{Z}, \end{cases}$$

that is, the n^{th} roots of a are numbers

$$z_k = \sqrt[n]{|a|}\, e^{i\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)}, \quad k \in \mathbb{Z}. \tag{1.11}$$

It is easy to see that among these complex numbers there are exactly n different numbers. Indeed, the numbers $z_0, \ldots, z_{n-1}$ are different since their arguments

$$\varphi_0 = \frac{\theta}{n}, \quad \varphi_1 = \frac{\theta + 2\pi}{n}, \ \ \ldots\ \ , \varphi_{n-1} = \frac{\theta + 2\pi(n-1)}{n}$$

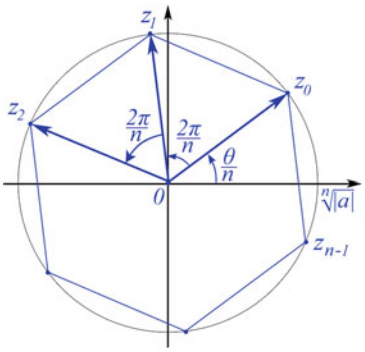

Fig. 1.3 The n^{th} roots of a complex number a

are various and differ from each other less than 2π. For any other number z_k, $k \notin \{0, \ldots, n-1\}$ there exist numbers $p \in \mathbb{Z}$ and $q \in \{0, 1, \ldots, n-1\}$ such that $k = pn + q$. This means that $z_k = z_q$.

Thus, the equation $z^n = a$ has n different roots $z_0, \ldots, z_{n-1}$, defined by the formula (1.11) and located at the vertices of a regular n-sided polygon inscribed in a circle of radius $\sqrt[n]{|a|}$ centered at the point 0 (Fig. 1.3).

1.2 Sequences in the Complex Plane: Extended Complex Plane

Since the modulus of a complex number is just the usual Euclidean norm in the vector space $\mathbb{R}^2$, it is natural to introduce the distance between two complex numbers as follows

$$d(z_1, z_2) := |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. In addition, we can naturally introduce the convergence of a sequence of complex numbers as the convergence of a sequence of the corresponding vectors in $\mathbb{R}^2$. We hope that the reader is familiar with the properties of convergent sequences from real analysis of several variables. Nevertheless, let us briefly recall the main definitions and properties.

Definition 1.3 A sequence $\{z_n = x_n + iy_n\}_{n\in\mathbb{N}}$ of complex numbers is said to converge to a complex number $a = \alpha + i\beta$ (denoted as $\lim\limits_{n\to+\infty} z_n = a$), if

$$\lim_{n\to+\infty} |z_n - a| = 0,$$

i.e., for every $\varepsilon > 0$, there exists an integer N such that

$$|z_n - a| < \varepsilon \quad \text{for all } n \geq N.$$

▲

From Definition 1.3 follows a statement, which is offered to the reader as an exercise.

Exercise 1.2 Prove that a sequence $\{z_n = x_n + iy_n\}_{n\in\mathbb{N}}$ converges to the complex number $a = \alpha + i\beta$ if and only if

$$\lim_{n\to+\infty} x_n = \alpha \quad \text{and} \quad \lim_{n\to+\infty} y_n = \beta.$$

Definition 1.4 It is said that a sequence $\{z_n\}_{n\in\mathbb{N}}$ of complex numbers converges to infinity ($\lim\limits_{n\to+\infty} z_n = \infty$), if

$$\lim_{n\to+\infty} |z_n| = +\infty,$$

i.e., for every $R > 0$, there exists an integer N such that

$$|z_n| > R \quad \text{for all} \;\; n \geq N.$$

▲

The symbol "∞" is called the *point at infinity.*

Definition 1.5 The set $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the extended complex plane.

Obviously, that each sequence in $\overline{\mathbb{C}}$ contains a convergent subsequence. This is called the *principle of compactness* in $\overline{\mathbb{C}}$. The point at infinity does not participate in algebraic operations, i.e. it cannot be multiplied or added to complex numbers. In real analysis, points labeled $+\infty$ and $-\infty$ produce the two-point compactification of the set of real numbers.

Geometric Interpretation of $\overline{\mathbb{C}}$
Consider the space

$$\mathbb{R}^3 = \big\{(\xi, \eta, \zeta) : \; \xi \in \mathbb{R}, \; \eta \in \mathbb{R}, \; \zeta \in \mathbb{R}\big\},$$

in which the ξ-axis coincides with the real axis, η-axis coincides with the imaginary axis, and ζ-axis is perpendicular to the complex plane (Fig. 1.4). The sphere

$$\mathbf{S} := \left\{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \; \xi^2 + \eta^2 + \left(\zeta - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}\right\}$$

is tangent to the complex plane at the origin. The point $N = (0, 0, 1)$, which lies on the sphere, will be called the “north pole”. Define a mapping $p\colon \overline{\mathbb{C}} \mapsto \mathbf{S}$ as

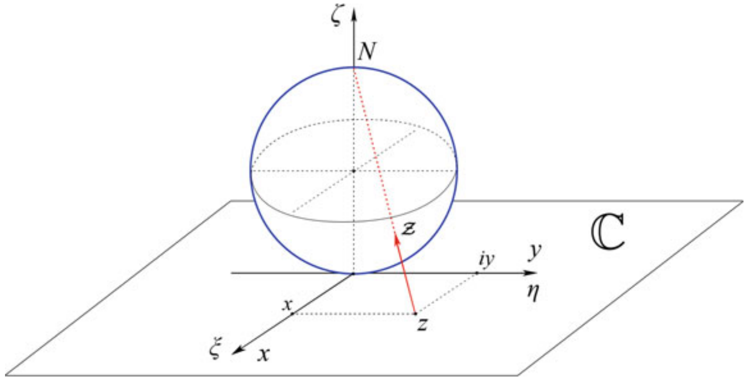

Fig. 1.4 Riemann sphere

follows: to each point $z \in \mathbb{C}$ we assign an intersection point $\mathcal{Z}(\xi, \eta, \zeta)$ where the line between z and N meets the sphere $\mathbf{S}$ apart from N (Fig. 1.4), that is

$$\mathbb{C} \ni z \overset{p}{\longmapsto} \mathcal{Z}(\xi, \eta, \zeta) := \Big(\mathbf{S} \cap [z, N]\Big) \setminus \{N\}.$$

Obviously, if $\lim\limits_{n\to+\infty} z_n = \infty$, then the images $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ on the sphere approach to N. Therefore, it is naturally to determine p at the point at infinity as follows: $\infty \overset{p}{\longmapsto} N$. The mapping $p : \overline{\mathbb{C}} \mapsto \mathbf{S}$ is called the *stereographic projection*.

Let us examine properties of p. Obviously, this is a one-to-one mapping. To explicitly define the stereographic projection, we exclude the variable t from the parametric equations of the segment $[N, z]$: $\xi = tx$, $\eta = ty$, $\zeta = 1 - t$, where $t \in [0, 1]$, and as a result we obtain formulas for the inverse mapping p^{-1} :

$$x = \frac{\xi}{1-\zeta}, \qquad y = \frac{\eta}{1-\zeta}. \tag{1.12}$$

Since the coordinates of the point $\mathcal{Z}(\xi, \eta, \zeta)$ satisfy the relation

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4} \iff \xi^2 + \eta^2 = \zeta(1-\zeta),$$

then

$$x^2 + y^2 = \frac{\xi^2 + \eta^2}{(1-\zeta)^2} = \frac{\zeta}{1-\zeta} \implies \zeta = \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

From the last equation and formulas (1.12) we get formulas for the stereographic projection:

$$\xi = \frac{x}{1+x^2+y^2}, \qquad \eta = \frac{y}{1+x^2+y^2}, \qquad \zeta = \frac{x^2+y^2}{1+x^2+y^2}. \tag{1.13}$$

It follows from (1.12) and (1.13) that $p : \overline{\mathbb{C}} \mapsto \mathbf{S}$ is a *homeomorphism* (by definition, it is a one-to-one, onto continuous mapping with a continuous inverse).

Using the map p, we can identify the extended complex plane $\overline{\mathbb{C}}$ with the sphere $\mathbf{S}$. After this identification, the sphere $\mathbf{S}$ is called the *Riemann sphere*, or the sphere of complex numbers.

Exercise 1.3 Prove that under the stereographic projection an arbitrary circle or straight line on $\overline{\mathbb{C}}$ maps to a circle on $\mathbf{S}$, and the angle between curves in $\overline{\mathbb{C}}$ is equal to the angle between the images these curves on $\mathbf{S}$.

1.3 Complex-Valued Functions of a Real Variable

Consider a function $f : \mathbb{R} \mapsto \mathbb{C}$. Such a complex-valued function of a real variable can be represented as $f(t) = u(t) + iv(t)$, $t \in \mathbb{R}$, where $u(t) := \mathrm{Re}(f(t))$ and $v(t) := \mathrm{Im}(f(t))$. Thus, we see that each function $f : \mathbb{R} \mapsto \mathbb{C}$ can be viewed as a vector-function $\binom{u}{v}$ from $\mathbb{R}$ in $\mathbb{R}^2$ due to the geometrical interpretation of the set of complex numbers. Therefore, such concepts as the limit of a function, continuity, uniform continuity and many other properties of vector-functions of a real variable are automatically transferred to such functions. Let us recall some of them.

Definition 1.6 A number $A = \alpha + i\beta$ is the limit of a function $f : \mathbb{R} \mapsto \mathbb{C}$ at a point $t_0 \in \mathbb{R}$ (denoted as $\lim\limits_{t \to t_0} f(t) = A$), if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(t) - A| = \sqrt{(u(t)-\alpha)^2 + (v(t)-\beta)^2} < \varepsilon \quad \text{whenever} \quad |t - t_0| < \delta.$$

▲

As in the proof of the assertion in Exercise 1.2, the following statement can be easily proved.

Proposition 1.1 *The limit of a function $f : \mathbb{R} \mapsto \mathbb{C}$ exists at a point $t_0 \in \mathbb{R}$ and it is equal to $A = \alpha + i\beta$ if and only if there exist the limits of its real and imaginary parts and they are equal to α and β, respectively, i.e.,*

$$\lim_{t \to t_0} f(t) = A \iff \lim_{t \to t_0} u(t) = \alpha \ \textit{and} \ \lim_{t \to t_0} v(t) = \beta.$$

Definition 1.7 A function $f\colon [a,b] \mapsto \mathbb{C}$ is called continuous on the closed interval $[a,b] \subset \mathbb{R}$ (denoted as $f \in C([a,b])$), if for all $t_0 \in [a,b]$

$$\lim_{t \to t_0} f(t) = f(t_0).$$

The corresponding one-sided limits are considered at the endpoints a and b.

Definition 1.8 The derivative of a function $f\colon \mathbb{R} \mapsto \mathbb{C}$ at a point $t_0 \in \mathbb{R}$ (denoted by $f'(t_0)$) is called the limit

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}, \tag{1.14}$$

provided that it exists. ▲

Suppose the limit (1.14) exists. Then, according to Proposition 1.1

$$\begin{aligned} f'(t_0) &= \lim_{t \to t_0} \left(\frac{u(t) - u(t_0)}{t - t_0} + i \frac{v(t) - v(t_0)}{t - t_0} \right) \\ &= \lim_{t \to t_0} \frac{u(t) - u(t_0)}{t - t_0} + i \lim_{t \to t_0} \frac{v(t) - v(t_0)}{t - t_0} = u'(t_0) + i v'(t_0). \end{aligned}$$

Obviously, the reverse chain of equalities is also true. Thus, the following statement is correct.

Proposition 1.2 *The derivative of a function* $f\colon \mathbb{R} \mapsto \mathbb{C}$ *at* $t_0 \in \mathbb{R}$ *exists if and only if the derivatives of its real and imaginary parts exist at* t_0.

Example 1.2 The function $f(t) = \exp(it)$, $t \in \mathbb{R}$, has the derivative at each point and $f'(t) = i \exp(it)$. Indeed, for any $t \in \mathbb{R}$

$$(\exp(it))' = (\cos t + i \sin t)' = -\sin t + i \cos t = i(\cos t + i \sin t) = i \exp(it).$$

Due to Proposition 1.2 the equality $f'(t_0) = u'(t_0) + i v'(t_0)$ can be taken as an equivalent definition of the derivative of a complex-valued function of a real variable. We will apply this approach to define the integral of a complex-valued function of a real variable.

Definition 1.9 Let $f(t) = u(t) + iv(t),\ t \in [a, b]$, and the functions u and v be Riemann-integrable on the segment $[a, b]$.

$$\int_a^b f(t)\,dt \overset{def}{=} \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

▲

Exercise 1.4 Prove that Definition 1.9 is equivalent to the definition of the integral introduced through the limit of the Riemann sums of f, i.e.,

$$\int_a^b f(t)\,dt = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(\tau_k)\,\Delta t_k,$$

where $a = t_0 < t_1 < \ldots < t_n = b,\ \Delta t_k := t_k - t_{k-1},\ t_{k-1} \le \tau_k \le t_k,\ \ \Delta = \max_{k\in\{1,\ldots,n\}} \Delta t_k$.

It is easy to check the following properties of integrals of complex-valued functions of a real variable:

(1) $\forall\,\lambda, \mu \in \mathbb{C} \quad \displaystyle\int_a^b (\lambda f(t) + \mu g(t))\,dt = \lambda\int_a^b f(t)\,dt + \mu\int_a^b g(t)\,dt;$

(2) $\forall\, c \in (a, b) \quad \displaystyle\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt;$

(3) if F is the antiderivative of f, i.e., $F'(t) = f(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a);$$

(4) $\displaystyle\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$

Exercise 1.5 Using Exercise 1.4, prove the fourth property.

Remark 1.1 Not all properties of real-valued functions are automatically carried over to complex-valued functions of a real argument. For instance, the statement of the mean value theorem is incorrect. This fact is easy to check for such a continuous function: $e^{it},\ t \in [0, 2\pi]$. Evidently that $e^{it} \neq 0$ for all $t \in [0, 2\pi]$. Therefore, on the one hand, assuming that the mean value theorem holds, we have that $\int_0^{2\pi} e^{it}\,dt \neq 0$. On the other hand,

$$\int_0^{2\pi} e^{it}\,dt = \int_0^{2\pi} \cos t\,dt + i\int_0^{2\pi} \sin t\,dt = 0.$$

Exercise 1.6 Show that the statements of Rolle's theorem and Cauchy's mean value theorem are also incorrect for complex-valued functions of a real variable. Recall that Rolle's theorem states the following: if a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$. The second theorem establishes the relationship between the derivatives of two functions. Let functions f and g be continuous on $[a, b]$, differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is a point $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Remark 1.2 Since it is impossible to introduce an order relation for complex numbers, the Weierstrass theorem for complex-valued functions of a real variable is formulated as follows: for a continuous function $f : [a, b] \mapsto \mathbb{C}$, its modulus reaches its largest and smallest value on the closed interval $[a, b]$.

1.4 Curves in the Complex Plane

A curve is a geometric concept, the exact and at the same time quite general definition of which presents significant difficulties and is given in various branches of mathematics and textbooks in different ways. For those branches in which methods of the theory of functions dominate, the natural definition of a curve is to define it by parametric equations. In this text, we will take this approach and give the following definition of a curve and its elements.

Definition 1.10 A curve in $\mathbb{C}$ (in $\overline{\mathbb{C}}$) is called a continuous complex-valued function of a real variable: $z = \gamma(t)$, $t \in [a, b] \subset \mathbb{R}$.

Moreover, the points $\gamma(a)$ and $\gamma(b)$ are called the initial and end points of γ, respectively. The curve γ is said to be closed if $\gamma(a) = \gamma(b)$. ▲

Remark 1.3 In the notation of a curve $z = \gamma(t)$, $t \in [a, b]$, or $\gamma\colon [a, b] \mapsto \mathbb{C}$, we will always mean that the closed interval $[a, b]$ is a real closed interval, i.e., $[a, b] \subset \mathbb{R}$.

The image of such a continuous function is also often called a curve. In the course "Complex Analysis" it is convenient to distinguish between these concepts in order to better understand some new definitions and theorem proofs. The image of γ, i.e., the set $\gamma\big([a, b]\big)$, is called the *trace* of the curve γ and is denoted by E_γ.

Each curve specifies an *orientation* that can be interpreted as the direction of movement of the point $\gamma(t)$ along the trace E_γ from its initial point to its end as the parameter t increases from a to b.

Example 1.3 Let $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, and $\gamma(t) = z_1 + t(z_2 - z_1)$, $t \in [0, 1]$. The initial point of this curve is $z_1 = \gamma(0)$, the end point is $z_2 = \gamma(1)$. We will denote its trace by $[z_1, z_2]$ and refer to it as the segment joining z_1 and z_2.

Separating the real and imaginary parts in the equality $z = \gamma(t)$, we find the parametric equations, which are called a *parametrization* of the curve γ, namely $x = \mathrm{Re}(\gamma(t))$, $y = \mathrm{Im}(\gamma(t))$, where the parameter $t \in [a, b]$.

Example 1.4 Let $z = \gamma_1(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$x + iy = \cos t + i \sin t \iff \begin{cases} x = \cos t, \\ y = \sin t, \end{cases} \qquad t \in [0, 2\pi].$$

The last two equations determine a parametrization of this curve, whose trace is the unit circle centered at the origin. It is a closed curve that starts at point $(1, 0)$ and is oriented counterclockwise.

It is easy to see that the curve $z = \gamma_2(t) = e^{i2\pi t}$, $t \in [0, 1]$, has the same trace and orientation as the curve γ_1 from Example 1.4. For such curves, we will give the following definition.

Definition 1.11 Two curves

$$z = \gamma_1(t), \quad t \in [a_1, b_1], \quad \text{and} \quad z = \gamma_2(\tau), \quad \tau \in [a_2, b_2],$$

are called equivalent ($\gamma_1 \sim \gamma_2$), if there exists a real-valued function $\tau = \mu(t)$, $t \in [a_1, b_1]$, such that

(1) $\mu \in C([a_1, b_1])$ and it is strictly increasing on $[a_1, b_1]$;
(2) $\mu(a_1) = a_2, \quad \mu(b_1) = b_2$;
(3) $\gamma_1(t) = \gamma_2(\mu(t))$ for all $t \in [a_1, b_1]$.

Exercise 1.7 Prove that this relation between two curves is the equivalence relation, i.e., it is reflexive, symmetric and transitive.

Therefore, a curve can be understood as the corresponding equivalence class. It is clear that equivalent curves have the same trace and orientation.

Example 1.5 The curve γ_1 from Example 1.4 and the curve

$$z = \gamma_2(\tau) = e^{i2\pi\tau}, \ \tau \in [0, 1],$$

are equivalent. To show this we need to take the function $\tau = \mu(t) = t/2\pi$, $t \in [0, 2\pi]$, and verify the conditions from Definition 1.11.

Exercise 1.8 Prove that the curve γ_1 from Example 1.4 and the curve $z = \gamma_3(\tau) = e^{-i\tau}$, $\tau \in [0, 2\pi]$, are not equivalent.

Definition 1.12 A point z_0 is called a self-intersection point of a curve $z = \gamma(t)$, $t \in [a, b]$, if there are $t_1 \neq t_2$, $\{t_1, t_2\} \subset [a, b]$ such that

$$\gamma(t_1) = \gamma(t_2) = z_0.$$

If a curve γ is closed, then the point $\gamma(a) = \gamma(b)$ is not considered a self-intersection point. ▲

A curve without self-intersection points is called *simple* and a closed simple curve is said to be a *Jordan curve*.

Let $z = \gamma(t)$, $t \in [a, b]$, be a Jordan curve in $\mathbb{C}$. Then the Jordan curve theorem asserts that the trace E_γ divides the complex plane into an "interior" region, denoted by $\text{int}(\gamma)$, bounded by the trace, and an "exterior" region, denoted by $\text{ext}(\gamma)$ (Fig. 1.5), i.e.,

$$\mathbb{C} \setminus E_\gamma = \text{int}(\gamma) \cup \text{ext}(\gamma).$$

Intuitively, the statement of this theorem is obvious and there is no trouble verifying it when a curve is given explicitly. A rigorous proof of the general result is rather difficult, and we refer the reader to a topology text, e.g., [15]. The proof of the Jordan curve theorem for smooth Jordan curves can be found in [13, §4.8].

We will say that a Jordan curve γ has the *positive orientation* (denoted by γ^+) if its interior remains on the left when traversing its trace for increasing values of the parameter t from a to b (Fig. 1.5). Otherwise, it is *negatively oriented* (γ^-).

Definition 1.13 A curve $z = \gamma(t)$, $t \in [a, b]$, is called smooth, if γ is continuously differentiable on $[a, b]$, i.e., $\gamma \in C^1([a, b])$ and

$$\gamma'(t) \neq 0 \quad \text{for all} \quad t \in [a, b]. \tag{1.15}$$

If γ is a closed curve, the condition $\gamma'(a) = \gamma'(b)$ must also be satisfied. ▲

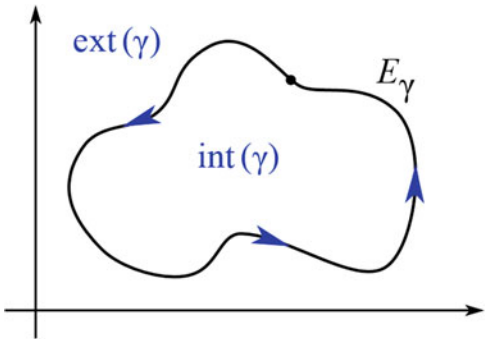

Fig. 1.5 The interior and exterior of a Jordan curve with the positive orientation

Let us find out the geometric meaning of (1.15). It is equivalent to $x'(t) + iy'(t) \neq 0$ for all $t \in [a, b]$. Since $\big(x'(t), y'(t)\big)$ is the tangent vector to E_γ at the point $\gamma(t)$, then the condition (1.15) means that at each point of E_γ there is a nonzero tangent vector that changes continuously.

Definition 1.14 A curve $z = \gamma(t)$, $t \in [a, b]$, is called piecewise smooth, if there is a partition $a = a_0 < a_1 < \ldots < a_n = b$ of $[a, b]$ such that for each $k \in \{0, 1, \ldots, n-1\}$ the curve $z = \gamma(t),\ t \in [a_k, a_{k+1}]$, is smooth.

Definition 1.15 A curve $z = \gamma(t)$, $t \in [a, b]$, is called rectifiable, if

- γ is differentiable on $[a, b]$ except, possibly, at a countable set of points and
- there exists a finite integral

$$\ell_\gamma := \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

The value ℓ_γ is called the length of γ.

An example of a piecewise smooth curve is a broken line. Note that a piecewise smooth curve is rectifiable.

Example 1.6 The curve

$$z = \gamma_5(t) = t^3 + it^2, \quad t \in [-1, 1],$$

is simple, however it is not smooth and piecewise smooth (Fig. 1.6).

The curve

$$z = \gamma_6(t) = \cos 2t \exp(it), \quad t \in [0, 2\pi] \quad \text{(four petal rose curve)}$$

is closed non-Jordan smooth curve that has the self-intersection point at the origin.

The curve

$$z = \gamma_7(t) = t\big(1 + i \sin \tfrac{1}{t}\big), \quad t \in \big[-\tfrac{1}{\pi}, \tfrac{1}{\pi}\big],$$

is simple and non-rectifiable, so it is not piecewise-smooth.

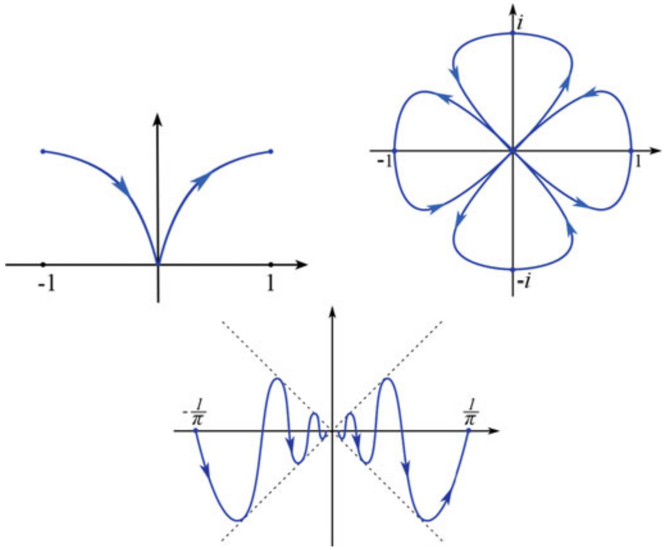

Fig. 1.6 The traces of curves γ_5 (left), γ_6 (right) and γ_7 (from below)

1.5 Basic Topological Concepts of the Complex Plane

Let us now recall some topological concepts. An *open disk* of radius $r > 0$ centered at a point $a \in \mathbb{C}$ is a set of all points of distance less than r from a, i.e.,

$$B_r(a) := \{z \in \mathbb{C} \colon\ |z - a| < r\}.$$

It is also called the *r-neighborhood* of a. The R-neighborhood of the point at infinitely is the set

$$B_R(\infty) := \{z \in \mathbb{C} \colon\ |z| > R\} \cup \{\infty\}.$$

Definition 1.16 A set $D \subset \overline{\mathbb{C}}$ is called open, if each point of D is contained in D together with some of its r-neighborhood, i.e.,

$$\forall\, z_0 \in D\ \ \exists\, r > 0 \quad \text{such that} \quad B_r(z_0) \subset D.$$

Definition 1.17 Let $D \subset \overline{\mathbb{C}}$ and $z_0 \in \overline{\mathbb{C}}$. The point z_0 is called a limit point of the set D if every r-neighborhood of z_0 contains at least one point of D different from z_0 itself, i.e.,

$$\forall\, r > 0\ \ \exists\, z \in D \quad \text{such that} \quad z \in B_r(z_0) \setminus \{z_0\}.$$

Definition 1.18 A set $D \subset \overline{\mathbb{C}}$ is called closed if it contains all its limit points.

Example 1.7 Let $D := \mathbb{Z}$. Then, the set D is closed in $\mathbb{C}$, since the set of its limit points is empty and $\emptyset \subset D$. However, it is not closed in $\overline{\mathbb{C}}$ since it does not contain its limit point ∞.

The joining to a set $D \subset \overline{\mathbb{C}}$ all its limit points is called the *closure* of D and denoted by $\overline{D}$. For example, the closure of the open disk $B_r(a)$ is the closed disk

$$\overline{B_r(a)} := \{z \in \mathbb{C} : |z - a| \leq r\}.$$

A set $D \subset \overline{\mathbb{C}}$ is said to be *path-connected* if for any two distinct points in D there is a curve whose trace belongs to D and connects these points (starting at one point and ending at the other).

A set $D \subset \overline{\mathbb{C}}$ is called a *domain* if it is open and path-connected.

Definition 1.19 Let D be a domain in $\overline{\mathbb{C}}$. The set $\partial D := \overline{D} \setminus D$ is called the boundary of D.

Exercise 1.9 Prove that the boundary of a domain is the closed set.

There are several approaches to introducing the concept of simply connected domains: a domain is simply connected if its fundamental group is trivial; a domain D in $\mathbb{R}^m$ is simply connected if any closed curve in D is homotopic to a point in this domain (see Definition 4.4 and Exercise 4.4); one can define simply connectedness through the general concept of connectedness of a set in a topological space. In this course an easier-to-understand definition of simply connectedness is proposed.

Definition 1.20 A domain D is said to be simply connected (also called 1-connected) in $\mathbb{C}$ if for any Jordan curve γ, whose trace belongs to D, the interior of γ is fully contained in D, i.e., $\text{int}(\gamma) \subset D$.

A domain D is said to be simply connected in $\overline{\mathbb{C}}$ if for any Jordan curve γ, whose trace belongs to D and $\infty \notin E_\gamma$, obligatorily either $\text{int}(\gamma) \subset D$ or $\text{ext}(\gamma) \subset D$.

Domains that are not simply connected are called multiply connected. ▲

Intuitively, a simply connected domain is a domain "without holes".

Definition 1.21 The connectedness order of a domain $D \subset \overline{\mathbb{C}}$ is the number of path-connected closed components of the boundary ∂D, which do not intersect.

Example 1.8 Consider the domain $D_1 = \{z : |z| > 1\}$. Obviously, it is multiply connected in $\mathbb{C}$, since the interior of the circle $\{z : |z| = 2\}$ is not a subset of D_1. In $\overline{\mathbb{C}}$ its connectedness order is 2, because the boundary of D_1 has two path-connected closed components that do not intersect, namely $\{z : |z| = 1\}$ and $\{\infty\}$.

Example 1.9 Due to the second part of Definition 1.20 the domain $D_2 = D_1 \cup \{\infty\}$ is 1-connected in $\overline{\mathbb{C}}$.

Example 1.10 The connectedness order of the domain

$$D_3 = \{z : \ |z| < 2\} \setminus \left(\bigcup_{k=1}^{N} \{z : \ z = x + \frac{i}{2^k}, \ x \in [\tfrac{1}{4}, \tfrac{3}{4}]\} \right)$$

is equal to $N + 1$, where $N \in \mathbb{N}$.