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Complex Analysis

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Taras Mel'nyk
Faculty of Mathematics and Mechanics
Taras Shevchenko National University
of Kyiv
Kyiv, Ukraine

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To Warriors of Light

Preface

The proposed textbook contains theoretical material that corresponds to the educational program "Complex Analysis" of training specialists of the educational qualification level "Bachelor" for students of the Faculty of Mechanics and Mathematics of the Taras Shevchenko National University of Kyiv.

Being a student of Lviv State University, where the world-famous school of functional analysis was founded, headed by Professor Stefan Banach, as well as a graduate student of Moscow State University, where I attended Professor Shabbat's course on Complex Analysis of Several Variables [10], I was given the opportunity to observe the best teaching traditions of these famous old universities. Taking them into account, I developed a two-semester course "Complex Analysis" in 1993 and since then I have been teaching this course regularly at the Faculty of Mechanics and Mathematics of the Taras Shevchenko National University of Kyiv.

During this time, the prepared lectures have been expanded and modified. Since 2004, they have been presented in electronic form (in Ukrainian) on the website of the Department of Mathematical Physics: http://www.matfiz.univ.kiev.ua/books. Then two new chapters and many figures were added to this electronic version. Several books [1, 2, 4, 9, 11] were useful guides in preparing lecture notes and then for the Ukrainian version of the textbook [5].

Since then, this textbook has been read and downloaded from my Research Gate webpage more than 2500 times by readers from different countries, although it was published in Ukrainian. I was also encouraged by the positive feedback from many readers and former students of mine who now teach at various universities (e.g., Professor Oleksandr Misiats used my lectures to teach a course on complex analysis for masters at the Courant Institute of Mathematical Sciences), and many of them recommended the publication of this book in English.

The English version is a substantial extension of the Ukrainian one. Some important new theorems and their proofs are added, as well as many new examples, exercises, and figures. I am very grateful to Springer Publisher for supporting my proposal and publishing this textbook.

What Advantages Does This Book Offer over Competitive Titles and What Is Unique About It?

- There are three concepts for constructing the theory of analytic functions, which are associated with Cauchy, Riemann, and Weierstrass, respectively. In most textbooks, the theory is based on the Cauchy or Weierstrass approach. Cauchy's approach focuses on the integral calculus of functions of a complex variable, and the key topics are contour integration and the Cauchy integral formula. For Weierstrass' approach, power series are central, and the key topic is spaces of analytic functions represented as power series. As a result, the concept of conformal mappings, which occupies a central place in complex analysis, appears only in the middle or at the end of textbooks (courses).

In the textbook, I adhere to the Riemann concept, in which the differentiability of complex-valued functions of a complex variable is central. This approach makes it possible to introduce the concept of conformal mappings at the beginning (Chap. 2) and to study properties of such mappings, including their hydromechanical interpretation and geometric meaning of the modulus and argument of the derivative. In addition, the material in the textbook is selected in such a way as to simplify the proof of subsequent theorems. This leads to a significant reduction in the volume of the textbook.

- I was surprised to find that many textbooks limit themselves to linear-fractional, power, exponential, and logarithmic functions only. For example, the Joukowsky function, which is used in many applications, is not considered at all. Chapter 3 examines in detail the properties of many elementary analytic functions, including trigonometric and hyperbolic functions, and their inverse functions. It is shown how to construct Riemann surfaces of these inverse functions. Such a detailed analysis of the properties of various elementary analytic functions allows students to better understand the general properties of analytic functions.
- There are many topics in this book that are often missing in other texts, namely the conformal mapping criterion and its proof; the equivalence of three approaches to the construction of the theory of analytic functions; the notion of an integral for an analytic function along an arbitrary curve (not necessarily piecewise-smooth) is extended thanks to the introduction and study of properties of an antiderivative along a curve; the theory of global analytic functions and their Riemann surfaces; the inverse function theorem in the general case, Lagrange-Bürmann and Puiseux series.
- The Complex Analysis course is a natural continuation of the theory of real functions. Therefore, the tutorial contains many examples that compare properties of analytic functions of a complex variable and real-valued functions, and also show their differences.
- Many textbooks contain a list of problems and exercises after each section, most of which are typical and standard. As there are many taskbooks for this course (e.g., [3, 6, 8, 12, 14]), the author has taken a different approach. Problems, many of them invented by the author, are presented in the textbook either immediately after the definition of a new concept in order to better understand it or immediately after the theorem that must be applied to solve it. These are theoretical tasks and their purpose is to help students actively and informally assimilate the material, as well as to illustrate the difficult points of the theory. Their solutions do not require sophisticated calculations, but only a good understanding. I like to call them "problems solved in three lines."

- In addition, the textbook contains many full-color figures that successfully illustrate the essence and basic properties of theoretical concepts.

Description of the Contents

The textbook contains a brief but fairly complete exposition of the main ideas of the theory of functions of a complex variable, with clear and rigorous proofs, the presence of which is mandatory in textbooks for mathematics departments. It consists of nine chapters.

The first chapter is introductory and introduces complex numbers and their various forms. We then look at the complex plane, the extended complex plane with its geometric interpretation, and their basic topological concepts. Complex-valued functions of a real variable and various curves in the complex plane are considered in more detail.

The second chapter introduces analytic functions as functions that are differentiable with respect to a complex variable. This leads to the proof of the Cauchy-Riemann theorem and to the concept of harmonic conjugate functions. We also define conformal mappings and study properties of such mappings, including their hydromechanical interpretation and the geometric significance of the modulus and argument of the derivative.

Chapter 3 examines in detail the properties of many elementary analytic functions and their inverse. These inverse functions turn out to be multivalued functions. Therefore, we first introduce the empirical concept of a Riemann surface for such functions and show how to construct Riemann surfaces for these inverse functions. A rigorous topological approach for Riemann surfaces is given in Chap. 8.

The theory of integration of complex-valued functions of a complex variable along a curve is considered in Chap. 4. Here we prove the Cauchy-Goursat theorem for triangles, the general Cauchy theorem for homotopic curves and corollaries to this theorem, as well as Cauchy's integral formula. In addition, the notion of an integral for an analytic function along an arbitrary curve (not necessarily piecewise-smooth) is expanded thanks to the introduction and study of properties of an antiderivative along a curve. Theorems on the existence of a local antiderivative, an antiderivative along a curve, and an antiderivative in the whole domain are proved.

Chapter 5 presents the most important application of the Cauchy integral formula, namely the proof that an analytic function in a disk can be expanded in a power series. As a result, we get simple and nice proofs of Liouville's theorem, the fundamental theorem of algebra, Morera's theorem, and the equivalence of three approaches to the construction of the theory of analytic functions. The theorem on the uniqueness of analytic functions coinciding on a certain sequence is also proved. This theorem allows one to characterize isolated zeros of an analytic function and their concentration, and as a consequence, it is easy to prove the theorem about the factorization of polynomials.

Chapter 6 deals with Laurent series. The properties of such series are studied and it is proved that an analytic function in an annulus can be expanded into a Laurent series. The connection between Laurent series and Fourier series is also demonstrated. Isolated singularities of analytic functions are classified and the behavior of analytic functions in neighborhoods of singularities, including the singularity at infinity, is studied. The chapter ends with the classification of analytic functions with respect to their isolated singularities and the proof of theorem on a meromorphic function.

Residue theory is considered in Chap. 7. Here various formulas for the calculation of residues are proved and a wide variety of applications of this theory are demonstrated, namely different methods for calculating integrals, expansions of meromorphic functions into the series of elementary fractions, and expansions of entire functions into infinite products. In addition, logarithmic residues are defined, the argument principle and Rouché's theorem are proved. Using the latter, we derive simple sufficient conditions for the conformity of a function, Hurwitz's theorem, and other corollaries.

The theory of analytic continuation is covered in Chap. 8. Here such important theorems as the principle of analytic continuation by continuity, the Schwarz reflection principle and the monodromy theorem are proved. The topological approach is used to present the theory of global analytic functions (such a function is a set of all analytic function elements obtained from some initial element by analytic continuation along all possible curves) and their Riemann surfaces.

The purpose of Chap. 9, devoted to the qualitative properties of analytic functions, is to prove the Riemann mapping theorem in the general case. Along the way, we prove the open mapping theorem, the maximum modulus principle, Schwarz's lemma, the inverse function theorem, and Montel's theorem. We also discuss the Lagrange-Bürmann formula and Puiseux series, and deduce the theorem on conformal automorphisms of canonical domains.

For Which Courses Would the Textbook Be Suitable?

The theory of complex-valued functions of a complex variable belongs in a basic course of mathematics faculties as well as physics and some engineering departments of many universities. Therefore, the book can be used in a two-semester course for undergraduate mathematics majors, a one-semester course for physics or engineering specialties, or a one-semester course for first-year graduate students in mathematics. Essential prerequisites include basic courses in mathematical analysis of real-valued functions of one and several variables as well as courses in linear algebra and elementary topology.

There are several topics in this textbook that can be in one or another advanced course; notably, the theory of global analytic functions and the general approach to the theory of Riemann surfaces of global analytic functions (Sects. 8.3–8.6) can be included in advanced course in topology; Lagrange-Bürmann formula and Puiseux series (Sect. 9.2) can be in approximation theory or asymptotic analysis; conformal isomorphisms and automorphisms and Montel's theorem (Sects. 9.3 and 9.4) will find their place in an advanced course in functional analysis.

There are, of course, many other interesting topics in the theory of complex analysis. The interested reader can familiarize himself with them, for example, in books [4, 10, 13].

Please send your feedback and suggestions on the content of the textbook to the email address: melnyk@imath.kiev.ua

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Taras Mel'nyk

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It gives me great pleasure to thank those who have helped me to write this textbook. First, I would like to thank Dr. Remi Lodh at Springer Heidelberg for his support and cooperation, and for his motivational letters, which have been of more value to me than he can know.

It took me more than two years to write this book, and much of that time coincided with the full-scale war unleashed by the Russian regime. Some sections were written in a bomb shelter in Kyiv during air and rocket attacks on the city. That is why I dedicate this book to all those who are part of the fight against this brutal and terrible aggression.

The writing of this book took place in parallel with my research at the University of Stuttgart, following a persuasive invitation from Professor Christian Rohde in May 2022, and supported first by the Humboldt Foundation until the end of February 2023, and then by the German Research Foundation (the research project SFB 1313, Number 327154368). I am therefore very grateful to a remarkable number of people who have made my family's stay in Stuttgart safe, pleasant, and, for me, productive.

I am sincerely grateful to my former graduate student Andriy Popov for transforming many of my figures into electronic format. Special thanks are due to Professor Andriy Olenko of La Trobe University and Professor Oleksandr Misiats of Virginia Commonwealth University, who read the first three chapters and provided a number of helpful comments and suggestions.

Instructions for Readers

In the text, you will come across the following symbols:

The equality sign with "def" above ($\stackrel{def}{=}$) means that the left-hand side is being defined by the right-hand side.
The equality sign with ":" before ($:=$) means that the left-hand side is the designation for the right-hand side.
□ indicates the end of the proof;
▲ indicates the end of a definition, remark, or example where appropriate.

Also, when a term is defined for the first time outside a formal "Definition," the word is italicized.

In addition to the generally accepted symbols

Symbol	Meaning
$\exists$	There exists
$\forall$	For all
!	Unique
$\Longrightarrow$	Implies
$\Longleftrightarrow$	Is equivalent to

which are often used when presenting mathematical arguments in statements, definitions, and proofs of theorems, the following symbols are also often found in the text:

Ω	domain (an open and path-connected set) in the complex plane
$\overline{\Omega}$	the closure of Ω
$\partial\Omega$	the boundary of Ω
$\mathcal{A}(\Omega)$	ring of analytic functions in Ω
$B_r(a)$	open disk of radius $r > 0$ centered at a point a
$\overline{B_r(a)}$	closed disk
E_γ	trace of a curve γ
$f_n \overset{M}{\rightrightarrows} f$	uniform convergence on a set M of a function sequence $\{f_n\}_{n\in\mathbb{N}}$

Contents

1 Complex Numbers and Complex Plane

Abstract

In this chapter we recall some concepts from basic courses in mathematical analysis of real-valued functions of one and several variables as well as from a course in linear algebra, namely complex numbers and their various forms, arithmetic operations on them (Sect. 1.1), and basic topological notions in the vector space $\mathbb{R}^2$ (Sect. 1.5). The novel notion of the stereographic projection in Sect. 1.2 provides a geometric interpretation of the extended complex plane. Complex-valued functions of a real variable and various curves in the complex plane are considered in more detail in Sects. 1.3 and 1.4, respectively.

1.1 Complex Numbers

A number is the basic concept of mathematics, which evolved throughout the history of humankind. The emergence and formation of this concept went hand in hand with the emergence and development of mathematics. Practical human activities, on the one hand, and internal needs of mathematics, on the other, determined the development of the concept of numbers.

The necessity of counting objects led to the emergence of the concept of the set of natural numbers ($\mathbb{N}$). Starting with natural numbers, the number system expanded in response to the need to describe quantities that could not be accommodated within the existing (previous) number system. As a result, sets of integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), and real numbers ($\mathbb{R}$) appeared in mathematics such that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Complex numbers arose from the need to find solutions of polynomial equations, for example, $x^2 + 1 = 0$. The first written mention of complex numbers as square roots of negative numbers can be found in Girolamo Cardano's book in 1545. For nearly two centuries, complex numbers remained mysterious, had a poor reputation and were generally not considered legitimate. The active use of complex numbers

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in calculations began with works of Leonard Euler (1707–1783), in particular, with his famous formula (1.9) introduced in 1748. The first systematic description of complex numbers, arithmetic operations on them and their geometric interpretation was conducted by Carl Gauss (1777–1855) in his memoir "Theoria residuorum biquadraticorum" (1828, 1832). The term "complex number" is due to C. Gauss in 1831. The 2000-year and engaging history of complex numbers is presented in [7].

Definition 1.1 The set $\mathbb{C}$ of complex numbers is the set of ordered pairs (x, y) of real numbers x and y, equipped with algebraic operations of addition and multiplication:

$$(x_1, y_1) + (x_2, y_2) \stackrel{def}{=} (x_1 + x_2,\ y_1 + y_2), \tag{1.1}$$

$$(x_1, y_1) \cdot (x_2, y_2) \stackrel{def}{=} (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2). \tag{1.2}$$

▲

It is clear that two complex numbers (x_1, y_1) and (x_2, y_2) are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. From the definition it follows that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0), \qquad (x_1, 0) \cdot (x_2, 0) = (x_1x_2, 0). \tag{1.3}$$

Thus, addition and multiplication on complex numbers of the form $(x, 0)$ coincide with the corresponding algebraic operations on real numbers. Therefore, we can identify each real number x with the complex number $(x, 0)$, i.e., $\mathbb{R} \ni x := (x, 0) \in \mathbb{C}$, and after this identification one can state that $\mathbb{R} \subset \mathbb{C}$. In addition, one can verify that for any real number a

$$a \cdot (x, y) = (a, 0) \cdot (x, y) = (ax, ay). \tag{1.4}$$

The complex number $(0, 1)$ is called the *imaginary unit* and is denoted by the Latin letter i. It is easy to check that

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1 \quad \text{and} \quad (0, y) = (0, 1) \cdot (y, 0) = iy.$$

Based on these notations, any complex number can be represented as

$$(x, y) = (x, 0) + (0, y) = x \cdot (1, 0) + y \cdot (0, 1) = x + iy,$$

which is called the *algebraic form* of a complex number. The algebraic form of a complex number is usually denoted by one letter $z := x + iy$. Moreover, the number x is called the *real part* of the complex number z and is denoted $\mathrm{Re}(z)$, while the number y is called the *imaginary part* of z and is denoted $\mathrm{Im}(z)$.

The *conjugate* of a complex number $z = x + iy$ is the complex number $\overline{z} := x - iy$. The *modulus* or *absolute value* of z is defined by

$$|z| := \sqrt{x^2 + y^2}.$$

Note that $|z| = |\overline{z}|$ and $z \cdot \overline{z} = x^2 + y^2 = |z|^2$.

Subtraction and division of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are defined as follows:

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2), \qquad \frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} \ (z_2 \neq 0). \tag{1.5}$$

The set of complex numbers with respect to the introduced operations forms a *field*, i.e., it is an *Abelian group*[1] with respect to addition with $0 = (0, 0)$ as the additive identity; the nonzero elements in $\mathbb{C}$ form an Abelian group with respect to multiplication with $1 = (1, 0)$ as the multiplicative identity; and multiplication distributes over addition.

Exercise 1.1 Prove that all these field properties are fulfilled.

From (1.3) it follows that the field of complex numbers includes the field of real numbers as a subfield. The reader is invited to make sure that all extensions of the field $\mathbb{R}$ obtained by joining the root of the equation $x^2 + 1 = 0$ to it are isomorphic to the field $\mathbb{C}$.

Based on Definition 1.1, (1.1) and (1.4) we can assert that the set of complex numbers is a *real vector space*,[2] or more precisely, the vector space $\mathbb{R}^2$. This makes the complex numbers a Cartesian plane (coordinate plane), called the *complex plane*. Clearly that the real numbers lie on the horizontal x-axis, called the *real axis*, and the y-axis is called the *imaginary axis* of the complex plane. This allows to give the geometric interpretation of complex numbers and arithmetic operations defined on $\mathbb{C}$ and, conversely, to express some geometric properties and constructions in terms of complex numbers. For instance, conjugation is the reflection symmetry with respect to the real axis; multiplication by -1 is the central symmetry about the origin.

[1] Recall that a *group* is a set of elements together with a binary operation on this set such that the following three requirements, known as group axioms, are satisfied: the binary operation is associative, there is a unique identity with respect to this operation, and every element of this set has an inverse with respect to this operation. In an Abelian group, the binary operation is additionally commutative.

[2] It is a set of objects called vectors, which may be added together and multiplied by real numbers (scalar multiplication). This set is an Abelian group under addition, and scalar multiplication has the following properties: $x(\mathbf{u} + \mathbf{v}) = x\mathbf{u} + x\mathbf{v}$, $(x + y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$, $(xy)\mathbf{v} = x(y\mathbf{v})$, and $1\mathbf{v} = \mathbf{v}$ for all $x, y \in \mathbb{R}$ and all vectors $\mathbf{u}$ and $\mathbf{v}$.

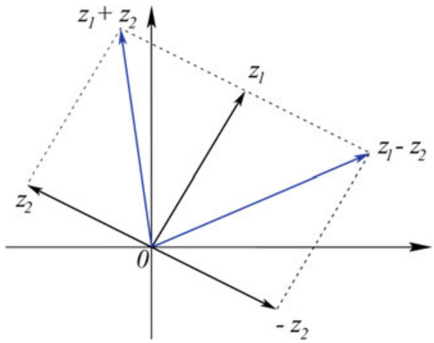

Fig. 1.1 Geometric interpretation of the sum and subtraction two complex numbers

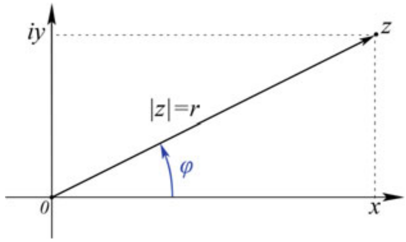

Fig. 1.2 The argument of z

Furthermore, sum and subtraction of two complex numbers coincides with the sum and subtraction of the corresponding vectors in $\mathbb{R}^2$ (Fig. 1.1). The absolute value of a complex number is the length of the corresponding vector (the usual Euclidean norm) in the vector space $\mathbb{R}^2$.

Other Forms of Writing Complex Numbers

It is known that the position of the point (x, y) is also determined by the pair (r, φ), where r is the distance of (x, y) to the origin and φ is the counterclockwise angle (measured in radians) between the positive x-axis and the ray from the origin through (x, y); the values r and φ are called the *polar coordinates* of (x, y), and

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{1.6}$$

Substituting these relations into the algebraic form of a complex number $z = x + iy$ yields the *trigonometric form:*

$$z = |z|\,(\cos\varphi + i\,\sin\varphi), \tag{1.7}$$

where the angle φ is called the *argument* of z (Fig. 1.2).

Note that the argument of each nonzero complex number z is defined ambiguously, and up to the term $2\pi k$ ($k \in \mathbb{Z}$); in addition, there exists a unique angle $\varphi_0 \in (-\pi, \pi]$ such that $\varphi = \varphi_0 + 2\pi k$. This angle φ_0 is called the *principal value* of the argument of z and is denoted $\arg(z)$. The set of all arguments of z is denoted by

$$\mathrm{Arg}(z) := \{\varphi_0 + 2\pi k : \ k \in \mathbb{Z}\}.$$

The argument of 0 is not defined. The principal value of the argument of z can be considered as a real-valued function defined on $\mathbb{C} \setminus \{0\}$ and it can be expressed from the formulas (1.6) in terms of the inverse trigonometric function arctan :

$$\arg(z) = \begin{cases} \arctan \frac{y}{x}, & \text{if } \ x > 0; \\ \pi + \arctan \frac{y}{x}, & \text{if } \ x < 0, \ \ y > 0; \\ \pi, & \text{if } \ x < 0, \ \ y = 0; \\ -\pi + \arctan \frac{y}{x}, & \text{if } \ x < 0, \ \ y < 0. \end{cases} \tag{1.8}$$

Example 1.1 It is easy to verify that

- the principal value of the argument of each positive number x $(y = 0)$ is zero, and the set of all arguments of x is $\mathrm{Arg}(x) = \{2\pi k \colon k \in \mathbb{Z}\}$;
- $\arg(1 - i) = -\frac{\pi}{4}$, and $\mathrm{Arg}(1 - i) = \{-\frac{\pi}{4} + 2\pi k \colon k \in \mathbb{Z}\}$;
- $\arg(-3) = \pi$, and $\mathrm{Arg}(-3) = \{\pi + 2\pi k \colon k \in \mathbb{Z}\}$.

Let us define the exponential function of an imaginary number $i\alpha$ by the following way:

$$e^{i\alpha} \overset{def}{=} \cos\alpha + i \sin\alpha \quad (\alpha \in \mathbb{R}), \tag{1.9}$$

which is known as Euler's formula (the proof is given in Example 5.4). From (1.9) it is clear that $|e^{i\alpha}| = 1$. In addition, it is easy to verify that

$$\begin{aligned} e^{i\alpha_1} \cdot e^{i\alpha_2} &= (\cos\alpha_1 + i \sin\alpha_1)(\cos\alpha_2 + i \sin\alpha_2) \\ &= \big(\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2\big) + i\big(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2\big) \\ &= \cos(\alpha_1 + \alpha_2) + i \sin(\alpha_1 + \alpha_2) \\ &= e^{i(\alpha_1 + \alpha_2)}. \end{aligned} \tag{1.10}$$

In (1.10) we used the addition formulas for sine and cosine. Similarly, it is proved that

$$\left(e^{i\alpha}\right)^n = e^{in\alpha}, \qquad \frac{e^{i\alpha_1}}{e^{i\alpha_2}} = e^{i(\alpha_1 - \alpha_2)}.$$

Using Euler's formula (1.9), we get from (1.7) the *exponential form* of a complex number: $z = |z|\, e^{i\varphi}$. This form well illustrates the essence of multiplication and division of complex numbers. If $z_1 = |z_1|\, e^{i\varphi_1}$ and $z_2 = |z_2|\, e^{i\varphi_2}$, then

$$z_1 \cdot z_2 = |z_1||z_2|\, e^{i(\varphi_1+\varphi_2)}, \qquad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\, e^{i(\varphi_1-\varphi_2)} \ \ (z_2 \neq 0).$$

Thus, when multiplying (respectively dividing) two complex numbers, their moduli are multiplied (resp. divided):

$$|z_1 \cdot z_2| = |z_1||z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

and arguments are added (resp. subtracted):

$$\varphi_1 + \varphi_2 \in \mathrm{Arg}(z_1 \cdot z_2), \qquad \varphi_1 - \varphi_2 \in \mathrm{Arg}(\frac{z_1}{z_2}).$$

Definition 1.2 A complex number z is called an n^{th} root of a complex number a, if $z^n = a$. Here, $n \in \mathbb{N}$ and $a \neq 0$. ▲

Let us derive a formula for finding n^{th} roots of a complex number $a = |a|\, e^{i\theta}$ ($\theta \in (-\pi, \pi)$). If $z = |z|\, e^{i\varphi}$ is an n^{th} root of a, then according to the definition

$$|z|^n\, e^{in\varphi} = |a|\, e^{i\theta} \iff \begin{cases} |z|^n = |a|, \\ n\varphi = \theta + 2\pi k, \quad k \in \mathbb{Z}, \end{cases}$$

whence

$$\begin{cases} |z| = \sqrt[n]{|a|}, \\ \varphi_k = \frac{\theta+2\pi k}{n}, \quad k \in \mathbb{Z}, \end{cases}$$

that is, the n^{th} roots of a are numbers

$$z_k = \sqrt[n]{|a|}\, e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}, \quad k \in \mathbb{Z}. \tag{1.11}$$

It is easy to see that among these complex numbers there are exactly n different numbers. Indeed, the numbers $z_0, \ldots, z_{n-1}$ are different since their arguments

$$\varphi_0 = \frac{\theta}{n}, \quad \varphi_1 = \frac{\theta + 2\pi}{n}, \ \ \ldots \ \ , \varphi_{n-1} = \frac{\theta + 2\pi(n-1)}{n}$$

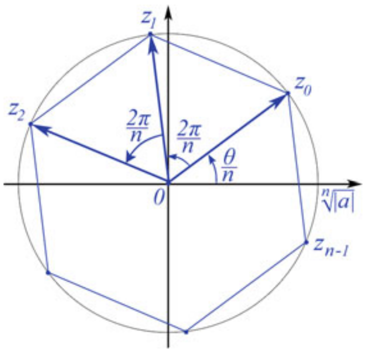

Fig. 1.3 The n^{th} roots of a complex number a

are various and differ from each other less than 2π. For any other number z_k, $k \notin \{0, \ldots, n-1\}$ there exist numbers $p \in \mathbb{Z}$ and $q \in \{0, 1, \ldots, n-1\}$ such that $k = pn + q$. This means that $z_k = z_q$.

Thus, the equation $z^n = a$ has n different roots $z_0, \ldots, z_{n-1}$, defined by the formula (1.11) and located at the vertices of a regular n-sided polygon inscribed in a circle of radius $\sqrt[n]{|a|}$ centered at the point 0 (Fig. 1.3).

1.2 Sequences in the Complex Plane: Extended Complex Plane

Since the modulus of a complex number is just the usual Euclidean norm in the vector space $\mathbb{R}^2$, it is natural to introduce the distance between two complex numbers as follows

$$d(z_1, z_2) := |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. In addition, we can naturally introduce the convergence of a sequence of complex numbers as the convergence of a sequence of the corresponding vectors in $\mathbb{R}^2$. We hope that the reader is familiar with the properties of convergent sequences from real analysis of several variables. Nevertheless, let us briefly recall the main definitions and properties.

Definition 1.3 A sequence $\{z_n = x_n + iy_n\}_{n\in\mathbb{N}}$ of complex numbers is said to converge to a complex number $a = \alpha + i\beta$ (denoted as $\lim\limits_{n\to+\infty} z_n = a$), if

$$\lim_{n\to+\infty} |z_n - a| = 0,$$

i.e., for every $\varepsilon > 0$, there exists an integer N such that

$$|z_n - a| < \varepsilon \quad \text{for all } n \geq N.$$

▲

From Definition 1.3 follows a statement, which is offered to the reader as an exercise.

Exercise 1.2 Prove that a sequence $\{z_n = x_n + iy_n\}_{n\in\mathbb{N}}$ converges to the complex number $a = \alpha + i\beta$ if and only if

$$\lim_{n\to+\infty} x_n = \alpha \quad \text{and} \quad \lim_{n\to+\infty} y_n = \beta.$$

Definition 1.4 It is said that a sequence $\{z_n\}_{n\in\mathbb{N}}$ of complex numbers converges to infinity ($\lim\limits_{n\to+\infty} z_n = \infty$), if

$$\lim_{n\to+\infty} |z_n| = +\infty,$$

i.e., for every $R > 0$, there exists an integer N such that

$$|z_n| > R \quad \text{for all} \;\; n \geq N.$$

▲

The symbol "∞" is called the *point at infinity.*

Definition 1.5 The set $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the extended complex plane.

Obviously, that each sequence in $\overline{\mathbb{C}}$ contains a convergent subsequence. This is called the *principle of compactness* in $\overline{\mathbb{C}}$. The point at infinity does not participate in algebraic operations, i.e. it cannot be multiplied or added to complex numbers. In real analysis, points labeled $+\infty$ and $-\infty$ produce the two-point compactification of the set of real numbers.

Geometric Interpretation of $\overline{\mathbb{C}}$
Consider the space

$$\mathbb{R}^3 = \big\{(\xi, \eta, \zeta) : \;\xi \in \mathbb{R}, \;\; \eta \in \mathbb{R}, \;\; \zeta \in \mathbb{R}\big\},$$

in which the ξ-axis coincides with the real axis, η-axis coincides with the imaginary axis, and ζ-axis is perpendicular to the complex plane (Fig. 1.4). The sphere

$$\mathbf{S} := \left\{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \; \xi^2 + \eta^2 + \left(\zeta - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}\right\}$$

is tangent to the complex plane at the origin. The point $N = (0, 0, 1)$, which lies on the sphere, will be called the "north pole". Define a mapping $p\colon \overline{\mathbb{C}} \mapsto \mathbf{S}$ as

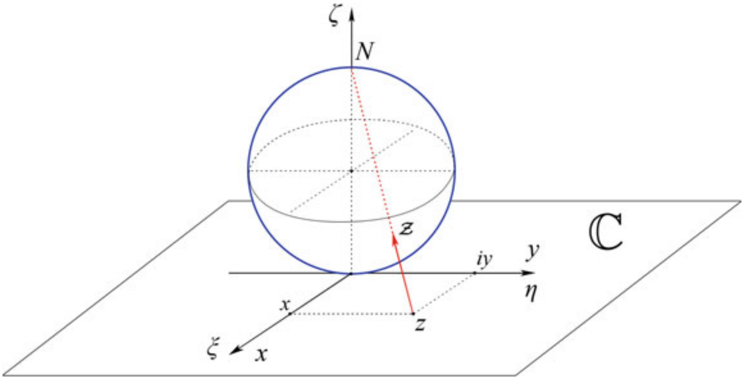

Fig. 1.4 Riemann sphere

follows: to each point $z \in \mathbb{C}$ we assign an intersection point $\mathcal{Z}(\xi, \eta, \zeta)$ where the line between z and N meets the sphere $\mathbf{S}$ apart from N (Fig. 1.4), that is

$$\mathbb{C} \ni z \overset{p}{\longmapsto} \mathcal{Z}(\xi, \eta, \zeta) := \Big(\mathbf{S} \cap [z, N]\Big) \setminus \{N\}.$$

Obviously, if $\lim\limits_{n \to +\infty} z_n = \infty$, then the images $\{\mathcal{Z}_n\}_{n \in \mathbb{N}}$ on the sphere approach to N. Therefore, it is naturally to determine p at the point at infinity as follows: $\infty \overset{p}{\longmapsto} N$. The mapping $p : \overline{\mathbb{C}} \mapsto \mathbf{S}$ is called the *stereographic projection*.

Let us examine properties of p. Obviously, this is a one-to-one mapping. To explicitly define the stereographic projection, we exclude the variable t from the parametric equations of the segment $[N, z]$: $\xi = tx$, $\eta = ty$, $\zeta = 1 - t$, where $t \in [0, 1]$, and as a result we obtain formulas for the inverse mapping p^{-1} :

$$x = \frac{\xi}{1 - \zeta}, \qquad y = \frac{\eta}{1 - \zeta}. \tag{1.12}$$

Since the coordinates of the point $\mathcal{Z}(\xi, \eta, \zeta)$ satisfy the relation

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4} \iff \xi^2 + \eta^2 = \zeta(1 - \zeta),$$

then

$$x^2 + y^2 = \frac{\xi^2 + \eta^2}{(1 - \zeta)^2} = \frac{\zeta}{1 - \zeta} \implies \zeta = \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

From the last equation and formulas (1.12) we get formulas for the stereographic projection:

$$\xi = \frac{x}{1 + x^2 + y^2}, \qquad \eta = \frac{y}{1 + x^2 + y^2}, \qquad \zeta = \frac{x^2 + y^2}{1 + x^2 + y^2}. \tag{1.13}$$

It follows from (1.12) and (1.13) that $p : \overline{\mathbb{C}} \mapsto \mathbf{S}$ is a *homeomorphism* (by definition, it is a one-to-one, onto continuous mapping with a continuous inverse).

Using the map p, we can identify the extended complex plane $\overline{\mathbb{C}}$ with the sphere $\mathbf{S}$. After this identification, the sphere $\mathbf{S}$ is called the *Riemann sphere*, or the sphere of complex numbers.

Exercise 1.3 Prove that under the stereographic projection an arbitrary circle or straight line on $\overline{\mathbb{C}}$ maps to a circle on $\mathbf{S}$, and the angle between curves in $\overline{\mathbb{C}}$ is equal to the angle between the images these curves on $\mathbf{S}$.

1.3 Complex-Valued Functions of a Real Variable

Consider a function $f : \mathbb{R} \mapsto \mathbb{C}$. Such a complex-valued function of a real variable can be represented as $f(t) = u(t) + iv(t),\ t \in \mathbb{R}$, where $u(t) := \text{Re}(f(t))$ and $v(t) := \text{Im}(f(t))$. Thus, we see that each function $f : \mathbb{R} \mapsto \mathbb{C}$ can be viewed as a vector-function $\binom{u}{v}$ from $\mathbb{R}$ in $\mathbb{R}^2$ due to the geometrical interpretation of the set of complex numbers. Therefore, such concepts as the limit of a function, continuity, uniform continuity and many other properties of vector-functions of a real variable are automatically transferred to such functions. Let us recall some of them.

Definition 1.6 A number $A = \alpha + i\beta$ is the limit of a function $f : \mathbb{R} \mapsto \mathbb{C}$ at a point $t_0 \in \mathbb{R}$ (denoted as $\lim\limits_{t \to t_0} f(t) = A$), if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(t) - A| = \sqrt{(u(t) - \alpha)^2 + (v(t) - \beta)^2} < \varepsilon \quad \text{whenever} \quad |t - t_0| < \delta.$$

▲

As in the proof of the assertion in Exercise 1.2, the following statement can be easily proved.

Proposition 1.1 *The limit of a function $f : \mathbb{R} \mapsto \mathbb{C}$ exists at a point $t_0 \in \mathbb{R}$ and it is equal to $A = \alpha + i\beta$ if and only if there exist the limits of its real and imaginary parts and they are equal to α and β, respectively, i.e.,*

$$\lim_{t \to t_0} f(t) = A \iff \lim_{t \to t_0} u(t) = \alpha \ \textit{and} \ \lim_{t \to t_0} v(t) = \beta.$$

Definition 1.7 A function $f\colon [a,b] \mapsto \mathbb{C}$ is called continuous on the closed interval $[a,b] \subset \mathbb{R}$ (denoted as $f \in C([a,b])$), if for all $t_0 \in [a,b]$

$$\lim_{t\to t_0} f(t) = f(t_0).$$

The corresponding one-sided limits are considered at the endpoints a and b.

Definition 1.8 The derivative of a function $f\colon \mathbb{R} \mapsto \mathbb{C}$ at a point $t_0 \in \mathbb{R}$ (denoted by $f'(t_0)$) is called the limit

$$\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0}, \tag{1.14}$$

provided that it exists. ▲

Suppose the limit (1.14) exists. Then, according to Proposition 1.1

$$\begin{aligned} f'(t_0) &= \lim_{t\to t_0} \left(\frac{u(t) - u(t_0)}{t - t_0} + i \frac{v(t) - v(t_0)}{t - t_0} \right) \\ &= \lim_{t\to t_0} \frac{u(t) - u(t_0)}{t - t_0} + i \lim_{t\to t_0} \frac{v(t) - v(t_0)}{t - t_0} = u'(t_0) + iv'(t_0). \end{aligned}$$

Obviously, the reverse chain of equalities is also true. Thus, the following statement is correct.

Proposition 1.2 *The derivative of a function* $f\colon \mathbb{R} \mapsto \mathbb{C}$ *at* $t_0 \in \mathbb{R}$ *exists if and only if the derivatives of its real and imaginary parts exist at* t_0.

Example 1.2 The function $f(t) = \exp(it),\ t \in \mathbb{R}$, has the derivative at each point and $f'(t) = i\exp(it)$. Indeed, for any $t \in \mathbb{R}$

$$(\exp(it))' = (\cos t + i \sin t)' = -\sin t + i \cos t = i(\cos t + i \sin t) = i \exp(it).$$

Due to Proposition 1.2 the equality $f'(t_0) = u'(t_0) + iv'(t_0)$ can be taken as an equivalent definition of the derivative of a complex-valued function of a real variable. We will apply this approach to define the integral of a complex-valued function of a real variable.

Definition 1.9 Let $f(t) = u(t) + iv(t),\ t \in [a, b]$, and the functions u and v be Riemann-integrable on the segment $[a, b]$.

$$\int_a^b f(t)\,dt \overset{def}{=} \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

▲

Exercise 1.4 Prove that Definition 1.9 is equivalent to the definition of the integral introduced through the limit of the Riemann sums of f, i.e.,

$$\int_a^b f(t)\,dt = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(\tau_k)\,\Delta t_k,$$

where $a = t_0 < t_1 < \ldots < t_n = b,\ \Delta t_k := t_k - t_{k-1},\ t_{k-1} \le \tau_k \le t_k,\ \Delta = \max_{k\in\{1,\ldots,n\}} \Delta t_k$.

It is easy to check the following properties of integrals of complex-valued functions of a real variable:

(1) $\forall\,\lambda, \mu \in \mathbb{C} \quad \int_a^b (\lambda f(t) + \mu g(t))\,dt = \lambda\int_a^b f(t)\,dt + \mu\int_a^b g(t)\,dt;$

(2) $\forall\, c \in (a, b) \quad \int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt;$

(3) if F is the antiderivative of f, i.e., $F'(t) = f(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a);$$

(4) $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$

Exercise 1.5 Using Exercise 1.4, prove the fourth property.

Remark 1.1 Not all properties of real-valued functions are automatically carried over to complex-valued functions of a real argument. For instance, the statement of the mean value theorem is incorrect. This fact is easy to check for such a continuous function: $e^{it},\ t \in [0, 2\pi]$. Evidently that $e^{it} \ne 0$ for all $t \in [0, 2\pi]$. Therefore, on the one hand, assuming that the mean value theorem holds, we have that $\int_0^{2\pi} e^{it}\,dt \ne 0$. On the other hand,

$$\int_0^{2\pi} e^{it}\,dt = \int_0^{2\pi} \cos t\,dt + i\int_0^{2\pi} \sin t\,dt = 0.$$

Exercise 1.6 Show that the statements of Rolle's theorem and Cauchy's mean value theorem are also incorrect for complex-valued functions of a real variable. Recall that Rolle's theorem states the following: if a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$. The second theorem establishes the relationship between the derivatives of two functions. Let functions f and g be continuous on $[a, b]$, differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is a point $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Remark 1.2 Since it is impossible to introduce an order relation for complex numbers, the Weierstrass theorem for complex-valued functions of a real variable is formulated as follows: for a continuous function $f : [a, b] \mapsto \mathbb{C}$, its modulus reaches its largest and smallest value on the closed interval $[a, b]$.

1.4 Curves in the Complex Plane

A curve is a geometric concept, the exact and at the same time quite general definition of which presents significant difficulties and is given in various branches of mathematics and textbooks in different ways. For those branches in which methods of the theory of functions dominate, the natural definition of a curve is to define it by parametric equations. In this text, we will take this approach and give the following definition of a curve and its elements.

Definition 1.10 A curve in $\mathbb{C}$ (in $\overline{\mathbb{C}}$) is called a continuous complex-valued function of a real variable: $z = \gamma(t)$, $t \in [a, b] \subset \mathbb{R}$.

Moreover, the points $\gamma(a)$ and $\gamma(b)$ are called the initial and end points of γ, respectively. The curve γ is said to be closed if $\gamma(a) = \gamma(b)$. ▲

Remark 1.3 In the notation of a curve $z = \gamma(t)$, $t \in [a, b]$, or $\gamma\colon [a, b] \mapsto \mathbb{C}$, we will always mean that the closed interval $[a, b]$ is a real closed interval, i.e., $[a, b] \subset \mathbb{R}$.

The image of such a continuous function is also often called a curve. In the course "Complex Analysis" it is convenient to distinguish between these concepts in order to better understand some new definitions and theorem proofs. The image of γ, i.e., the set $\gamma\big([a, b]\big)$, is called the *trace* of the curve γ and is denoted by E_γ.

Each curve specifies an *orientation* that can be interpreted as the direction of movement of the point $\gamma(t)$ along the trace E_γ from its initial point to its end as the parameter t increases from a to b.

Example 1.3 Let $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, and $\gamma(t) = z_1 + t(z_2 - z_1)$, $t \in [0, 1]$. The initial point of this curve is $z_1 = \gamma(0)$, the end point is $z_2 = \gamma(1)$. We will denote its trace by $[z_1, z_2]$ and refer to it as the segment joining z_1 and z_2.

Separating the real and imaginary parts in the equality $z = \gamma(t)$, we find the parametric equations, which are called a *parametrization* of the curve γ, namely $x = \mathrm{Re}(\gamma(t))$, $y = \mathrm{Im}(\gamma(t))$, where the parameter $t \in [a, b]$.

Example 1.4 Let $z = \gamma_1(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$x + iy = \cos t + i \sin t \Longleftrightarrow \begin{cases} x = \cos t, \\ y = \sin t, \end{cases} \qquad t \in [0, 2\pi].$$

The last two equations determine a parametrization of this curve, whose trace is the unit circle centered at the origin. It is a closed curve that starts at point $(1, 0)$ and is oriented counterclockwise.

It is easy to see that the curve $z = \gamma_2(t) = e^{i2\pi t}$, $t \in [0, 1]$, has the same trace and orientation as the curve γ_1 from Example 1.4. For such curves, we will give the following definition.

Definition 1.11 Two curves

$$z = \gamma_1(t), \quad t \in [a_1, b_1], \quad \text{and} \quad z = \gamma_2(\tau), \quad \tau \in [a_2, b_2],$$

are called equivalent ($\gamma_1 \sim \gamma_2$), if there exists a real-valued function $\tau = \mu(t)$, $t \in [a_1, b_1]$, such that

(1) $\mu \in C([a_1, b_1])$ and it is strictly increasing on $[a_1, b_1]$;
(2) $\mu(a_1) = a_2, \quad \mu(b_1) = b_2$;
(3) $\gamma_1(t) = \gamma_2(\mu(t))$ for all $t \in [a_1, b_1]$.

Exercise 1.7 Prove that this relation between two curves is the equivalence relation, i.e., it is reflexive, symmetric and transitive.

Therefore, a curve can be understood as the corresponding equivalence class. It is clear that equivalent curves have the same trace and orientation.

Example 1.5 The curve γ_1 from Example 1.4 and the curve

$$z = \gamma_2(\tau) = e^{i2\pi\tau}, \ \tau \in [0, 1],$$

are equivalent. To show this we need to take the function $\tau = \mu(t) = t/2\pi$, $t \in [0, 2\pi]$, and verify the conditions from Definition 1.11.

Exercise 1.8 Prove that the curve γ_1 from Example 1.4 and the curve $z = \gamma_3(\tau) = e^{-i\tau}$, $\tau \in [0, 2\pi]$, are not equivalent.

Definition 1.12 A point z_0 is called a self-intersection point of a curve $z = \gamma(t)$, $t \in [a, b]$, if there are $t_1 \neq t_2$, $\{t_1, t_2\} \subset [a, b]$ such that

$$\gamma(t_1) = \gamma(t_2) = z_0.$$

If a curve γ is closed, then the point $\gamma(a) = \gamma(b)$ is not considered a self-intersection point. ▲

A curve without self-intersection points is called *simple* and a closed simple curve is said to be a *Jordan curve*.

Let $z = \gamma(t)$, $t \in [a, b]$, be a Jordan curve in $\mathbb{C}$. Then the Jordan curve theorem asserts that the trace E_γ divides the complex plane into an "interior" region, denoted by $\operatorname{int}(\gamma)$, bounded by the trace, and an "exterior" region, denoted by $\operatorname{ext}(\gamma)$ (Fig. 1.5), i.e.,

$$\mathbb{C} \setminus E_\gamma = \operatorname{int}(\gamma) \cup \operatorname{ext}(\gamma).$$

Intuitively, the statement of this theorem is obvious and there is no trouble verifying it when a curve is given explicitly. A rigorous proof of the general result is rather difficult, and we refer the reader to a topology text, e.g., [15]. The proof of the Jordan curve theorem for smooth Jordan curves can be found in [13, §4.8].

We will say that a Jordan curve γ has the *positive orientation* (denoted by γ^+) if its interior remains on the left when traversing its trace for increasing values of the parameter t from a to b (Fig. 1.5). Otherwise, it is *negatively oriented* (γ^-).

Definition 1.13 A curve $z = \gamma(t)$, $t \in [a, b]$, is called smooth, if γ is continuously differentiable on $[a, b]$, i.e., $\gamma \in C^1([a, b])$ and

$$\gamma'(t) \neq 0 \quad \text{for all } \ t \in [a, b]. \tag{1.15}$$

If γ is a closed curve, the condition $\gamma'(a) = \gamma'(b)$ must also be satisfied. ▲

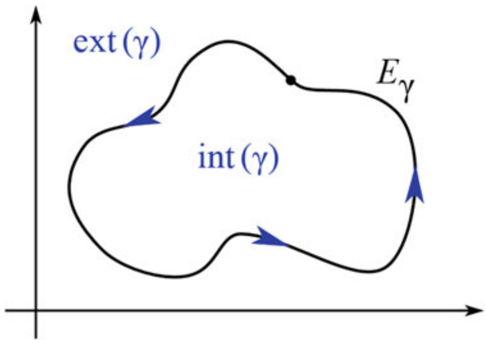

Fig. 1.5 The interior and exterior of a Jordan curve with the positive orientation

Let us find out the geometric meaning of (1.15). It is equivalent to $x'(t) + iy'(t) \neq 0$ for all $t \in [a, b]$. Since $\big(x'(t), y'(t)\big)$ is the tangent vector to E_γ at the point $\gamma(t)$, then the condition (1.15) means that at each point of E_γ there is a nonzero tangent vector that changes continuously.

Definition 1.14 A curve $z = \gamma(t)$, $t \in [a, b]$, is called piecewise smooth, if there is a partition $a = a_0 < a_1 < \ldots < a_n = b$ of $[a, b]$ such that for each $k \in \{0, 1, \ldots, n-1\}$ the curve $z = \gamma(t),\ t \in [a_k, a_{k+1}]$, is smooth.

Definition 1.15 A curve $z = \gamma(t)$, $t \in [a, b]$, is called rectifiable, if

- γ is differentiable on $[a, b]$ except, possibly, at a countable set of points and
- there exists a finite integral

$$\ell_\gamma := \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

The value ℓ_γ is called the length of γ.

An example of a piecewise smooth curve is a broken line. Note that a piecewise smooth curve is rectifiable.

Example 1.6 The curve

$$z = \gamma_5(t) = t^3 + it^2, \quad t \in [-1, 1],$$

is simple, however it is not smooth and piecewise smooth (Fig. 1.6).

The curve

$$z = \gamma_6(t) = \cos 2t \exp(it), \quad t \in [0, 2\pi] \quad \text{(four petal rose curve)}$$

is closed non-Jordan smooth curve that has the self-intersection point at the origin.

The curve

$$z = \gamma_7(t) = t\big(1 + i \sin \tfrac{1}{t}\big), \quad t \in \big[-\tfrac{1}{\pi}, \tfrac{1}{\pi}\big],$$

is simple and non-rectifiable, so it is not piecewise-smooth.

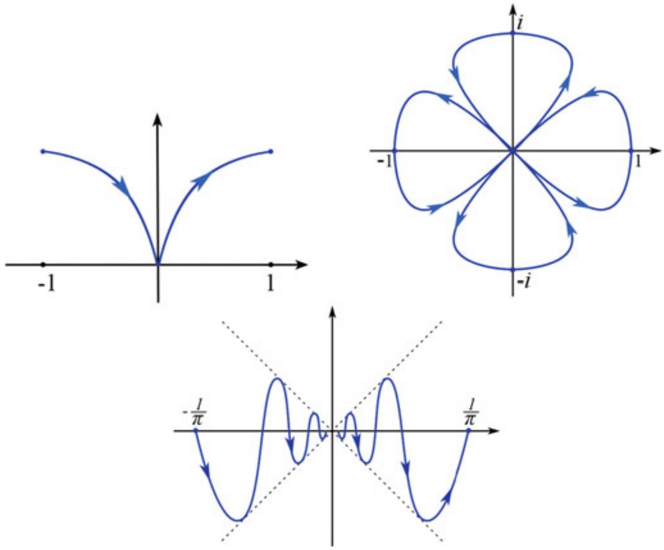

Fig. 1.6 The traces of curves γ_5 (left), γ_6 (right) and γ_7 (from below)

1.5 Basic Topological Concepts of the Complex Plane

Let us now recall some topological concepts. An *open disk* of radius $r > 0$ centered at a point $a \in \mathbb{C}$ is a set of all points of distance less than r from a, i.e.,

$$B_r(a) := \{z \in \mathbb{C} : \ |z - a| < r\}.$$

It is also called the *r-neighborhood* of a. The R-neighborhood of the point at infinitely is the set

$$B_R(\infty) := \{z \in \mathbb{C} : \ |z| > R\} \cup \{\infty\}.$$

Definition 1.16 A set $D \subset \overline{\mathbb{C}}$ is called open, if each point of D is contained in D together with some of its r-neighborhood, i.e.,

$$\forall z_0 \in D \ \exists r > 0 \quad \text{such that} \quad B_r(z_0) \subset D.$$

Definition 1.17 Let $D \subset \overline{\mathbb{C}}$ and $z_0 \in \overline{\mathbb{C}}$. The point z_0 is called a limit point of the set D if every r-neighborhood of z_0 contains at least one point of D different from z_0 itself, i.e.,

$$\forall r > 0 \ \exists z \in D \quad \text{such that} \quad z \in B_r(z_0) \setminus \{z_0\}.$$

Definition 1.18 A set $D \subset \overline{\mathbb{C}}$ is called closed if it contains all its limit points.

Example 1.7 Let $D := \mathbb{Z}$. Then, the set D is closed in $\mathbb{C}$, since the set of its limit points is empty and $\emptyset \subset D$. However, it is not closed in $\overline{\mathbb{C}}$ since it does not contain its limit point ∞.

The joining to a set $D \subset \overline{\mathbb{C}}$ all its limit points is called the *closure* of D and denoted by $\overline{D}$. For example, the closure of the open disk $B_r(a)$ is the closed disk

$$\overline{B_r(a)} := \{z \in \mathbb{C} : \ |z - a| \leq r\}.$$

A set $D \subset \overline{\mathbb{C}}$ is said to be *path-connected* if for any two distinct points in D there is a curve whose trace belongs to D and connects these points (starting at one point and ending at the other).

A set $D \subset \overline{\mathbb{C}}$ is called a *domain* if it is open and path-connected.

Definition 1.19 Let D be a domain in $\overline{\mathbb{C}}$. The set $\partial D := \overline{D} \setminus D$ is called the boundary of D.

Exercise 1.9 Prove that the boundary of a domain is the closed set.

There are several approaches to introducing the concept of simply connected domains: a domain is simply connected if its fundamental group is trivial; a domain D in $\mathbb{R}^m$ is simply connected if any closed curve in D is homotopic to a point in this domain (see Definition 4.4 and Exercise 4.4); one can define simply connectedness through the general concept of connectedness of a set in a topological space. In this course an easier-to-understand definition of simply connectedness is proposed.

Definition 1.20 A domain D is said to be simply connected (also called 1-connected) in $\mathbb{C}$ if for any Jordan curve γ, whose trace belongs to D, the interior of γ is fully contained in D, i.e., $\text{int}(\gamma) \subset D$.

A domain D is said to be simply connected in $\overline{\mathbb{C}}$ if for any Jordan curve γ, whose trace belongs to D and $\infty \notin E_\gamma$, obligatorily either $\text{int}(\gamma) \subset D$ or $\text{ext}(\gamma) \subset D$.

Domains that are not simply connected are called multiply connected. ▲

Intuitively, a simply connected domain is a domain "without holes".

Definition 1.21 The connectedness order of a domain $D \subset \overline{\mathbb{C}}$ is the number of path-connected closed components of the boundary ∂D, which do not intersect.

Example 1.8 Consider the domain $D_1 = \{z : \ |z| > 1\}$. Obviously, it is multiply connected in $\mathbb{C}$, since the interior of the circle $\{z : \ |z| = 2\}$ is not a subset of D_1. In $\overline{\mathbb{C}}$ its connectedness order is 2, because the boundary of D_1 has two path-connected closed components that do not intersect, namely $\{z : \ |z| = 1\}$ and $\{\infty\}$.

Example 1.9 Due to the second part of Definition 1.20 the domain $D_2 = D_1 \cup \{\infty\}$ is 1-connected in $\overline{\mathbb{C}}$.

Example 1.10 The connectedness order of the domain

$$D_3 = \{z: \ |z| < 2\} \setminus \Big(\bigcup_{k=1}^{N} \{z: \ z = x + \frac{i}{2^k}, \ x \in [\tfrac{1}{4}, \tfrac{3}{4}]\}\Big)$$

is equal to $N + 1$, where $N \in \mathbb{N}$.

2 Analytic Functions

Abstract

In this chapter and onwards, we study properties of complex-valued functions of a complex variable. It turns out that every complex-valued function is determined by the corresponding vector function from $\mathbb{R}^2$ into $\mathbb{R}^2$. This fact enables us to obtain some properties of complex-valued functions from the first section. Fundamentally new is the notion of differentiability of complex-valued functions, which we introduce in Sect. 2.2, although it formally coincides with the standard definition (from calculus) of the differentiability of real functions of one real variable. Complex-valued differentiable functions, which we will call analytic functions, have many remarkable and unexpected properties that do not exist for real-valued differentiable functions. For example, a complex-valued differentiable function necessarily has derivatives of all orders, and many of its properties are determined by its values on arbitrary sets that have a limit point inside. These functions are of great importance both in various branches of mathematics and in many applications. The study of their properties is the main goal of complex analysis. In this section, we prove a criterion for the differentiability of complex-valued functions, which includes equivalence to the Cauchy–Riemann equations. They are a system of two partial differential equations that relate the real and imaginary parts of a complex-valued function. This leads to the concept of conjugate harmonic functions in Sect. 2.3. In addition, using some properties of conjugate harmonic functions, the hydrodynamic interpretation of analytic functions is given in Sect. 2.4. The chapter ends with Sect. 2.5, which introduces conformal mappings as analytic functions with a nonzero derivative. It turns out that a conformal function at a point z_0 preserves angles between curves at z_0 and equally stretches all curves starting at z_0. These two properties of a conformal function are characterized by the argument and the modulus of its derivative at z_0, respectively.

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2.1 Structure of Complex-Valued Functions of a Complex Variable

Let Ω be a subset of $\overline{\mathbb{C}}$. Consider a function $f\colon \Omega \mapsto \overline{\mathbb{C}}$. Such functions are called complex-valued functions of a complex variable. For them, the notation $w = f(z)$, $z \in \Omega$, will also be used. To visualize such mappings, we will consider two copies of the complex plane: $\overline{\mathbb{C}}_z$ and $\overline{\mathbb{C}}_w$. The relationship between the complex variables w and z can be described by two real-valued functions of two real variables:

$$w = f(z) \iff u + iv = \operatorname{Re}\big(f(x+iy)\big) + i\operatorname{Im}\big(f(x+iy)\big)$$

$$\iff \begin{cases} u = \operatorname{Re}\big(f(x+iy)\big) =: u(x,y), \\ v = \operatorname{Im}\big(f(x+iy)\big) =: v(x,y). \end{cases}$$

Example 2.1 Consider the function $\omega = z^2 - i\bar{z},\ z \in \mathbb{C}$. Separating the real and imaginary parts in this equality, we get

$$u + iv = (x+iy)^2 - i(x-iy) = x^2 - y^2 - y + i(2xy - x).$$

Thus, $u(x,y) = x^2 - y^2 - y$ and $v(x,y) = 2xy - x$.

Definition 2.1 Let $\Omega \subset \mathbb{C},\ f\colon \Omega \mapsto \overline{\mathbb{C}},\ A = \alpha + i\beta \in \mathbb{C}$, and $z_0 = x_0 + iy_0$ be a limit point of the set Ω. We say that the limit of the function f at z_0 is equal to A as z approaches z_0, denoted by

$$\lim_{\Omega \ni z \to z_0} f(z) = A,$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ such that when $z \in \Omega$ and $0 < |z - z_0| < \delta$, then $|f(z) - A| < \varepsilon$, i.e.,

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, z \in \Omega : \quad 0 < |z - z_0| < \delta \implies |f(z) - A| < \varepsilon.$$

▲

It is very important to understand that $f(z)$ tends to A no matter what direction z approaches z_0. For example, it is easy to show that there is no limit of the function $f(z) = \frac{\bar{z}}{z}$ as $z \to 0$. Indeed, if $z = x \in \mathbb{R} \setminus \{0\}$, then $f(x) = \frac{x}{x} = 1$, so the limit of f as $z = x \to 0$ along the real axis is 1. On the other hand, on the imaginary axis we have $f(iy) = \frac{-iy}{iy} = -1$, and the limit of f as $z = iy \to 0$ is -1. Since these two limits disagree, the limit of f as $z \to 0$ does not exist.

Now let us try to understand how the limit of a function f at z_0 is related to the limits of its real and imaginary parts. The following statement holds.

Proposition 2.1 *Let* $f(z) = u(x, y)+iv(x, y)$, $z = x+iy \in \Omega$, *and* $z_0 = x_0+iy_0$ *be a limit point of* Ω; $A = \alpha + i\beta$.

There exists $\lim\limits_{\Omega\ni z\to z_0} f(z) = A$ *if and only if there exist*

$$\lim_{x\to x_0,\ y\to y_0} u(x, y) = \alpha \quad \textit{and} \quad \lim_{x\to x_0,\ y\to y_0} v(x, y) = \beta.$$

The proof follows directly from Definition 2.1 and the equalities

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

$$|f(z) - A| = \sqrt{(u(x, y) - \alpha)^2 + (v(x, y) - \beta)^2}.$$

Definition 2.2 Let every point of a set $\Omega \subset \mathbb{C}$ be a limit point of Ω and $z_0 \in \Omega$.

- A function $f: \Omega \mapsto \mathbb{C}$ is said to be continuous at z_0 if

$$\lim_{\Omega\ni z\to z_0} f(z) = f(z_0).$$

- A function $f: \Omega \mapsto \mathbb{C}$ is called continuous in Ω if it is continuous at every point of Ω.

The set of all continuous functions on Ω is denoted by $C(\Omega)$. ▲

Proposition 2.1 allows transferring some properties of limits of functions of two real variables to functions of a complex variable. In particular, a function f is continuous at a point $z_0 = x_0 + iy_0$ if and only if the functions u and v are continuous at the point (x_0, y_0). From this we obtain theorems on the continuity of the sum, product, and division of two continuous functions of a complex variable.

Since each function $f: \mathbb{C} \mapsto \mathbb{C}$ can be viewed as a vector-function $\binom{u}{v}$ from $\mathbb{R}^2$ in $\mathbb{R}^2$, many properties of such vector-functions are automatically transferred to complex-valued functions of a complex variable. Let us recall some of them.

Theorem 2.1 *Let* K *be a path-connected and compact set (bounded and closed set) in* $\mathbb{C}$, $\omega = f(z)$, $z \in K$. *If* $f \in C(K)$, *then*

1. *the modulus of* f *is bounded on* K, *i.e.,*

$$\exists\, M > 0 \quad \forall\, z \in K: \quad |f(z)| \le M;$$

2. *its modulus takes its minimum and its maximum on* K, *i.e.,*

$$\exists\, \{z_1, z_2\} \in K \quad \forall\, z \in K: \quad |f(z_1)| \le |f(z)| \le |f(z_2)|;$$

3. *it is uniformly continuous on* K, *i.e., for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|f(z_1) - f(z_2)| < \varepsilon \quad \textit{for all } \{z_1, z_2\} \in K \;\; \textit{such that } |z_1 - z_2| < \delta.$$

Remark 2.1 We will say that a complex-valued function is bounded on a set if its modulus is bounded on that set.

2.2 Differentiability of Complex-Valued Functions of a Complex Variable

Hereinafter, we denote by Ω a domain (open and path-connected set) in the complex plane $\mathbb{C}$. If Ω is a domain in $\overline{\mathbb{C}}$, this will be specified.

Definition 2.3 The derivative of a given function $f\colon \Omega \mapsto \mathbb{C}$ at a point $z_0 \in \Omega$ (denoted by $f'(z_0)$) is called the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided this limit exists. ▲

With derivatives there are several alternative notations, for example,

$$f'(z_0) = \frac{df(z)}{dz}\bigg|_{z=z_0} = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

From Definition 2.3 and the limit properties, it follows that all formulas for calculating derivatives that are known in the course of mathematical analysis (derivative of the sum, product, division and superposition) are transferred to complex-valued functions of a complex variable.

Example 2.2 Let us show that $\left(z^n\right)' = n\, z^{n-1}$ for any $z \in \mathbb{C}$ and for any $n \in \mathbb{N}$. Using the binomial formula, we deduce

$$\begin{aligned}\frac{(z + \Delta z)^n - z^n}{\Delta z} &= \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} (\Delta z)^k - z^n}{\Delta z} \\ &= nz^{n-1} + \frac{n(n-1)}{2} z^{n-2} \Delta z + \ldots + (\Delta z)^{n-1}.\end{aligned}$$

Thus,

$$\left(z^n\right)' = \lim_{\Delta z\to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} = nz^{n-1} \quad \text{for all } z \in \mathbb{C}.$$

Definition 2.4 A function $f\colon \Omega \mapsto \mathbb{C}$ is said to be differentiable at a point $z_0 \in \Omega$ if its increment $\Delta f(z_0) := f(z_0 + \Delta z) - f(z_0)$ can be represented as

$$\Delta f(z_0) = A \cdot \Delta z + \overline{o}(\Delta z) \quad \text{as} \ \ \Delta z \to 0,$$

where A is a complex number; in this case, the linear part of the increment is called the differential of f at z_0 and is denoted by

$$df(z_0) \overset{def}{=} A \cdot \Delta z =: A \cdot dz,$$

and the second term has a higher order of smallness with respect to Δz and is denoted by $\overline{o}(\Delta z)$. ▲

The symbol $\overline{o}(\Delta z)$, pronounced "little *oh* of Δz" is one of the Landau symbols that are used to symbolically express the behavior of some function with respect to another, in our case with respect to Δz as $\Delta z \to 0$. By definition, one says that $g(\Delta z) = \overline{o}(\Delta z)$ as $\Delta z \to 0$, if

$$\lim_{\Delta z \to 0} \frac{g(\Delta z)}{\Delta z} = 0. \tag{2.1}$$

Remark 2.2 It is obvious that the limit (2.1) is equivalent to

$$\lim_{|\Delta z| \to 0} \frac{|g(\Delta z)|}{|\Delta z|} = 0.$$

Thus, $g(\Delta z) = \overline{o}(\Delta z)$ as $\Delta z \to 0$, if and only if $g(\Delta z) = \overline{o}(|\Delta z|)$ as $|\Delta z| \to 0$.

Remark 2.3 Differentiable complex-valued functions of a complex value are sometimes called complex-differentiable or $\mathbb{C}$-differentiable. The authors of many textbooks, and I myself, are of the opinion that there is no need to complicate the name of the concept when it is clear that complex-valued functions of a complex variable are being considered.

In the same way as for real functions of a real variable, such a statement is proved.

Proposition 2.2 *A function $f\colon \Omega \mapsto \mathbb{C}$ is differentiable at a point $z_0 \in \Omega$ if and only if there exists $f'(z_0)$. In addition, $df(z_0) = f'(z_0)dz$.*

It is easy to see that if a function f is differentiable at a point z_0, then f is continuous at z_0.

As noted above, the continuity of a complex-valued function at a point $z_0 = x_0 + iy_0$ is equivalent to the continuity at the point (x_0, y_0) of its real and imaginary parts. Such a statement does not exist for a differentiable function of a complex variable, although this statement holds for a complex-valued function of a real variable (see

Proposition 1.2). The connection between a differentiable function of a complex variable and the differentiability of its real and imaginary parts is established by the following theorem.

Theorem 2.2 *Let* $f(z) = u(x, y) + iv(x, y)$, $z = x + iy \in \Omega$, *and* $z_0 \in \Omega$. *The function* f *is differentiable at the point* $z_0 = x_0 + iy_0$ *if and only if*

(1) the functions u *and* v *are differentiable at the point* (x_0, y_0),
(2) the Cauchy–Riemann equations are satisfied at (x_0, y_0) :

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \textit{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.2}$$

Moreover, the derivative of f *at* z_0 *is given by*

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \tag{2.3}$$

Proof

Necessity Due to Proposition 2.2 the function f has the derivative at the point z_0. Therefore, there exist the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \tag{2.4}$$

where $\Delta z = \Delta x + i\Delta y$, and we can compute this limit by letting Δz approach zero from any direction in the complex plane.

Putting first $\Delta z = \Delta x$ in this limit and taking Proposition 2.1 into account, we deduce

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right] \\ &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned} \tag{2.5}$$

On the other hand, if Δz approaches zero vertically, i.e., $\Delta z = i\Delta y$ in the limit (2.4), we find

$$\begin{aligned} f'(z_0) &= \lim_{i\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \right] \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) follow the Cauchy–Riemann equations (2.2) and formulas (2.3).

It remains to show that the functions u and v are differentiable at the point (x_0, y_0). From Definition 2.4, Proposition 2.2 and (2.2) we get

$$\begin{aligned} \Delta f(z_0) &= \Delta u(x_0, y_0) + i\Delta v(x_0, y_0) = f'(z_0) \cdot \Delta z + \overline{o}(\Delta z) \\ &= \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \right) (\Delta x + i\Delta y) + \varepsilon_1 + i\varepsilon_2 \\ &= \left(\frac{\partial u}{\partial x}(x_0, y_0)\, \Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\, \Delta y + \overline{o}\left(\sqrt{(\Delta x)^2 + (\Delta y)^2} \right) \right) \\ &\quad + i\left(\frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + \overline{o}\left(\sqrt{(\Delta x)^2 + (\Delta y)^2} \right) \right), \end{aligned} \tag{2.7}$$

where $\varepsilon_1 := \operatorname{Re}\big(\overline{o}(\Delta z)\big)$, $\varepsilon_2 := \operatorname{Im}\big(\overline{o}(\Delta z)\big)$, and based on Remark 2.1 it is easy to verify that

$$\varepsilon_k = \overline{o}(|\Delta z|) = \overline{o}\left(\sqrt{(\Delta x)^2 + (\Delta y)^2} \right), \quad k = 1, 2.$$

Equating the real and imaginary parts in (2.7), we obtain

$$\Delta u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + \overline{o}\left(\sqrt{(\Delta x)^2 + (\Delta y)^2} \right), \tag{2.8}$$

$$\Delta v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + \overline{o}\left(\sqrt{(\Delta x)^2 + (\Delta y)^2} \right), \tag{2.9}$$

as $\Delta x \to 0$, $\Delta y \to 0$. The relations (2.8) and (2.9) imply that u and v are differentiable at the point (x_0, y_0).

Sufficiency Since the functions u and v are differentiable at (x_0, y_0), this means that the relations (2.8) and (2.9) hold. Multiplying (2.9) by the imaginary unit i and summing with (2.8), and taking the Cauchy–Riemann equations and Remark 2.1 into account, we derive

$$\begin{aligned}
&\Delta u(x_0, y_0) + i\Delta v(x_0, y_0) \\
&\qquad = \frac{\partial u}{\partial x}(x_0, y_0)\big(\Delta x + i\Delta y\big) + \frac{\partial v}{\partial x}(x_0, y_0)\big(-\Delta y + i\Delta x\big) + \overline{o}(|\Delta z|) \\
&\qquad = \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right)\Delta z + \overline{o}(\Delta z) \quad \text{as} \quad \Delta z \to 0.
\end{aligned}$$

This relation means that the function f is differentiable at the point z_0. Formulas (2.3) follow from Proposition 2.2 and (2.2).

□

Remark 2.4 For the first time, the Cauchy–Riemann equations were obtained in the works of d'Alembert (1752) and Euler (1755) on fluid dynamics. However, the implications of these conditions in terms of the differentiability of complex-valued functions of a complex variable were not identified. About 70 years later, in papers by Cauchy and then by Riemann, a clear definition of the differentiability of such functions was given.

Theorem 2.2 shows that differentiable functions of a complex variable cannot be identified with differentiable vector-valued functions from $\mathbb{R}^2 \mapsto \mathbb{R}^2$ (differentiability of the latter is equivalent to differentiability of each component). Obviously, the set of differentiable functions of a complex variable is narrower than the set of differentiable vector-valued functions from $\mathbb{R}^2 \mapsto \mathbb{R}^2$.

This distinction between the two concepts of differentiability leads to the fact that differentiable functions of a complex variable have significantly different properties. Because of these properties, the theory of differentiable functions of a complex variable has wide applications both in various branches of mathematics and directly in many other areas of natural science.

Theorem 2.2 also highlights that only existence of partial derivatives of real-valued functions u and v satisfying the Cauchy–Riemann equations at (x_0, y_0) does not ensure differentiability of the function $f = u + iv$ at $z_0 = (x_0, y_0)$. The functions u and v are required to be differentiable at (x_0, y_0) as functions on $\mathbb{R}^2$. This condition is stronger than existence of partial derivatives.

The Cauchy-Riemann equations provide us with a direct way to test the differentiability of a function and calculate its derivative.

Example 2.3 Consider the function $f(z) = \frac{3}{2}z - \frac{1}{2}\bar{z},\ z \in \mathbb{C}$. It is easy to find that

$$u(x, y) = \operatorname{Re} f(x + iy) = x, \qquad v(x, y) = \operatorname{Im} f(x + iy) = 2y.$$

Since the first Cauchy-Riemann equation is not satisfied ($\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 2$), the function f is not differentiable at any point of the complex plane.

However, the corresponding vector-function $\begin{pmatrix} x \\ 2y \end{pmatrix} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is differentiable at each point of $\mathbb{R}^2$.

Example 2.4 Using Euler's formula (1.9), the exponential function of a complex variable is defined as

$$e^z = e^{x+iy} \stackrel{def}{=} e^x \, e^{iy} = e^x \cos y + i e^x \sin y \quad \text{for all } \ z \in \mathbb{C}.$$

The real and imaginary parts of e^z are as follows

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

These real-valued functions are differentiable at each point of $\mathbb{R}^2$ and

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x} \quad \text{for all } \ (x, y) \in \mathbb{R}^2.$$

Therefore, according to Theorem 2.2, the function e^z is differentiable in the complex plane and

$$(e^z)' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^z \quad \text{for all } \ z \in \mathbb{C}. \tag{2.10}$$

Definition 2.5 A function f is called analytic in a domain $\Omega \subset \mathbb{C}$, if it is differentiable at every point of this domain.

Definition 2.6 A function f is called analytic at a point $z_0 \in \mathbb{C}$, if it is differentiable in some neighborhood of this point.

Definition 2.7 A function f is called analytic at ∞, if the function $g(z) := f(\frac{1}{z})$ is analytic at zero.

This definition allows us to consider analytic functions on $\overline{\mathbb{C}}$. Note that the notion of a derivative in "∞" is meaningless.

The set of all analytic functions in a domain Ω is denoted by $\mathcal{A}(\Omega)$.

Exercise 2.1 Prove that the set $\mathcal{A}(\Omega)$ forms a ring with respect to the operations of adding and multiplying two functions, i.e., it is an Abelian group with respect to addition, and multiplication distributes over addition.

Example 2.5 Consider a function $f(z) = |z|^2$, $z \in \mathbb{C}$. Its real and imaginary parts are $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. In addition, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Thus, due to Theorem2.2, the function f is differentiable only at one point $z = 0$ and it is not analytic at any point of the complex plane.

Exercise 2.2 Prove that if f is analytic in a domain and if $|f|$ is constant there, then f is constant.

Definition 2.8 An analytic function whose domain is the whole complex plane is called an entire function.

Examples 2.2 and 2.4 show that e^z and z^n are entire functions.

2.3 Conjugate Harmonic Functions

Let $f \in \mathcal{A}(\Omega)$ and $f(z) = u(x, y) + iv(x, y)$, $z = x + iy \in \Omega$. In Sect. 5.3 we will prove that analytic functions have derivatives of all orders. This, in particular, implies that the functions u and v have partial derivatives of all orders with respect to x and y. Therefore, without loss of generality, we assume that $\{u, v\} \subset C^2(\Omega)$ in this chapter. Then the following relations follow from the Cauchy-Riemann equations (2.2):

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} & \text{in } \Omega, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} & \text{in } \Omega, \end{cases} \implies \begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} & \text{in } \Omega, \end{cases} \implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \Omega.$$

Similar calculations show that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ in Ω. In the mathematical literature, the sum of such second derivatives is called the *Laplace operator* or *Laplacian*, for which the following notation is introduced:

$$\Delta v := \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

Definition 2.9 A real-valued function $g \in C^2(\Omega)$ is called harmonic in Ω if it is a solution the Laplace equation in Ω, i.e., $\Delta g = 0$ in Ω.

Definition 2.10 An ordered pair $\langle h, g \rangle$ of harmonic functions in Ω satisfying the Cauchy-Riemann equations is called a conjugate pair of harmonic functions in Ω.

Thus, if a function f is analytic in Ω, then its real and imaginary parts form the conjugate pair $\langle u, v \rangle$ of harmonic functions in Ω.

Obviously, if $\langle u, v\rangle$ is a conjugate pair of harmonic functions in Ω, then by Theorem 2.2 the function $f(z) := u(x, y)+iv(x, y),\ z = x+iy \in \Omega$, is analytical in Ω. As a result, the following statement follows from the above considerations.

Proposition 2.3 *A function $f(z) = u(x, y) + iv(x, y),\ z = x + iy \in \Omega$, for which $\{u, v\} \subset C^2(\Omega)$, is analytic in Ω if and only if $\langle u, v\rangle$ is a conjugate pair of harmonic functions in Ω.*

A natural question arises whether it is possible to restore an analytical function of a complex variable for a given real or imaginary part, i.e., if u is a real-valued and harmonic function in Ω, does there exist an analytic function f in Ω such that $\mathrm{Re} f = u$ in Ω? The answer is given by Proposition 2.3 and the following theorem.

Theorem 2.3 *Let u be a harmonic function in a simply connected domain Ω. Then there exists a function v, which is determined up to an additive constant, such that $\langle u, v\rangle$ is a conjugate pair of harmonic functions in Ω.*

Before the proof, we recall some facts from the course on mathematical analysis of several variables. A differential form is called an *exact form* if it is the exterior derivative of another differential form. A differential form $w = g_1(x, y)dx + g_2(x, y)dy$ with coefficients of class C^1 is an exact form in a simply connected domain Ω if and only if $\dfrac{\partial g_1}{\partial y} = \dfrac{\partial g_2}{\partial x}$ in Ω. If the last condition is satisfied, then there exists a function $v \in C^2(\Omega)$, which is defined up to an additive constant, such that

$$dv = w \iff \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = g_1\,dx + g_2\,dy.$$

Proof Consider the differential form $w = -\dfrac{\partial u}{\partial y}dx + \dfrac{\partial u}{\partial x}dy$. Taking into account that u is harmonic, we have

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \quad \text{in } \Omega.$$

Since Ω is simply connected, the differential form w is exact. This means that there is a function $v \in C^2(\Omega)$, which is defined up to an additive constant, such that $dv = w$ in Ω, i.e.,

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{in } \Omega \quad \iff \begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \end{cases} \quad \text{in } \Omega.$$

Thus, the Cauchy-Riemann equations in the domain Ω are satisfied for the functions u and v. In addition, from the last relations it follows that

$$\begin{cases} \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \\ \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}, \end{cases} \implies \Delta v = 0 \text{ in } \Omega.$$

Therefore, $\langle u, v \rangle$ is a conjugate pair of harmonic functions in Ω. □

Example 2.6 Find an analytic function in $\mathbb{C}$ with the real part

$$u(x, y) = x^2 - y^2 - x.$$

Since u belongs to $C^2(\mathbb{R}^2)$ and $\Delta u = 2 - 2 = 0$ in the simply connected domain $\Omega = \mathbb{C}$, by Theorem 2.3, there is a harmonic function v such that $f = u + iv$ is analytic in $\mathbb{C}$.

It follows from the first equation in (2.2) that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x - 1.$$

Therefore,

$$v(x, y) = 2xy - y + \varphi(x),$$

where φ is some differentiable function. Then, $\frac{\partial v}{\partial x} = 2y + \varphi'(x)$ and from the second equation in (2.2) we get

$$-2y - \varphi'(x) = -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -2y \implies \varphi'(x) = 0 \implies \varphi \equiv c,$$

where $c \in \mathbb{R}$. Thus, $v(x, y) = 2xy - y + c$ and

$$f(x+iy) = x^2 - y^2 - x + i(2xy - y + c) = (x+iy)^2 - (x+iy) + ic = z^2 - z + ic.$$

Exercise 2.3 Prove that if f and $\overline{f}$ are analytic functions in $\mathbb{C}$, then f is constant in $\mathbb{C}$.

Many properties of harmonic functions resemble those of analytic functions and will be discussed in the following chapters of this textbook. Harmonic functions occur in problems of electric, magnetic and gravitational potentials, in problems of steady-state temperatures and in problems of hydrodynamics.

2.4 Hydrodynamic Interpretation of Analytical Functions

Let $f \in \mathcal{A}(\Omega)$ and $f(z) = u(x, y) + iv(x, y),\ z = x + iy \in \Omega$. From results obtained in Sect. 2.3 it follows that $\langle u, v \rangle$ is a conjugate pair of harmonic functions in Ω. We define a plane-parallel vector field

$$\mathbf{V}(x, y) := \left(\frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y), 0 \right), \qquad (x, y) \in \Omega;$$

in this case, the function u is called the *potential* of the vector field $\mathbf{V}$, and the function f is called the *complex potential* of $\mathbf{V}$.

It is easy to verify that

$$\operatorname{div} \mathbf{V} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \Delta u = 0 \quad \text{in } \Omega.$$

This means that the vector field $\mathbf{V}$ is *solenoidal* (or incompressible) in Ω (no sources and drains).

Calculating

$$\operatorname{curl} \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial \mu} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \end{vmatrix} = \left(0, 0, \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right) = \mathbf{0} \quad \text{in } \Omega,$$

we conclude that the vector field $\mathbf{V}$ is *irrotational* (or curl-free) in Ω.

Thus, every analytical function f in a domain Ω is the complex potential of some plane-parallel, solenoidal, and irrotational vector field in Ω.

Now let a solenoidal and irrotational vector field

$$\mathbf{V} = \big(\varphi_1(x, y), \varphi_2(x, y), 0\big)$$

be given in a simply connected domain Ω. We assume that the functions φ_1 and φ_2 belong to the space $C^1(\Omega)$ and $\mathbf{V} \neq \mathbf{0}$ in Ω.

Since $\mathbf{V}$ is irrotational,

$$\operatorname{curl} \mathbf{V} = \mathbf{0} \iff \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial \mu} \\ \varphi_1 & \varphi_2 & 0 \end{vmatrix} = \mathbf{0} \implies \frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial x} \text{ in } \Omega.$$

Therefore, taking into account the simply connectedness of the domain Ω, one can assert that the differential form $w_1 = \varphi_1 dx + \varphi_2 dy$ is an exact form in Ω. Then, due to Theorem 2.3 there exists a function $u \in C^2(\Omega)$ such that $du = w_1$. This means that

$$\frac{\partial u}{\partial x} = \varphi_1, \qquad \frac{\partial u}{\partial y} = \varphi_2 \quad \text{in } \Omega, \tag{2.11}$$

i.e., u is the potential of the vector field $\mathbf{V}$.

Since $\mathbf{V}$ is solenoidal,

$$\operatorname{div} \mathbf{V} = 0 \implies \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} = 0 \iff -\frac{\partial \varphi_2}{\partial y} = \frac{\partial \varphi_1}{\partial x} \quad \text{in } \Omega.$$

Thus, the differential form $w_2 = -\varphi_2 dx + \varphi_1 dy$ is an exact form in Ω, which means there is a function $v \in C^2(\Omega)$ such that $dv = w_2$, whence

$$\frac{\partial v}{\partial x} = -\varphi_2, \qquad \frac{\partial v}{\partial y} = \varphi_1 \quad \text{in } \Omega. \tag{2.12}$$

It follows from (2.11) and (2.12) that the functions u and v satisfy the Cauchy–Riemann equations in Ω. Therefore, one can determine a function $f := u + iv$, which, based on Theorem 2.2, will be analytic in Ω.

Therefore, any plane-parallel, solenoidal, irrotational vector field in a simply connected domain can be associated with an analytical function, which is the complex potential for that field.

Let us see what v means in physical terms. Consider a curve that is implicitly given by the equation $v(x, y) = \text{const}$. Then according to (2.12),

$$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\varphi_2}{\varphi_1}.$$

This means that the vector $\left(1, \frac{dy}{dx}, 0\right) = \left(1, \frac{\varphi_2}{\varphi_1}, 0\right)$, which is the tangent vector to the curve $v(x, y) = \text{const}$, is collinear to $\mathbf{V} = (\varphi_1, \varphi_2, 0)$. Consequently, the curve $v(x, y) = \text{const}$ is the motion trajectory of particles of a fluid flow, the velocity vector of which coincides with $\mathbf{V}$. The function v is called the *stream function* of the vector field $\mathbf{V}$.

2.5 Conformal Mappings: Geometric Meaning of the Modulus and Argument of the Derivative

Here, we take a closer look at the mapping properties of analytic functions.

Definition 2.11 A function $f: \Omega \mapsto \mathbb{C}$ at a point $z_0 \in \Omega$ is called conformal at z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$. The function f is said to be conformal in the domain Ω if it is analytic in Ω and conformal at every point of Ω. ▲

Let f be conformal at a point $z_0 \in \mathbb{C}$. Then for any smooth simple curve $z = \gamma(t),\ t \in [a, b]$, with the origin at $z_0 = \gamma(a)$, the limit

$$\lim_{z \to z_0,\ z \in E_\gamma} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)| \neq 0 \tag{2.13}$$

exists, since f has the derivative at z_0 and the limit does not depend on how z tends to z_0. Thus, this limit is independent of γ. On the other hand,

$$\lim_{z \to z_0,\ z \in E_\gamma} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \lim_{t \to a} \frac{|f(\gamma(t)) - f(\gamma(a))|}{|\gamma(t) - \gamma(a)|},$$

and this limit can be interpreted as a stretch coefficient (a scale factor) of the curve γ at the point z_0 under the mapping f.

Hence, a conformal function at z_0 stretches equally any smooth simple curve emanating from z_0, and the equality (2.13) expresses the *geometric meaning of the modulus of the derivative* $f'(z_0)$: this is the stretch coefficient at the point z_0 under the mapping f.

This mapping property of the conformal function f can be commented as follows: f stretches small circles centered at the point z_0 of radius $|\Delta z| = |z - z_0|$ in circles with center at $\omega_0 = f(z_0)$ of radius $|f'(z_0)|\,|\Delta z|$ up to a value $\overline{o}(|\Delta z|)$ (Fig. 2.1). Indeed,

$$|\omega - \omega_0| = |f(z) - f(z_0)| = |f'(z_0)\,\Delta z + \overline{o}(\Delta z)| \sim |f'(z_0)|\,|\Delta z| \quad \text{as } |\Delta z| \to 0.$$

Definition 2.12 A mapping f is called a mapping with equal stretch at a point z_0 if it stretches equally any smooth simple curve outgoing from z_0, i.e., for any smooth simple curve $z = \gamma(t),\ t \in [a, b]$, with the origin at $z_0 = \gamma(a)$, the limit

$$\lim_{t \to a} \frac{|f(\gamma(t)) - f(\gamma(a))|}{|\gamma(t) - \gamma(a)|} \tag{2.14}$$

exists, is independent of γ, and does not equal to zero.

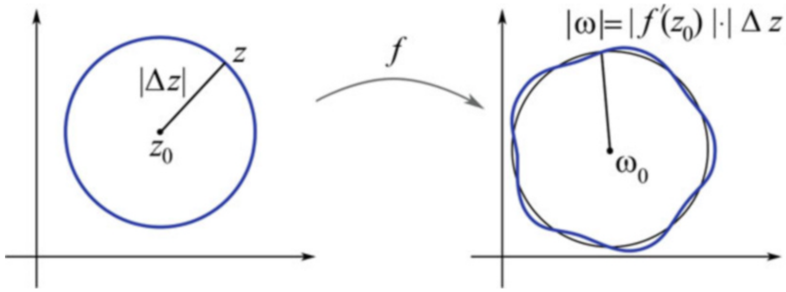


Fig. 2.1 Mapping small circles by a conformal mapping f

Example 2.7 The function $f(z) = 2\overline{z}$ is not analytic at any point of the complex plane. However, its stretch coefficient of any smooth simple curve equals 2. Indeed,

$$\lim_{t\to a} \frac{|f(\gamma(t)) - f(\gamma(a))|}{|\gamma(t) - \gamma(a)|} = 2 \lim_{t\to a} \frac{|\overline{\gamma(t)} - \overline{\gamma(a)}|}{|\gamma(t) - \gamma(a)|} = 2 \lim_{t\to a} \frac{|\overline{\gamma(t) - \gamma(a)}|}{|\gamma(t) - \gamma(a)|} = 2.$$

Example 2.8 It is easy to verify that the stretch coefficient of any smooth simple curve under the mapping $f(z) = x + i2y$ at any point of the complex plane is equal to 1 in the horizontal direction, and it is 2 in the vertical direction. Hence, this function is not a mapping with equal stretch at any point of the complex plane.

Example 2.9 Let f be conformal at a point $z_0 = x_0 + iy_0 \in \mathbb{C}$ $(f(z) = u(x, y) + iv(x, y))$. Then the Jacobian of the corresponding vector-valued function

$$\begin{pmatrix} u \\ v \end{pmatrix} : \Omega \longmapsto \mathbb{R}^2$$

at the point (x_0, y_0) in view of the Cauchy-Riemann equations is equal to

$$J(u, v)|_{(x_0,y_0)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z_0)|^2 > 0. \tag{2.15}$$

It is known from vector calculus that the Jacobian J of the vector-valued function $\binom{u}{v}$ is the linear stretch coefficient of infinitesimal areas.

From example 2.9 and the theorem on the preservation of a domain under a continuously differentiable mapping $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$, whose Jacobian is not equal to zero, the following statement follows.

Proposition 2.4 (Open Mapping Property) *A conformal function f in a domain Ω maps this domain into a domain in $\mathbb{C}$.*

A stronger statement is proved in Sect. 9.1. Example 2.9 and the inverse function theorem for a continuously differentiable mapping $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$, whose Jacobian is not equal to zero, lead to the following statement.

Proposition 2.5 *A conformal function f at a given point z_0 is a one-to-one mapping in some neighborhood of z_0. In addition, the inverse function is conformal at the point $w_0 = f(z_0)$ and by the chain rule*

$$\left(f^{-1}\right)'(w_0) = \frac{1}{f'(z_0)}. \tag{2.16}$$

The question of how to explicitly find the inverse function is discussed in Sect. 9.2.

Let us now clarify the geometric meaning of the argument of the derivative of a conformal function f at a point z_0.

It is known that for a smooth curve $z = \gamma(t) = x(t) + iy(t),\ t \in [a, b]$, with $\gamma(a) = z_0$, the value $\gamma'(a) = (x'(a), y'(a))$ is the *tangent vector* to the curve at z_0, and $\arg(\gamma'(a))$ is the angle of inclination of this vector to the real axis. Then for the curve $\widetilde{\gamma}(t) = f(\gamma(t))$, $t \in [a, b]$, we have

$$\widetilde{\gamma}'(a) = f'(z_0) \cdot \gamma'(a) \implies \operatorname{Arg} \widetilde{\gamma}'(a) = \arg f'(z_0) + \arg \gamma'(a). \tag{2.17}$$

Equality (2.17) is understood as follows: one of the arguments of the complex number $\widetilde{\gamma}'(a)$ is equal to the sum $\arg(f'(z_0)) + \arg(\gamma'(a))$.

From (2.17) it follows that $\arg(f'(z_0))$ is the angle by which you need to rotate the tangent vector to γ to get the angle of inclination of the tangent vector to the image of this curve at the point $\omega_0 = f(z_0)$, in other words: $\arg(f'(z_0))$ is the *angle of rotation of an arbitrary smooth curve emanating from z_0 under the mapping f* (Fig. 2.2).

Consider another smooth curve $z = \mu(t)$, $t \in [a, b]$, emanating from z_0. By $\widetilde{\mu}$ we denote its image under the mapping f, i.e., $\widetilde{\mu}(t) = f(\mu(t))$, $t \in [a, b]$. Similarly, we deduce

$$\widetilde{\mu}'(a) = f'(z_0) \cdot \mu'(a) \implies \operatorname{Arg} \widetilde{\mu}'(a) = \arg f'(z_0) + \arg \mu'(a). \tag{2.18}$$

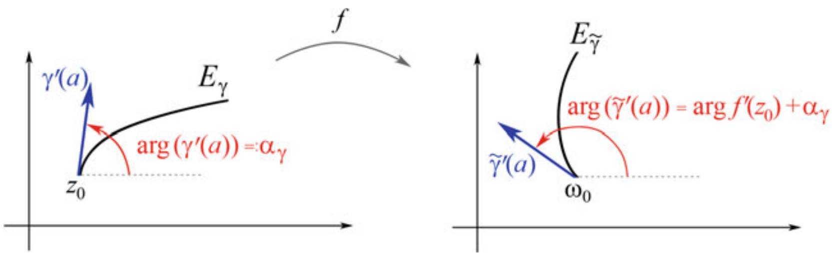


Fig. 2.2 Angle of rotation of a curve γ under a conformal mapping f

It follows from (2.17) and (2.18) that

$$\begin{aligned}\measuredangle(\widetilde{\mu}, \widetilde{\gamma})|_{\omega_0} &:= \operatorname{Arg} \widetilde{\mu}\,'(a) - \operatorname{Arg} \widetilde{\gamma}\,'(a) \\ &= \arg(\mu'(a)) - \arg(\gamma'(a)) =: \measuredangle(\mu, \gamma)|_{z_0}. \end{aligned} \tag{2.19}$$

Here, the symbol $\measuredangle(\mu, \gamma)|_{z_0}$ denotes the oriented angle between the curves μ and γ at z_0; by definition, it is equal to the oriented angle between the vectors $\mu'(a)$ and $\gamma'(a)$, i.e., $\arg(\mu'(a)) - \arg(\gamma'(a))$.

Equality (2.19) means that the angle between the curves μ and γ at z_0 is equal to the angle between their images under the conformal mapping f both in magnitude and in the direction of readout. These properties of f are called *angle-preserving and orientation-preserving* at the point z_0.

Example 2.10 The function from Example 2.7 is angle-preserving, but is not orientation-preserving. It reflects any smooth simple curve across the x-axis and then stretches it by 2.

The function from Example 2.8 is not angle-preserving, but preserves the orientation since the Jacobian of the vector-valued function $\binom{x}{2y} \colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ is positive.

Example 2.11 Let $f(z) = z^2$. Then $f'(z) = 2z$ and $f'(0) = 0$. Thus, f in not conformal at 0. Let us show that f is not angle-preserving at 0.

Indeed, consider two curves (their traces are segments) emanating from the origin:

$$\gamma_1(t) = t\, e^{i\alpha} \quad \text{and} \quad \gamma_2(t) = t\, e^{i\beta}, \quad t \in [0, 1].$$

Then, $\measuredangle(\gamma_2, \gamma_1)|_{z=0} = \beta - \alpha$. But the angle between their images

$$\widetilde{\gamma}_1(t) = t^2 e^{i2\alpha} \quad \text{and} \quad \widetilde{\gamma}_2(t) = t^2 e^{i2\beta}, \quad t \in [0, 1],$$

is equal to $\measuredangle(\widetilde{\gamma}_2, \widetilde{\gamma}_1)|_{\omega=0} = 2(\beta - \alpha)$.

Let us summarize the above, proving the main theorem characterizing conformal mappings.

Theorem 2.4 (Conformal Mapping Criterion) *Let* $f \colon \Omega \mapsto \mathbb{C}$, $f(z) = u(x, y) + iv(x, y)$, $z = x + iy \in \Omega$. *The function* f *is conformal in the domain* Ω *if and only if*

(1) the real-valued functions u *and* v *are differentiable in* Ω;
(2) f *is a mapping with equal stretch, angle-preserving and orientation-preserving at any point of* Ω.

Proof The necessity follows from the considerations above in this paragraph. Let us prove the sufficiency. By Definition 2.12, the limit (2.14) exists, does not depend on a curve and is not equal to zero for each point $z_0 = x_0 + iy_0$ of the domain Ω. Then the following limit

$$\lim_{\Delta z\to 0}\frac{|\Delta f(z_0)|^2}{|\Delta z|^2} = \lim_{\Delta z\to 0}\frac{|\Delta u + i\Delta v|^2}{|\Delta x + i\Delta y|^2} = \lim_{\Delta z\to 0}\frac{(\Delta u)^2 + (\Delta v)^2}{(\Delta x)^2 + (\Delta y)^2} =: K \neq 0,$$

exists, where the symbol Δ refer to corresponding increments (see (2.8), (2.9)).

Putting first $\Delta z = \Delta x$ ($\Delta y = 0$) in this limit and taking into account the first condition of the theorem, we find

$$\begin{aligned} K = &\lim_{\Delta x\to 0}\frac{(u(x_0+\Delta x, y_0) - u(x_0, y_0))^2 + (v(x_0+\Delta x, y_0) - v(x_0, y_0))^2}{(\Delta x)^2} \\ =&\Big(\frac{\partial u}{\partial x}(x_0, y_0)\Big)^2 + \Big(\frac{\partial v}{\partial x}(x_0, y_0)\Big)^2. \end{aligned} \tag{2.20}$$

By the same way we find the limit in the case $\Delta x = 0$:

$$K = \Big(\frac{\partial u}{\partial y}(x_0, y_0)\Big)^2 + \Big(\frac{\partial v}{\partial y}(x_0, y_0)\Big)^2. \tag{2.21}$$

In the case $\Delta x = \Delta y$ we get

$$K = K + \frac{\partial u}{\partial x}(x_0, y_0)\,\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0)\,\frac{\partial v}{\partial y}(x_0, y_0). \tag{2.22}$$

The Eqs. (2.20), (2.21), and (2.22) lead to the system

$$\begin{cases} \big(\frac{\partial u}{\partial x}\big)^2 + \big(\frac{\partial v}{\partial x}\big)^2 = \big(\frac{\partial u}{\partial y}\big)^2 + \big(\frac{\partial v}{\partial y}\big)^2 \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \end{cases}$$

Since $K \neq 0$, one of the derivatives in this system is not equal to zero. We can regard that $\frac{\partial v}{\partial y}(x_0, y_0) \neq 0$. Then

$$\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\frac{\partial u}{\partial x}(x_0, y_0)\,\frac{\partial u}{\partial y}(x_0, y_0)}{\frac{\partial v}{\partial y}(x_0, y_0)}. \tag{2.23}$$

Substituting this expression in the first equality of the system, we find

$$\frac{\partial u}{\partial x}(x_0, y_0) = \pm\frac{\partial v}{\partial y}(x_0, y_0). \tag{2.24}$$

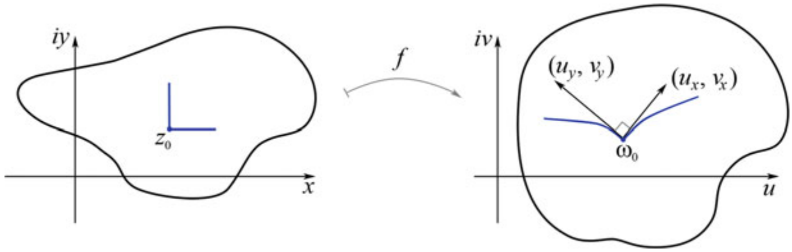


Fig. 2.3 An angle-preserving map f

Thus, we conclude from (2.23) and (2.24) that either the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{in} \quad \Omega \tag{2.25}$$

hold or opposite to them, namely

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \text{in} \quad \Omega. \tag{2.26}$$

Let us draw from the point z_0 two segments parallel to the coordinate axes, which belong to the domain Ω (Fig. 2.3). Recall that $z_0 = (x_0, y_0)$ is arbitrary point from Ω.

Since f is angle-preserving, the images of these segments will be perpendicular curves emanating from $\omega_0 = f(z_0)$. The vectors

$$\left(\frac{\partial u}{\partial x}(x_0, y_0), \frac{\partial v}{\partial x}(x_0, y_0)\right) \quad \text{and} \quad \left(\frac{\partial u}{\partial y}(x_0, y_0), \frac{\partial v}{\partial y}(x_0, y_0)\right)$$

are the tangent vectors to these curves at ω_0, respectively.

Turning the vector $(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x})$ at the angle $\frac{\pi}{2}$ counterclockwise and taking into account that f is orientation-preserving, we find that the directions of the vectors $(-\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x})$ and $(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y})$ coincide (obviously, these are nonzero vectors). Therefore,

$$0 < \left(-\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x}\right) \cdot \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = J(u, v). \tag{2.27}$$

If Eq. (2.26) are satisfied, then from (2.27) we get a contradiction:

$$\begin{aligned} 0 &< \frac{\partial u}{\partial x}(x_0, y_0)\frac{\partial v}{\partial y}(x_0, y_0) - \frac{\partial v}{\partial x}(x_0, y_0)\frac{\partial u}{\partial y}(x_0, y_0) \\ &= -\left(\frac{\partial v}{\partial y}(x_0, y_0)\right)^2 - \left(\frac{\partial v}{\partial x}(x_0, y_0)\right)^2 < 0. \end{aligned}$$

Thus, the Cauchy-Riemann equations (2.25) hold. Then, taking into account the first condition of the theorem and Theorem 2.2, we conclude that $f \in \mathcal{A}(\Omega)$. From (2.27), based on (2.15), it follows that

$$0 < J(u, v) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2.$$

This means that f is conformal in Ω. □

We now give a definition of the conformality of a function in the case when a point or its image is the point at infinity.

Definition 2.13 Let $z_0 = \infty$ and $f(z_0) \neq \infty$. A function f is said to be conformal at z_0 if the function $g(z) := f\left(\frac{1}{z}\right)$ is conformal at 0.

Let $z_0 \neq \infty$ and $f(z_0) = \infty$. A function f is said to be conformal at z_0 if the function $g(z) := \frac{1}{f(z)}$ is conformal at z_0.

Let $z_0 = \infty$ and $f(z_0) = \infty$. A function f is said to be conformal at z_0 if the function $g(z) := \frac{1}{f(\frac{1}{z})}$ is conformal at 0.

Definition 2.14 An analytic function $f : \Omega \mapsto \mathbb{C}$ is called univalent in Ω if it is injective in the domain Ω, i.e., for any two different points z_1 and z_2 from Ω we have that $f(z_1) \neq f(z_2)$.

In this case, Ω is referred to as the domain of univalence of f. ▲

One of the most important properties of a univalent function is the following: if $f : \Omega \mapsto \mathbb{C}$ is univalent, then the derivative of f is never zero, i.e., f is conformal in Ω. This statement will be proved in Theorem 7.8. It should be noted here that this statement does not hold for real-valued smooth functions, e.g., $f(x) = x^3, \ x \in \mathbb{R}$.

The main theorem on conformal mappings is the following Riemann theorem, the proof of which will be presented in Sect. 9.5.

Theorem 2.5 (Riemann Mapping Theorem) *Let Ω and G be two arbitrary simply connected domains in $\overline{\mathbb{C}}$ whose boundaries contain more than one point. Then for arbitrary points $z_0 \in \Omega$ and $\omega_0 \in G$ and any real number $\alpha \in (-\pi, \pi]$ there exists a unique bijective conformal mapping $f : \Omega \longmapsto G$ such that $\omega_0 = f(z_0)$ and* $\arg f'(z_0) = \alpha$.

At first glance, the statement of this theorem seems implausible. Simply connected domains in the complex plane can be very complicated. For instance, there are bounded domains such that the boundary is a nowhere-differentiable fractal curve of infinite length. And the fact that such a domain can be mapped onto a regular unit disk in an angle-preserving manner sounds counterintuitive.

Therefore, conformal mappings are invaluable for solving problems in engineering and physics that can be expressed in terms of functions of a complex variable, such as for example, boundary value problems involving Laplace's equation in complicated domains. By choosing an appropriate mapping, the inconvenient geometry of such a domain can be transformed into a much more convenient one.

To summarise what has been said so far in this section, conformal mappings preserve the shape of any sufficiently small figure, possibly rotating and scaling it (but not reflecting it).

Elementary Analytic Functions

3

Abstract

Conformal mappings are of immense importance in various branches of mathematics and in many applications. To solve many problems, one needs to be able to construct a bijective conformal mapping from one domain onto another in the complex plane. In this chapter we study how to construct such bijective conformal mappings. We will consider various elementary analytic functions, find domains of univalence and images of these domains. In addition, for many elementary analytic functions in $\mathbb{C}$ we find their inverses, which in some cases turn out to be multivalued. We introduce the first (intuitive) concept of a Riemann surface for multivalued functions and show how to construct Riemann surfaces for the inverses of elementary analytic functions. As a result of these studies, we will establish facts that are incorrect in real analysis, for example, we can calculate the logarithm of negative numbers and solve the equation

$$\sin z = 2.$$

3.1 Linear and Fractional-Linear Functions and Their Simplest Properties

Since the largest domain in the complex plane $\mathbb{C}$ is $\mathbb{C}$ itself, it is natural to start our study of conformal mappings by considering the analytic functions from $\mathbb{C}$ to itself, which are one-to-one and onto.

Definition 3.1 A function of the form

$$w = az + b,$$

where $\{a, b\} \subset \mathbb{C}$ and $a \neq 0$, is called a linear function of a complex variable.

T. Mel'nyk, *Complex Analysis*, https://doi.org/10.1007/978-3-031-39615-1_3

Since $w' = a \neq 0$, a linear function is entire, conformal and univalent in the complex plane $\mathbb{C}$. One can easily check that

$$z = \frac{w - b}{a}$$

is the inverse linear mapping from $\mathbb{C}$ onto $\mathbb{C}$. In Sect. 9.3, we will prove the following statement: *every conformal and univalent mapping from* $\mathbb{C}$ *onto* $\mathbb{C}$ *is a linear function.*

It is obvious that $\lim_{z\to\infty}(az + b) = \infty$. This suggests that each linear function can be determined at the point at infinity by setting it equal to ∞. Let us show the conformality of a linear function in ∞. In accordance with Definition 2.13, one should consider the function

$$g(z) = \frac{1}{az^{-1} + b} = \frac{z}{a + bz}$$

and check its conformality at zero: $g'(0) = \frac{1}{a} \neq 0.$

Having written the number a in the exponential form $a = |a| \cdot e^{i\ \arg(a)}$, the function $w = az + b$ can be represented as

$$w = e^{i\ \arg(a)}|a|\, z + b,$$

whence it is visible that each linear function is the composition of three mappings:

(1) $\xi = |a|z$ (homothety centered at the origin and with ratio $|a|$);
(2) $\tau = e^{i\ \arg(a)}\xi$ (rotation around the origin at the angle $\arg(a)$);
(3) $w = \tau + b$ (translation by the vector b).

Example 3.1 Figure 3.1 shows that the square

$$K_1 = \{z = x + iy :\ x \in (0, 1),\ y \in (0, 1)\}$$

is mapped by the linear function

$$\omega = \left(\sqrt{2} + i\sqrt{2}\right)z - i\sqrt{2} = 2e^{i\frac{\pi}{4}}z - i\sqrt{2}$$

onto the square $\{\omega = u + iv :\ |u| + |v| < \sqrt{2}\}$. ▲

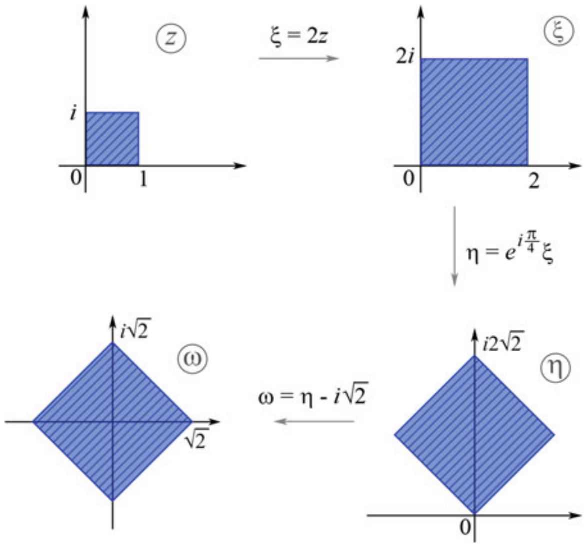


Fig. 3.1 The image of the square K_1

Also in Sect. 9.3, we will prove that conformal and univalent mappings from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ are a special class of functions of the form

$$w = \frac{az + b}{cz + d}, \qquad \{a, d, b, c\} \subset \mathbb{C}.$$

One additional restriction $ad - bc \neq 0$ is needed to ensure that such a function is neither identically constant nor meaningless. It turns out that such functions have many interesting properties. We will explore them in detail in this and the next three sections.

Definition 3.2 A fractional-linear function is a mapping of the form

$$w = \frac{az + b}{cz + d},$$

where $\{a, d, b, c\} \subset \mathbb{C}$ and $ad - bc \neq 0$.

When the coefficient $c = 0$, then the fractional-linear function becomes linear, some of the properties of which were studied above. Therefore, in this section we assume that $c \neq 0$. Since

$$\lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c} \quad \text{and} \quad \lim_{z \to -\frac{d}{c}} \frac{az + b}{cz + d} = \infty,$$

it is naturally to extend the fractional-linear function to $\overline{\mathbb{C}}$ by continuity as follows:

$$w = \mathfrak{F}(z) := \begin{cases} \dfrac{az+b}{cz+d}, & z \in \mathbb{C} \setminus \{-\frac{d}{c}\}, \\ \infty, & z = -\frac{d}{c}, \\ \dfrac{a}{c}, & z = \infty. \end{cases}$$

Remark 3.1 Thus, every fractional-linear function is a continuous mapping from $\overline{\mathbb{C}}$ in $\overline{\mathbb{C}}$.

Theorem 3.1 *A fractional-linear function is a homeomorphism from* $\overline{\mathbb{C}}$ *onto* $\overline{\mathbb{C}}$.

Proof Recall that a homeomorphism is, by definition, a one-to-one, onto continuous mapping with a continuous inverse. First, let us show that $\mathfrak{F}$ is bijection, namely

$$\forall\, w \in \overline{\mathbb{C}} \quad \exists !\, z \in \overline{\mathbb{C}} : \quad \mathfrak{F}(z) = w.$$

To prove this, it suffices to show that the equation $\mathfrak{F}(z) = w$ for z has only one root in $\overline{\mathbb{C}}$. Indeed,

$$\frac{az+b}{cz+d} = w \quad \Longrightarrow \quad z = \frac{-dw+b}{cw-a} \quad \text{if } w \in \mathbb{C} \setminus \left\{\frac{a}{c}\right\};$$

in addition, if $w = \infty$, then $z = -\frac{d}{c}$; and if $w = \frac{a}{c}$, then $z = \infty$.

Thus, for each fractional-linear mapping there is an inverse which is also a fractional-linear mapping determined by the formula

$$\mathfrak{F}^{-1}(w) := \begin{cases} \dfrac{-dw+b}{cw-a}, & w \in \mathbb{C} \setminus \{\frac{a}{c}\}, \\ \infty, & w = \frac{a}{c}, \\ -\dfrac{d}{c}, & w = \infty. \end{cases} \tag{3.1}$$

Remark 3.1 completes the proof. □

Theorem 3.2 *A fractional-linear function is conformal in* $\overline{\mathbb{C}}$.

Proof If $z \notin \{-\frac{d}{c}, \infty\}$, then

$$w' = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

If $z = -\frac{d}{c}$, then according to Definition 2.13 one should consider the function $g(z) = \frac{1}{\mathfrak{F}(z)} = \frac{cz+d}{az+b}$ and find its derivative at $z = -\frac{d}{c}$:

$$g'(z) = \frac{cb - ad}{(az + b)^2}\Big|_{z=-\frac{d}{c}} = \frac{c^2}{cb - ad} \neq 0.$$

If $z = \infty$, then one should consider the function

$$g(z) := \mathfrak{F}\left(\frac{1}{z}\right) = \frac{\frac{a}{z} + b}{\frac{c}{z} + d} = \frac{a + bz}{c + dz}$$

and find its derivative at $z = 0$:

$$g'(z) = \frac{bc - ad}{(c + dz)^2}\Big|_{z=0} = \frac{bc - da}{c^2} \neq 0.$$

□

3.2 Group and Circular Properties of Fractional-Linear Functions

By $\boldsymbol{\Lambda}$ we denote the set of all fractional-linear mappings and define a binary operation on this set as the composition of fractional-linear mappings:

$$\mathfrak{F}_2 \circ \mathfrak{F}_1 := \mathfrak{F}_2\big(\mathfrak{F}_1\big).$$

Exercise 3.1 Let there be given two fractional-linear mappings

$$\xi = \mathfrak{F}_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad w = \mathfrak{F}_2(\xi) = \frac{a_2 \xi + b_2}{c_2 \xi + d_2},$$

where $a_1 d_1 - c_1 b_1 \neq 0$ and $a_2 d_2 - c_2 b_2 \neq 0$. Prove that their composition is a fractional-linear mapping

$$w = \big(\mathfrak{F}_2 \circ \mathfrak{F}_1\big)(z) = \frac{az + b}{cz + d},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad ad - bc \neq 0.$$

Theorem 3.3 (The Group Property) *The set* $(\boldsymbol{\Lambda}, \circ)$ *is a noncommutative group.*

Proof Let us verify the group axioms:

- associativity

$$\forall \{\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3\} \subset \boldsymbol{\Lambda}: \quad \mathfrak{F}_3 \circ \big(\mathfrak{F}_2 \circ \mathfrak{F}_1\big) = \big(\mathfrak{F}_3 \circ \mathfrak{F}_2\big) \circ \mathfrak{F}_1$$

  follows from Exercise 3.1 and associativity of matrix multiplication;
- the identity mapping $E(z) = z$ is the identity element in this group:

$$\mathfrak{F} \circ E = E \circ \mathfrak{F} = \mathfrak{F};$$

- the existence of the inverse element follows from (3.1).

To check non-commutativity, consider two fractional-linear mappings: $\mathfrak{F}_1(z) = \frac{1}{z}$ and $\mathfrak{F}_2(z) = z + 2$. Then

$$\mathfrak{F}_2 \circ \mathfrak{F}_1 = \frac{1}{z} + 2 \neq \mathfrak{F}_1 \circ \mathfrak{F}_2 = \frac{1}{z+2}.$$

The theorem is proved. □

Theorem 3.4 (The Circular Property) *Each fractional-linear function maps a circle in $\overline{\mathbb{C}}$ onto a circle in $\overline{\mathbb{C}}$.*

It should be noted here that by a circle in $\overline{\mathbb{C}}$ we mean either a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ together with the point $\{\infty\}$.

Proof Each fractional-linear function $w = \mathfrak{F}(z) = \frac{az+b}{cz+d}$, $(ad - bc \neq 0,\ c \neq 0)$ can be represented as follows

$$\mathfrak{F}(z) = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{ad - bc}{c^2(z + \frac{d}{c})} =: A + \frac{B}{z + C},$$

where

$$A = \frac{a}{c}, \quad B = -\frac{ad - bc}{c^2}, \quad C = \frac{d}{c}.$$

Thus, $\mathfrak{F}(z) = \mathfrak{F}_3\big(\mathfrak{F}_2(\mathfrak{F}_1(z))\big)$, where

$$\mathfrak{F}_1 : z \mapsto z + C, \quad \mathfrak{F}_2 : z \mapsto \frac{1}{z}, \quad \mathfrak{F}_3 : z \mapsto A + Bz, \quad z \in \mathbb{C}.$$

The functions $\mathfrak{F}_1$ and $\mathfrak{F}_3$ are linear and map circles onto circles, since every linear function is a composition of homothety, rotation and translation (see Sect. 3.1). So, it remains to prove the theorem statement for $\mathfrak{F}_2$.

A general equation of a circle in the coordinate plane is

$$E(x^2 + y^2) + F_1 x + F_2 y + G = 0,$$

where $\{E, F_1, F_2, G\} \subset \mathbb{R}$, $E^2 + F_1^2 + F_2^2 + G^2 \neq 0$. It covers both the equation of a (standard) circle and the equation of a straight line, as well as a point (a circle with zero radius) and the empty set.

Since $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$ and $x^2 + y^2 = z\overline{z}$, the equation of a circle, written in a complex variable, has the form

$$Ez\overline{z} + Fz + \overline{F}\,\overline{z} + G = 0,$$

where $F = \frac{1}{2}(F_1 - iF_2)$, $\overline{F} = \frac{1}{2}(F_1 + iF_2)$. The mapping $w = \frac{1}{z}$ transforms a circle into a curve whose equation is

$$E\frac{1}{w} \cdot \frac{1}{\overline{w}} + F\frac{1}{w} + \overline{F}\frac{1}{\overline{w}} + G = 0 \iff Gw\overline{w} + \overline{F}w + F\overline{w} + E = 0,$$

but this is the equation of a circle in the complex plane. □

It turns out that a fractional-linear function is uniquely determined by the images of any three different points in $\overline{\mathbb{C}}$.

Theorem 3.5 *There is only one fractional-linear function $\mathfrak{F}$ that maps three different given points $\{z_1, z_2, z_3\} \subset \overline{\mathbb{C}}$ to three different given points $\{w_1, w_2, w_3\} \subset \overline{\mathbb{C}}$, i.e., $\mathfrak{F}(z_k) = w_k$ for $k \in \{1, 2, 3\}$.*

This fractional-linear mapping is defined by the formula

$$\frac{w - w_1}{w - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}. \tag{3.2}$$

Proof

1. Let us first consider the case when all the given complex numbers are finite, i.e., $\{z_1, z_2, z_3\} \subset \mathbb{C}$ and $\{w_1, w_2, w_3\} \subset \mathbb{C}$.

Having found w from (3.2), we check that (3.2) defines a fractional-linear function $\mathfrak{F}$. In addition, it is easy to verify that $\mathfrak{F}(z_k) = w_k$, $k \in \{1, 2, 3\}$. Let us prove the uniqueness of such a function.

Assume that there is another fractional-linear mapping $\mathfrak{F}_1$ such that $\mathfrak{F}_1(z_k) = w_k$ for $k \in \{1, 2, 3\}$. Then, based on Theorem 3.3, we get the following three relations:

$$\mathfrak{F}(z_k) = \mathfrak{F}_1(z_k) \iff \mathfrak{F}_1^{-1}\big(\mathfrak{F}(z_k)\big) = z_k \iff \frac{az_k + b}{cz_k + d} = z_k \tag{3.3}$$

for $k \in \{1, 2, 3\}$. Here, $\{a, b, c, d\} \subset \mathbb{C}$ and $ad - cd \neq 0$. It follows from (3.3) that the quadratic equation $cz^2 + (d - a)z + b = 0$ or the linear equation $(d - a)z + b = 0$ (if $c = 0$) has three different roots z_1, z_2, z_3 :

$$cz_k^2 + (d - a)z_k + b = 0 \quad \text{for all } \ k \in \{1, 2, 3\}. \tag{3.4}$$

This is possible only when $c = 0,\ d = a,\ b = 0$, i.e.,

$$\mathfrak{F}_1^{-1}\big(\mathfrak{F}(z)\big) = z \ \ (\forall\, z \in \mathbb{C}) \quad \iff \quad \mathfrak{F}_1 = \mathfrak{F}.$$

2. If one of the points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ coincides with the point $\{\infty\}$, the corresponding numerator and denominator in (3.2), where this point appears, must be replaced by 1 and then repeat the previous reasonings. For example, if $z_1 = \infty$ and $w_3 = \infty$, then the corresponding fractional-linear function is represented by the formula

$$\frac{w - w_1}{w - w_2} \cdot \frac{1 - \frac{w_1}{w_3}}{1 - \frac{w_1}{w_3}} = \frac{\frac{z}{z_1} - 1}{z - z_2} \cdot \frac{z_3 - z_2}{\frac{z_3}{z_1} - 1} \quad \iff \quad \frac{w - w_1}{w - w_2} = \frac{z_3 - z_2}{z - z_2}.$$

The theorem is proved.

□

Remark 3.2 The expression

$$\frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

is called the cross-ratio of four points $z,\ z_1,\ z_2,\ z_3$, and the equality (3.2) means invariance of the cross-ratio of four points under a fractional-linear mapping.

Remark 3.3 It follows from the proof of Theorem 3.5 that a fractional-linear map $\mathfrak{F} \neq E$ can have no more than two fixed different points z_1, z_2, i.e., $\mathfrak{F}(z_k) = z_k$, $k = 1, 2$. In this case, thanks to (3.2) this mapping it is given by the formula

$$\frac{w - z_1}{w - z_2} = A\, \frac{z - z_1}{z - z_2}$$

if $z_1 \neq \infty$ and $z_2 \neq \infty$, or $w - z_1 = A(z - z_1)$ if $z_2 = \infty$. Here the coefficient $A \in \mathbb{C} \setminus \{1\}$.

Depending on the coefficient A, such mappings are called *hyperbolic* fractional-linear mappings if A is positive and $A \neq 1$; *elliptic* if $A = e^{i\theta}$, $\theta \neq 2\pi n$; and *loxodromic* if $A = |A|e^{i\alpha}$, $|A| \neq 1$, $\alpha \neq 2\pi n$, where $n \in \mathbb{Z}$.

A fractional-linear function is called a *parabolic* fractional-linear mapping if it has only one fixed point, i.e., the quadratic Eq. (3.4) has zero discriminant (fixed points are coincided).

Exercise 3.2 Prove that every parabolic fractional-linear mapping can be represented in the form

$$\frac{1}{w - z_1} = \frac{1}{z - z_1} + \beta$$

if $z_1 \neq \infty$, or $w = z + \beta$ if $z_1 = \infty$. Here, $\beta \in \mathbb{C}, \quad \beta \neq 0$.

Corollary 3.1 *Let γ_1 and γ_2 be two circles in $\overline{\mathbb{C}}$. Then there exists a fractional-linear mapping $\mathfrak{F}$ such that $\gamma_2 = \mathfrak{F}(\gamma_1)$.*

To prove this, one needs to take three different points on one and the other circle, and then use the formula (3.2) and Theorem 3.4.

Corollary 3.2 *Let B_1 and B_2 be two disks in $\overline{\mathbb{C}}$. Then there exists a fractional-linear mapping $\mathfrak{F}$ such that $B_2 = \mathfrak{F}(B_1)$.*

Proof Recall that by a disk in $\overline{\mathbb{C}}$ we mean either a disk in $\mathbb{C}$ or its exterior in $\overline{\mathbb{C}}$ or a half plane together with the point $\{\infty\}$.

Let us take three different finite points z_1, z_2, z_3 on the boundary of the disk B_1 so that when they are successively traversed from z_1 trough z_2 to z_3, the disk remains to the left. By the same way we choose three different points w_1, w_2, w_3 on the boundary of B_2 (Fig. 3.2). Then a fractional-linear mapping $\mathfrak{F}$, given by (3.2), is required. Let's check it out.

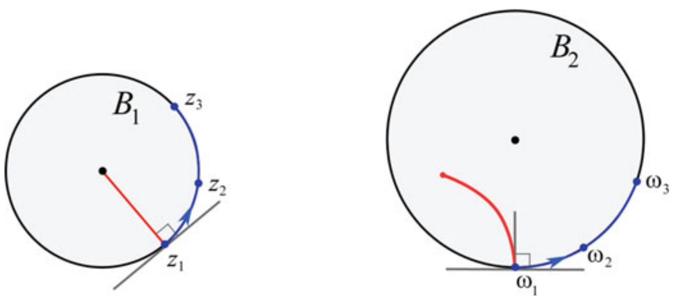


Fig. 3.2 Disks B_1 and B_2

Since each linear fractional mapping is conformal in $\overline{\mathbb{C}}$, then, based on statement Preposition 2.4, Theorem 3.1 and Corollary 3.1, the image of B_1 under the mapping $\mathfrak{F}$ can be either B_2 or the complement to $\overline{B_2}$ in $\overline{\mathbb{C}}$. Let us show that only the first option is possible.

Obviously that the angle between the arc $(\widetilde{z_1, z_2, z_3})$ of the circle ∂B_1 and the segment drawn from the point z_1 to the center of the disk B_1 is equal to $\frac{\pi}{2}$ (Fig. 3.2). Considering that each fractional-linear function is angle-preserving (it is conformal), the image of this segment will be an circular arc outgoing from w_1 and forming a right angle with the arc $(\widetilde{w_1, w_2, w_3})$ counterclockwise. This means that $\mathfrak{F}(B_1) = B_2$. □

It follows from this proof that fractional-linear mappings preserve the orientation of the boundaries. It turns out that this is also true for other conformal mappings.

Theorem 3.6 (Principle of Preserving Boundaries and Their Orientations [11]) *Let Ω and G be bounded simply connected domains with smooth boundaries. If a function f conformally and univalently maps Ω onto G, then*

(1) the function f can be continuously extended to $\overline{\Omega}$;
(2) this extended function is an orientation-preserving bijection between $\partial\Omega$ and ∂G.

3.3 Preservation of Symmetric Points by Fractional-Linear Mappings

In this section, we will discover another interesting property of fractional-linear transformations: it turns out that they are symmetry preserving.

Definition 3.3 Two points z_1 and z_1^* are called symmetric with respect to the circle $\Gamma = \{z : |z - a| = R\}$ (denoted by $z_1^* = \mathrm{Inv}_\Gamma(z_1)$) if

(1) $\arg(z_1^* - a) = \arg(z_1 - a)$,
(2) $|z_1^* - a| \cdot |z_1 - a| = R^2$.

By definitionition, we assume that $\infty = \mathrm{Inv}_\Gamma(a)$. ▲

The first condition in Definition 3.3 means that the points z_1 and z_1^* lie on the same ray emanating from the point a, the second one means that these points are on opposite sides of the circle Γ, or on Γ (in this case $z_1^* = z_1 \in \Gamma$). Obviously, if $z_1^* = \mathrm{Inv}_\Gamma(z_1)$, then $z_1 = \mathrm{Inv}_\Gamma(z_1^*)$.

From Definition 3.3 it follows that

$$z_1^* - a = k\,(z_1 - a), \tag{3.5}$$

where $k > 0$. Therefore,

$$k = \frac{|z_1^* - a|}{|z_1 - a|} = \frac{R^2}{|z_1 - a|^2} = \frac{R^2}{(z_1 - a) \cdot \overline{(z_1 - a)}} = \frac{R^2}{(z_1 - a) \cdot (\overline{z}_1 - \overline{a})}.$$

Substituting this expression for k in (3.5), we obtain

$$z_1^* = a + \frac{R^2}{\overline{z}_1 - \overline{a}} \quad \Longleftrightarrow \quad z_1 = a + \frac{R^2}{\overline{z_1^*} - \overline{a}}.$$

Using this formula, we define the function

$$z^* = \mathrm{Inv}_\Gamma(z) := \begin{cases} a + \dfrac{R^2}{\overline{z} - \overline{a}}, & \text{if } z \in \mathbb{C} \setminus \{a\}, \\ \infty, & \text{if } z = a, \\ a, & \text{if } z = \infty, \end{cases}$$

which is called *inversion* (symmetry) with respect to the circle Γ.

Example 3.2 Let $a = 0$ and $R = 1$. Then the inversion with respect to the unit circle $\Gamma_1 = \{z : |z| = 1\}$ is given with the formula

$$\mathrm{Inv}_{\Gamma_1}(z) = \frac{1}{\overline{z}}.$$

From this equality it follows that the fractional-linear function

$$w = \frac{1}{z} = \overline{\frac{1}{\overline{z}}}$$

is the composition of the inversion with respect to the circle Γ_1 and the symmetry with respect to the real axis.

Lemma 3.1 *Two different points z_1 and z_1^* are symmetric with respect to the circle $\Gamma = \{z : |z - a| = R\}$ if and only if every circle γ in $\overline{\mathbb{C}}$, passing through the points z_1 and z_1^*, intersects Γ orthogonally.*

Proof

Necessity Let $z_1^* = \mathrm{Inv}_\Gamma(z_1)$ and γ be a circle in $\overline{\mathbb{C}}$ passing through z_1 and z_1^*. Let us draw a tangent line to γ from the point a and denote the point of tangency by P. Then the intersecting chord theorem states that $|P - a|^2 = |z_1^* - a| \cdot |z_1 - a|$. Since the points z_1 and z_1^* are symmetric with respect to Γ, we have that $|P - a| = R$.

Thus, the point P is on Γ and the segment $[a, P]$ is a radius of the circle Γ. This means that the circle γ intersects Γ orthogonally.

Sufficiency Let an arbitrary circle in $\overline{\mathbb{C}}$, passing through the points z_1 and z_1^*, intersects Γ orthogonally. Then the line (a special case of circles in $\overline{\mathbb{C}}$), which passes through these points, must also intersect the circle Γ at a right angle. This is possible only when this line passes through the center of the circle Γ, that is, through the point a.

Moreover, the points z_1 and z_1^* lie on a ray emanating from the point a and on opposite sides of Γ, since otherwise the circle of radius $\frac{1}{2}|z_1^* - z_1|$, which passes through these points, cannot intersect the circle Γ orthogonally. Thus, $\arg(z_1^* - a) = \arg(z_1 - a)$.

It remains to verify the second condition of Definition 3.3. Now let γ be a circle in $\mathbb{C}$ that passes through the points z_1 and z_1^* and intersects the circle Γ at a right angle. Let us denote by P one of the intersection points. Then the radius $[a, P]$ is a segment of the tangent line to the circle γ, and hence, based on the intersecting chord theorem, we have

$$R^2 = |P - a|^2 = |z_1^* - a| \cdot |z_1 - a|.$$

The lemma is proved.

□

Remark 3.4 The right part of the statement of Lemma 3.1 can be taken as a new definition of symmetric points with respect to a circle. This definition is more general because it includes the case when Γ is a straight line (then it is the symmetry of points relative to this line).

Theorem 3.7 (Symmetric Points Preservation Property) *Let Γ be a circle in $\overline{\mathbb{C}}$ and $z_1^* = \mathrm{Inv}_\Gamma(z_1)$. Then for any fractional-linear mapping $\mathfrak{F}$ we have that*

$$w_1^* = \mathrm{Inv}_{\mathfrak{F}(\Gamma)}(w_1),$$

where $w_1 = \mathfrak{F}(z_1)$ and $w_1^* = \mathfrak{F}(z_1^*)$; that is, symmetric points with respect to the circle Γ under a fractional-linear mapping become symmetrical points relative to the image of this circle.

Proof Note that based on Theorem3.4, $\mathfrak{F}(\Gamma)$ is a circle in $\overline{\mathbb{C}}$. Let $\widetilde{\gamma}$ be a circle in $\overline{\mathbb{C}}$ passing through the points w_1^* and w_1. Then $\gamma := \mathfrak{F}^{-1}(\widetilde{\gamma})$ is a circle that passes through the points z_1 and z_1^*, and therefore, according to the preliminary lemma, the circle γ intersects Γ at a right angle.

Considering the conformality of fractional-linear mappings, the circle $\mathfrak{F}(\gamma) = \widetilde{\gamma}$ must intersect the circle $\mathfrak{F}(\Gamma)$ also orthogonally. Then, by Lemma 3.1, we conclude that $w_1^* = \mathrm{Inv}_{\mathfrak{F}(\Gamma)}(w_1)$. □

3.4 Fractional-Linear Isomorphisms and Automorphisms

Definition 3.4 Two domains Ω and Ω^* in $\overline{\mathbb{C}}$ are called fractional-linear isomorphic if there exists a fractional-linear mapping $\mathfrak{F}$ such that $\Omega^* = \mathfrak{F}(\Omega)$. In this case, the mapping $\mathfrak{F}\colon \Omega \mapsto \Omega^*$ is called the fractional-linear isomorphism of Ω onto Ω^*. ▲

Obviously, that the inverse function $\mathfrak{F}^{-1}$ maps the domain Ω^* onto Ω.

Proposition 3.1 *Each fractional-linear isomorphism of the half-plane* $\{z\colon \operatorname{Im} z > 0\}$ *onto the unit disk* $B_1 := \{w\colon\ |w| < 1\}$ *can be represented in the form*

$$w = e^{i\alpha}\,\frac{z-a}{z-\overline{a}}, \tag{3.6}$$

where $\operatorname{Im} a > 0$ *and* α *is a real number.*

Proof Let $\mathfrak{F}$ be a fractional-linear isomorphism of $\{z\colon\ \operatorname{Im} z > 0\}$ onto B_2. The existence of at least one such isomorphism follows from Corollary 3.2. Then there exists a unique point $a \in B_1$ such that $\mathfrak{F}(a) = 0$. Due to Theorem 3.7 we have that $\mathfrak{F}(\overline{a}) = \infty$.

Putting $z_1 = a$, $z_2 = \overline{a}$, $w_1 = 0$, $w_2 = \infty$ in (3.2), we get

$$\frac{w}{1}\cdot\frac{1}{w_3} = \frac{z-a}{z-\overline{a}}\cdot\frac{z_3-\overline{a}}{z_3-a} \quad\Longrightarrow\quad w = A\,\frac{z-a}{z-\overline{a}}\,, \tag{3.7}$$

where A is a complex number.

Let us show that $|A| = 1$. Since points of the real axis under the mapping $\mathfrak{F}$ go to points on the circle $\{w : |w| = 1\}$, we have for all $x \in \mathbb{R}$ that

$$1 = |\mathfrak{F}(x)| \iff 1 = |A|\,\frac{|x-a|}{|x-\overline{a}|} = |A| \quad\Longrightarrow\quad A = e^{i\alpha}\ (\alpha \in \mathbb{R}).$$

It is easy to check that a mapping of the form (3.6) is a fractional-linear isomorphism of the upper half-plane onto B_1. □

Definition 3.5 Fractional-linear isomorphism of a domain Ω onto itself is called a fractional-linear automorphism of Ω.

Obviously, the set of all fractional-linear automorphisms of a domain Ω forms a subgroup of the group $(\boldsymbol{\Lambda}, \circ)$.

Proposition 3.2 *Each fractional-linear automorphism of the unit disk $B_1 = \{z \in \mathbb{C} : |z| < 1\}$ can be represented in the form*

$$w = e^{i\beta} \frac{z-b}{1-z\overline{b}}, \tag{3.8}$$

where $|b| < 1$ and β is a real number.

Proof Let $\mathfrak{F}$ be a fractional-linear automorphism of the disk B_1. Then there exists a unique point $b \in B_1$ such that $\mathfrak{F}(b) = 0$. Due to Theorem 3.7 we have that $\mathfrak{F}(b^*) = \infty$, where

$$b^* = \mathrm{Inv}_{\partial B_1}(b) = \frac{1}{\overline{b}}.$$

Substituting $z_1 = b$, $z_2 = \frac{1}{\overline{b}}$, $w_1 = 0$, $w_2 = \infty$ in (3.2), we get

$$w = A \frac{z-b}{z-\frac{1}{\overline{b}}} = A_1 \frac{z-b}{1-z\overline{b}}.$$

Since $\mathfrak{F}(1) \in \partial B_1$, we have the following

$$|\mathfrak{F}(1)| = 1 \iff 1 = |A_1| \frac{|1-b|}{|1-\overline{b}|} = |A_1| \implies A_1 = e^{i\beta} \quad (\beta \in \mathbb{R}).$$

It is easy to check that a mapping of the form (3.8) is a fractional-linear automorphism of the disk B_1. □

Exercise 3.3 Let $\mathfrak{F}$ be a fractional-linear automorphism of the unit disk B_1. It is known that

$$\mathfrak{F}\Big(\frac{1}{2} + \frac{i}{2}\Big) = -\frac{i}{8}.$$

Reasonably find the value of $\mathfrak{F}\big(\sqrt{2}\, e^{i\frac{\pi}{4}}\big)$.

Exercise 3.4 Prove that each fractional-linear automorphism of the half-plane $\{z : \operatorname{Im} z > 0\}$ can be represented in the form

$$w = \frac{az+b}{cz+d},$$

where a, b, c, d are real numbers and $ad - cb > 0$.

For more properties of fractional-linear automorphisms, see Sect. 9.3.

3.5 Power Functions with Natural Exponents

Consider the power function

$$\omega = f(z) := z^n, \quad z \in \mathbb{C} \quad (n \in \mathbb{N},\ n \geq 2).$$

It follows from Example 2.2 that $f'(z) = nz^{n-1}$ for any $z \in \mathbb{C}$. Thus, $f \in \mathcal{A}(\mathbb{C})$ and f is conformal in $\mathbb{C} \setminus \{0\}$.

Let us find domains of univalence (see Definition 2.14) of f. Let $z_1 \neq z_2$ and $z_1^n = z_2^n$. Then

$$|z_1|^n e^{in\alpha_1} = |z_2|^n e^{in\alpha_2} \quad \Longrightarrow \quad \begin{cases} |z_1| = |z_2|, \\ \alpha_1 = \alpha_2 + \frac{2\pi k}{n}, \quad k \in \mathbb{Z}, \end{cases}$$

where $\alpha_1 \in \operatorname{Arg}(z_1)$, $\alpha_2 \in \operatorname{Arg}(z_2)$.

Thus, a domain Ω is the domain of univalence for the function z^n if it does not contain any pair of complex numbers with the same moduli and with the principal arguments that differ by $\frac{2\pi}{n}$. In particular, the function $\omega = z^n$ is univalent in the corner

$$\mathcal{K}_\beta := \left\{ z \in \mathbb{C} : \ \beta < \operatorname{Arg}(z) < \beta + \frac{2\pi}{n} \right\} \quad (\beta \in \mathbb{R}).$$

Here, by $\operatorname{Arg}(z)$ we mean one of arguments of z, which satisfies this inequality. Let us find the image of $\mathcal{K}_\beta$ under the power function f.

It is easy to check that f maps the ray

$$z = t\, e^{i\alpha}, \quad t \in (0, +\infty),$$

into the ray $z = t^n e^{in\alpha}$, $t \in (0, +\infty)$ (Fig. 3.3).

Now if α changes from β to $\beta + \frac{2\pi}{n}$, then these rays will fill out the domain $\mathbb{C} \setminus \{\omega\colon \operatorname{Arg}(w) = n\beta\}$ (Fig. 3.4). This means that $\omega = z^n$ conformally and univalently maps the corner $\mathcal{K}_\beta$ onto $\mathbb{C} \setminus \{\omega : \ \operatorname{Arg}(\omega) = n\beta\}$.

In particular, if $\beta = 0$ and $n = 2$, then the image of the upper half-plane under $w = z^2$ is the complex plane without the positive real half-axis.

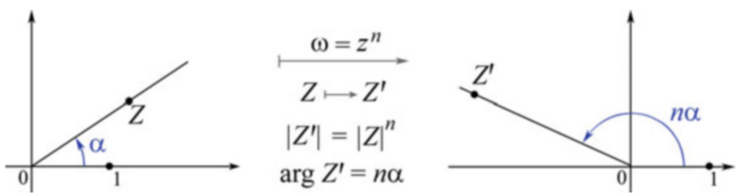


Fig. 3.3 The image of a ray under the power function z^n

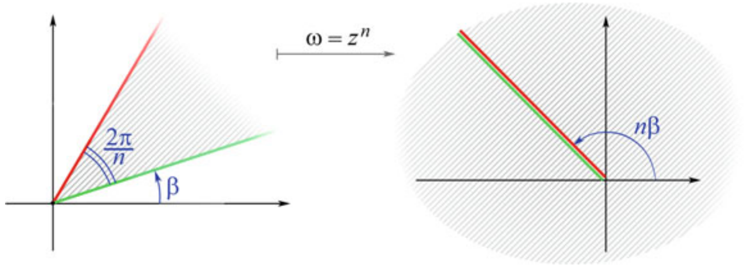


Fig. 3.4 The image of the corner $\mathcal{K}_\beta$ under the power function z^n

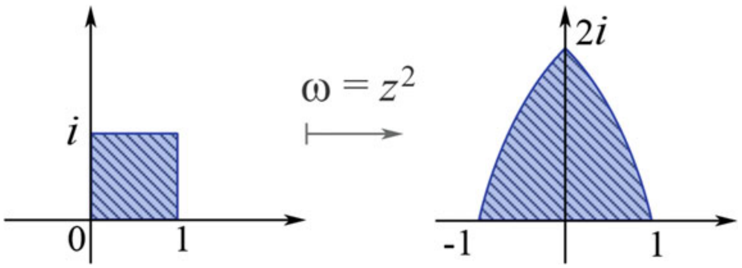


Fig. 3.5 The image of the square K_1 under $w = z^2$

Example 3.3 Let us find the image of the unit square K_1 from Example 3.1 under the function $\omega = z^2$.

Since the square K_1 is symmetric with respect to the straight line $y = x$, its image will be symmetric with respect to the imaginary axis, because $\omega = z^2$ sends the points $|a|\exp(i\varphi)$ and $|a|\exp(i(\frac{\pi}{2} - \varphi))$ (here $\varphi \in (0, \frac{\pi}{2})$) respectively to the points

$$|a|^2 \exp(i2\varphi) \quad \text{and} \quad |a|^2 \exp(i(\pi - 2\varphi)).$$

It is easy to verify that the function $\omega = z^2$ maps the segment $[0, 1]$ onto itself, and the segment $[0, 1 + i]$ onto $[0, 2i]$.

Substituting the parameterization $z = 1+iy,\ y \in [0, 1]$, of the segment $[1, 1+i]$ in $u + iv = z^2$, we find its image that is a part of the parabola

$$u = 1 - \frac{v^2}{4}, \qquad v \in [0, 2].$$

So the square K_1 is mapped by $\omega = z^2$ onto a domain bounded by parabolas

$$u = 1 - \frac{v^2}{4} \quad \text{and} \quad u = -1 + \frac{v^2}{4},$$

and the real axis (Fig. 3.5).

3.6 The Inverse to a Power Function and Its Riemann Surface

First, consider the function $w = z^2$. It is clear that there is no single-valued inverse function for $z^2 : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$, because every point from $\overline{\mathbb{C}}$, except for the points 0 and ∞, has two inverse images, i.e., the function $z = \sqrt{w}$ is two-valued. It is inconvenient to work with multi-valued functions because it is not clear, for example, how to introduce the concept of continuity, differentiation, etc. The German mathematician B. Riemann (1826–1866) in his dissertation proposed the following approach for transforming a multi-valued function into a single-valued one:

> it is necessary to separate points that have more than two images and consider them on different sheets of a surface, which is now called a *Riemann surface*.

In this section and next one we will get acquainted with the concept of a Riemann surface on examples of some functions, and the abstract approach will be considered in Sect. 8.5. First, let us implement this approach for the two-valued function $z = \sqrt{w}$. Consider the upper and lower half-planes

$$D_0 = \{z \in \mathbb{C} : \ 0 < \arg z < \pi\} \quad \text{and} \quad D_1 = \{z \in \mathbb{C} : \ -\pi < \arg z < 0\}\,.$$

The previous section shows that the function $w = f(z) = z^2$ conformally and univalently maps D_0 and D_1 onto the domain $E = \{w \in \mathbb{C} : \ 0 < \operatorname{Arg} w < 2\pi\}$.

Therefore, for each of these mappings there is a unique inverse function $f_k^{-1} : E \longmapsto D_k$ ($k \in \{0, 1\}$; Fig. 3.6), i.e.,

$$f(f_k^{-1}(w)) = w \quad \text{for all} \ \ w \in E, \quad \text{and} \quad f_k^{-1}(f(z)) = z \quad \text{for all} \ \ z \in D_k.$$

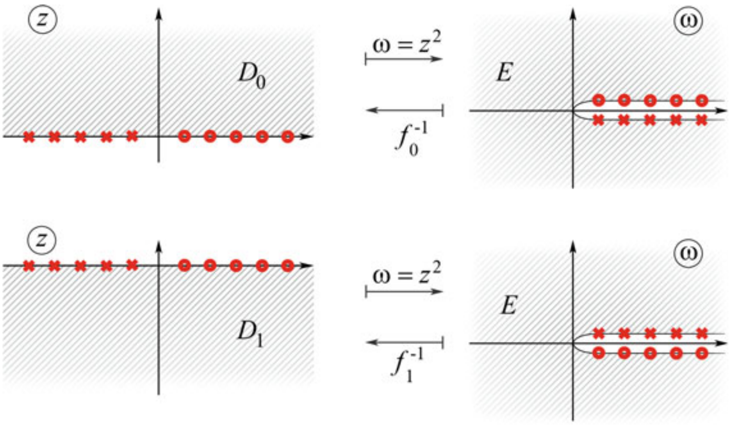


Fig. 3.6 Branches f_0^{-1} and f_1^{-1} of the inverse function $z = \sqrt{w}$

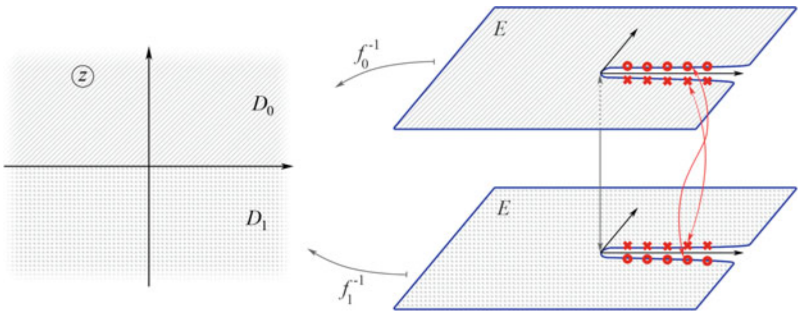


Fig. 3.7 Construction of the Riemann surface of the function $\sqrt{w}$

Using (1.11), we get that

$$f_k^{-1}(w) = \sqrt{|w|}\, e^{i\left(\frac{\operatorname{Arg} w}{2} + \pi k\right)} \quad \text{for all } \ w \in E \quad (0 < \operatorname{Arg} w < 2\pi).$$

Since

$$\frac{df_k^{-1}(w)}{dw} = \frac{1}{f'(z)}\bigg|_{z=f_k^{-1}(w)} = \frac{1}{2 f_k^{-1}(w)} \neq 0 \quad \text{for all } \ w \in E,$$

the function f_k^{-1} is conformal and univalent in E $(k \in \{0,\ 1\})$.

Take two sheets (instances) of the domain E and place them one above the other (Fig. 3.7). The sheet from which the function f_0^{-1} acts is called the 0th-sheet; and the sheet from which the function f_1^{-1} acts is called the 1st-sheet. The point $w = 0$ from the 0th-sheet is identified with the point $w = 0$ from the 1st-sheet, because they have the same inverse image under the mapping $w = f(z) = z^2$; the same for $w = \infty$.

To understand how to glue (identify) points from different edges of the cuts of these two sheets, one should find the limits of the functions f_0^{-1} and f_1^{-1} as w tends from each sheet to positive x from above and below. Since

$$\lim_{w \to x>0,\ \operatorname{Im} w>0} f_0^{-1}(w) = \sqrt{x} = \lim_{w \to x>0,\ \operatorname{Im} w<0} f_1^{-1}(w),$$

points of the upper edge of the 0th-sheet cut must be glued (identified) with the corresponding points of the lower edge of the 1st-sheet cut. Since

$$\lim_{w \to x>0,\ \operatorname{Im} w<0} f_0^{-1}(w) = -\sqrt{x} = \lim_{w \to x>0,\ \operatorname{Im} w>0} f_1^{-1}(w),$$

points of the lower edge of the 0th-sheet cut must be glued with the corresponding points of the upper edge of the 1st-sheet cut.

The surface thus glued is called a Riemann surface of $\sqrt{w}$ and is denoted by $\mathfrak{R}_{\sqrt{w}}$ (Fig. 3.8). The function $\sqrt{w}$ becomes single-valued on its Riemann surface and conformal everywhere except for the points 0 and ∞, because the two values that the root assigns to each nonzero complex number are now images of two different points lying on different sheets above this number. Positive numbers are no exception, since over them there is no self-intersection of the sheets of the Riemann surface.

Example 3.4 Consider, for example, the complex number -2. Over it there are two different points -2_0 and -2_1 lying on the 0th-sheet and 1st-sheet, respectively. Since $\sqrt{w} = f_0^{-1}(w)$ on the 0th-sheet and $\sqrt{w} = f_1^{-1}(w)$ on the 1th-sheet, we find

$$\sqrt{-2_0} = f_0^{-1}(-2) = \sqrt{|-2|}\, e^{i\frac{\pi}{2}} = \sqrt{2}\, i,$$

$$\sqrt{-2_1} = f_1^{-1}(-2) = \sqrt{|-2|}\, e^{i(\frac{\pi}{2}+\pi)} = -\sqrt{2}\, i.$$

▲

Each of the points 0 and ∞ is called a *first-order branch point*. The order of a branch point indicates the additional number of sheets of the Riemann surface that must be traversed around this point in order to return to the original position (Fig. 3.8).

The functions f_0^{-1} and f_1^{-1} are called *analytic branches* of the single-valued function $\sqrt{w}$ given on its Riemann surface $\mathfrak{R}_{\sqrt{w}}$. In $3D$-space, we will not able to draw the Riemann surface of $z = \sqrt{w}$ without self-intersection, but schematically it can be represented as shown in Fig. 3.8.

Similarly, the Riemann surface of the function $\sqrt[n]{w}$ $(n \in \mathbb{N}, n > 2)$ is constructed. To do this, we consider the angles

$$D_k = \left\{ z \in \mathbb{C} : \frac{2\pi k}{n} < \operatorname{Arg} z < \frac{2\pi(k+1)}{n} \right\}, \quad k \in \{0, 1, \ldots, n-1\},$$

that are domains of univalence for the function $w = f(z) = z^n$. It conformally and univalently maps each of these corners onto the domain $E =$

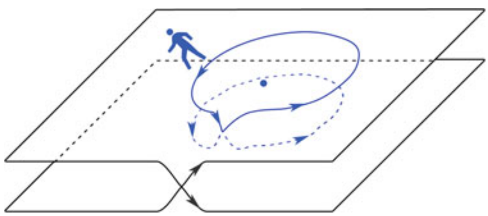


Fig. 3.8 The Riemann surface $\mathfrak{R}_{\sqrt{w}}$

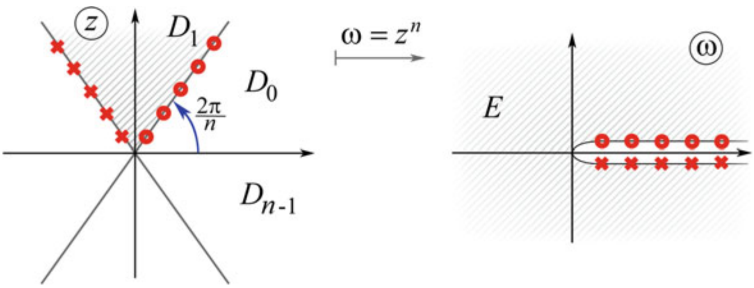


Fig. 3.9 The image of D_k by the function $w = z^n$

$\{w \in \mathbb{C}\colon 0 < \operatorname{Arg} w < 2\pi\}$ (Fig. 3.9). Therefore, for each of these mappings there is a unique inverse function $f_k^{-1}: E \mapsto D_k$, i.e.,

$$f(f_k^{-1}(w)) = w \quad \text{for all } w \in E, \quad \text{and} \quad f_k^{-1}(f(z)) = z \quad \text{for all } z \in D_k;$$

and

$$f_k^{-1}(w) = \sqrt[n]{|w|}\, e^{i(\frac{\operatorname{Arg} w}{n} + \frac{2\pi k}{n})} \quad \text{for all } w \in E \quad (0 < \operatorname{Arg} w < 2\pi).$$

Since

$$\frac{df_k^{-1}(w)}{dw} = \frac{1}{f'(z)}\Big|_{z=f_k^{-1}(w)} = \frac{f_k^{-1}(w)}{n\,w} \neq 0 \quad \text{for all } w \in E,$$

the function f_k^{-1} is conformal and univalent in E ($k \in \{0, 1, \ldots, n-1\}$).

We take n sheets of E and place them on top of each other above the complex plane. The sheet from which the function f_k^{-1} acts is called the kth-sheet. Using the functions $\{f_k^{-1}\}_{k=0}^{n-1}$, these sheets must be glued together to form a continuous function.

The points $w = 0$ from these sheets are identified because they have the same inverse image under the mapping $w = z^n$; the same for $w = \infty$. Since for each $k \in \{1, \ldots, n-2\}$

$$\lim_{w \to x>0,\ \operatorname{Im} w>0} f_k^{-1}(w) = \sqrt[n]{x}\, e^{i\frac{2\pi k}{n}} = \lim_{w \to x>0,\ \operatorname{Im} w<0} f_{k-1}^{-1}(w)$$

and

$$\lim_{w \to x>0,\ \operatorname{Im} w<0} f_k^{-1}(w) = \sqrt[n]{x}\, e^{i\frac{2\pi(k+1)}{n}} = \lim_{w \to x>0,\ \operatorname{Im} w>0} f_{k+1}^{-1}(w),$$

points of the upper edge of the kth-sheet cut must be glued with the corresponding points of the lower edge of the $(k-1)$th-sheet cut, and points of the lower edge of

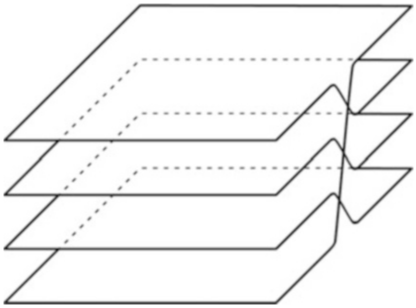


Fig. 3.10 Riemann surface $\mathfrak{R}_{\sqrt[4]{w}}$

the kth-sheet cut must be glued with the corresponding points of the upper edge of the $(k+1)$th-sheet cut. Since

$$\lim_{w\to x>0,\ \operatorname{Im} w<0} f_{n-1}^{-1}(w) = \sqrt[n]{x} = \lim_{w\to x>0,\ \operatorname{Im} w>0} f_0^{-1}(w),$$

points of the lower edge of the $(n-1)$th-sheet must be glued with the corresponding points of the upper edge of the 0-sheet.

The surface thus glued is called a Riemann surface of the function $\sqrt[n]{w}$ and is denoted by $\mathfrak{R}_{\sqrt[n]{w}}$ (Fig. 3.10). The function $\sqrt[n]{w}$ becomes single-valued on its Riemann surface and conformal everywhere except for the points 0 and ∞ that are branch points of order $n-1$. The functions $\{f_k^{-1}\}_{k=0}^{n-1}$ are called analytic branches of the single-valued function $\sqrt[n]{w}$ given on its Riemann surface $\mathfrak{R}_{\sqrt[n]{w}}$. Now above each nonzero complex number there are n different points lying on different sheets of the Riemann surface $\mathfrak{R}_{\sqrt[n]{w}}$.

3.7 Exponential Function, Logarithmic Function and Its Riemann Surface

The exponential function is already defined in Example 2.4:

$$w = e^z = e^{x+iy} \overset{def}{=} e^x(\cos y + i \sin y), \quad z \in \mathbb{C},$$

which also shows that it is an entire function and $(e^z)' = e^z \neq 0$ for all $z \in \mathbb{C}$. Due to (1.10) we get $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

A new property of the exponential function is its periodicity with the main period $2\pi i$. Indeed, $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$ for all $z \in \mathbb{C}$. On the other hand, if we assume that there is another period $T = T_1 + iT_2$, then for all $z \in \mathbb{C}$

$$e^{z+T} = e^z \implies e^{T_1} e^{iT_2} = 1 \implies T_1 = 0 \quad \text{and} \quad T_2 = 2\pi k, \quad k \in \mathbb{Z},$$

i.e., $T = i2\pi k, \quad k \in \mathbb{Z} \setminus \{0\}$.

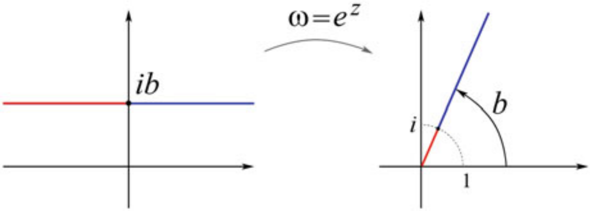


Fig. 3.11 The image of a horizontal line by the function e^z

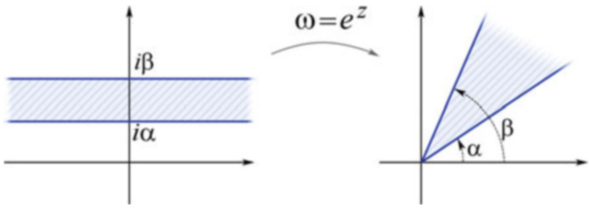


Fig. 3.12 The image of a horizontal strip by the function e^z

Consequently, a domain is a domain of univalence for e^z if it does not contain any pair of points connected by the relation $z_1 - z_2 = 2\pi ik$, $k \in \mathbb{Z} \setminus \{0\}$. In particular, the horizontal strips

$$D_k := \{z : -\pi + 2k\pi < \operatorname{Im} z < \pi + 2\pi k\}, \quad k \in \mathbb{Z}.$$

are domains of univalence for the exponential function.

Let us find the image of the horizontal strip $\{z\colon \alpha < \operatorname{Im} z < \beta\}$ $(0 < \beta - \alpha < 2\pi)$ under the mapping e^z. It is easy to check that the exponential function maps a horizontal line

$$z = x + ib, \quad x \in (-\infty, +\infty),$$

into the ray $w = e^x\, e^{ib}$, $x \in (-\infty, +\infty)$ (see Fig. 3.11).

Now if b changes from α to β, then e^z conformally and univalently maps the strip $\{z\colon \alpha < \operatorname{Im} z < \beta\}$ onto the corner $\{w\colon \alpha < \operatorname{Arg} w < \beta\}$ (Fig. 3.12).

Example 3.5 Similarly, we verify that the image of a vertical segment $z = a + iy$, $y \in [-\pi, \pi]$, by $w = e^z$ is the circle $w = e^a e^y$, $y \in [-\pi, \pi]$, of radius e^a centered at the origin (Fig. 3.13); here $a \in \mathbb{R}$. Thus, the image of the closed rectangle $[a, b] \times [-i\pi, i\pi]$ by $w = e^z$ is the annulus $\{w : \ e^a < |w| < e^b\}$.

From the above it follows that for all $k \in \mathbb{Z}$ the exponential function $f(z) := e^z$ conformally and univalently maps the horizontal strip D_k onto the domain

$$E_1 = \{w\colon \ -\pi < \arg(w) < \pi\},$$

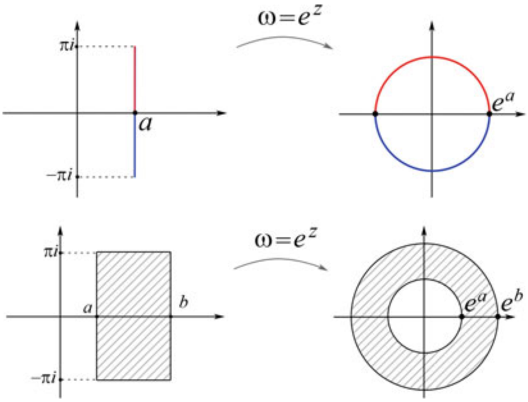


Fig. 3.13 Images of a segment and rectangle by the function e^z

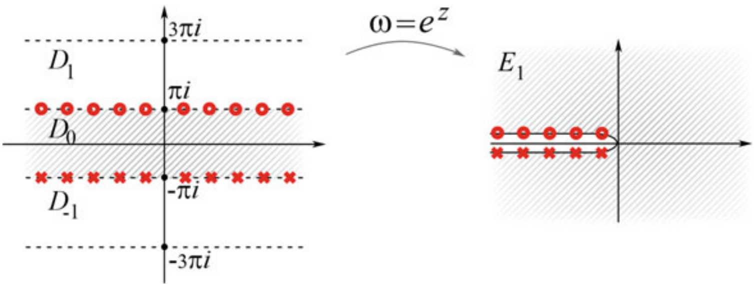


Fig. 3.14 The image of the strip D_k by the function e^z

moreover, the lower part of the boundary of D_k is mapped onto the lower edge of the cut $(-\infty, 0]$, and the upper one is mapped onto the upper edge of this cut (Fig. 3.14).

This means that for each such mapping there is an inverse $f_k^{-1}\colon E_1 \mapsto D_k$, i.e., for all integer k

$$f(f_k^{-1}(w)) = w \quad \text{for all } w \in E_1, \quad \text{and} \quad f_k^{-1}(f(z)) = z \quad \text{for all } z \in D_k.$$

In addition, since

$$\frac{d\, f_k^{-1}(w)}{dw} = \frac{1}{\left(e^z\right)'\big|_{z=f_k^{-1}(w)}} = \frac{1}{w} \neq 0 \quad \text{for all } w \in E_1, \tag{3.9}$$

the function f_k^{-1} is conformal and univalent in E_1; $k \in \mathbb{Z}$.

To deduce formula for the inverse function f_k^{-1}, one should find the unique root $z \in D_k$ of the equation $e^z = w$, where $w \in E_1$. From the equality of two complex numbers we derive

$$e^z = w \iff e^x\, e^{iy} = |w|\, e^{i\arg(w)} \implies x = \log|w| \quad \text{and} \quad y = \arg(w) + 2\pi k$$

$$\implies z = x + iy = \log|w| + i\,\big(\arg(w) + 2\pi k\big).$$

Thus,

$$f_k^{-1}(w) = \log|w| + i\ \arg(w) + 2\pi i k, \quad w \in E_1.$$

The functions $\{f_k^{-1}\}_{k\in\mathbb{Z}}$ are called analytical branches of the multi-valued logarithmic function

$$\mathrm{Log}\,(w) := \log|w| + i\,\mathrm{Arg}(w), \quad w \in \mathbb{C} \setminus 0.$$

Recall that $\mathrm{Arg}\, w$ is the set of all arguments of a complex number w.

The analytical branch $f_0^{-1}(w) = \log|w| + i\ \arg(w)$ is called the *principal branch* of Log.

Remark 3.5 As follows from (1.8), the function $\arg(w)$, $w = u + iv \in E_1$, is differentiable in E_1 as a function of two real variables u and v, and it has a jump 2π as w crosses the negative real axis:

$$\lim_{w\to x<0,\ \mathrm{Im}\, w>0} \arg(w) = \pi \quad \text{and} \quad \lim_{w\to x<0,\ \mathrm{Im}\, w<0} \arg(w) = -\pi.$$

Thus, the values of each branch f_k^{-1} jump by $2\pi i$ when crossing the negative real axis.

Exercise 3.5 By using the Cauchy–Riemann Theorem 2.2, check the differentiability of each branch f_k^{-1} in E_1.

To construct the Riemann surface of Log we take a denumerable set of E_1-sheets and place them on top of each other over the complex plane. The sheet from which the function f_k^{-1} acts is called the kth-sheet. Using the functions $\{f_k^{-1}\}_{k\in\mathbb{Z}}$, these sheets must be glued together to form a continuous function. Since for each $k \in \mathbb{Z}$

$$\lim_{w\to x<0,\ \mathrm{Im}\, w>0} f_k^{-1}(w) = \log|x| + i(\pi + 2\pi k) = \lim_{w\to x<0,\ \mathrm{Im}\, w<0} f_{k+1}^{-1}(w)$$

and

$$\lim_{w\to x<0,\ \mathrm{Im}\, w<0} f_k^{-1}(w) = \log|x| + i(-\pi + 2\pi k) = \lim_{w\to x<0,\ \mathrm{Im}\, w>0} f_{k-1}^{-1}(w),$$

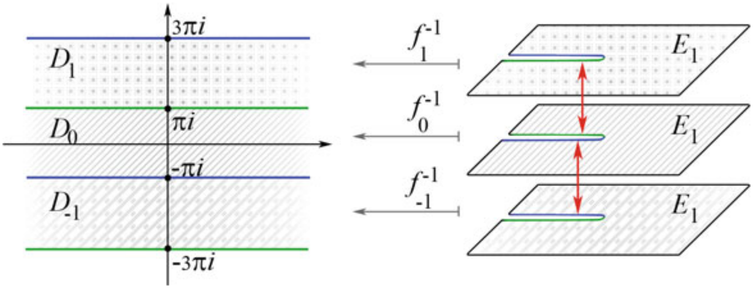
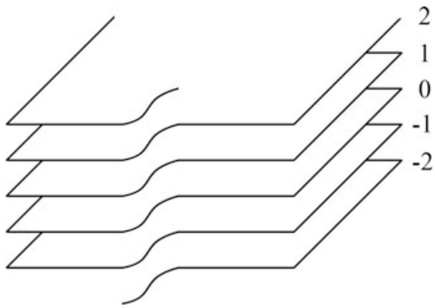


Fig. 3.15 Construction of the Riemann surface of the function Log

Fig. 3.16 The Riemann surface $\mathfrak{R}_{\mathrm{Log}}$



points of the upper edge of the kth-sheet cut must be glued with the corresponding points of the lower edge of the $(k+1)$th-sheet cut, and points of the lower edge of the kth-sheet cut must be glued with the corresponding points of the upper edge of the $(k-1)$th-sheet cut (Fig. 3.15).

At the points $w = 0$ and $w = \infty$ the logarithm is not defined and, therefore, its Riemann surface is not contain points above $w = 0$ and $w = \infty$. The surface thus glued is a Riemann surface of the function Log and is denoted by $\mathfrak{R}_{\mathrm{Log}}$ (infinite helical surface, Fig. 3.16). The function $\mathrm{Log}\, w$ becomes single-value on $\mathfrak{R}_{\mathrm{Log}}$ and conformal everywhere except for the points 0 and ∞ that are branch points of infinite order (or *logarithmic branch points*).

Example 3.6 Consider the complex number -2. There are countably many different points $\{-2_k\}_{k\in\mathbb{Z}}$ above -2, each of them lies on the corresponding kth-sheet. Since $\mathrm{Log}\, w = f_k^{-1}(w)$ on the kth-sheet, we find

$$\mathrm{Log}(-2_k) = f_k^{-1}(-2) = \log|-2| + i\ \arg(-2) + 2\pi i k = \log 2 + i\pi + 2\pi i k.$$

3.8 Joukowsky Function

Here we study an interesting function named after Nikolai Joukowsky (1847–1921), the founder of hydro and aeromechanics. He used this function to derive the formula for aircraft wing lift (1906), to develop the vortex theory of propellers, and to determine optimal wing and propeller blade profiles. This function was used to theoretically calculate the possibility of performing the "loop" aerobatic manoeuvre, first performed by P. Nesterov in Kyiv (1913).

The Joukowsky (or Zhukovsky) function is a mapping of the form

$$w = J(z) := \frac{1}{2}\left(z + \frac{1}{z}\right), \quad z \in \mathbb{C} \setminus \{0\}.$$

By continuity, we define J at zero, namely, since $\lim\limits_{z\to 0} J(z) = \infty$, we set

$$J(0) \overset{def}{=} \infty.$$

Obviously, $J(z) = J(1/z)$ for all $z \in \overline{\mathbb{C}}$. Since

$$J'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) \quad (z \neq 0)$$

and the derivative of the function $g(z) = \frac{1}{J(z)} = \frac{2z}{1+z^2}$ at 0 is equal to 2, the Joukowsky function is conformal in $\overline{\mathbb{C}} \setminus \{\pm 1\}$.

Let there exist two different numbers $z_1 \neq z_2$ such that $J(z_1) = J(z_2)$, i.e., $z_1 - z_2 + \frac{1}{z_1} - \frac{1}{z_2} = 0$. From here we deduce

$$(z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right) = 0 \quad \Longrightarrow \quad z_1 z_2 = 1. \tag{3.10}$$

Thus, some domain is a domain of univalence for the Joukowsky function if it does not contain any pair of points that satisfy the relation (3.10). It is easy to understand that a domain of univalence of J cannot contain points ± 1, because in arbitrary neighborhoods of these points there are always different points satisfying (3.10). Obviously, the following domains: $\{z : |z| > 1\}$, $\{z : |z| < 1\}$, $\{z : \operatorname{Im} z > 0\}$ and $\{z : \operatorname{Im} z < 0\}$ are domains of univalence for the Joukowsky function.

Let us derive the so-called transition formulas for the Joukowsky function, which will help us to find images of various curves and regions. If the function J sends a point $z = r\,e^{i\varphi}$ to a point $w = u + iv$, then

$$w = J(z) \iff u + iv = \frac{1}{2}\big(r + \frac{1}{r}\big)\cos\varphi + i\frac{1}{2}\big(r - \frac{1}{r}\big)\sin\varphi$$

$$\iff u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi \quad \text{and} \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\varphi. \tag{3.11}$$

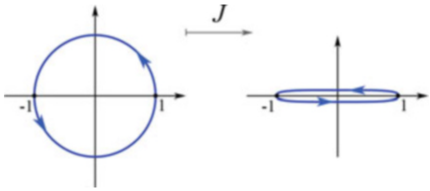


Fig. 3.17 The image of the unit circle

Consider the circle

$$K_R := \{z\colon z = Re^{i\varphi},\ \varphi\ \in\ [0, 2\pi]\}.$$

If $R = 1$, then it follows from (3.11) that $u = \cos\varphi$ and $v = 0$. Thus, the image of the unit circle is the segment $[-1, 1]$ that is traversed twice when φ changes from 0 to 2π (Fig. 3.17).

If $R \neq 1$, the image of K_R under the mapping J is the ellipse

$$\left\{ w = u + iv\colon\ \frac{u^2}{\left[\frac{1}{2}\left(R + R^{-1}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(R - R^{-1}\right)\right]^2} = 1 \right\}$$

with the semi-major axes

$$a = \frac{1}{2}\left(R + R^{-1}\right) \quad \text{and} \quad b = \frac{1}{2}\left|R - R^{-1}\right|,$$

and with foci at ± 1. However, for $R > 1$, the orientation of the ellipse is preserved (this means that the upper semicircle is mapped to the upper half-ellipse, and the lower semicircle is mapped to the lower half-ellipse (see the upper part of Fig. 3.18), and when $R < 1$, the orientation is reversed (this means that the upper semicircle

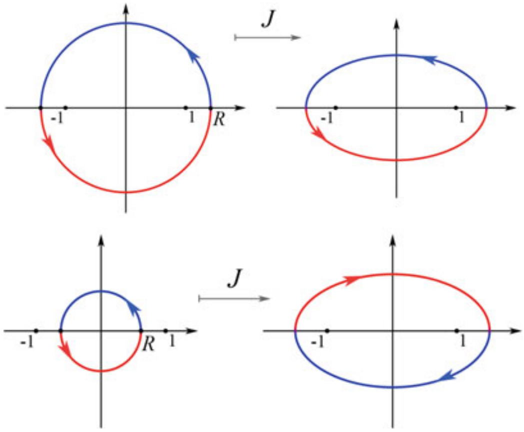


Fig. 3.18 Images of circles by the Joukowsky function

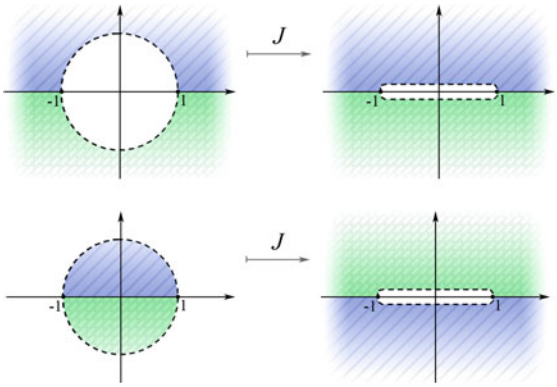


Fig. 3.19 Images of the exterior and interior of the unit disk by J

is mapped to the lower half-ellipse, and the lower semicircle into the upper half-ellipse).

Let's have a look at where the Joukowsky function maps its univalence domains: $\{z : |z| > 1\}$, $\{z : |z| < 1\}$, $\{z : \operatorname{Im} z > 0\}$ and $\{z : \operatorname{Im} z < 0\}$.

Since $\{z\colon |z| > 1\} = \bigcup_{R\in(1,+\infty)} K_R$, the image of the exterior of the unit disk by the Joukowsky function is $\mathbb{C}\setminus[-1, 1]$; moreover, knowing the images of semicircles (see Fig. 3.18), we have

$$\{z : |z| > 1, \operatorname{Im} z > 0\} \overset{J}{\longmapsto} \{w : \operatorname{Im} w > 0\}$$

and

$$\{z : |z| > 1, \operatorname{Im} z < 0\} \overset{J}{\longmapsto} \{w : \operatorname{Im} w < 0\}$$

(see the upper part of Fig. 3.19).

Since $J(z) = J(\frac{1}{z})$, the image of the unit disk $\{z : |z| < 1\}$ by the Joukowsky function is also $\mathbb{C} \setminus [-1, 1]$, but now (see the lower part of Fig. 3.19)

$$\{z : |z| < 1, \operatorname{Im} z > 0\} \overset{J}{\longmapsto} \{w : \operatorname{Im} w < 0\}$$

and

$$\{z : |z| < 1, \operatorname{Im} z < 0\} \overset{J}{\longmapsto} \{w : \operatorname{Im} w > 0\}.$$

To find the image of the upper half-plane, we present it as a union of three sets, the images of which were found above: $\{z : |z| > 1, \operatorname{Im} z > 0\}$, $\{z : |z| < 1, \operatorname{Im} z > 0\}$ and $\{z : |z| = 1, \operatorname{Im} z > 0\}$. Thus,

$$\{z : \operatorname{Im} z > 0\} \overset{J}{\longmapsto} \mathbb{C} \setminus \big\{(-\infty, -1] \cup [1, +\infty)\big\} =: E_2 \tag{3.12}$$

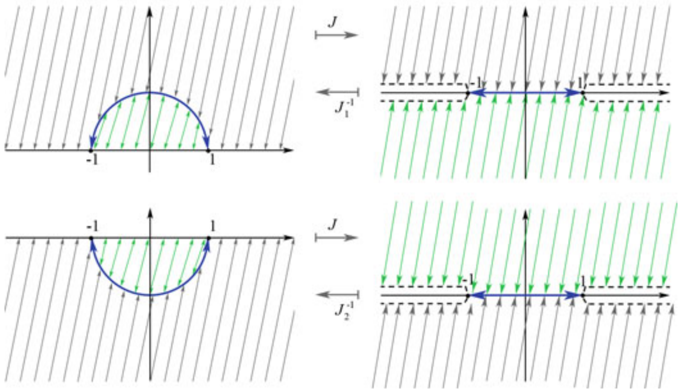


Fig. 3.20 Images of the upper and lower half-planes by J

(the complex plane with cuts along the rays $(-\infty, -1]$ and $[1, +\infty)$; see the upper part of Fig. 3.20). Similarly, we find that (the lower part of Fig. 3.20)

$$\{z : \operatorname{Im} z < 0\} \overset{J}{\longmapsto} E_2. \tag{3.13}$$

Exercise 3.6 Find a conformal and univalent function that maps the domain $\{z : |z| < 1, \ 0 < \arg(z) < \frac{\pi}{2}\}$ onto the unit disk $B_1(0)$.

The mappings (3.12) and (3.13) are one-to-one. The inverse mappings

$$J_1^{-1} \colon E_2 \longmapsto \{z \colon \operatorname{Im} z > 0\} \quad \text{and} \quad J_2^{-1} \colon E_2 \longmapsto \{z \colon \operatorname{Im} z < 0\}$$

are called analytical branches of the multi-valued inverse function

$$J^{-1}(w) = w + \sqrt{w^2 - 1}.$$

Exercise 3.7 With the help of J_1^{-1} and J_2^{-1} construct the Riemann surface of the multi-value function $J^{-1}(w) = w + \sqrt{w^2 - 1}$.

Now we find where Joukowsky's function maps a ray

$$\{z = r\, e^{i\alpha} \colon \ r \in (0, +\infty)\}.$$

By (3.11), we conclude that the image of this ray is the curve

$$\left\{ w = u + iv = \frac{1}{2}\Big(r + \frac{1}{r}\Big) \cos\alpha \ + i\frac{1}{2}\Big(r - \frac{1}{r}\Big) \sin\alpha : \ \ r \in (0, +\infty) \right\}.$$

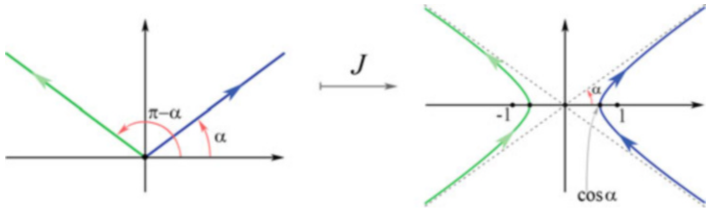


Fig. 3.21 Images of rays under Joukowsky function

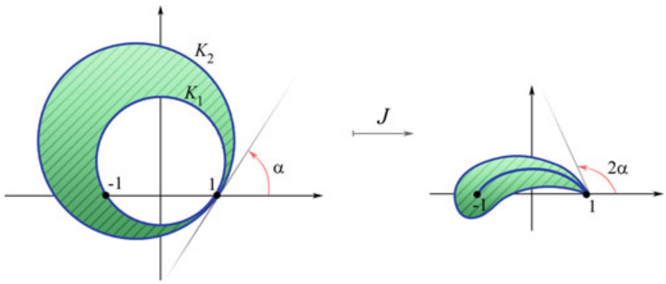


Fig. 3.22 Joukowsky wing profile (on the right)

Here, α is a fixed value from $(0, \pi)$ This is the right branch of the hyperbola

$$\frac{u^2}{\cos^2 \alpha} - \frac{v^2}{\sin^2 \alpha} = 1$$

with the foci at the points ± 1 if $\alpha \in (0, \frac{\pi}{2})$; this is the left branch of this hyperbola if $\alpha \in (\frac{\pi}{2}, \pi)$; and this is the imaginary axis if $\alpha = \frac{\pi}{2}$. It should be noted that if the parameter r changes from 0 to $+\infty$, then the point on these branches moves from the bottom to top (Fig. 3.21).

Exercise 3.8 Suppose that two circles K_1 and K_2 pass through the point $z = 1$ at an angle α to the real axis, and the circle K_1 also passes through the point $z = -1$ and lies inside the circle K_2 (Fig. 3.22).

Prove that the image of the domain $\operatorname{int}(K_2) \cap \operatorname{ext}(K_1)$ under the Joukowsky function is a so called "Joukowsky wing profile" (Fig. 3.22).

3.9 Trigonometric and Hyperbolic Functions and Their Inverses

From the Euler formula (1.9) we derive

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These relations are used to define the trigonometric functions of a complex variable:

$$\cos z \overset{def}{=} \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z \overset{def}{=} \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}, \tag{3.14}$$

$$\tan z \overset{def}{=} \frac{\sin z}{\cos z} = -i\frac{e^{i2z} - 1}{e^{i2z} + 1}, \quad z \in \mathbb{C} \setminus \left\{\frac{\pi}{2} + \pi n : n \in \mathbb{Z}\right\}, \tag{3.15}$$

$$\cot z \overset{def}{=} \frac{\cos z}{\sin z} = i\frac{e^{i2z} + 1}{e^{i2z} - 1}, \quad z \in \mathbb{C} \setminus \{\pi n : n \in \mathbb{Z}\}. \tag{3.16}$$

From these formulas and properties of the exponential function it follows that cos and sin are 2π-periodic functions, and tan and cot are π-periodic. Using (3.14)–(3.16), it can be verified that all formulas for the trigonometric functions of a real argument remain true for a complex argument as well.

Applying (2.10), one finds the derivatives of $\sin z$ and $\cos z$, namely

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z \quad \text{for all } z \in \mathbb{C}.$$

Therefore, $\sin z$ and $\cos z$ are entire functions. Then the quotient rule states

$$(\tan z)' = \frac{1}{\cos^2 z} \quad \text{for all } z \in \mathbb{C} \setminus \left\{\frac{\pi}{2} + \pi n : n \in \mathbb{Z}\right\},$$

$$(\cot z)' = -\frac{1}{\sin^2 z} \quad \text{for all } z \in \mathbb{C} \setminus \{\pi n : n \in \mathbb{Z}\}.$$

It is easy to see that the trigonometric functions are connected with the hyperbolic functions

$$\cosh z \overset{def}{=} \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z \overset{def}{=} \frac{e^{z} - e^{-z}}{2},$$

$$\tanh z \overset{def}{=} \frac{\sinh z}{\cosh z} = \frac{e^{2z} - 1}{e^{2z} + 1}, \qquad \coth z \overset{def}{=} \frac{\cosh z}{\sinh z} = \frac{e^{2z} + 1}{e^{2z} - 1} \tag{3.17}$$

through the following relations:

$$\cosh(iz) = \cos z, \qquad \cos(iz) = \cosh z,$$

$$\sinh(iz) = i\sin z, \qquad \sin(iz) = i\sinh z,$$

$$\tanh(iz) = i\tan z, \qquad \tan(iz) = i\tanh z.$$

Hence, cosh and sinh are $2\pi i$-periodic functions, and tanh and coth are πi-periodic.

A new property of the trigonometric functions $\sin z$ and $\cos z$ is that they are unbounded, which means that their moduli are unbounded functions. Indeed,

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

$$\Longrightarrow \ |\sin z| = \sqrt{\sin^2 x + \sinh^2 y} = |\sinh y| \sqrt{1 + \frac{\sin^2 x}{\sinh^2 y}} \ \sim \ \frac{1}{2} e^{|y|},$$

as $|y| \to +\infty$. Similarly, we show that $|\cos z| \sim \frac{1}{2} e^{|y|}$ as $|y| \to +\infty$.

Remark 3.6 It turns out that every nonconstant entire function is unbounded. We will prove this statement in Sect. 5.2.

Since $\cos z = \sin(z + \frac{\pi}{2})$, we continue to consider the function

$$w = f(z) := \sin z = \sin x \cosh y + i \cos x \sinh y, \quad z = x + iy \in \mathbb{C}. \tag{3.18}$$

Let there exist two different numbers $z_1 \neq z_2$ such that $\sin z_1 = \sin z_2$. Then $2 \sin \frac{z_1 - z_2}{2} \cos \frac{z_1 + z_2}{2} = 0$, which means

$$z_1 - z_2 = 2\pi n, \ \ n \in \mathbb{Z} \setminus \{0\}, \quad \text{or} \quad z_1 + z_2 = \pi + 2\pi k, \ \ k \in \mathbb{Z}. \tag{3.19}$$

Recalling the geometric interpretation of the sum and subtraction of complex numbers, it follows from (3.19) that the domains

$$D_k := \left\{ z : \ -\frac{\pi}{2} + \pi k < \operatorname{Re} z < \frac{\pi}{2} + \pi k \right\}, \quad k \in \mathbb{Z},$$

are domains of univalence for the sine. Let us find their images.

If sin sends a point $z = x + iy$ to a point $w = u + iv$, we derive from (3.18) the transition formulas

$$u = \sin x \ \cosh y, \quad v = \cos x \ \sinh y. \tag{3.20}$$

By (3.20), we conclude that the image of a vertical straight line

$$l_\alpha = \{z = \alpha + iy : \ y \in (-\infty, +\infty)\},$$

where $\alpha \in (0, \frac{\pi}{2})$ (see Fig. 3.23), is the right branch of the hyperbola

$$\begin{cases} u = \sin\alpha \cosh y, \\ v = \cos\alpha \sinh y, \end{cases} \ y \in \mathbb{R}, \ \Longrightarrow \ \frac{u^2}{(\sin\alpha)^2} - \frac{v^2}{(\cos\alpha)^2} = 1,$$

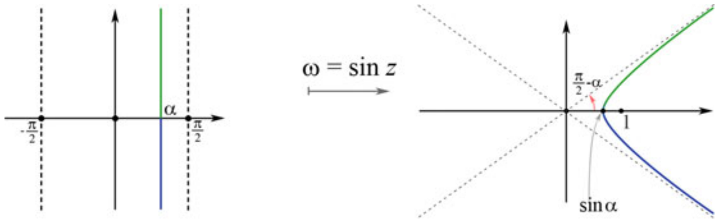


Fig. 3.23 The image of a vertical line by mapping the sine function

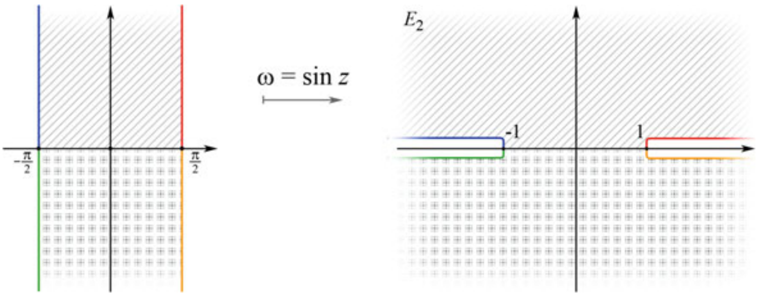


Fig. 3.24 The image of the strip D_0 by mapping the sine function

and the left branch if $\alpha \in (-\frac{\pi}{2}, 0)$, and this is the imaginary axis if $\alpha = 0$. Moreover, the upper part of the line is mapped onto the upper part of the corresponding branch of the hyperbola, and the lower part into the lower one.

Since $D_0 = \bigcup_{\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})} l_\alpha$ and the right (left) branch of the hyperbola is shrunk into the cut $[1, +\infty)$ ($(-\infty, -1]$) as $\alpha \to \frac{\pi}{2} - 0$ ($\alpha \to -\frac{\pi}{2} + 0$), the image of the vertical strip D_0 by the function sin is the complex plane with the cuts $(-\infty, -1]$ and $[1, +\infty)$ along the real axis (Fig. 3.24).
Such a domain was faced in the previous paragraph and was denoted by E_2 (see (3.12)). Also note that the part of the strip D_0 lying in the upper half-plane $\{z : \operatorname{Im} z > 0\}$ is mapped onto the upper half-plane $\{w : \operatorname{Im} w > 0\}$, and the lower one is mapped onto the lower half-plane, respectively.

Using the formula $\sin z = (-1)^k \sin(z + k\pi)$, we can assert that for each $k \in \mathbb{Z}$ the function $f(z) = \sin z$ conformally and univalently maps the vertical strip D_k onto E_2. However,

- for $k = 2p$ the part of the strip D_k lying in the upper half-plane $\{z : \operatorname{Im} z > 0\}$ is mapped onto the upper half-plane $\{w : \operatorname{Im} w > 0\}$, and the lower one, respectively, onto the lower half-plane;
- for $k = 2p - 1$ the part of D_k lying in the upper half-plane is mapped onto the lower half-plane, and the lower one, respectively, onto the upper half-plane (Fig. 3.25).

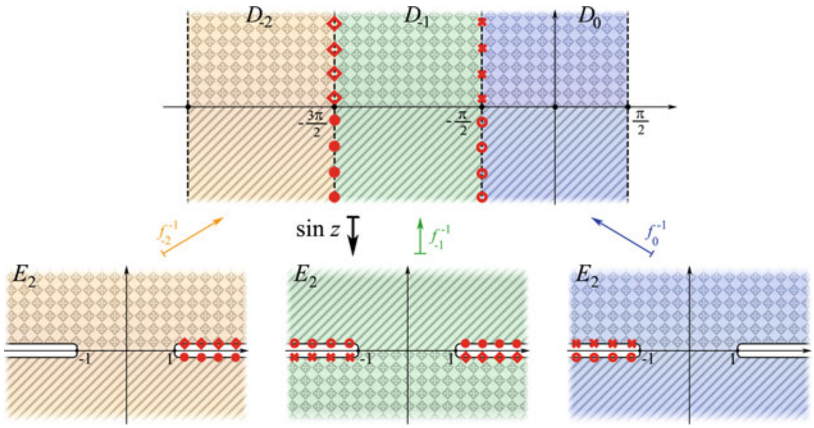


Fig. 3.25 The images of the strips D_k by mapping the sine function

Thus, for each $k \in \mathbb{Z}$ there exists an inverse $f_k^{-1}\colon E_2 \mapsto D_k$ such that

$$f(f_k^{-1}(w)) = w \quad \text{for all} \quad w \in E_2, \qquad \text{and} \qquad f_k^{-1}(f(z)) = z \quad \text{for all} \quad z \in D_k,$$

$$\frac{df_k^{-1}(w)}{dw} = \frac{1}{\cos z}\bigg|_{z=f_k^{-1}(w)} = \frac{1}{\sqrt{1-w^2}} \neq 0 \quad \text{for all} \quad w \in E_2.$$

Therefore, the function f_k^{-1} is conformal and univalent in E_2. Obviously, that $f_k^{-1}(0) = \pi k$. The functions $\{f_k^{-1}\}_{k\in\mathbb{Z}}$ are called analytical branches of the multi-valued function $z = \operatorname{Arcsin} w$. To find the formula for Arcsin, you need to solve the equation

$$\sin z = w \iff \frac{e^{iz} - e^{-iz}}{2i} = w \iff (e^{iz})^2 - 2iwe^{iz} - 1 = 0$$

$$\implies e^{iz} = iw + \sqrt{1-w^2} \implies z = -i\,\operatorname{Ln}(iw + \sqrt{1-w^2}) =: \operatorname{Arcsin} w.$$

In the same way, one can find other functions that are inverse to both trigonometric and hyperbolic functions, for example,

$$\operatorname{Arccos} w = -i\,\operatorname{Ln}(w + \sqrt{w^2-1}), \qquad \operatorname{Arctan} w = \frac{1}{2i}\operatorname{Ln}\left(\frac{1+iw}{1-iw}\right),$$

$$\operatorname{Arcsinh} w = \operatorname{Ln}(w + \sqrt{1+w^2}), \qquad \operatorname{Arctanh} w = \frac{1}{2}\operatorname{Ln}\left(\frac{1+w}{1-w}\right).$$

To construct the Riemann surface of Arcsin we take a denumerable set of E_2-sheets and place them on top of each other over the complex plane. The sheet from which the function f_k^{-1} acts is called the kth sheet. Using the functions $\{f_k^{-1}\}_{k\in\mathbb{Z}}$, these sheets must be glued together to form a continuous function.

Take a sheet with an odd number $k = 2p - 1$. Considering how the sine maps the upper and lower parts of the vertical strip D_k (see Fig. 3.25), we get the rule for gluing the E_2-sheets. Since

$$\lim_{w\to x<-1,\ \mathrm{Im}\, w<0} f_k^{-1}(w) = \lim_{w\to x<-1,\ \mathrm{Im}\, w>0} f_{k+1}^{-1}(w)$$

and

$$\lim_{w\to x<-1,\ \mathrm{Im}\, w>0} f_k^{-1}(w) = \lim_{w\to x<-1,\ \mathrm{Im}\, w<0} f_{k+1}^{-1}(w),$$

points of the lower edge of the left kth sheet cut must be glued with the corresponding points of the upper edge of the left $(k+1)$th sheet cut, and points of the upper edge of the left kth sheet cut must be glued with the corresponding points of the lower edge of the left $(k+1)$th sheet cut.

Similarly, since

$$\lim_{w\to x>1,\ \mathrm{Im}\, w<0} f_k^{-1}(w) = \lim_{w\to x>1,\ \mathrm{Im}\, w>0} f_{k-1}^{-1}(w)$$

and

$$\lim_{w\to x>1,\ \mathrm{Im}\, w>0} f_k^{-1}(w) = \lim_{w\to x>1,\ \mathrm{Im}\, w<0} f_{k-1}^{-1}(w),$$

points of the lower edge of the right kth sheet cut must be glued with the corresponding points of the upper edge of the right $(k-1)$th sheet cut, and point of the upper edge of the right kth sheet cut must be glued with the corresponding points of the lower edge of the right $(k-1)$th sheet cut.

The surface thus glued is the Riemann surface of Arcsin and is denoted by $\mathfrak{R}_{\mathrm{Arcsin}}$. The function $\mathrm{Arcsin}\, w$ becomes single-value on $\mathfrak{R}_{\mathrm{Arcsin}}$ and conformal everywhere except for the points ± 1 and ∞. Over the points 1 and -1 of the complex plane there is a denumerable set of first-order branch points, respectively. The point ∞ is a logarithmic branch point. The scheme of the transition between the sheets of the Riemann surface is shown in Fig. 3.26, and a part of the Riemann surface is shown in Fig. 3.27.

It turns out that the trigonometric functions cos and sin and the hyperbolic functions cosh and sinh can be represented as the corresponding compositions of the Joukowsky function and the exponential function, namely

$$\cos z = J(e^{iz}), \qquad \sin z = -J(i\, e^{iz}),$$
$$\cosh z = J(e^{z}), \qquad \sinh z = -i J(i\, e^{z}).$$

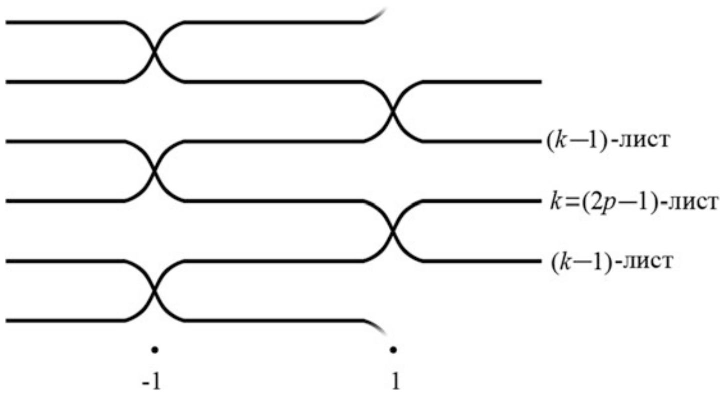


Fig. 3.26 Transition scheme between sheets of the Riemann surface of Arcsin

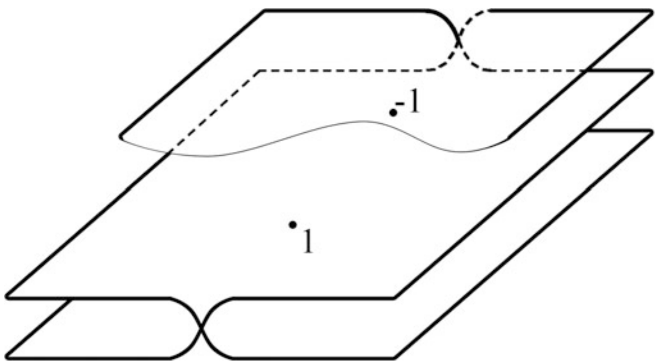


Fig. 3.27 A part of the Riemann surface of Arcsin

Therefore, to find the images of domains mapped by these functions, the properties of the exponential function and the Joukowsky function are used.

To find the images of domains when mapped by the functions tan, cot, tanh and coth, the formulas (3.15), (3.16) and (3.17) are used.

Example 3.7 The function

$$w = \tan z = -i\,\frac{e^{2iz} - 1}{e^{2iz} + 1}$$

is the composition of the linear function $\xi = 2iz$, the exponential function $\eta = e^{\xi}$, and the fractional-linear function

$$w = -i\,\frac{\eta - 1}{\eta + 1}.$$

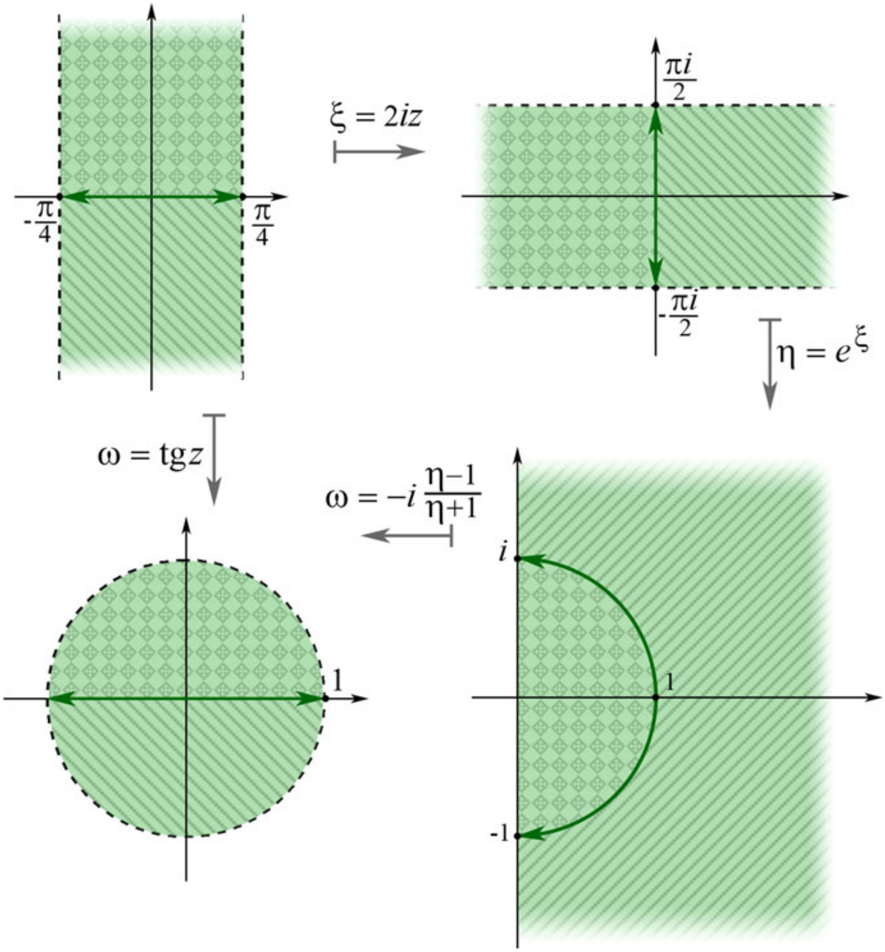


Fig. 3.28 The image of a vertical strip by mapping the function $w = \tan z$

Therefore, the tangent function maps the vertical strip

$$\left\{z : -\frac{\pi}{4} < \operatorname{Re} z < \frac{\pi}{4}\right\}$$

onto the unit disk $B_1(0)$, as shown in Fig. 3.28.

Integration of Functions of a Complex Variable 4

Abstract

In the previous two chapters, it was shown that analytic complex-valued functions enjoy excellent differentiability properties that their real counterparts do not share. It is well known that differentiation and integration are mutually inverse operations and they are the main concerns of calculus. To continue on, the next logical step is to consider the integration in the complex plane, as initiated by the French mathematician Augustin-Louis Cauchy (1789–1857). Integration is impossible without the concept of an antiderivative, which becomes much more complicated in complex analysis. For example, it turns out that there are analytic functions in some domains that have no antiderivatives. In this chapter, we will introduce a new concept of an antiderivative along a curve and study its properties. We will also show that the beauty of complex integration also goes far beyond real analysis and prove very important and interesting theorems.

4.1 Line Integrals and Their Simplest Properties

Definition 4.1 Let f be a continuous function in a domain Ω and let $z = \gamma(t)$, $t \in [a, b]$, be a piecewise smooth curve, whose trace belongs to Ω, i.e., $E_\gamma \subset \Omega$. The integral of the function f along the curve γ is defined as follows

$$\int_\gamma f(z)\,dz \overset{def}{=} \int_a^b f(\gamma(t))\,\gamma'(t)\,dt. \tag{4.1}$$

▲

The integrand function in the right-hand side of (4.1) is a complex-valued function of a real variable. Under the definition hypotheses, its real and imaginary

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parts have only a finite number of first-order breakpoints of the kind, and are therefore Riemann-integrable on $[a, b]$.

Let's analyze several examples.

Example 4.1 Compute the integral $\int_\gamma \overline{z}\, dz$, where

(1) the curve γ is the straight line from 0 to $1 - i$,
(2) γ is the broken line connecting in order the numbers 0, 1 and $1 - i$.

Solution 1 We take the parametrization $\gamma(t) = t(1-i),\ t \in [0, 1]$. Then $\gamma' = 1-i$ and the line integral is

$$\int_\gamma \overline{z}\, dz = \int_0^1 t(1+i)(1-i)\, dt = 1.$$

Solution 2 Now we parameterize the the broken line by

$$\gamma(t) = \begin{cases} t, & t \in [0, 1], \\ 1 - i(t-1), & t \in [1, 2]. \end{cases}$$

So,

$$\gamma'(t) = \begin{cases} 1, & t \in [0, 1], \\ -i, & t \in (1, 2], \end{cases}$$

and the integral becomes

$$\int_\gamma \overline{z}\, dz = \int_0^1 t\, dt + \int_1^2 (1 + i(t-1))(-i)\, dt = \frac{1}{2} + \frac{1}{2} - i = 1 - i.$$

Example 4.2 Let $\gamma\colon [a, b] \mapsto \mathbb{C}$ be a piecewise smooth curve; $n \in \mathbb{Z} \setminus \{-1\}$, and moreover, if $n < 0$ then we also assume that $\{0\} \notin E_\gamma$. Then

$$\int_\gamma z^n\, dz = \int_a^b (\gamma(t))^n\, \gamma'(t)\, dt = \frac{1}{n+1}\Big(\gamma^{n+1}(b) - \gamma^{n+1}(a)\Big).$$

Example 4.3 Let $z = \gamma(t) = a + re^{it},\ t \in [0, 2\pi]$, where $a \in \mathbb{C}$, $r > 0$; $n \in \mathbb{Z}$.

By Definition 4.1, we find

$$\int_\gamma (z-a)^n\, dz = r^{n+1} i \int_0^{2\pi} e^{i(n+1)t}\, dt = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \neq -1. \end{cases}$$

Important
Example 4.1 shows that the value of $\int_\gamma \overline{z}\, dz$ depends on the curve along which the integration takes place.

On the other hand, the integral of the function z^n $(n \neq -1)$ in Example 4.2 is not depend on the integration curve and is determined only by its beginning and end. And if γ is closed, then $\int_\gamma z^n\, dz = 0$.

From Example 4.3 it follows that there exists a function whose integral, even over a closed curve, is not equal to zero.

Why this happens, and when line integrals do not depend on the integration curve, but only on its start and end points, is what we need to find out in this chapter.

Remark 4.1 Separating the real and imaginary parts in

$$\gamma'(t) = x'(t) + iy'(t) \quad \text{and} \quad f(\gamma(t)) = u\big(x(t), y(t)\big) + iv\big(x(t), y(t)\big),$$

the integral (4.1) can be represented as the sum of two curvilinear integrals of the second kind

$$\int_\gamma f(z)\, dz = \int_\gamma u\, dx - v\, dy \; + \; i \int_\gamma u\, dy + v\, dx. \tag{4.2}$$

Now we list the main properties of line integrals of complex-valued functions. Later in this section, we assume that integrand functions are continuous and curves are piecewise smooth.

1. Linearity. For any $\{\lambda, \mu\} \in \mathbb{C}$

$$\int_\gamma \big(\lambda\, f(z) \, + \, \mu\, g(z)\big)\, dz = \lambda \int_\gamma f(z)\, dz \; + \; \mu \int_\gamma g(z)\, dz.$$

The proof follows directly from Definition 4.1 and the linearity of the Riemann integral.

2. Additivity. Let $z = \gamma_1(t)$, $t \in [a, b]$, and $z = \gamma_2(t)$, $t \in [b, c]$, are piecewise smooth curves, for which $\gamma_1(b) = \gamma_2(b)$. The union of these curves is called the curve

$$\gamma_1 \cup \gamma_2 := \begin{cases} \gamma_1(t), & t \in [a, b], \\ \gamma_2(t), & t \in [b, c]. \end{cases} \tag{4.3}$$

Then

$$\int_{\gamma_1 \cup \gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

The proof follows directly from Definition 4.1 and the additivity of the Riemann integral.

3. Orientability. Let $z = \gamma(t),\ t \in [a, b]$, is a given curve. Denote by γ^- the curve

$$\gamma^-(\tau) := \gamma(a + b - \tau), \quad \tau \in [a, b].$$

It is easy to see that the curves γ and γ^- have the same trace, but opposite orientations (the initial and end points are switched).

Then

$$\int_{\gamma^-} f(z)\,dz = -\int_{\gamma} f(z)\,dz. \tag{4.4}$$

Proof Since $(\gamma^-)'_\tau(\tau) = \gamma'_t(a + b - \tau) \cdot (-1)$,

$$\begin{aligned}\int_{\gamma^-} f(z)\,dz &= \int_a^b f\big(\gamma^-(\tau)\big)\,(\gamma^-)'_\tau(\tau)\,d\tau \\ &= \int_a^b f(\gamma(a+b-\tau)) \cdot \gamma'_t(a+b-\tau) \cdot (-1)\,d\tau = \Big\langle\, t = a + b - \tau \,\Big\rangle \\ &= -\int_a^b f(\gamma(t)) \cdot \gamma'_t(t)\,dt = -\int_\gamma f(z)\,dz.\end{aligned}$$

□

This property also follows from (4.2) and the fact that curvilinear integrals of the second kind change sign when changing the orientation of the curve.

4. Invariance. If a curve $z = \gamma_1(t),\ t \in [a_1, b_1]$, is equivalent to a curve $z = \gamma_2(\tau),\ \tau \in [a_2, b_2]$ (see Definition 1.11; moreover, the function μ, realizing this equivalence is assumed to be continuously differentiable), then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Proof By the definition of the integral,

$$\int_{\gamma_2} f(z)\,dz = \int_{a_2}^{b_2} f(\gamma_2(\tau))(\gamma_2)'_\tau(\tau)\,d\tau.$$

Substituting $\tau = \mu(t)$ in the integral and taking into account that $\gamma_2(\mu(t)) = \gamma_1(t),\ t \in [a_1, b_1]$, we get

$$\int_{a_2}^{b_2} f(\gamma_2(\tau))(\gamma_2)'_\tau(\tau)\,d\tau = \int_{a_1}^{b_1} f\big(\gamma_2(\mu(t)\big)\ \underbrace{(\gamma_2)'_\tau\big(\mu(t)\big)\,\mu'_t(t)}_{\overset{\shortparallel}{\gamma_1'(t)}}\,dt$$

$$= \int_{a_1}^{b_1} f(\gamma_1(t))\cdot\gamma_1'(t)\,dt = \int_{\gamma_1} f(z)\,dz.$$

□

Remark 4.2 Since equivalent curves have the same trace and the same orientation, it is often, if it does not cause misunderstanding, one speaks of the integral over the trace and indicates its orientation. So, the result of Example 4.3 can be rewritten as

$$\int\limits_{\{|z-a|=r\}^+} (z-a)^n\,dz = \begin{cases} 2\pi i, & n = -1,\\ 0, & n \neq -1.\end{cases}$$

Here "+" under the integral sign indicates the positive orientation of the circle.

5. Estimation of the integral. The following inequality holds:

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,dl, \tag{4.5}$$

where $dl = \sqrt{(x'(t))^2 + (y'(t))^2}dt = |\gamma'(t)|dt$ is the arc length differential; on the right in (4.5) there is a curvilinear integral of the first kind of the function $|f|$ along the curve γ.

Proof Let us denote by $\mathcal{I} := \int_\gamma f(z)\,dz$ and write this number in the exponent form $\mathcal{I} = |\mathcal{I}|e^{i\theta}$. Then

$$|\mathcal{I}| = e^{-i\theta}\mathcal{I} = \int_\gamma e^{-i\theta} f(z)\,dz = \int_a^b e^{-i\theta} f(\gamma(t))\cdot\gamma'(t)\,dt.$$

Since the integral on the right is a real nonnegative number and $|\mathrm{Re}z| \leq |z|$, then

$$|\mathcal{I}| = \int_a^b \mathrm{Re}\Big(e^{-i\theta} f(\gamma(t))\cdot\gamma'(t)\Big)\,dt \leq \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt = \int_\gamma |f|\,dl.$$

□

The estimate (4.5) implies the corollary.

Corollary 4.1 *If the modulus of a function f is bounded by a constant M on the trace of a curve γ, i.e., $\exists\, M > 0\ \forall\, z \in E_\gamma : \ |f(z)| \le M$, then*

$$\left|\int_\gamma f\,dz\right| \le \int_\gamma |f|\,dl \le M \int_\gamma dl = M\,\ell_\gamma,$$

where ℓ_γ is the length of γ.

4.2 An Antiderivative: Cauchy-Goursat Theorem

As with functions of a real variable, we introduce the following definition of an antiderivative (a primitive) for a function f defined in a domain $\Omega \subset \mathbb{C}$.

Definition 4.2 Let $F \in \mathcal{A}(\Omega)$ and $f \in C(\Omega)$. The function F is called an antiderivative of f in Ω if

$$F'(z) = f(z) \quad \text{for all } \ z \in \Omega.$$

▲

Obviously, if F is an antiderivative of f in Ω, then for any complex number c the function $F + c$ is also an antiderivative of f in Ω.

Let F_1 and F_2 be two antiderivatives of f in Ω. Then $F_1 - F_2 =: G \in \mathcal{A}(\Omega)$ and certainly $G'(z) = 0$ for all $z \in \Omega$. Denote by u and v the real and respectively the imaginary part of the function G. Using Theorem 2.2, the last equality can be rewritten as

$$0 = G'(z) = \frac{\partial u(x,y)}{\partial x} + i\,\frac{\partial v(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} - i\,\frac{\partial u(x,y)}{\partial y},$$

whence

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = \dfrac{\partial u}{\partial y}(x,y) = 0 \quad \text{for all } \ (x,y) \in \Omega, \\[2mm] \dfrac{\partial v}{\partial x}(x,y) = \dfrac{\partial v}{\partial y}(x,y) = 0 \quad \text{for all } \ (x,y) \in \Omega, \end{cases} \quad \Longrightarrow \quad \begin{cases} u \equiv c_1, \\ v \equiv c_2. \end{cases}$$

Thus, $G \equiv c := c_1 + i c_2$.

Theorem 4.1 (The Cauchy–Goursat Theorem for Triangles) *If $f \in \mathcal{A}(\Omega)$, then for any triangle $\triangle$, which together with its closure belongs to the domain Ω (we write this fact as follows $\triangle \subseteq \Omega$), we have*

$$\int_{\partial^+\triangle} f(z)\,dz = 0,$$

where $\partial^+\triangle$ is the positively oriented triangle boundary.

Proof The proof is by contradiction. Suppose that there exist a positive number M and a triangle $\triangle_0 \subseteq \Omega$ such that

$$\left|\int_{\partial^+\triangle_0} f(z)\,dz\right| = M > 0. \tag{4.6}$$

Divide the triangle $\triangle_0$ by the middle lines into four triangles $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ with positively oriented boundaries as in Fig. 4.1. Then

$$\int_{\partial^+\triangle_0} f(z)\,dz = \sum_{k=1}^{4}\int_{\partial^+\triangle_k} f(z)\,dz, \tag{4.7}$$

because on the right side of (4.7) the integrals along the middle lines are taken twice, but in opposite directions, and by (4.4) their sum is zero.

It follows from (4.6) and (4.7) that there exists a number $k \in \{1, 2, 3, 4\}$ (let for definiteness $k = 1$) such that

$$\left|\int_{\partial^+\triangle_1} f(z)\,dz\right| \geq \frac{M}{4}.$$

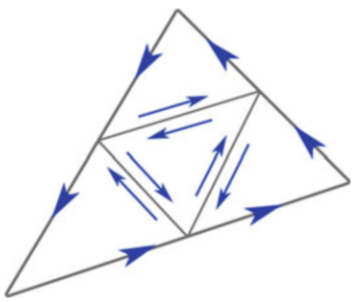

Fig. 4.1 Subdividing of a triangle

Let us do the same procedure with triangle $\triangle_1$ and subdivide it into four triangles $\triangle_{11}$, $\triangle_{12}$, $\triangle_{13}$, $\triangle_{14}$. It possible to find one of them, call it $\triangle_{11}$, for which

$$\left|\int_{\partial^+\triangle_{11}} f(z)\,dz\right| \ \geq\ \frac{M}{4^2}.$$

Evidently, that $\triangle_{11} \subset \triangle_1$.

Continuing the same considerations, we obtain a sequence of nested triangles

$$\triangle_0 \supset \triangle_1 \supset \triangle_{11} \supset \triangle_{111} \ldots$$

such that for the nth triangle $\triangle^{(n)} := \triangle_{\underbrace{11\ldots1}_{n}}$ the inequality

$$\left|\int_{\partial^+\triangle^{(n)}} f(z)\,dz\right| \ \geq\ \frac{M}{4^n} \tag{4.8}$$

holds and the intersection of their closures is some point

$$\{z_0\} = \bigcap_{n=0}^{+\infty} \overline{\triangle^{(n)}},$$

which obviously belongs to the domain Ω. Moreover, the perimeter $\ell(\triangle^{(n)})$ is equal to

$$\ell(\triangle^{(n)}) = \frac{1}{2}\ell(\triangle^{(n-1)}) = \ldots = \frac{1}{2^n}\ell(\triangle_0). \tag{4.9}$$

Since f is differentiable at z_0, the following statement is satisfied: for any $\varepsilon > 0$ there are a number $\delta > 0$ and a function α such that for all $z \in B_\delta(z_0)$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \alpha(z)(z - z_0) \quad \text{and} \quad |\alpha(z)| < \varepsilon. \tag{4.10}$$

From the sequence $\{\triangle^{(n)}\}_{n\in\mathbb{N}_0}$ we choose a triangle $\triangle^{(k)}$ that belongs to the disk $B_\delta(z_0)$. Then, on the one hand, the inequality (4.8) holds for this triangle. On the other hand, based on (4.10) and (4.9) we have

$$\frac{M}{4^k} \leq \left|\int_{\partial^+\triangle^{(k)}} f(z)\,dz\right| = \left|f(z_0)\int_{\partial^+\triangle^{(k)}} dz + f'(z_0)\int_{\partial^+\triangle^{(k)}} (z - z_0)\,dz\right.$$
$$\left. + \int_{\partial^+\triangle^{(k)}} \alpha(z)(z - z_0)dz\right| = \left|\int_{\partial^+\triangle^{(k)}} \alpha(z)(z - z_0)dz\right|$$

$$\leq \varepsilon\,\ell(\triangle^{(k)}) \int_{\partial\triangle^{(k)}} dl = \varepsilon\big(\ell(\triangle^{(k)})\big)^2 = \varepsilon\left(\frac{\ell(\triangle^{(k)})}{2^k}\right)^2 = \varepsilon\frac{\big(\ell(\triangle^{(k)})\big)^2}{4^k}.$$

Here, in the first line, the integrals of 1 and $z - z_0$ along the boundary of $\triangle^{(k)}$ are equal to zero (see Example 4.2). Thus,

$$0 < M \leq \varepsilon\left(\ell(\partial\triangle_0)\right)^2$$

for any $\varepsilon > 0$. It means that $M = 0$. But this is a contradiction. □

4.3 Local Existence of an Antiderivative: Antiderivative Along a Curve

The question on the existence of an antiderivative in the whole domain is more complicated and will be considered in the next section. For now, we just note that not every analytic function has an antiderivative in the whole domain (see Remark 4.6). This statement does not agree with the fact from real analysis. Indeed, it is known that any continuous function φ on an interval has the primitive $F(x) = \int_{x_0}^{x} \varphi(t)\,dt$.

Theorem 4.2 (On the Local Existence of an Antiderivative) *Let $f \in \mathcal{A}(\Omega)$. Then there is an antiderivative of f in an arbitrary disk $B_r(a) \subset \Omega$, which is determined by the formula*

$$F(z) := \int_{[a,z]} f(\xi)\,d\xi, \quad z \in B_r(a). \tag{4.11}$$

In (4.11) the integral is taken along the segment $[a, z]$ from a to z.

Proof Let us consider an arbitrary disk $B_r(a) \subset \Omega$, then fix a complex number $z \in B_r(a)$ and take any complex number $\triangle z$ such that $z + \triangle z \in B_r(a)$.

Evidently, the closure of the triangle $\triangle_{a,z,z+\triangle z}$ with vertices at the points a, z and $z + \triangle z$ belongs to the domain Ω. Therefore, due to Theorem 4.1,

$$\int_{\partial^+\triangle_{a,z,z+\triangle z}} f(\xi)\,d\xi = 0,$$

wherefrom

$$\int_{[a,z]} f(\xi)\,d\xi + \int_{[z,z+\triangle z]} f(\xi)\,d\xi + \int_{[z+\triangle z,a]} f(\xi)\,d\xi = 0.$$

According to the notation (4.11) and the orientability property, the last equality can be rewritten as

$$F(z+\triangle z) - F(z) = \int_{[z,z+\triangle z]} f(\xi)\, d\xi.$$

Dividing this equality by $\triangle z$ and subtracting $f(z)$, we get

$$\left| \frac{F(z+\triangle z) - F(z)}{\triangle z} - f(z) \right| = \frac{1}{|\triangle z|} \cdot \left| \int_{[z,z+\triangle z]} (f(\xi) - f(z))\, d\xi \right|. \tag{4.12}$$

Here we use the identity

$$f(z) = \frac{1}{\triangle z} \int_{[z,z+\triangle z]} f(z)\, d\xi.$$

Since f is continuous, the following statement holds: for any $\varepsilon > 0$ there is a positive number δ that for all $\triangle z \in \mathbb{C}$ such that $|\triangle z| < \delta$ and for all $\xi \in [z, z+\triangle z]$ we have

$$|f(\xi) - f(z)| < \varepsilon.$$

Therefore, taking (4.12) into account, we get

$$\left| \frac{F(z+\triangle z) - F(z)}{\triangle z} - f(z) \right| \leq \frac{1}{|\triangle z|} \int_{[z,z+\triangle z]} \left|f(\xi) - f(z)\right| dl < \varepsilon.$$

This means that

$$\lim_{\triangle z \to 0} \frac{F(z+\triangle z) - F(z)}{\triangle z} = f(z).$$

In view of the fact that z is an arbitrary point in the disk $B_r(a)$, the function F defined by the formula (4.11) is a primitive of f in $B_r(a)$. □

Remark 4.3 The formula (4.11) cannot always be extended for the whole domain Ω, for example, if Ω is not simply connected. Therefore, the equality (4.11) is said to define only the local antiderivative for the function f.

Remark 4.4 When proving this theorem, two facts were used, namely

- continuity of f,
- and for an arbitrary triangle $\triangle$ that $\triangle \subseteq \Omega$, it is necessary $\int_{\partial^+\triangle} f\, d\xi = 0$.

Therefore, instead of the analyticity of f in Theorem 4.2, we can require the above properties for the function f to prove this theorem. ▲

Now consider the intermediate concept of antiderivative between the notions of local antiderivative and antiderivative in the whole domain, namely the antiderivative along a curve.

Definition 4.3 Let $f \in C(\Omega)$ and $\gamma\colon [a,b] \mapsto \mathbb{C}$ be a curve, whose trace E_γ belongs to the domain Ω.

A continuous function $\Psi : [a,b] \mapsto \mathbb{C}$ is called an antiderivative of the function f along the curve γ, if in some neighborhood of each point on E_γ there is an antiderivative of f, whose restriction to the corresponding part of the trace E_γ coincides with Ψ, i.e.,

$$\forall\, t_0 \in [a,b]\ \exists\, \delta > 0\ \exists\, r > 0\ \exists\, F_{t_0} \in \mathcal{A}\big(B_r(\gamma(t_0))\big) \text{ such that}$$

$$\begin{cases} F'_{t_0}(z) = f(z) & \text{for all } z \in B_r(\gamma(t_0)) \subset \Omega, \\ F_{t_0}(\gamma(t)) = \Psi(t) & \text{for all } t \in (t_0 - \delta, t_0 + \delta) \cap [a,b]. \end{cases}$$

Important
This definition does not require the existence of an antiderivative of f in the whole domain Ω. Moreover, the antiderivative Ψ along the curve γ does not necessarily have a derivative (it is only continuous).

Remark 4.5 It is easy to understand that if there is an antiderivative F of f in Ω, then for an arbitrary curve $\gamma\colon [a,b] \mapsto \Omega$ the function $F(\gamma(t))$, $t \in [a,b]$, is an antiderivative of f along the curve γ.

Example 4.4 Let us find an antiderivative of $\cos z$ along the curve

$$\gamma(t) = t + i|t|, \quad t \in [-1,1].$$

Since $\sin z$ is an antiderivative of $\cos z$ in $\mathbb{C}$, the function

$$\Psi(t) = \sin(t + i|t|) = \sin t \cosh |t| + i \cos t \sinh |t|, \quad t \in [-1,1],$$

is an antiderivative of $\cos z$ along γ. Evidently, $\Psi \notin C^1([-1,1])$.

Theorem 4.3 (The Existence of an Antiderivative Along a Curve) *If $f \in \mathcal{A}(\Omega)$, then there exists an antiderivative of the function f along any curve γ whose trace belongs to the domain Ω.*

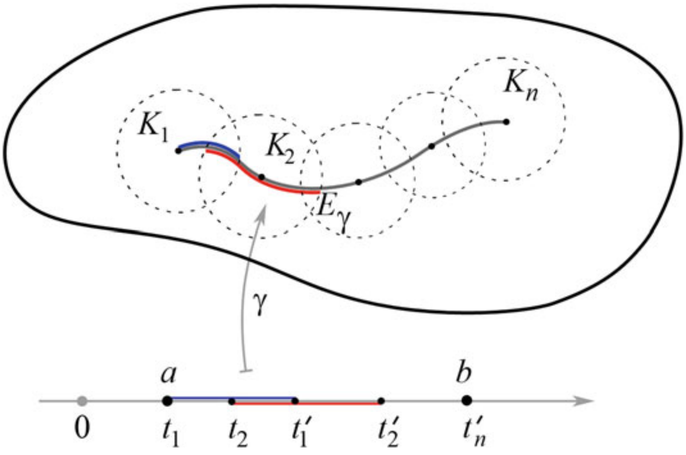

Fig. 4.2 Construction an antiderivative along a curve

This antiderivative is determined up to an additive constant.

Proof Let $z = \gamma(t)$, $t \in [a, b]$, be a curve, whose trace E_γ is in Ω. Then, due to Theorem 4.2, for every point on E_γ there exists a disk centered at that point, which belongs to Ω and in which there exists a primitive of f. Thus, we have obtained an infinite cover of E_γ by disks. Since E_γ is a compact, from this cover one can choose a finite subcover $\{K_1, \dots, K_n\}$. According to this subcover, we divide the segment $[a, b]$ by the segments $\left\{I_m := [t_m, t'_m]\right\}_{m=1}^{n}$ as follows:

$$t_1 = a, \quad t'_n = b, \quad t_m < t_{m+1} < t'_m, \quad \gamma(I_m) = E_\gamma \cap \overline{K_m}$$

(see Fig. 4.2). A more general case, for example, when a curve has self-intersection points, is considered in [4, 9].

Now we fix an antiderivative F_1 of f in the disk K_1. If we consider an arbitrary antiderivative of f in the disk K_2, then at the intersection of $K_1 \cap K_2$ these primitives can differ only by some constant, since they are two antiderivatives of the function f in $K_1 \cap K_2$. Therefore, there exists a unique antiderivative F_2 of f in K_2 such that $F_2 \equiv F_1$ in $K_2 \cap K_1$. Continuing these considerations, in each disk K_m we choose the unique antiderivative F_m of f such that

$$F_m \equiv F_{m-1} \quad \text{in} \ \ K_m \cap K_{m-1}, \quad m \in \{2, \dots, n\}.$$

Now we can determine the function

$$\Psi(t) := F_m(\gamma(t)), \quad t \in I_m, \ \ m \in \{1, \dots, n\}. \tag{4.13}$$

Then Ψ is an antiderivative of the function f along the curve γ. In fact, $\Psi \in C([a, b])$ by construction. Also, for every $\tau_0 \in [a, b]$ there is a disk $B_r(\gamma(\tau_0))$ belonging to some disk K_{m_0} from the subcover $\{K_1, \dots, K_n\}$, and there is an antiderivative F_{m_0} of f in $B_r(\gamma(\tau_0))$ for which, thanks to (4.13), we have

$$F_{m_0}(\gamma(t)) = \Psi(t), \quad t \in (\tau_0 - \delta, \tau_0 + \delta) \cap [a, b]$$

for some $\delta > 0$.

Now we prove the second part. Let Ψ_1 and Ψ_2 be two antiderivative of f along γ. Consider the function $\varphi(t) := \Psi_1(t) - \Psi_2(t),\ t \in [a, b]$. Let us show that it is locally constant. Then, taking into account its continuity on $[a, b]$, this means that $\varphi \equiv \text{const}$ on $[a, b]$.

Based on Definition 4.3, we have the following statement:

$$\forall\, t_0 \in [a, b]\ \exists\, \delta > 0\ \exists\, r > 0\ \exists \text{ antiderivatives } F_{t_0}^{(1)} \text{ and } F_{t_0}^{(2)}$$

of f in the disk $B_r(\gamma(t_0))$ such that

$$\Psi_1(t) = F^{(1)}(\gamma(t)), \quad \Psi_2(t) = F^{(2)}(\gamma(t)), \quad t \in I_\delta := (t_0 - \delta, t_0 + \delta) \cap [a, b].$$

Since $F_{t_0}^{(1)} = F_{t_0}^{(2)} + C$ in $B_r(\gamma(t_0))$, the function φ is constant in I_δ. □

Theorem 4.4 (Analog of the Newton-Leibniz Formula) *Let $\gamma \colon [a, b] \mapsto \Omega$ be a piecewise smooth curve. If a function f is continuous in the domain Ω and has an antiderivative Ψ along γ, then*

$$\int_\gamma f(z)\, dz = \Psi(b) - \Psi(a). \tag{4.14}$$

Proof Let us first consider the case when γ is a smooth curve and its trace belongs to some disk $K \subset \Omega$, in which there exists an antiderivative F of f. Then the composition $F(\gamma(t)),\ t \in [a, b]$, is an antiderivative of the function f along the curve γ, and from the second part of Theorem 4.3 it follows that

$$\Psi(t) = F(\gamma(t)) + C, \quad t \in [a, b].$$

Since $F' = f$ in K and γ is a smooth curve,

$$\Psi'(t) = F'(\gamma(t)) \cdot \gamma'(t) = f(\gamma(t)) \cdot \gamma'(t) \quad \text{for all } t \in [a, b].$$

Therefore,

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \Psi'(t)\, dt = \Psi(b) - \Psi(a). \tag{4.15}$$

In the general case, by virtue of the assumptions of the theorem and Definition 4.3, the curve γ can be decomposed into a finite number of smooth curves

$$\gamma_m : [\mu_m, \mu_{m+1}] \mapsto \Omega, \quad m \in \{0, 1, \dots, n-1\} \quad (a = \mu_0 < \dots < \mu_n = b),$$

so that the trace of each curve γ_m belongs to a disk K_m in which there exists an antiderivative of f.

Then, due to (4.15) we get

$$\int_\gamma f(z)\,dz = \sum_{m=0}^{n-1} \int_{\gamma_m} f(z)\,dz = \sum_{m=0}^{n-1} \Big(\Psi(\mu_{m+1}) - \Psi(\mu_m)\Big) = \Psi(b) - \Psi(a).$$

□

Remark 4.6 This theorem makes it possible to verify that not every analytic function in a domain has an antiderivative in that domain.

Consider, for example, the function

$$f(z) = \frac{1}{z}, \quad z \in \Omega = \{z : 0 < |z| < 2\}.$$

Obviously, $f \in \mathcal{A}(\Omega)$. Assume that there is an antiderivative F of f in Ω.

Take the circle $\gamma(t) = e^{it}$, $t \in [-\pi, \pi]$ ($E_\gamma \subset \Omega$). Then the function $F\big(\gamma(t)\big)$, $t \in [-\pi, \pi]$, is an antiderivative of f along γ, and according to Theorem 4.4,

$$\int_\gamma z^{-1}\,dz = F(e^{i\pi}) - F(e^{-i\pi}) = F(-1) - F(-1) = 0.$$

On the other hand, given the result of Example 4.3, we have

$$\int_\gamma z^{-1}\,dz = 2\pi i. \tag{4.16}$$

This contradiction indicates that there is no antiderivative of f in Ω.

However, the function f has an antiderivative along γ thanks to Theorem 4.3. Due to (3.9) it is equal to

$$\Psi(t) = \ln|e^{it}| + i\arg(e^{it}) + i2\pi k = it + i2\pi k, \quad t \in [-\pi, \pi],$$

for any fixed $k \in \mathbb{Z}$. Therefore, using (4.14), we get the result

$$\int_\gamma z^{-1}\,dz = \Psi(\pi) - \Psi(-\pi) = 2\pi i$$

that coincides with (4.16).

Remark 4.7 The Newton-Leibniz formula (4.14) and Theorem 4.3 make it possible to generalize the concept of the integral for an analytic function f in a domain Ω along an arbitrary curve $\gamma\colon [a, b] \mapsto \Omega$. Recall that Definition 4.1 requires the piecewise smoothness of γ. Since every analytic function f has an antiderivative Ψ along any continuous curve γ, the integral of f along γ can be determined as follows

$$\int_\gamma f(z)\, dz \overset{def}{=} \Psi(b) - \Psi(a). \tag{4.17}$$

Based on Remark 4.5 and Theorem 4.4, we have the statement.

Proposition 4.1 *If a function $f\colon \Omega \mapsto \mathbb{C}$ is continuous in Ω and has an antiderivative F in Ω, then for any two points $\{z_1, z_2\} \subset \Omega$*

$$\int_{\widetilde{z_1,z_2}} f(\xi)\, d\xi = F(z_2) - F(z_1),$$

where $\widetilde{z_1, z_2}$ is an arbitrary piecewise smooth curve with initial point z_1 and the end point z_2, and its trace is in the domain Ω.

Taking Remark 4.7 and Proposition 4.1 into account and using the equalities

$$fg' = (fg)' - f'g, \qquad \int_{\widetilde{z_1 z_2}} \big(f(\xi)g(\xi)\big)'\, d\xi = f(z_2)g(z_2) - f(z_1)g(z_1),$$

we deduce the statement.

Proposition 4.2 *If $f, g \in \mathcal{A}(\Omega)$, then for any $\{z_1, z_2\} \subset \Omega$*

$$\int_{\widetilde{z_1,z_2}} f(\xi)\, g'(\xi)\, d\xi = f(z_2)g(z_2) - f(z_1)g(z_1) - \int_{\widetilde{z_1,z_2}} f'(\xi)\, g(\xi)\, d\xi,$$

where $\widetilde{z_1, z_2}$ is an arbitrary curve with initial point z_1 and the end point z_2, and its trace is in Ω.

It follows from the above statements that the methods and formulas for integrating complex-valued functions of a complex variable remain the same as for functions of a real variable.

Example 4.5 For any $z_1, z_2 \in \mathbb{C}$

$$\int_{\widetilde{z_1,z_2}} e^\xi\, d\xi = e^{z_2} - e^{z_1}.$$

Sometimes curvilinear integrals of the second kind can be calculated using the Newton-Leibniz formula (4.14).

Example 4.6 Let $\gamma(t)$, $t \in [a, b]$, be a piecewise smooth curve in $\mathbb{R}^2$ with initial point (0, 0) and the end point (1, 1). Then due to (4.2)

$$\int_\gamma \sin x \cosh y \, dx - \cos x \sinh y \, dy = \operatorname{Re}\Big(\int_\gamma \sin z \, dz\Big)$$

$$= \operatorname{Re}\big(-\cos(\gamma(b)) + \cos(\gamma(a))\big) = \operatorname{Re}\big(-\cos(1+i) + \cos 0\big) = 1 - \cos 1 \cosh 1.$$

Exercise 4.1 Find an antiderivative for the function $\cosh z$ along the segment $[0, 1+i]$ with the orientation from 0 to $1+i$, and by the Newton-Leibniz formula calculate

$$\operatorname{Re}\Big(\int_{[0,1+i]} \cosh z \, dz\Big).$$

4.4 The Cauchy Integral Theorem and Corollaries

From the formula (4.17) and proving Theorem 4.3 it is visible that the integral of an analytical function along a curve will not change when the curve is continuously deformed so that its ends remain in place and its trace remains in the subcover $\{K_1, \dots, K_n\}$. How to understand the continuous deformation of a curve? For this purpose, we recall some definitions and facts from differential geometry. We will assume that curves considered in this section are defined on the closed interval $I := [0, 1]$; this can always be done with the admissible change of a variable without leaving the curve equivalence class.

Definition 4.4 (Homotopic Curves)

- Two curves

  $$\gamma_0 : I \mapsto \Omega \quad \text{and} \quad \gamma_1 : I \mapsto \Omega$$

  with the same start point $a = \gamma_0(0) = \gamma_1(0)$ and end point $b = \gamma_0(1) = \gamma_1(1)$ $(a \neq b)$ are called homotopic in a domain Ω (denoted as $\gamma_0 \approx \gamma_1$ in Ω) if there exists a continuous map $\varphi : I \times I \mapsto \Omega$ with the following properties (Fig. 4.3):

  (1) $\varphi(0, t) = \gamma_0(t)$ and $\varphi(1, t) = \gamma_1(t)$ for all $t \in I$;
  (2) $\varphi(s, 0) = a$ and $\varphi(s, 1) = b$ for all $s \in I$.

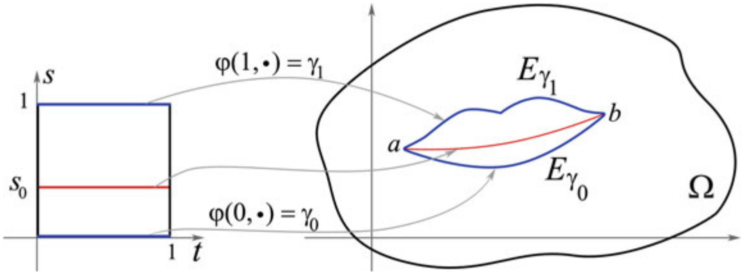

Fig. 4.3 Homotopic curves

- Similarly, two closed curves $\gamma_0 \colon I \mapsto \Omega$ and $\gamma_1 \colon I \mapsto \Omega$ are said to be homotopic in a domain Ω, if the previous first condition is satisfied and the second one reads as follows:

$$\varphi(s, 0) = \varphi(s, 1) \quad \text{for all } s \in I.$$

If $\gamma_1 \equiv \text{const} = \gamma_1(0) \in \Omega$ (γ_1 is a constant curve), then we write that $\gamma_0 \approx 0$ in Ω.

Remark 4.8 The first part of Definition 4.4 means that for each $s \in I$ the curve

$$\gamma_s := \varphi(s, t), \quad t \in I, \qquad \text{(in red color in Fig. 4.3)} \tag{4.18}$$

has the same initial and end points ($\gamma_s(0) = a$ and $\gamma_s = b$), its trace belongs to the domain Ω, and in addition, the family$\{\gamma_s\}_{s\in I}$ is a continuous deformation of γ_0 into γ_1 inside of Ω such that the endpoints are fixed during deformation.

The same interpretation applies to the second part of Definition 4.4, but now all curves in the family$\{\gamma_s\}_{s\in I}$ must be closed, and the start and end points, which now coincide, can move in Ω without leaving Ω.

If γ_1 is a constant curve, i.e., its trace is a point in the domain Ω, then we say that the curve γ_0 can be continuously deformed to this point while remaining in Ω, and γ_0 is said to be *null-homotopic* in Ω.

Example 4.7 Let Ω be *a convex domain*, i.e, any two points in Ω can be connected by a segment that entirely belongs to Ω. Then, any two closed curves γ_0 and γ_1, whose traces belong to Ω, are homotopic in Ω.

Indeed, the homotopic function that deforms γ_0 into γ_1 is as follows

$$\varphi(s, t) = \gamma_0(t) + s\big(\gamma_1(t) - \gamma_0(t)\big), \quad (s, t) \in I \times I.$$

Example 4.8 Any closed curve $\gamma\colon I \mapsto \mathbb{C}$ is homotopic in $\mathbb{C}$ to a point $a \in \mathbb{C}$ via the homotopy

$$\varphi(s,t) = s\,a + (1-s)\,\gamma(t), \quad (s,t) \in I \times I.$$

Example 4.9 The curve $\gamma(t) = e^{i2\pi t},\ t \in I$, is not null-homotopic in $\mathbb{C} \setminus \{0\}$.

Exercise 4.2 Prove that the homotopy relation of curves is an equivalence relation, that is, it is reflexive, symmetric, and transitive.

Exercise 4.3 Prove if $\gamma_1 \sim \gamma_2$ (see Definition 1.11) and $\gamma_2 \approx \gamma_3$ in Ω, then $\gamma_1 \approx \gamma_3$ in Ω.

Exercises 4.2 and 4.3 show that all curves in a domain Ω with the same endpoints (or closed curves) can be divided into homotopy classes, and equivalent curves fall into the same homotopy class.

Exercise 4.4 Prove that a domain in $\mathbb{C}$ is simply connected (see Definition 1.20) if and only if an arbitrary closed curve is null-homotopic in this domain.

Exercise 4.5 Prove that an arbitrary closed curve is null-homotopic in a domain if and only if two arbitrary curves with the same initial and end points are homotopic in that domain.

Theorem 4.5 (Homotopy Version of the Cauchy Integral Theorem) *If a function $f\colon \Omega \mapsto \mathbb{C}$ is analytic in a domain Ω and $\gamma_0 \approx \gamma_1$ in Ω, then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz. \tag{4.19}$$

Proof Let $\varphi\colon I \times I \mapsto \Omega$ be a homotopy between γ_0 and γ_1. By Theorem 4.3, for any $s_0 \in I$ there is an antiderivative Ψ_{s_0} of the function f along the curve $\gamma_{s_0}(t) := \varphi(s_0, t),\ t \in I$. In addition, it follows from the construction of the antiderivative Ψ_{s_0} that there is an $\varepsilon_0 > 0$ such that a curvilinear strip

$$\mathcal{U}_{\varepsilon_0}(\gamma_{s_0}) := \{z \in \mathbb{C}\colon \ \operatorname{dist}\big(z;\, E_{\gamma_{s_0}}\big) < \varepsilon_0\}$$

belongs to the union of the disks $\{K_1, \ldots, K_n\}$ (see the Proof of Theorem 4.3 and Fig. 4.2). Here, $\operatorname{dist}\big(z;\, E_{\gamma_{s_0}}\big)$ is the distance from z to the trace $E_{\gamma_{s_0}}$ of γ_{s_0}.

Due to the uniform continuity of φ, there is a positive number δ_0 such that for all $s \in (s_0 - \delta_0, s_0 + \delta_0) \cap I$

$$|\gamma_s(t) - \gamma_{s_0}(t)| < \varepsilon_0 \quad \text{for all } \ t \in I,$$

i.e., the trace E_{γ_s} belongs to the curvilinear strip $\mathcal{U}_{\varepsilon_0}(\gamma_{s_0})$. Moreover, this means that we can use the same functions $\{F_1, \ldots, F_n\}$ to determine an antiderivative Ψ_s of the function f along the curve γ_s as for defining Ψ_{s_0} (see the Proof of Theorem 4.3). As a result, by using the formula (4.17), we get

$$\int_{\gamma_{s_0}} f(z)\,dz = \int_{\gamma_s} f(z)\,dz \quad \text{for all} \quad s \in (s_0 - \delta_0, s_0 + \delta_0) \cap I. \tag{4.20}$$

Since $I = [0, 1]$ is compact, it can be covered by a finite number of intervals, on each of which equalities (4.20) hold. Then, starting from $s_0 = 0$, in a finite number of steps we arrive at the equality (4.19). □

Corollary 4.2 *If* $f \in \mathcal{A}(\Omega)$ *and* $\gamma \approx 0$ *in* Ω, *then* $\int_\gamma f(z)\,dz = 0$.

Proof Since $\gamma \approx 0$ in Ω, there exists a constant curve $\gamma_1 \equiv \text{const} = a \in \Omega$ such that $\gamma \approx \gamma_1$ in Ω. Then due to Theorem 4.5, we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz = \int_0^1 f(\gamma_1(t))\,\underbrace{\gamma_1'(t)}_{\substack{\shortparallel \\ 0}}\,dt = 0.$$

□

Corollary 4.3 *If* $f \in \mathcal{A}(\Omega)$ *and the domain* Ω *is simply connected, then for an arbitrary closed curve* $\gamma\colon I \mapsto \Omega$

$$\int_\gamma f(z)\,dz = 0.$$

The proof follows directly from Exercise 4.4 and Corollary 4.2.

Definition 4.5 A function f is analytic in the closure of a domain Ω ($f \in \mathcal{A}(\overline{\Omega})$), if there is a domain $G \supset \overline{\Omega}$ such that $f \in \mathcal{A}(G)$.

Corollary 4.4 *If* $f \in \mathcal{A}(\overline{\Omega})$ *and the domain* Ω *is bounded and simply connected, then*

$$\int_{\partial^+\Omega} f(z)\,dz = 0.$$

Proof By Definition 4.5, there is a domain $G \supset \overline{\Omega}$ where $f \in \mathcal{A}(G)$. Since Ω is simply connected and bounded, the curve, whose trace coincides with $\partial\Omega$, is null-homotopic in G. Next we should apply Corollary 4.2. □

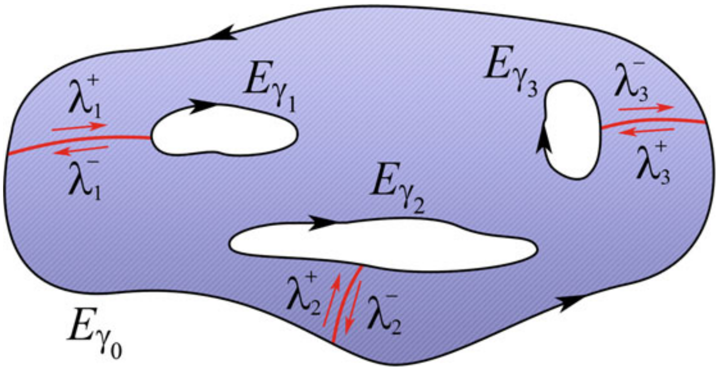

Fig. 4.4 Cuts of a domain Ω

Corollary 4.5 *Let Ω be a bounded domain and $\partial\Omega = \bigcup_{k=0}^{n} E_{\gamma_k}$, where*

- $\{\gamma_k\}_{k=0}^n$ *are Jordan curves whose traces are pairwise disjoint,*
- *for any $k \in \{1, \dots, n\}$ the trace E_{γ_k} belongs to the interior of γ_0,*
- *for any $k \in \{0, 1, \dots, n\}$ the orientation of γ_k coincides with the positive orientation of the boundary $\partial\Omega$.*

If $f \in \mathcal{A}(\overline{\Omega})$, then

$$\int_{\partial^+\Omega} f(z)\, dz = 0.$$

Remark 4.9 The *positive orientation of the boundary* $\partial^+\Omega$ of a bounded domain Ω is such an orientation of the closed curves whose traces form the domain boundary, in which the domain Ω always remains on the left when traversing the traces.

Figure 4.4 shows a domain Ω with the positive orientation of the boundary, which is described in conditions of Corollary 4.5. Note that the curve γ_0 is oriented counterclockwise, and all other curves $\{\gamma_k\}_{k=1}^n$ are clockwise.

Proof There exists a domain $G \supset \overline{\Omega}$ such that $f \in \mathcal{A}(G)$. Let λ_k^- and λ_k^+ be oppositely oriented curves whose traces coincide and connect E_{γ_k} with E_{γ_0} (Fig. 4.4); here $k \in \{1, \dots, n\}$.

Then with the help of the curves $\{\gamma_k\}_{k=0}^n$ and $\{\lambda_k^\pm\}_{k=1}^n$ we construct the closed curve

$$\Lambda := \gamma_0 \cup (\lambda_1^- \cup \gamma_1 \cup \lambda_1^+) \cup (\lambda_2^- \cup \gamma_2 \cup \lambda_2^+) \cup \dots \cup (\lambda_n^- \cup \gamma_n \cup \lambda_n^+)$$

with the positive orientation. It is easy to see that the curve Λ is null-homotopic in G. Therefore, considering Corollary 4.2, we have

$$0 = \int\limits_{\Lambda} f\,dz = \int\limits_{\partial^+\Omega} f(z)\,dz + \sum_{k=1}^{n} \underbrace{\Big(\int\limits_{\lambda_k^+} f(z)\,dz + \int\limits_{\lambda_k^-} f(z)\,dz\Big)}_{=0} = \int\limits_{\partial^+\Omega} f(z)\,dz.$$

The corollary is proved. □

Remark 4.10 The condition of Corollary 4.5 for the function f can be weaken, namely $f \in \mathcal{A}(\Omega) \cap C(\overline{\Omega})$. However, in this case the curves $\{\gamma_k\}$ must be piecewise smooth. Then also $\int_{\partial^+\Omega} f(z)\,dz = 0$.

Theorem 4.6 (On the Global Existence of an Antiderivative) *If $f \in \mathcal{A}(\Omega)$ and the domain Ω is simply connected, then there exists an antiderivative of f in Ω.*

Proof Since Ω is simply connected, an arbitrary closed curve whose trace belongs to Ω is null-homotopic in Ω. According to Exercise 4.5, this means that any curves with the same endpoints, whose traces are in Ω, are homotopic.

Then, by Theorem 4.5, the integral of f along a curve depends only on the initial and end points of the curve, but not on the curve itself. Therefore, we can determine the single-valued function

$$F(z) := \int_{\widetilde{a,z}} f(\xi)\,d\xi, \quad z \in \Omega, \tag{4.21}$$

where $\widetilde{a,z}$ is an arbitrary curve with initial point $a \in \Omega$ and the end point z and its trace is in Ω.

Now we fix arbitrary $z \in \Omega$ and take any complex number $\triangle z$ such that the segment $[z, z+\triangle z] \subset \Omega$. Then the curve

$$\Gamma := (\widetilde{a,z}) \cup [z, z+\triangle z] \cup (\widetilde{z+\triangle z, a})$$

is closed and its trace belongs to Ω (Fig. 4.5).

Corollary 4.3 gives that

$$\int_{\Gamma} f(\xi)\,d\xi = 0,$$

or

$$\int_{\widetilde{a,z}} f(\xi)\,d\xi + \int_{[z,z+\Delta z]} f(\xi)\,d\xi + \int_{\widetilde{z+\Delta z,a}} f(\xi)\,d\xi = 0,$$

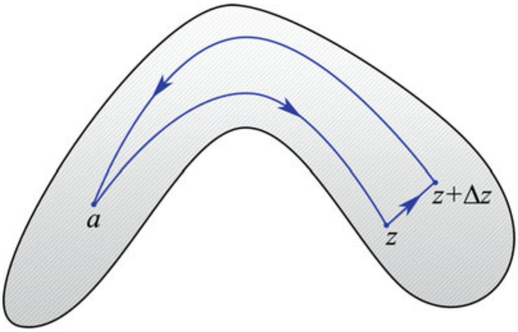

Fig. 4.5 The curve Γ

whence, using (4.21), we get

$$F(z+\Delta z) - F(z) = \int_{[z,z+\Delta z]} f(\xi)\,d\xi.$$

Further, as in Theorem 4.2 (on the local existence of an antiderivative), we prove that the derivative of F exists and $F'(z) = f(z)$. □

When proving the theorem, we used only the continuity of the function f and the vanishing of the integral along any closed curve. Therefore, the corollary is true.

Corollary 4.6 *If a function f is continuous in a domain Ω and the integral of f along any closed piecewise smooth curve, whose trace belongs to Ω, is equal to zero, then the function*

$$F(z) := \int_{\widetilde{a,z}} f(\xi)\,d\xi, \quad z \in \Omega,$$

is an antiderivative of f in Ω. Here $\widetilde{a,z}$ is an arbitrary piecewise smooth curve with initial point $a \in \Omega$ and the end point z and its trace is in Ω.

It is clear that if a continuous function f in a domain Ω has an antiderivative in Ω, then the integral of f along an arbitrary closed piecewise smooth curve whose trace lies in Ω is equal to zero. Thus, we have the following theorem on necessary and sufficient conditions for the existence of an antiderivative in the whole domain.

Theorem 4.7 *Let a function f be continuous in a domain Ω. There exists an antiderivative of f in Ω if and only if the integral of f along an arbitrary closed piecewise smooth curve whose trace lies in Ω is equal to zero.*

Returning to Remark 4.6, we see that the sufficient condition of Theorem 4.7 does not hold for the function z^{-1} in the domain $\{z : 0 < |z| < 2\}$, since the integral (4.16) does not vanish.

4.5 The Cauchy Integral Formula

From the Cauchy integral theorem we derive an integral representation for an analytic function, which has many important consequences.

Theorem 4.8 (The Cauchy Integral Formula) *Let Ω be a bounded domain and $\partial\Omega = \bigcup_{k=0}^{n} E_{\gamma_k}$, where*

- $\{\gamma_k\}_{k=0}^{n}$ *are Jordan curves whose traces are pairwise disjoint,*
- *for any $k \in \{1, \ldots, n\}$ the trace E_{γ_k} belongs to the interior of γ_0,*
- *for any $k \in \{0, 1, \ldots, n\}$ the orientation of γ_k coincides with the positive orientation of the boundary $\partial\Omega$.*

If $f \in \mathcal{A}(\overline{\Omega})$, then the following integral representation takes place:

$$f(z) = \frac{1}{2\pi i} \int_{\partial^+\Omega} \frac{f(\xi)}{\xi - z}\, d\xi \quad \text{for any } z \in \Omega. \tag{4.22}$$

Important
The formula (4.22) expresses a very interesting fact: the value of an analytic function in a domain is completely determined by its values on the boundary of that domain. Therefore, this representation is very often used in both theoretical and applied problems.

Proof Let us fix any $z \in \Omega$. Clearly, that there is a positive number r_0 such that for all $r \in (0, r_0)$ the closed disk $\overline{B_r(z)}$ belongs to the domain Ω. Consider the function

$$\frac{f(\xi)}{\xi - z}, \quad \xi \in \Omega_r := \Omega \backslash \overline{B_r(z)}.$$

Obviously, that this function is analytic in $\overline{\Omega_r}$. Then Corollary 4.5 yields

$$0 = \int_{\partial^+\Omega_r} \frac{f(\xi)}{\xi - z}\, d\xi = \int_{\partial^+\Omega} \frac{f(\xi)}{\xi - z}\, d\xi - \int_{\partial^+ B_r(z)} \frac{f(\xi)}{\xi - z}\, d\xi. \tag{4.23}$$

Here the positive orientation of $\partial\Omega_r$ means that the circle $\partial B_r(z)$ is oriented clockwise, and this orientation is opposite to the positive orientation of the boundary of the disk $B_r(z)$. Therefore, the minus appeared before the last integral in (4.23). It follows from (4.23) that for all $r \in (0, r_0)$

$$\frac{1}{2\pi i} \int_{\partial^+\Omega} \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} \int_{\partial^+ B_r(z)} \frac{f(\xi)}{\xi - z}\, d\xi. \tag{4.24}$$

Since $f \in C(\overline{\Omega})$, we have that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \xi \in B_\delta(z): \quad |f(\xi) - f(z)| < \varepsilon.$$

Then, thanks to Example 4.3, for an arbitrary $r \in (0, \ \min(\delta, r_0))$

$$\left| f(z) - \frac{1}{2\pi i} \int_{\partial^+ B_r(z)} \frac{f(\xi)}{\xi - z}\, d\xi \right| = \left| \frac{1}{2\pi i} \int_{\partial^+ B_r(z)} \frac{f(z) - f(\xi)}{\xi - z}\, d\xi \right|$$

$$\leq \frac{1}{2\pi} \int_{\partial B_r(z)} \frac{|f(\xi) - f(z)|}{|\xi - z|} dl < \frac{1}{2\pi} \varepsilon \cdot \frac{1}{r} \cdot 2\pi r = \varepsilon.$$

This means that

$$\lim_{r \to 0} \frac{1}{2\pi i} \int_{\partial^+ B_r(z)} \frac{f(\xi)}{\xi - z}\, d\xi = f(z).$$

Taking this fact into account and passing to the limit in (4.24), we obtain the Cauchy integral formula (4.22). □

The Cauchy formula remains correct if $f \in \mathcal{A}(\Omega) \cap C(\overline{\Omega})$ and the boundary of the domain Ω consists of a finite number of piecewise smooth closed curves.

It follows from Corollary 4.5 that

$$\frac{1}{2\pi i} \int_{\partial^+ \Omega} \frac{f(\xi)}{\xi - z}\, d\xi = 0 \quad \text{for all } \ z \notin \overline{\Omega}. \tag{4.25}$$

Using the Cauchy integral formula, it is sometimes possible to calculate integrals from analytic functions along closed curves.

Example 4.10 Let us find the integral

$$I := \int_\gamma \frac{\sin \xi}{\xi^2 + 4}\, d\xi,$$

where $z = \gamma(t) = i + 2\exp(it), \ t \in [0, 2\pi]$.

The integral can be rewritten as follows

$$I = \frac{1}{4i} \int\limits_\gamma \left(\frac{1}{\xi - 2i} - \frac{1}{\xi + 2i} \right) \sin \xi \, d\xi = \frac{1}{4i} \int\limits_\gamma \frac{\sin \xi}{\xi - 2i}\, d\xi - \frac{1}{4i} \int\limits_\gamma \frac{\sin \xi}{\xi + 2i}\, d\xi.$$

The point $2i$ belongs to the interior of γ and $-2i \notin \operatorname{int}(\gamma)$. Thus, thanks to (4.22) and (4.25) we have

$$I = \frac{\pi}{2}\sin(2i) = i\frac{\pi}{2}\sinh 2.$$

Theorem 4.9 (Mean Value Theorem) *Let $f \in \mathcal{A}(\Omega)$. Then for any disk $B_R(a)$, which together with its closure belongs to the domain Ω, the value of f at the center of this disk is equal to the mean value of this function taken around the disk boundary $\partial B_R(a)$, i.e.*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + R\,e^{it})\,dt. \tag{4.26}$$

Proof For any $a \in \Omega$ there is a positive number R such that $\overline{B_R(a)} \subset \Omega$. Then, by virtue of the Cauchy integral formula, we have

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_{\partial^+ B_R(a)} \frac{f(\xi)}{\xi - a}\,d\xi = \left\langle \xi = a + Re^{it},\ t \in [0, 2\pi]\right\rangle \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a + R\,e^{it})}{R\,e^{it}} \cdot Rie^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(a + R\,e^{it})\,dt. \end{aligned}$$

The theorem is proved. □

This theorem shows once again that analytic functions are very "nice functions" (the value of an analytic function at each point is closely related to the values of this function at neighboring points). In the next sections, using this fact, we will prove many of their other remarkable properties.

Exercise 4.6 Does there exist an entire function that

- on the circle $\{z\colon |z - 1 + i| = 4\}$ takes the value $3i$,
- and on the circle $\{z\colon |z - 1 + i| = 3\}$ takes the value $1 + i$?

Exercise 4.7 Let f be an analytic function in $\{z\colon \operatorname{Im} z \geq 0\}$ and $f(x) = 2 - i$ for all $x \in \mathbb{R}$. Find the value of f at the points $1 + i$ and $2 + i$.

Exercise 4.8 Using Theorem 4.9 and Exercise 2.2, prove *the maximum modulus principle*, which states that if f is an analytic function, then its modulus cannot have a strict local maximum in the domain of f.

In a different way, this principle, as well as other properties of the modulus of an analytic function, will be proved in Sect. 9.1.

Complex Power Series

5

Abstract

The main goal of this chapter is to show that analytic functions can be represented as infinite power series. The key to proving this theorem is the Cauchy integral formula established in the previous section. Here we generalize this formula for derivatives and prove the surprising fact that derivatives of analytic functions can be calculated by integration. Conversely, we will establish that the sum of a complex power series is an analytic function in the open disk where this series converges. This fact is then used to prove that an analytic function is infinitely differentiable. In addition, the reader can familiarize himself with the proofs of such remarkable statements as Liouville's theorem, Maurer's theorem, and the equivalence of three approaches to the construction of the theory of analytic functions. In the last section, applications of power series representations lead us to a statement about the coincidence of analytic functions when they coincide on some sequence, and to a statement characterising the isolated zeros of an analytic function and their concentration.

5.1 Basic Definitions and Properties of Function Series and Power Series

The main properties of infinite complex series are the same as real ones. Nevertheless, we briefly recall the fundamental definitions and some properties.

Definition 5.1 Two sequences of complex numbers, one of which is a sequence $\{a_n\}_{n\in\mathbb{N}}$, and the second is determined from the previous one as follows

$$\left\{S_n := \sum_{k=1}^{n} a_k\right\}_{n\in\mathbb{N}}$$

T. Mel'nyk, *Complex Analysis*, https://doi.org/10.1007/978-3-031-39615-1_5

are called a series of complex numbers and are denoted by one symbol

$$\sum_{k=1}^{+\infty} a_k. \tag{5.1}$$

The sequence $\{S_n\}_{n\in\mathbb{N}}$ is said to be the sequence of partial sums of the series (5.1). If $\{S_n\}_{n\in\mathbb{N}}$ converges to a number $S \in \mathbb{C}$, then we say that the series (5.1) is convergent and the number S is the sum of (5.1); in this case, one also writes

$$S = \sum_{k=1}^{+\infty} a_k.$$

If there is no finite limit of $\{S_n\}_{n\in\mathbb{N}}$, then the series (5.1) is called divergent.

Example 5.1 The series $\sum_{k=1}^{+\infty} \left(\frac{i}{2}\right)^k$ is convergent since

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \left(\frac{i}{2}\right)^k = \lim_{n\to+\infty} \frac{i}{2} \frac{\left(\frac{i}{2}\right)^n - 1}{\frac{i}{2} - 1} = \frac{i}{2-i} = -\frac{1}{5} + \frac{2}{5}i,$$

and its sum is equal to $-\frac{1}{5} + \frac{2}{5}i$.

From Exercise 1.2 it follows that a series is convergent if and only if the corresponding series of real and imaginary parts are convergent. Similarly, as for real series, we prove that the sum and difference of two convergent series are convergent, and the necessary condition for $\sum_{k=1}^{+\infty} a_k$ to be convergent is that $\lim_{k\to+\infty} a_k = 0$.

A series $\sum_{k=1}^{+\infty} a_k$ is said to be *absolutely convergent* if $\sum_{k=1}^{+\infty} |a_k|$ converges; a series is said to be *conditionally convergent* if it converges, but it does not converge absolutely. It is easy to verify that an absolutely convergent series converges.

Let $\{f_n : \Omega \mapsto \mathbb{C}\}_{n\in\mathbb{N}}$ be a sequence of functions. For each $z \in \Omega$ consider the series

$$\sum_{n=1}^{+\infty} f_n(z) \tag{5.2}$$

and its sequence of partial sums $\{S_n(z) := \sum_{k=1}^{n} f_n(z)\}_{n\in\mathbb{N}}$.

Definition 5.2 The set of such numbers $z \in \Omega$ for which the series (5.2) converges is called the convergence set of the function series (5.2).

Definition 5.3 A function series $\sum_{n=1}^{+\infty} f_n(z)$ is called uniformly convergent on a set $M \subset \Omega$ to a function $f : M \mapsto \mathbb{C}$, if the sequence of its partial sums $\{S_n(z),\ z \in \Omega\}_{n\in\mathbb{N}}$ uniformly converges to the function f on M ($S_n \overset{M}{\rightrightarrows} f$), i.e.,

$$\forall\, \varepsilon > 0\, \exists\, n_0 \in \mathbb{N}\, \forall\, n \geq n_0\, \forall z \in M: \ \ |f(z) - S_n(z)| = \left| \sum_{k=n+1}^{+\infty} f_k(z) \right| < \varepsilon.$$

As in real analysis, we establish the theorem.

Theorem 5.1 (Weierstrass Criterion for Uniform Convergence) *Let for a sequence of functions* $\{f_n : \Omega \mapsto \mathbb{C}\}_{n\in\mathbb{N}}$ *the following conditions are satisfied:*

(1) $\forall\, n \in \mathbb{N}\ \ \exists\, a_n \in \mathbb{R}_+ : \quad \sup\limits_{z\in\Omega} |f_n(z)| \leq a_n$;
(2) the series $\sum_{n=1}^{+\infty} a_n$ *is convergent.*

Then the function series $\sum\limits_{n=1}^{+\infty} f_n(z)$ *converges uniformly and absolutely on* Ω.

Also, without any changes one can prove the theorem about the continuity of the sum of a function series and theorem about the term-by-term integration of a uniformly convergent function series.

Definition 5.4 Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers and $z_0 \in \mathbb{C}$. A power series centered at z_0 is called a function series of the form

$$\sum_{n=0}^{+\infty} c_n(z - z_0)^n. \tag{5.3}$$

The constants $\{c_n\}_{n\in\mathbb{N}}$ are called the coefficients of this power series. ▲

For the power series, we define the value

$$\frac{1}{R} := \limsup_{n\to+\infty} \sqrt[n]{|c_n|}. \tag{5.4}$$

Obviously, that $0 \leq R \leq +\infty$.

Theorem 5.2 (Cauchy–Hadamard Theorem) *If* $R = +\infty$, *then the series (5.3) is absolutely convergent in* $\mathbb{C}$; *when* $R = 0$ *the series (5.3) converges only at the point* $z = z_0$ *and diverges if* $z \neq z_0$.

If $R \in (0, +\infty)$, *the following statements hold:*

(1) the series (5.3) converges absolutely at every point

$$z \in B_R(z_0) = \{z \in \mathbb{C}\colon |z - z_0| < R\};$$

(2) the series (5.3) diverges at every point $z \notin \overline{B_R}(z_0)$;
(3) the series (5.3) converges uniformly on any compact set $K \subset B_R(z_0)$.

The proof is carried out in exactly the same way as for real power series. The disk $B_R(z_0)$ is called the *disk of convergence* of the power series (5.3) and the number R is called the *convergence radius* of (5.3).

Note that the convergence set of a power series may also contain points that lie on the boundary of the disk of convergence.

Example 5.2 Let us consider the following power series:

$$(A)\ \sum_{n=1}^{+\infty} z^n, \qquad (B)\ \sum_{n=1}^{+\infty} \frac{z^n}{n}, \qquad (C)\ \sum_{n=1}^{+\infty} \frac{z^n}{n^2}.$$

By using (5.4), we conclude that $B_1(0) = \{z\colon |z| < 1\}$ is the disk of convergence for these series. However,

- the series (A) diverges for all $z \in \partial B_1(0)$ (the necessary convergence condition is not fulfilled);
- the series (B) diverges at the point $z = 1$ (this is the harmonic series) and for the other points $z = e^{it} \in \partial B_1(0) \setminus \{1\},\ t \in (0, 2\pi)$, this series

$$\sum_{n=1}^{+\infty} \frac{e^{int}}{n} = \sum_{n=1}^{+\infty} \frac{\cos(nt)}{n} + i \sum_{n=1}^{+\infty} \frac{\sin(nt)}{n}$$

  converges only conditionally since its real and imaginary parts conditionally converge due to the Dirichlet criterion:

  ✓ the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ converges monotonically to zero,
  ✓ and the sequence of partial sums of the series $\sum_{n=1}^{+\infty} \cos(nt)$ is bounded for each $t \in (0, 2\pi)$ (the same for $(\sum_{n=1}^{+\infty} \sin(nt)\,)$);

- the series (C) absolutely converges for all $z \in \partial B_1(0)$ based on the Weierstrass criterion $\left(\left|\frac{z^n}{n^2}\right| \le \frac{1}{n^2} \text{ for all } z \in \partial B_1(0)\right)$.

5.2 Expansion of a Differentiated Function Into a Power Series

One of the main theorems of complex analysis is the following theorem.

Theorem 5.3 *Let Ω be a domain in $\mathbb{C}$ and $f \in \mathcal{A}(\Omega)$. Then, in arbitrary disk $K := B_R(z_0) \subset \Omega$, the function f can be represented as the sum of the power series*

$$f(z) = \sum_{n=0}^{+\infty} c_n (z - z_0)^n \quad \textit{for all} \;\; z \in K, \tag{5.5}$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{\{|\xi - z_0| = r\}^+} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi\,, \qquad \forall\, r \in (0, R). \tag{5.6}$$

Proof Take any point $z_0 \in \Omega$ and a such positive number R that the disk $K = B_R(z_0)$ belongs to Ω. Then, taking into account the Cauchy integral formula (4.22), we have for any $z \in K$ that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{\xi - z} d\xi\,,$$

where $\gamma_r = z_0 + r\, \exp(it)$, $t \in [0, 2\pi]$, and r is a number from the interval $\big(|z - z_0|, R\big)$.

The function $(\xi - z)^{-1}$, $\xi \in E_{\gamma_r}$, can be represented as sum of the following series (the sum of an infinite geometric series):

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)\left(1 - \frac{z - z_0}{\xi - z_0}\right)} = \frac{1}{\xi - z_0} \sum_{n=0}^{+\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n .$$

This series uniformly converges on E_{γ_r}, since

$$\frac{|z - z_0|}{|\xi - z_0|} = \frac{|z - z_0|}{r} =: q < 1 \quad \text{for all} \;\; \xi \in E_{\gamma_r}, \quad \text{and} \quad \sum_{n=0}^{+\infty} q^n = \frac{1}{1 - q}.$$

Then the series

$$\frac{1}{2\pi i} \frac{f(\xi)}{\xi - z} = \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot (z - z_0)^n, \tag{5.7}$$

which is obtained from the previous one by the multiplication with the bounded function $\frac{1}{2\pi i} f(\xi)$, $\xi \in E_{\gamma_r}$, also converges uniformly on E_{γ_r}.

Therefore, according to the theorem on the term-by-term integration of a function series, we can integrate the series (5.7) term by term. As a result, we obtain the representation (5.5), whose coefficients are defined by (5.6). It should be noted here that integrals in (5.6) are independent of $r \in (0, R)$ due to the Cauchy integral Theorem 4.5. □

Remark 5.1 The series (5.5) is called *a power series representation* of f around the point z_0. It is clear that the radius of convergence of (5.5) is not less than the distance from z_0 to the boundary of Ω.

Remark 5.2 It follows from Theorem 5.3 that the expansion of an analytic function in a power series around a given point is a necessary condition for the analyticity of this function at this point (see Definition 2.6).

Corollary 5.1 (Cauchy's Inequalities for the Coefficients) *Let $f \in \mathcal{A}(\overline{B_r(z_0)})$. If f is bounded by a constant M on $\partial B_r(z_0)$, then the coefficients of the power series (5.5) satisfy the inequalities*

$$|c_n| \leq \frac{M}{r^n} \quad \textit{for all } \; n \in \mathbb{N}_0. \tag{5.8}$$

Proof From (5.6), due to the boundedness of f and (4.5), we get

$$|c_n| = \left| \frac{1}{2\pi i} \int\limits_{\partial^+ B_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right| \leq \frac{1}{2\pi} \int\limits_{\partial B_r(z_0)} \frac{|f(\xi)|}{|\xi - z_0|^{n+1}} dl \leq \frac{M}{r^n}.$$

□

Theorem 5.4 (Liouville's Theorem) *If an entire function f is bounded, then it is constant.*

Proof Using Theorem 5.3, for any $R > 0$

$$f(z) = \sum_{n=0}^{+\infty} c_n z^n \quad \text{for all } \; z \in \overline{B_R(0)}.$$

Since f is bounded in $\mathbb{C}$, i.e. $\exists\, M > 0\; \forall\, z \in \mathbb{C} : |f(z)| \leq M$, it follows from (5.8) that

$$|c_n| \leq \frac{M}{R^n} \quad \text{for all } \; n \in \mathbb{N}_0.$$

Seeing that R can be taken arbitrarily large and $\lim_{R\to+\infty} \frac{M}{R^n} = 0$ if $n \in \mathbb{N}$, we have $c_n = 0$ for all $n \in \mathbb{N}$. This means that $f(z) \equiv c_0$ for all $z \in \mathbb{C}$. □

One of the elegant applications of Liouville's theorem is the following proof of the fundamental theorem of algebra.

Theorem 5.5 *Every polynomial* $P_n(z) := c_n z^n + c_{n-1}z^{n-1} + \ldots + c_1 z + c_0$ *with complex coefficient, where* $n \in \mathbb{N}$ *and* $c_n \neq 0$, *has at least one root.*

Proof Assume that $P_n(z) \neq 0$ for all $z \in \mathbb{C}$. Then, $f(z) := \frac{1}{P_n(z)}$ is an entire function. Moreover, f is bounded since $\lim_{|z|\to+\infty} |P_n(z)| = +\infty$. So, by Liouville's theorem, the function f is constant, which yields a contradiction because P_n is not constant. Hence, P_n must have a root in $\mathbb{C}$. □

Exercise 5.1 Prove that if $f \in \mathcal{A}(\overline{\mathbb{C}})$, then f is constant.

Exercise 5.2 Let $f \in \mathcal{A}(\mathbb{C})$. Prove that

(1) if $\lim_{z\to\infty} f(z) = \infty$, then the set $\{z \in \mathbb{C} : f(z) = 0\} \neq \emptyset$;
(2) if $\operatorname{Im} f(z) > 0$ for all $z \in \mathbb{C}$, then f is constant.

5.3 Analyticity of the Sum of a Power Series

It turns out that the converse claim to Theorem 5.3 also holds.

Theorem 5.6 *The sum*

$$f(z) := \sum_{n=0}^{+\infty} c_n (z-a)^n \tag{5.9}$$

is an analytic function in the convergence disk $B_R(a)$ *of the series (5.9).*

Moreover, the derivative of f *is calculated by the formula*

$$f'(z) = \sum_{n=1}^{+\infty} n\, c_n\, (z-a)^{n-1} \quad \textit{for all } z \in B_R(a). \tag{5.10}$$

Proof Since $B_R(a)$ is the disk of convergence of the series (5.9), the radius R is determined with the formula (5.4). It is easy to verify that this disk is also the disk of convergence for the following series:

$$\phi(z) := \sum_{n=1}^{+\infty} n\, c_n (z-a)^{n-1}. \tag{5.11}$$

Due to the Cauchy–Hadamard Theorem 5.2 and theorem on the continuity of the sum of a function series, the function ϕ is continuous in $B_R(a)$. In addition, the series (5.11) uniformly converges on $\partial\triangle$, where $\triangle$ is an arbitrary triangle that, together with its closure, belongs to $B_R(a)$. Using the theorem on termwise integration of a function series and the Cauchy-Goursat Theorem 4.1 for triangles, we get

$$\int_{\partial^+\triangle} \phi(z)dz = \sum_{n=1}^{+\infty} n\, c_n \int_{\partial^+\triangle} (z-a)^{n-1} dz = 0.$$

Then, according to Remark 4.4, the function ϕ has an antiderivative Ψ in the disk $B_R(a)$, which is defined by the formula $\Psi(z) = \int_{[a,z]} \phi(\xi)d\xi,\ z \in B_R(a)$.

On the other hand,

$$\int_{[a,z]} \phi(\xi)\, d\xi = \sum_{n=1}^{+\infty} n c_n \int_{[a,z]} (\xi - a)^{n-1}\, d\xi = \sum_{n=1}^{+\infty} c_n (z-a)^n = f(z) - c_0$$

for all $z \in B_R(a)$, whence we obtain that $f = c_0 + \Psi$ in $B_R(a)$. Thus, the function f is also an antiderivative for ϕ in $B_R(a)$, i.e., $f \in \mathcal{A}(B_R(a))$ and

$$f'(z) = \phi(z) = \sum_{n=1}^{+\infty} n\, c_n (z-a)^{n-1} \quad \text{for all } \ z \in B_R(a).$$

□

Corollary 5.2 *The derivative of an analytic function in a domain Ω is also analytic in this domain.*

Proof Let $f \in \mathcal{A}(\Omega)$. Consider an arbitrary point $z_0 \in \Omega$ and a disk $B_r(z_0)$ that belongs to Ω. Due to Theorem 5.3 the function f is represented as the sum of a power series

$$f(z) = \sum_{n=0}^{+\infty} c_n (z - z_0)^n, \quad z \in B_r(z_0). \tag{5.12}$$

Using the second claim of Theorem 5.6, the function f has the derivative that is obtained from (5.12) by termwise differentiation, i.e., the derivative f' is also represented as a power series in the same disk. Then we apply the first assertion of Theorem 5.6 to f' and obtain that $f' \in \mathcal{A}(B_r(z_0))$. Since z_0 is an arbitrary point of Ω, the function f' is analytic in Ω. □

Applying Corollary 5.2 to f' gives the statement.

Corollary 5.3 *Any analytic function in a domain Ω has derivatives of all orders in Ω.*

Corollary 5.2 implies the following necessary condition for the existence of an antiderivative.

Corollary 5.4 *If a continuous function f in a domain Ω has an antiderivative, then f is analytic in Ω.*

Remark 5.3 From Theorems 5.3 and 5.6 it follows that the representation of a function as the sum of a convergent power series in some disk is a necessary and sufficient condition for the analyticity of this function in this disk.

However, the convergence of a power series at points on the boundary of its disk of convergence is not related to the analyticity of its sum at those points. Indeed, let us return to Example 5.2.

The sum of the series (A) is equal to $\frac{z}{1-z}$ in the disk $B_1(0)$ and it diverges at each point of the boundary $\partial B_1(0)$. But the function $\frac{z}{1-z}$ is analytic in the domain $\mathbb{C} \setminus \{1\}$.

The series (C) absolutely converges to some function f in the closed disk $\overline{B_1(0)}$. Assuming that f is analytic at the point $z = 1$, by Corollary 5.2 the derivative f' must be also analytic at $z = 1$. Thus, there must be a finite limit

$$\lim_{z \to 1-0,\ \operatorname{Im} z = 0} f'(z) = f'(1).$$

However, since

$$f'(z) = \sum_{n=1}^{+\infty} \frac{z^{n-1}}{n} \quad \text{for all } \ z \in B_1(0),$$

$$\lim_{x \to 1-0} f'(x) = -\lim_{x \to 1-0} \frac{1}{x} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}(-x)^n}{n} = -\lim_{x \to 1-0} \frac{1}{x} \ln(1-x) = +\infty.$$

This contradiction indicates that f can not be analytic at $z = 1$.

5.4 Uniqueness of Power Series Expansions: Morera's Theorem

A natural question that arises is: if for a given function we somehow get a kind of power series expansion about a given point, will this be the only expansion? For example, is the right-hand side of the identity

$$z^2 + 1 = 2 + 2(z-1) + (z-1)^2 \tag{5.13}$$

the unique power representation of the function $z^2 + 1$ around the point 1? The answer is given by the following theorem.

Theorem 5.7 *If a function f is equal to the sum of a power series in a disk $B_r(z_0)$, then this series is its Taylor series, i.e., if*

$$f(z) = \sum_{n=0}^{+\infty} c_n (z - z_0)^n \quad \textit{for all } z \in B_r(z_0), \tag{5.14}$$

then

$$c_n = \frac{f^{(n)}(z_0)}{n\,!} \qquad \forall\, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{5.15}$$

where $f^{(n)}$ denotes the n-th order derivative $\frac{d^n f}{dz^n}$.

Proof Theorem 5.6 implies that f in analytic in the disk $B_r(z_0)$ and Corollary 5.3 implies that f has derivatives of all orders in this disk. In addition, from the formula (5.10) it follows that for any $k \in \mathbb{N}$ the kth order derivative

$$f^{(k)}(z) = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} c_n (z - z_0)^{n-k} \quad \text{for all } z \in B_r(z_0). \tag{5.16}$$

Taking $z = z_0$ in (5.16), we find $f^{(k)}(z_0) = k!\, c_k$. That is, we get the formula (5.15) for the power series coefficients. □

Remark 5.4 Sometimes one can find such formulations of this theorem:

- *a power series is the Taylor series of its sum*;
- *a function can be represented by a power series only in one way.*

Example 5.3 Using (5.15), it is easy to check that the coefficients of the power representation of $z^2 + 1$ around 1 are as follows: $c_0 = c_1 = 2$, $c_2 = 1$, and $c_n = 0$ for $n > 2$ (compare with (5.13)).

It is often inexpedient to calculate coefficients using (5.15) or impractical formulas (5.6). Based on Theorem 5.7 and with the help of known formulas for the coefficients of the Taylor series of elementary real functions, it is possible to write down the power representations of the corresponding elementary complex-valued entire functions:

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n\,!}, \qquad \sin z = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)\,!}, \qquad \cos z = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)\,!}.$$

Example 5.4 From the representations above, the following identity follows:

$$\cos z + i\ \sin z = \sum_{n=0}^{+\infty}(-1)^n \frac{z^{2n}}{(2n)\,!} + i \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)\,!}$$
$$= 1 + i\,z - \frac{z^2}{2!} - i\,\frac{z^3}{3!} + \frac{z^4}{4!} + i\,\frac{z^5}{5!} - \ldots$$
$$= 1 + i\,z + \frac{(i\,z)^2}{2!} + \frac{(i\,z)^3}{3!} + \frac{(i\,z)^4}{4!} + \frac{(i\,z)^5}{5!} + \ldots = e^{iz}$$

for all $z \in \mathbb{C}$. This identity coincides with Euler's formula (1.9) when z is a real number. ▲

Comparing formulas (5.6) and (5.15) for finding power series coefficients, we get

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial^+ B_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi, \quad n \in \mathbb{N}_0; \tag{5.17}$$

here $r \in (0, R)$ and R is the convergence radius of (5.14).

What is interesting is that equalities (5.17) make it possible to estimate the derivative of an arbitrary order of an analytic function through the value of this function on the boundary of a disk.

Proposition 5.1 *Let* $f \in \mathcal{A}(\overline{B_r(z_0)})$. *Then*

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\int_{\partial B_r(z_0)} \frac{|f(\xi)|}{r^{n+1}}\,dl \le \frac{n!}{r^n}\max_{\xi\in\partial B_r(z_0)}|f(\xi)| \quad \textit{for all}\ \ n \in \mathbb{N}.$$

Exercise 5.3 Let $f \in \mathcal{A}(\mathbb{C})$ and there exist numbers $A > 0$, $B > 0$ and $n \in \mathbb{N}$ such that for all $r \in (0, +\infty)$:

$$\max_{|z|=r}|f(z)| \le A\,r^n + B.$$

Show that f is a polynomial whose degree is not greater than n.

Exercise 5.4 Let $P_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$, where $a_n \neq 0$, and

$$\max_{|z|=1}|P_n(z)| \le M.$$

Show that

$$|P_n(z)| \le M|z|^n \quad \text{for all}\ \ z,\ |z| \ge 1.$$

Lemma 5.1 (The Cauchy Integral Formula for Derivatives) *Let $f \in \mathcal{A}(\overline{\Omega})$ and the other conditions of Theorem 4.8 be satisfied. Then for any $n \in \mathbb{N}$ and any $z_0 \in \Omega$:*

$$f^{(n)}(z_0) = \frac{n\,!}{2\pi i} \int_{\partial^+\Omega} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi. \tag{5.18}$$

Proof Since $f \in \mathcal{A}(\overline{\Omega})$, there exists a domain $G \supset \overline{\Omega}$ such that $f \in \mathcal{A}(G)$. Fix any point $z_0 \in \Omega$ and take any circle $\beta_r = z_0 + r\,e^{it}$, $t \in [0, 2\pi]$, whose interior belongs in Ω. Obviously, the curve β_r is null-homotopic in G (see Definition 4.4).

Similarly, as in Corollary 4.5, we construct the closed curve Λ with the positive orientation (see Sect. 1.4), which is null-homotopic in G and whose trace does not contain the point z_0. Then $\Lambda \approx \beta_r$ in $G \setminus \{z_0\}$.

Now, based on the Cauchy integral Theorem 4.5 and the equality (5.17), just as in Corollary 4.5, we derive

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n\,!}{2\pi i} \int_{\beta_r} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi = \frac{n\,!}{2\pi i} \int_{\Lambda} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi \\ &= \frac{n\,!}{2\pi i} \int_{\partial^+\Omega} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi. \end{aligned}$$

□

Again, as in the case the Cauchy integral formula (see Example 4.1), it is sometimes possible to calculate integrals from analytic functions along closed curves with the help of the formula (5.18).

Example 5.5 Compute $\displaystyle\int_{\{|z|=2\}^+} \frac{\cos iz}{(z-1)^3}\, dz.$

Since $f(z) = \cos(iz)$ is analytic in $\overline{B_2(0)}$ and $z_0 = 1 \in B_2(0)$, we use (5.18) with $n = 2$ to get

$$\int_{\{|z|=2\}^+} \frac{\cos iz}{(z-1)^3}\, dz = \pi i\ f''(1) = \pi i\ \cos i = \pi i\ \cosh 1.$$

From mathematical analysis, it is known that any continuous function on a segment can be uniformly approximated by a polynomial with a given accuracy. A similar problem arises in complex analysis: can an analytic function on a compact set be approximated by a polynomial with a given accuracy? It is clear that if this compact set is a closed disk, then the problem is solved by Taylor polynomials. However, power series converge in disks, so Taylor polynomials are not suitable for approximating analytic functions in general domains. For simply connected domains, this problem was solved by the German mathematician Karl Runge (1856–1927) in 1885.

Theorem 5.8 (Runge's Theorem) *Let f be an analytic function in a simply connected domain Ω. Then for any compact set $K \subset \Omega$ and any $\varepsilon > 0$ there is a polynomial $P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ such that*

$$\max_{z \in K} |f(z) - P_n(z)| < \varepsilon.$$

We do not present the proof of the theorem. The reader can find it in [4, Vol. 3]. Runge's theorem has many applications in various branches of complex analysis; its generalizations are Walsh's theorem, Mergelyan's theorem and Keldysh–Lavrent'ev theorem (see [4]).

Now we can prove a converse statement of the Cauchy-Goursat Theorem 4.1. Surprisingly, the property described there is almost equivalent to analyticity.

Theorem 5.9 (Morera's Theorem) *Let f be a continuous function in a domain Ω. If the integral of f over the boundary of an arbitrary triangle, which together with its closure belongs to Ω, is zero, then the function f is analytic in Ω.*

Proof Take any point $a \in \Omega$ and any number $r > 0$ such that $B_r(a) \subset \Omega$. Based on Remark 4.4, the function

$$F(z) := \int_{[a,z]} f(\xi) d\xi, \quad z \in B_r(a),$$

is an antiderivative of f in $B_r(a)$. Corollary 5.4 implies that $f \in \mathcal{A}(B_R(a))$. Since the point a was chosen arbitrarily, $f \in \mathcal{A}(\Omega)$. □

As we will see further on, Morera's theorem is useful in proving many important facts. Now consider an example.

Example 5.6 Show that the function

$$f(z) = \int_c^d e^{-t}\, t^{z-1}\, dt \quad z \in \Omega_r := \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\},$$

where $0 < c < d < +\infty$, is analytic in the right half-plane Ω_r.

The first thing to note is that the function f is well defined and continuous in Ω_r because the integrand is a continuous function of two variables t and z in the domain of definition ($t^{z-1} = e^{(z-1)\log t}$).

Now take an arbitrary triangle $\triangle \subseteq \Omega_r$ and calculate the integral

$$\int_{\partial^+\triangle} f(z)\, dz = \int_{\partial^+\triangle} \left(\int_c^d e^{-t}\, t^{z-1}\, dt \right) dz$$
$$= \int_c^d e^{-t} \left(\int_{\partial^+\triangle} t^{z-1}\, dz \right) dt = \int_c^d 0\, dt = 0.$$

Here we used the analyticity of the function $t^{z-1} = e^{(z-1)\log t}$ with respect to the variable z and the Cauchy-Goursat Theorem 4.1. Then, Morera's theorem implies that $f \in \mathcal{A}(\Omega_r)$.

Exercise 5.5 Using the same approach, show that

$$g(z) = \int_1^{+\infty} \frac{t^{z-1}}{e^t - 1}\,dt \tag{5.19}$$

is an entire function. ▲

It is well known from mathematical analysis that the term-by-term differentiation of a functional series requires its convergence at some point and the uniform convergence of the corresponding series of derivatives. In complex analysis, the situation is simplified for functional series consisting of analytic functions. The following theorem holds.

Theorem 5.10 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of analytic functions in a domain Ω. If for any compact set $K \subset \Omega$ the series*

$$\sum_{n=1}^{+\infty} f_n(z)$$

converges uniformly on K to a function f, then $f \in \mathcal{A}(\Omega)$ and for all $k \in \mathbb{N}$

$$f^{(k)}(z) = \sum_{n=1}^{+\infty} f_n^{(k)}(z) \quad \textit{for all } z \in \Omega.$$

Proof Fix a point a in Ω and consider a closed disk $\overline{B_R(a)} \subset \Omega$. Since the series $\sum_{n=1}^{+\infty} f_n(z)$ converges uniformly on $\overline{B_R(a)}$ to f and each term of this series is a continuous function, the function $f \in C\big(\overline{B_R(a)}\big)$.

Now take any triangle $\triangle \subseteq B_R(a)$. Then, due to the uniform convergence and the Cauchy-Goursat Theorem 4.1 we have

$$\int_{\partial^+\Delta} f(z)\,dz = \int_{\partial^+\Delta} \sum_{n=1}^{+\infty} f_n(z)\,dz = \sum_{n=1}^{+\infty} \underbrace{\int_{\partial^+\Delta} f_n(z)\,dz}_{\overset{\shortparallel}{0}} = 0.$$

Based on Morera's Theorem 5.9, $f \in \mathcal{A}(B_R(a))$, and hence $f \in \mathcal{A}(\Omega)$, since a is an arbitrary point from Ω.

To prove the second statement, consider a circle $\gamma_r = a + re^{it}$, $t \in [0, 2\pi]$, such that $E_{\gamma_r} \subset B_R(a)$. Since the series

$$\frac{k\,!}{2\pi i} \sum_{n=1}^{+\infty} \frac{f_n(z)}{(z-a)^{k+1}} \qquad (k \in \mathbb{N})$$

converges uniformly on E_{γ_r} to $\frac{f(z)}{(z-a)^{k+1}}$, it can be integrated term by term. Recalling the formulas (5.17), we get

$$f^{(k)}(a) = \frac{k\,!}{2\pi i} \int\limits_{\gamma_r} \frac{f(z)}{(z-a)^{k+1}} dz = \sum_{n=1}^{+\infty} \frac{k\,!}{2\pi i} \int\limits_{\gamma_r} \frac{f_n(z)}{(z-a)^{k+1}} dz = \sum_{n=1}^{+\infty} f_n^{(k)}(a).$$

The theorem is proved. □

It follows from this theorem that analyticity is preserved in uniform limits, in contrast to differentiability in real analysis, where the uniform limit of differentiable functions may be nowhere differentiable.

Corollary 5.5 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of analytic functions in a domain Ω. If for any compact set $K \subset \Omega$ the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on K to a function f, then $f \in \mathcal{A}(\Omega)$ and for all $k \in \mathbb{N}$*

$$f^{(k)}(z) = \lim_{n\to+\infty} f_n^{(k)}(z) \quad \textit{for all} \ \ z \in \Omega.$$

Example 5.7 Using the Weierstrass criterion (see Theorem 5.1), it is easy to verify that the function series $\sum_{n=1}^{+\infty} n^{-z}$ converges absolutely and uniformly in $\{z \in \mathbb{C} \colon \mathrm{Re} \geq 1+\delta\}$, where δ is an arbitrary positive number. To apply this criterion, you need the inequality

$$|n^{-z}| = \left|e^{-(x+iy)\log n}\right| = e^{-x\log n} \leq \frac{1}{n^{1+\delta}},$$

and the well-known fact that the numerical series $\sum_{n=1}^{+\infty} n^{-(1+\delta)}$ converges.

Since for every $n \in \mathbb{N}$ the function $n^{-z} = e^{-z\log n}$ is entire, Theorem 5.10 says that the sum

$$\zeta(z) := \sum_{n=1}^{+\infty} \frac{1}{n^z} \tag{5.20}$$

is an analytic function in the half-plane $\{z \in \mathbb{C} \colon \mathrm{Re}\, z > 1\}$. This function is called the *Riemann zeta function*. Note that ζ has a singularity at $z = 1$ and $\lim_{\varepsilon\to 0} \zeta(1 +$

$\varepsilon) = +\infty$ since the partial sum $\sum_{n=1}^{N} n^{-1}$ of the harmonic series is equivalent to $\log N$ as $N \to +\infty$.

Example 5.8 The real-valued gamma function

$$\Gamma(x) = \int_0^{+\infty} e^{-t}\, t^{x-1}\, dt, \quad x \in (0, +\infty), \tag{5.21}$$

is studied in mathematical analysis, and it is well known that $\Gamma \in C^{\infty}(0, +\infty)$.

We now define the gamma function in the right half-plane Ω_r by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t}\, t^{z-1}\, dt, \quad z \in \Omega_r = \{\xi \in \mathbb{C} \colon \operatorname{Im} \xi > 0\}. \tag{5.22}$$

Since

$$|e^{-t}\, t^{z-1}| = |e^{-t}\, e^{(z-1)\log t}| = e^{-t}\, t^{x-1}, \tag{5.23}$$

the inequality

$$|\Gamma(z)| \le \int_0^{+\infty} e^{-t}\, t^{x-1}\, dt < +\infty$$

holds for all $z \in \Omega_r$, where $\operatorname{Re} z = x > 0$. Thus, the gamma function is correctly defined in Ω_r. In addition,

$$\Gamma(z) = \lim_{n \to +\infty} f_n(z), \quad \text{where } \ f_n(z) = \int_{\frac{1}{n}}^{n} e^{-t}\, t^{z-1}\, dt.$$

Every function f_n is analytic in Ω_r (see Example 5.6). Using (5.23), it is easy to show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on any compact $K \subset \Omega_r$ (even on every vertical strip $\{z \colon 0 < \alpha \le \operatorname{Re} z \le \beta < +\infty\}$). By Corollary 5.5, the gamma function is analytic in the right half-plane Ω_r.

Replacing the integration variable (nt to t), where $n \in \mathbb{N}$, we get

$$\int_0^{+\infty} e^{-nt}\, t^{z-1}\, dt = \frac{1}{n^z} \int_0^{+\infty} e^{-t}\, t^{z-1}\, dt = \frac{\Gamma(z)}{n^z},$$

from where, we deduce for any $z \in \{\xi \in \mathbb{C} \colon \ \operatorname{Re} \xi > 1\}$ that

$$\Gamma(z) \sum_{n=1}^{+\infty} \frac{1}{n^z} = \int_0^{+\infty} t^{z-1} \lim_{N \to +\infty} \sum_{n=1}^{N} e^{-nt}\, dt = \int_0^{+\infty} \frac{t^{z-1}}{e^t - 1}\, dt.$$

As a result, we obtain the relation

$$\Gamma(z)\,\zeta(z) = \int_0^{+\infty} \frac{t^{z-1}}{e^t - 1}\,dt, \quad z \in \{\xi : \operatorname{Re}\xi > 1\}, \tag{5.24}$$

between the gamma and zeta functions. These functions have many applications in physics, probability theory, and applied statistics; the Riemann zeta function plays a central role in analytic number theory. We will continue our study of these functions in Sect. 8.2. ▲

In this section, we have proved several equivalent statements about the analyticity of a function. We summarise them in the following theorem.

Theorem 5.11 *The following three statements are equivalent:*

(R) *a function f is differentiable in a disk $B_r(a)$;*

(C) $f \in C(B_r(a))$ *and for any triangle* $\triangle \subseteq B_r(a)$: $\int_{\partial^+\triangle} f(z)\,dz = 0$;

(W) $f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n$ *for all* $z \in B_r(a)$.

Remark 5.5 These three statements reflect three concepts in the development of the theory of analytic functions:

(1) functions are called Riemann analytic or simply analytic if they satisfy the (R) condition;
(2) functions are called Cauchy analytic or holomorphic if they satisfy the (C) condition;
(3) functions are called Weierstrass analytic or regular if they satisfy the (W) condition.

Thus there were three different starting points in the development of the theory of complex-valued functions of a complex variable, and this explains why, even today, different words are used, such as "analytic", "regular" and "holomorphic". Now we see that these are equivalent concepts and we prefer the term "analytic". ▲

The proof of Theorem 5.11 is based on already proved theorems. The equivalence of the statements (R) and (C) is ensured by the Cauchy-Goursat Theorem 4.1 and Morera's Theorem 5.9, and the equivalence of statements (R) and (W) is provided by Theorem 5.3 (on the expansion of a analitic function into a power series) and Theorem 5.6 (on the sum of a power series).

5.5 Uniqueness Theorem for Analytic Functions: Zeros of Analytic Functions

Let us now consider some applications of power series representations. We begin with the uniqueness theorem for analytic functions.

Theorem 5.12 *Suppose that f and g are analytic functions in a domain Ω and there exists a sequence of distinct points $\{z_n\}_{n\in\mathbb{N}} \subset \Omega$ such that*

- $\lim\limits_{n\to+\infty} z_n = a \in \Omega$ *and* $f(z_n) = g(z_n)$ *for all* $n \in \mathbb{N}$.

Then $f \equiv g$ in Ω.

This theorem shows one more difference between differentiable complex-valued functions and differentiable real-valued functions. It is easy to give many examples where two infinitely differentiable functions of a real variable may coincide on a certain segment of the domain of definition, but not be identically equal in the entire domain of definition. According to this theorem, if two analytic functions coincide on an arbitrary set that has a limit point in the domain of analyticity of these two functions, then they necessarily coincide identically in the entire domain.

Proof

1. Let us first show that these functions coincide in some neighborhood of the point a. Since $a \in \Omega$, there is a positive number R such that the disk $B_R(a)$ belongs to the domain Ω. By Theorem 5.3, we have

$$f(z) = \sum_{k=0}^{+\infty} c_k(z-a)^k \quad \text{and} \quad g(z) = \sum_{k=0}^{+\infty} d_k(z-a)^k$$

for all $z \in B_R(a)$. Consider the difference

$$h(z) := f(z) - g(z) = \sum_{k=0}^{+\infty} (c_k - d_k)(z-a)^k, \quad z \in B_R(a).$$

Since $\lim_{n\to+\infty} z_n = a$, for any $r_0 \in (0, R)$ there exists a number $n_0 \in \mathbb{N}$ such that

$$z_n \in B_{r_0}(a) \subset B_R(a) \quad \text{for all } n \geq n_0.$$

By the Cauchy-Hadamard Theorem 5.2, the series

$$\sum_{k=0}^{+\infty}(c_k - d_k)(z-a)^k$$

converges uniformly on $\overline{B_{r_0}(a)}$. Therefore, passing to the limit ($n \to +\infty$) in the equality

$$0 = h(z_n) = \sum_{k=0}^{+\infty}(c_k - d_k)(z_n - a)^k \qquad (n \geq n_0), \tag{5.25}$$

we find that $0 = h(a) = c_0 - d_0$, whence $c_0 = d_0$.

Now the equality (5.25) can be rewritten in the form

$$0 = h(z_n) = \sum_{k=1}^{+\infty}(c_k - d_k)(z_n - a)^{k-1}. \tag{5.26}$$

Passing to the limit in (5.26), we obtain $c_1 = d_1$. Continuing this process, we find $c_k = d_k$ for all $k \in \mathbb{N} \cup \{0\}$. This means that for all $z \in B_{r_0}(a)$

$$h(z) = 0 \iff f(z) = g(z).$$

2. Consider now an arbitrary point $b \in \Omega$. Let $\gamma(t)$, $t \in [0, 1]$, be a broken line whose trace E_γ is contained in Ω, starting at the point $a = \gamma(0)$ and ending at the point $b = \gamma(1)$.

Denote by $\delta := \text{dist}(E_\gamma, \partial\Omega)$ (the distance from E_γ to the boundary of Ω). It is obvious that $\delta > 0$. In what follows, we assume that the number r_0 chosen in the first item of the proof satisfies the inequality $r_0 < \delta$.

Since the set E_γ is compact, there are finitely many disks $\{B_{r_0}(a_j)\}_{j=0}^m$ covering E_γ, and we can assume that the center of the next disk is contained in the previous one and $a_0 = a,\ a_m = b$.

According to the just proved $h(z) \equiv 0$ for all $z \in B_{r_0}(a_0)$. Since the point $a_1 \in B_{r_0}(a_0)$, there exists an infinite sequence $\{z_n^{(1)}\}_{n\in\mathbb{N}}$ of distinct points in $B_{r_0}(a_1) \cap B_{r_0}(a_0)$ converging to a_1 and $h(z_n^{(1)}) = 0$ for all $n \in \mathbb{N}$. Similarly, as in the previous item, we show that $h \equiv 0$ in $B_{r_0}(a_1)$.

Continuing this procedure, after a finite number of steps we get $h(z) = 0$ for all $z \in B_{r_0}(b)$.

□

Based on this theorem, we can immediately conclude about the structure of the set of zeros $Z := \{z \in \Omega : f(z) = 0\}$ of an analytic function $f : \Omega \mapsto \mathbb{C}$.

Corollary 5.6 *Let $f \in \mathcal{A}(\Omega)$ and $f \not\equiv 0$. Then the following two cases are possible:*

(1) the set Z consists only of a finite number of isolated points;
(2) the set Z is countable, and if a is the limit point of E, then $a \in \partial\Omega$.

To prove this corollary, we should apply Theorem 5.12 to the functions f and $g \equiv 0$.

Let us consider an example showing that the second assertion of Corollary 5.6 is incorrect in real analysis.

Example 5.9 (Counterexample from Mathematical Analysis) Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \in (-1, 1) \setminus \{0\}; \\ 0, & x = 0. \end{cases}$$

This function is differentiable on the interval $(-1, 1)$ and is not identical to zero. However, the set of its zeros is countable and the limit point of the zeros, the point 0, belongs to $(-1, 1)$. ▲

Using the uniqueness theorem, one can prove functional identities in the complex plane, which are valid for real numbers. For example, let us show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$. Consider the entire functions

$$f(z) = \sin^2 z + \cos^2 z - 1 \quad \text{and} \quad g \equiv 0.$$

Since $f(x) = 0$ for all $x \in \mathbb{R}$, then by Theorem 5.12 $f(z) = 0$ for all $z \in \mathbb{C}$.

The next theorem shows the behavior of an analytic function in a neighborhood of its isolated zero.

Theorem 5.13 *Let $f \in \mathcal{A}(\Omega)$, $f \not\equiv 0$, and $z_0 \in \Omega$. If the point z_0 is a zero of f, then in some neighborhood of it the function f can be uniquely represented as*

$$f(z) = (z - z_0)^m \varphi(z), \tag{5.27}$$

where $m \in \mathbb{N}$ and the function φ is analytic and not equal to zero in this neighborhood.

Proof Consider a disk $B_R(z_0) \subset \Omega$. By Theorem 5.3,

$$f(z) = \sum_{k=0}^{+\infty} c_k (z - z_0)^k \quad \text{for all } z \in B_R(z_0). \tag{5.28}$$

Since $f \not\equiv 0$, not all coefficients in (5.28) are zero. Hence, there is a unique natural number m such that

$$c_0 = c_1 = \ldots = c_{m-1} = 0 \quad \text{and} \quad c_m \neq 0. \tag{5.29}$$

Then

$$f(z) = \sum_{k=m}^{+\infty} c_k (z - z_0)^k = (z - z_0)^m \sum_{n=0}^{+\infty} c_{n+m} (z - z_0)^n, \qquad z \in B_R(z_0).$$

Define the function

$$\varphi(z) := \sum_{n=0}^{+\infty} c_{n+m} (z - z_0)^n, \quad z \in B_R(z_0).$$

It is clear that as the sum of the power series the function $\varphi \in \mathcal{A}(B_R(z_0))$ and $\varphi(z_0) = c_m \neq 0$. This means that there exists such a number $r \in (0, R)$ that $\varphi(z) \neq 0$ for all $z \in B_r(z_0)$.

Thus, the function f has the form (5.27) in the disk $B_r(z_0)$. □

Definition 5.5 The number m in (5.27) is called the order (multiplicity) of the zero z_0 of the function f. If $m = 1$, then z_0 is called a simple zero of f.

The representation (5.27) shows that an analytic function can be factorized in a neighbourhood of its isolated zero like polynomials (see Theorem 5.14). Factorization of entire functions in the complex plane will be discussed in Sect. 7.6.

The following consequences follow from the proof of Theorem 5.13, namely from (5.29) and formulas (5.15).

Corollary 5.7 *Let $z_0 \in \Omega$ be a zero of multiplicity m for an analytic function f in Ω. Then*

$$f^{(k)}(z_0) = 0 \quad \textit{for all } k \in \{0, 1, \ldots, m-1\} \quad \textit{and } f^{(m)}(z_0) \neq 0.$$

Corollary 5.8 *An analytic function that is not identical to zero cannot have zeros of infinite order (a point z_0 is a zero of infinite order of an analytic function f, if $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}_0$).*

Example 5.10 (Counterexample) Consider the following function:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

It is easy to verify that $f \in C^\infty(\mathbb{R})$, $f \not\equiv 0$, but for all $n \in \mathbb{N}_0$: $f^{(n)}(0) = 0$. This counterexample shows that an infinitely differentiable non-zero real-valued function can have zeros of infinite order.

Exercise 5.6 Let $f, g \colon \Omega \mapsto \mathbb{C}$ be two analytical functions. Prove that if there is a point z_0 in the domain Ω such that

$$\frac{d^n f}{dz^n}(z_0) = \frac{d^n g}{dz^n}(z_0) \quad \text{for all} \;\; n \in \mathbb{N}_0,$$

then $f \equiv g$ in Ω.

Corollary 5.7 is useful for finding multiplicity of zeros of analytic functions.

Example 5.11 Find the order of the zero $z_0 = 1$ of the function

$$f(z) = \sin(z-1) - z + 1.$$

Direct calculations give that $f(1) = 0,\ f'(1) = 0,\ f''(1) = 0,\ f'''(1) = -1 \neq 0.$ So the multiplicity of this zero is 3.

In this case, it is easy to factorize this function

$$f(z) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(z-1)^{2n-1}}{(2n-1)\,!} - (z-1) = (z-1)^3 \varphi(z),$$

where $\varphi(z) = \sum_{n=2}^{+\infty} (-1)^{n-1} \frac{(z-1)^{2n-4}}{(2n-1)\,!}$, $\varphi(1) = -\frac{1}{6} \neq 0$. Thus, $z_0 = 1$ is indeed the third order zero of the function f.

Now we will prove a stronger assertion than in the fundamental theorem of algebra (Theorem 5.5).

Theorem 5.14 *The polynomial*

$$P_n(z) = z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0,$$

where $\{c_j\}_{j=0}^{n-1} \subset \mathbb{C}$, $n \in \mathbb{N}$, *has, counted with multiplicity, exactly n zeros.*

In addition, the following factorization of P_n *is valid:*

$$P_n(z) = (z - a_1)^{m_1} \cdot \ldots \cdot (z - a_k)^{m_k} \quad \textit{for all } z \in \mathbb{C}, \tag{5.30}$$

where $\{a_1, \ldots, a_k\}$ *are distinct zeros of the polynomial* P_n *of multiplicity* $m_1, \ldots,$ m_k*, respectively, and* $m_1 + \ldots + m_k = n$.

Proof Theorem 5.5 implies that P_n has as least one zero; denote it by a_1. Then, according to Theorem 5.13 and the Euclidean division of polynomials, there exists a unique number $m_1 \in \mathbb{N}$ and a polynomial P_{n-m_1} of degree $n - m_1$ such that

$$P_n(z) = (z - a_1)^{m_1} P_{n-m_1}(z), \quad \text{and} \ \ P_{n-m_1}(a_1) \neq 0.$$

If $n - m_1 \geq 1$, then again thanks to Theorem 5.5 there is a zero a_2 ($a_2 \neq a_1$) of the polynomial P_{n-m_1}. Repeating the previous considerations , we get

$$P_n(z) = (z - a_1)^{m_1} (z - a_2)^{m_2} P_{n-m_1-m_2}(z),$$

where $P_{n-m_1-m_2}(a_p) \neq 0$ for $p \in \{1, 2\}$. Continuing these considerations, we obtain (5.30). □

Definition 5.6 Let f be an analytic function at ∞ (see Definition 2.7). The point at infinity is called a zero of order m of the function f, if $z_0 = 0$ is a zero of order m of the function $g(z) = f\left(\frac{1}{z}\right)$.

Recall that by definition $f(\infty) = \lim_{z \to \infty} f(z)$.

Theorem 5.15 *Let* f *be an analytic function at* ∞ *and let* f *have a zero of order* m *at* ∞*. Then there exists a number* $r_1 > 0$ *and a unique function* $\psi \in \mathcal{A}\big(B_{r_1}(\infty)\big)$, $\psi(z) \neq 0$ *for all* $z \in B_{r_1}(\infty)$*, such that*

$$f(z) = \frac{\psi(z)}{z^m} \quad \textit{for all } \ z, \ |z| > r_1.$$

Proof By Definition 5.6, the point $z_0 = 0$ is a zero of multiplicity m of the function $g(z) = f\left(\frac{1}{z}\right)$. Then, due to Theorem 5.13, there exists a disk $B_r(0)$ and a unique function $\varphi \in \mathcal{A}(B_r(0))$, $\varphi(z) \neq 0$ for all $z \in B_r(0)$, such that

$$g(z) = z^m \varphi(z) \quad \text{for all } \ z \in B_r(0).$$

Returning to the function f, we find

$$f(z) = g\left(\tfrac{1}{z}\right) = \frac{1}{z^m}\,\varphi\left(\tfrac{1}{z}\right) \quad \text{for all } z,\ |z| > r_1,\ \text{where } r_1 = \tfrac{1}{r}.$$

It remains now to denote $\psi(z) := \varphi\left(\frac{1}{z}\right)$. □

6 Laurent Series: Isolated Singularities of Analytic Functions

Abstract

In this chapter, we continue the study of power series, but already their generalizations, namely power series containing terms $(z - z_0)^n$ with a negative integer n. These series were introduced by the French mathematician Pierre Laurent (1813–1854) in 1843. Laurent series are a valuable tool for studying the behavior of analytic functions near their isolated singularities, a classification of which is given here. It is noteworthy that, knowing the behavior of an analytic function near its singular points, one can determine its behavior in the entire domain, as well as calculate other characteristics associated with that function. As a result, it became possible to classify analytic functions according to their isolated singularities (Sect. 6.5). Interestingly, Laurent series have an equivalent relationship to Fourier series (Sect. 6.2), which have real applications in engineering (signal processing, spectroscopy, computer tomography, and many others).

6.1 Expansion of an Analytic Function Into a Laurent Series

It is often possible to find a series representation for a function involving both positive and negative powers of z. Consider, for example, the function

$$f(z) = \frac{1}{(1-z)(z+2)}, \qquad z \in \Omega := \{z: \ 1 < |z| < 2\}.$$

It can be presented as

$$f(z) = \frac{1}{3}\left(\frac{1}{1-z} + \frac{1}{z+2}\right), \qquad z \in \Omega := \{z: \ 1 < |z| < 2\}.$$

T. Mel'nyk, *Complex Analysis*, https://doi.org/10.1007/978-3-031-39615-1_6

If $|z| > 1$, then the formula for the sum of an infinite geometric progression gives us

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{+\infty} \frac{1}{z^{n+1}} = -\sum_{n=1}^{+\infty} z^{-n} = -\sum_{n=-1}^{-\infty} z^n.$$

If $|z| < 2$, then in the same way we derive

$$\frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} z^n.$$

As a result,

$$f(z) = \frac{1}{3}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} z^n - \sum_{n=-1}^{-\infty} z^n\right) =: \sum_{n=-\infty}^{+\infty} c_n z^n \quad \text{for all } z \in \Omega, \tag{6.1}$$

where

$$c_n = \begin{cases} \dfrac{(-1)^n}{3 \cdot 2^{n+1}}, & n \in \mathbb{Z}, \quad n \geq 0; \\ -\dfrac{1}{3}, & n \in \mathbb{Z}, \quad n < 0. \end{cases}$$

Thus, the function f, which is analytic in Ω, has been expanded into a series in integer powers of z. Such series are often called generalized power series or Laurent series. Let us give a rigorous definition of such a series.

Definition 6.1 Let $z_0 \in \mathbb{C}$ and $\{c_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$. Laurent's series around the point z_0 is called the following expression:

$$(\mathtt{L}): \quad \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n.$$

The series $(\mathtt{L})$ is called convergent at a point z, if the functional series

$$(\mathtt{R}): \quad \sum_{n=0}^{+\infty} c_n (z - z_0)^n$$

and

$$(\mathtt{P}): \quad \sum_{n=-1}^{-\infty} c_n (z - z_0)^n = \sum_{n=1}^{+\infty} c_{-n} \frac{1}{(z - z_0)^n}$$

converge at z, and the sum of the Laurent series (L) is defined as follows:

$$\sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n := \sum_{n=0}^{+\infty} c_n (z-z_0)^n + \sum_{n=1}^{+\infty} c_{-n} \frac{1}{(z-z_0)^n}.$$

The series (R) is named the regular part, and the series (P) is named the principal part of the Laurent series (L). ▲

Let us determine the sets of convergence of the series (R) and (P). The series (R) is a power series, so it is convergent in the disk $\{z : |z-z_0| < R\}$, where the number R is determined with the formula (5.4).

If we make the substitution $\eta = \frac{1}{z-z_0}$, then the series (P) is reduced to the power series $\sum_{n=1}^{+\infty} c_{-n}\, \eta^n$, and therefore the series (P) converges in the set $\{z : |z-z_0| > r\}$, where $r = \limsup_{n\to+\infty} \sqrt[n]{|c_{-n}|}$. Thus,

- if $r > R$, then by Definition 6.1 the series (L) is divergent;
- if $r = R$, then the series (L) can be both convergent and divergent at points on the circle $\{z\colon |z-z_0| = r\}$;
- if the inequality $r < R$ holds, then the series (L) converges in the annulus $\{z\colon r < |z-z_0| < R\}$.

Definition 6.2 The annulus $A := \{z :\ 0 \le r < |z-z_0| < R\}$, the inner radius of which is determined by the formula

$$r = \limsup_{n\to+\infty} \sqrt[n]{|c_{-n}|}, \tag{6.2}$$

and the outer radius by the formula

$$R = \left(\limsup_{n\to+\infty} \sqrt[n]{|c_n|}\right)^{-1} \tag{6.3}$$

is called the annulus of convergence of the Laurent series (L). ▲

The following statement follows from the Cauchy–Hadamard Theorem 5.2.

Proposition 6.1 *For the Laurent series* (L) *the following assertions hold:*

(1) the series (L) *absolutely converges at each point of the annulus A;*
(2) the series (L) *diverges at each point of the set $\mathbb{C} \setminus \overline{A}$;*
(3) the series (L) *can be both convergent and divergent at points on ∂A;*
(4) for any compact set $K \subset A$ the series (L) *converges uniformly on K.*

Again using Cauchy's integral formula, as in Theorem 5.3, we prove that an analytic function in an annulus can be represented as a Laurent series.

Theorem 6.1 *Let* $0 \le \rho_1 < \rho_2 \le +\infty$ *and* $A_{\rho_1,\rho_2} := \{z\colon \rho_1 < |z - z_0| < \rho_2\}$, *where* $z_0 \in \mathbb{C}$. *If* f *is an analytic function in the annulus* A_{ρ_1,ρ_2}, *then*

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n \quad \text{for all } z \in A_{\rho_1,\rho_2}, \tag{6.4}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\{|z-z_0|=\rho\}^+} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \qquad n \in \mathbb{Z}, \quad \forall\, \rho \in (\rho_1, \rho_2). \tag{6.5}$$

Proof Fix any $z \in A_{\rho_1,\rho_2}$. Obviously, there are numbers $r_1 > 0$ and $R_1 > 0$ such that $\rho_1 < r_1 < |z - z_0| < R_1 < \rho_2$. Then $\overline{A_{r_1,R_1}} \subset A_{\rho_1,\rho_2}$ and the function $f \in \mathcal{A}\big(\overline{A_{r_1,R_1}}\big)$. By Cauchy's integral formula (Theorem 4.8)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial^+ A_{r_1,R_1}} \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{\partial^+ B_{R_1}(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \int_{\partial^+ B_{r_1}(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi. \end{aligned} \tag{6.6}$$

The appearance of a minus before the last integral is explained in the same way as in the equality (4.23).

1. For any $\xi \in \partial B_{R_1}(z_0)$

$$\frac{|z - z_0|}{|\xi - z_0|} = \frac{|z - z_0|}{R_1} =: q < 1.$$

Therefore,

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}},$$

moreover, this series converges uniformly with respect to $\xi \in \partial B_{R_1}(z_0)$. This means that the series

$$\frac{1}{2\pi i} \frac{f(\xi)}{(\xi - z)} = \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \cdot (z - z_0)^n,$$

which obtained from the previous one by the multiplication with the bounded function $\frac{1}{2\pi i} f(\xi)$, $\xi \in \partial B_{R_1}(z_0)$, converges uniformly on the same circle as well. Hence, this series can be integrated term by term and we obtain

$$\frac{1}{2\pi i} \int_{\partial^+ B_{R_1}(z_0)} \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=0}^{+\infty} c_n (z - z_0)^n, \tag{6.7}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial^+ B_{R_1}(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi, \quad n \in \mathbb{Z},\ n \geq 0. \tag{6.8}$$

2. For $\xi \in \partial B_{r_1}(z_0)$ we have

$$\frac{|\xi - z_0|}{|z - z_0|} = \frac{r_1}{|z - z_0|} =: q_1 < 1.$$

And therefore,

$$\frac{1}{\xi - z} = -\frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\xi - z_0}{z - z_0}} = -\sum_{n=0}^{+\infty} \frac{(\xi - z_0)^n}{(z - z_0)^{n+1}},$$

and this series converges uniformly on $\partial B_{r_1}(z_0)$, as well as the series

$$\frac{1}{2\pi i} \frac{f(\xi)}{(\xi - z)} = -\sum_{n=1}^{+\infty} \frac{1}{2\pi i} f(\xi) \cdot (\xi - z_0)^{n-1} \cdot (z - z_0)^{-n};$$

here we shifted the summation index. Integrating this equality term by term and changing the summation index to the opposite, we get

$$-\frac{1}{2\pi i} \int_{\partial^+ B_{r_1}(z_0)} \frac{f(\xi}{\xi - z}\, d\xi = \sum_{n=-1}^{-\infty} c_n (z - z_0)^n, \tag{6.9}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial^+ B_{r_1}(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi, \quad n \in \mathbb{Z},\ n < 0. \tag{6.10}$$

From (6.6), (6.7) and (6.9) follows the representation (6.4). Based on Theorem 4.5, the equalities (6.8) and (6.10) imply (6.5).

□

Just as the Cauchy inequalities were proved for the coefficients of power series, we prove the Cauchy inequalities for the coefficients of Laurent series.

Corollary 6.1 *Let $f \in \mathcal{A}(A_{\rho_1,\rho_2})$ and*

$$\exists\, \rho_0 \in (\rho_1, \rho_2), \quad \exists\, M > 0 \quad \text{such that} \quad \max_{z \in \{\xi:\ |\xi - z_0| = \rho_0\}} |f(z)| \le M.$$

Then, $|c_n| \le \dfrac{M}{\rho_0^n}$ for all $n \in \mathbb{Z}$.

In general, the integral formulas (6.5) are not practical for calculating Laurent coefficients. Instead, various algebraic techniques are used, such as those described at the beginning of this section. However, to justify the fact that the result is a Laurent series, we need a theorem on the uniqueness of the expansion, which is proved below.

Theorem 6.2 *Let*

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n \quad \text{for all } z \in A = \{z :\ 0 \le r < |z - z_0| < R\}.$$

Then f is an analytic function in the annulus A and the coefficients are determined with the formulas

$$c_n = \frac{1}{2\pi i} \int_{\{|\xi - a| = \rho\}^+} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi, \quad n \in \mathbb{Z}, \tag{6.11}$$

where ρ is an arbitrary number from the interval (r, R).

Proof The analyticity of f in the annulus A follows from the definition of the sum of a Laurent series (see Definition 6.1) and the theorem on the analyticity of the sum of a power series (see Theorem 5.6).

Take any number $\rho \in (r, R)$. Then, according to the fourth item of Proposition 6.1, the series

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n$$

converges uniformly on the circle $\{z\colon |z - z_0| = \rho\}$.

Multiplying this series by $\frac{1}{(z-z_0)^{m+1}}$, where m is an arbitrary integer, we obtain the series

$$\frac{f(z)}{(z - a)^{m+1}} = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^{n-m-1},$$

that converges uniformly on the same circle. Integrating over the positively oriented circle and taking into account the results of Example 4.3, we get

$$\int_{\{|z-z_0|=\rho\}^+} \frac{f(z)}{(z-z_0)^{m+1}}\,dz = \sum_{n=-\infty}^{+\infty} c_n \int_{\{|z-z_0|=\rho\}^+} (z-z_0)^{n-m-1}\,dz = c_m 2\pi i$$

from which the formulas (6.11) follow. □

Based on this theorem we can state that the expansion (6.1) is indeed a Laurent series for the function $\frac{1}{(1-z)(z+2)}$ in the annulus $\{z :\ 1 < |z| < 2\}$.

6.2 Relationship Between Laurent Series and Fourier Series

Let $\varphi \in C^1(\mathbb{R})$ and φ be a 2π-periodic function. It is known from calculus that such a function can be expanded into a Fourier series

$$\varphi(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \big(a_n \cos(nt) + b_n \sin(nt)\big), \quad t \in \mathbb{R}, \tag{6.12}$$

moreover, this series converges uniformly on $\mathbb{R}$, and its coefficients are determined by the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\cos(nt)\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\sin(nt)\,dt.$$

By using the identities

$$\cos(nt) = \frac{e^{int} + e^{-int}}{2} \quad \text{and} \quad \sin(nt) = \frac{e^{int} - e^{-int}}{2i}, \tag{6.13}$$

we re-write the series (6.12) as follows

$$\varphi(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[\frac{a_n - ib_n}{2}e^{int} + \frac{a_n + ib_n}{2}e^{-int}\right] = \sum_{n=-\infty}^{+\infty} c_n e^{int}, \tag{6.14}$$

where

$$c_n = \begin{cases} \dfrac{a_n - ib_n}{2} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \varphi(t)e^{-int}dt, & n \in \mathbb{Z},\ n \geq 0; \\ \dfrac{a_{-n} + ib_{-n}}{2} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \varphi(t)e^{-int}dt, & n \in \mathbb{Z},\ n < 0. \end{cases}$$

The series (6.14) is the Fourier series (6.12) written in complex form.

Let us introduce a new variable $z = e^{it}$, $t \in [-\pi, \pi]$, in (6.14). Then $t = -i \ln z$. Denoting $f(z) := \varphi(-i \ln z)$, we get

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n \quad \text{for all } z, \ |z| = 1,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt = \frac{1}{2\pi i} \int_{\{|z|=1\}^+} \frac{f(z)}{z^{n+1}} dz.$$

This definition of the coefficients $\{c_n\}$ is consistent with the formulas (6.5) for Laurent coefficients. So, we can make the following conclusion.

The Fourier series of a function φ written in complex form is the Laurent series of the function $f(z) := \varphi(-i \ln z)$ on the unit circle centered at the point $z = 0$. Conversely, the Laurent series of any analytic function f restricted to the unit circle (if the convergence annulus contains this circle) is the Fourier series of the function $\varphi(t) := f(e^{it})$, $t \in [-\pi, \pi]$.

Sometimes a Fourier series expansion is easier to obtain through a Laurent series. Consider the following example.

Example 6.1 Expand the function

$$\varphi(t) = \frac{a \sin t}{1 - 2a \cos t + a^2}, \qquad t \in \mathbb{R}, \quad (a \in (-1, 1))$$

into a Fourier series.

Solution Obviously, φ is 2π-periodic. Using (6.13) for $n = 1$, we obtain

$$\varphi(t) = \frac{1}{2i} \frac{a(e^{it} - e^{-it})}{1 - a(e^{it} + e^{-it}) + a^2}.$$

Introducing a new variable $z = e^{it}$, $t \in [-\pi, \pi]$, we deduce

$$f(z) := \varphi(-i \ln z) = \frac{1}{2i} \frac{a\left(z - \frac{1}{z}\right)}{1 - a\left(z + \frac{1}{z}\right) + a^2} = \frac{1}{2i}\left(\frac{1}{1 - az} - \frac{1}{1 - a/z}\right).$$

Using the formula for the sum of an infinite geometric progression, we get

$$\frac{1}{1-az} = \sum_{n=0}^{+\infty} a^n z^n \quad \text{for } |z| < \frac{1}{|a|}, \quad \text{and} \quad \frac{1}{1-a/z} = \sum_{n=0}^{+\infty} \frac{a^n}{z^n} \quad \text{for } |z| > |a|.$$

Hence, the Laurent series for f looks as follows

$$f(z) = \sum_{n=0}^{+\infty} a^n \frac{1}{2i}(z^n - z^{-n}) \quad \text{for all } z,\ |a| < |z| < \frac{1}{|a|}.$$

Its restriction to the unit circle gives the Fourier series

$$\varphi(t) = f(e^{it}) = \sum_{n=0}^{+\infty} a^n \frac{e^{int} - e^{-int}}{2i} = \sum_{n=0}^{+\infty} a^n \sin(nt), \quad t \in [-\pi, \pi].$$

6.3 Isolated Singularities of Analytic Functions

It is well known that discontinuities of a real-valued function of a single real variable can be classified as removable, jump, infinite, or mixed. In this section, we give a classification of isolated singular points of analytic functions, show how one can characterize their type using Laurent series, and study the behavior of analytic functions near their singularities.

Definition 6.3 A point $z_0 \in \overline{\mathbb{C}}$ is a called an isolated singular point (or isolated singularity) of an analytic function f, if there exists a number $R > 0$ such that f is analytic in the punctured disk

$$\breve{B}_R(z_0) := \begin{cases} \{z\colon 0 < |z - z_0| < R\}, & \text{if } z_0 \neq \infty; \\ \{z\colon |z| > R\}, & \text{if } z_0 = \infty; \end{cases}$$

Depending on the behavior of the function f in a punctured neighborhood of the point z_0, three types of isolated singularities are distinguished.

Definition 6.4 (Classification of Singularities) An isolated singularity z_0 of an analytic function f is

- *a removable singularity* if there exists a finite $\lim_{z\to z_0} f(z)$;
- *a pole* if $\lim_{z\to z_0} f(z) = \infty$;
- *an essential singularity* if the limit of f as $z \to z_0$ does not exist.

Example 6.2 Consider examples of different types of isolated singular points:

- the point $z_0 = 0$ is removable for $f(z) = \dfrac{\sin z}{z}$ since $\lim\limits_{z\to 0} f(z) = 1$;
- the point $z_0 = -1$ is a pole for $f(z) = \dfrac{z}{1+z}$ since $\lim\limits_{z\to -1} f(z) = \infty$;
- the point $z_0 = \infty$ is essential for e^z. Indeed, since

$$\lim_{x\to+\infty} e^x = +\infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0,$$

the limit $\lim_{z\to\infty} e^z$ does not exist.

Important
Note that an analytic function can also have non-isolated singular points. For example, the function

$$f(z) = \left(\sin\frac{\pi}{z}\right)^{-1} \tag{6.15}$$

has poles at the points $\left\{a_n = \frac{1}{n}\right\}_{n\in\mathbb{Z}\setminus\{0\}}$, whose limit point is 0. Thus, the point 0 is a non-isolated singular point of f.

Next, we prove theorems showing the connection between the type of an isolated singular point of f and the form of the Laurent series for f around that point.

Theorem 6.3 *Let $f \in \mathcal{A}(\breve{B}_R(z_0))$, $z_0 \in \mathbb{C}$.*

The point z_0 is removable for the function f if and only if the Laurent series of f around z_0 has no principal part, i.e.,

$$f(z) = \sum_{n=0}^{+\infty} c_n(z-z_0)^n \quad \textit{for all } z \in \breve{B}_R(z_0). \tag{6.16}$$

Proof

Necessity Theorem 6.1 implies that f expands into a Laurent series in $\breve{B}_R(z_0)$, whose coefficients are determined by the formulas (6.5).

If $z_0 \in \mathbb{C}$ is removable for f, then there exists finite $\lim_{z\to z_0} f(z)$, and this means that f is bounded in a punctured neighborhood of the point z_0, i.e.,

$$\exists\, R_0 \in (0, R) \quad \exists\, M > 0 \quad \forall\, z \in \breve{B}_{R_0}(z_0): \quad |f(z)| \le M.$$

In virtue of Corollary 6.1, the coefficients of this Laurent series satisfy the inequalities

$$|c_n| \leq \frac{M}{\rho^n}$$

for all $\rho \in (0, R_0)$ and $n \in \mathbb{Z}$. But if n is negative, then

$$|c_n| \leq \frac{M}{\rho^n} \to 0 \quad \text{as} \quad \rho \to 0.$$

Hence, $c_n = 0$ for $n \in \mathbb{Z},\ n < 0$.

Sufficiency If the Laurent series of f around z_0 has no principal part (see (6.16)), then it is a power series and its sum is an analytic function in $B_R(z_0)$. This means that $\lim\limits_{z \to z_0} f(z) = c_0$.

□

The next corollary follows directly from the proof of Theorem 6.3.

Corollary 6.2 *A point $z_0 \in \mathbb{C}$ is removable for an analytic function f if and only if f is bounded in a punctured neighborhood of z_0.*

In addition, the function f can be extended by continuity at its removable singular point z_0, namely we set $f(z_0) := \lim_{z \to z_0} f(z)$, and as a result, f will be analytic in the whole disk $B_r(z_0)$.

Example 6.3 From the first item of Example 6.2 and Corollary 6.2 it follows that the function

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & \text{if } z \neq 0; \\ 1, & \text{if } z = 0, \end{cases} \tag{6.17}$$

is analytic in $\mathbb{C}$. In addition, its Laurent series is

$$f(z) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-2} \quad \text{for all } z \in \mathbb{C}.$$

Theorem 6.4 (About a Pole) *Let $f \in \mathcal{A}(\breve{B}_R(z_0))$ and $z_0 \in \mathbb{C}$.*

The point z_0 is a pole for the function f if and only if the principal part of the Laurent series of f around z_0 contains a finite number of nonzero terms, i.e., there exists a unique positive integer N such that

$$f(z) = \sum_{n=-N}^{+\infty} c_n (z - z_0)^n \quad \text{for all } z \in \breve{B}_R(z_0), \quad \text{and } c_{-N} \neq 0. \tag{6.18}$$

Proof

Necessity We get from Theorem 6.1 that f is expanded into a Laurent series in $\breve{B}_R(z_0)$, whose coefficients are determined by the formulas (6.5).

If z_0 is a pole, then $\lim_{z\to z_0} f(z) = \infty$, and this means that there exists a number $r_0 \in (0, R)$ such that $f(z) \neq 0$ for all $z \in \breve{B}_{r_0}(z_0)$. Hence,

$$\frac{1}{f} =: \varphi \in \mathcal{A}(\breve{B}_{r_0}(z_0)) \quad \text{and} \quad \lim_{z\to z_0} \varphi(z) = 0.$$

Thus, z_0 is a removable point for the function φ, which can be extended by continuity at the point z_0, and as a result, we get the analytic function in the whole disk $B_{r_0}(z_0)$; in addition, $\varphi(z_0) = 0$ and $\varphi(z) \neq 0$ for $z \in \breve{B}_{r_0}(z_0)$.

From Theorem 5.13 (about a zero of an analytic function) it follows that there exists a unique number $N \in \mathbb{N}$ and a unique function $\psi \in \mathcal{A}(B_{r_0}(z_0))$ such that

$$\varphi(z) = (z - z_0)^N \psi(z) \quad \text{and} \quad \psi(z) \neq 0 \quad \text{for all } z \in B_{r_0}(z_0). \tag{6.19}$$

Now consider the function $\frac{1}{\psi}$ that is also analytic in $B_{r_0}(z_0)$, and therefore, by Theorem 5.3, it can be expanded into the power series

$$\frac{1}{\psi(z)} = \sum_{n=0}^{+\infty} b_n (z - a)^n \quad \text{for all } z \in B_{r_0}(z_0), \quad \text{and } b_0 \neq 0. \tag{6.20}$$

From (6.19) and (6.20) we have that

$$f(z) = \frac{1}{(z - z_0)^N} \frac{1}{\psi(z)} = \sum_{n=-N}^{+\infty} c_n (z - z_0)^n, \quad z \in \breve{B}_{r_0}(z_0), \tag{6.21}$$

where $c_n = b_{n+N},\ c_{-N} = b_0 \neq 0$. Due to the uniqueness of the expansion into a Laurent series (Theorem 6.2), the representation (6.21) holds in $\breve{B}_R(z_0)$.

Sufficiency Let the Laurent series of f around z_0 be of the form (6.18). Then

$$\varphi(z) := (z - z_0)^N f(z) = c_{-N} + c_{-N+1}(z - z_0) + \ldots, \quad z \in \breve{B}_R(z_0).$$

By Theorem 6.3, this means that z_0 is removable for the function φ and $\lim_{z\to z_0} \varphi(z) = c_{-N} \neq 0$. Thus,

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{\varphi(z)}{(z - z_0)^N} = \infty,$$

that is, the point z_0 is a pole of the function f.

□

From the proof of Theorem 6.4 it follows the corollary.

Corollary 6.3 *The point $z_0 \in \mathbb{C}$ is a pole of an analytic function f in $\breve{B}_R(z_0)$ if and only if there exists a unique positive integer N and a unique function $\varphi \in \mathcal{A}(B_R(z_0))$, $\varphi \neq 0$ in $B_R(z_0)$, such that*

$$f(z) = \frac{\varphi(z)}{(z - z_0)^N} \quad \textit{for all } \ z \in \breve{B}_R(z_0). \tag{6.22}$$

Definition 6.5 The number N in (6.22) is named the order (multiplicity) of the pole z_0.

Comparing Corollary 6.3 and Theorem 5.13, we get the statement.

Corollary 6.4 *A point $z_0 \in \mathbb{C}$ is a pole of order N of an analytic function f if and only if z_0 is a zero of order N for the function $\frac{1}{f}$.*

Remark 6.1 Simple zero and simple pole are terms used for zeroes and poles of order $N = 1$.

Example 6.4 The function $\dfrac{\sin z}{z^4}$ has a pole of order 3 at the point $z_0 = 0$. Indeed, it can be represented as

$$\frac{\sin z}{z^4} = \frac{f(z)}{z^3} \quad \text{for all } \ z \in \breve{B}_{\frac{\pi}{2}}(0),$$

where the analytic function f is determined in (6.17) and $f \neq 0$ in $B_{\frac{\pi}{2}}(0)$. In addition, its Laurent series is of the form

$$\frac{\sin z}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1}{6}\frac{1}{z}}_{\textit{the principal part}} + \sum_{n=3}^{+\infty} \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-5} \quad \text{for all } \ z \in \mathbb{C} \setminus \{0\}.$$

Theorem 6.3 and Theorem 6.4 provide the following theorem.

Theorem 6.5 (About an Essential Singularity) *Let $f \in \mathcal{A}(\breve{B}_R(z_0))$ and $z_0 \in \mathbb{C}$. The point z_0 is essential for the function f if and only if the principal part of the Laurent series of f around z_0 contains infinitely many nonzero terms.*

Example 6.5 Clearly, the point $z_0 = 0$ is essential for $e^{\frac{1}{z}}$ ($\lim_{x\to 0-} e^{\frac{1}{x}} = 0$ and $\lim_{x\to 0+} e^{\frac{1}{x}} = +\infty$). Its Laurent series around 0 is of the form

$$e^{\frac{1}{z}} = \underbrace{\sum_{n=1}^{+\infty} \frac{1}{n!}\,\frac{1}{z^n}}_{\textit{the principal part}} + 1 \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \tag{6.23}$$

Isolated Singularity at Infinity

Theorems 6.3, 6.4, 6.5 need to be clarified when $z_0 = \infty$. Let $f \in \mathcal{A}(\breve{B}_R(\infty))$. Then the function $\varphi(\omega) := f\left(\frac{1}{\omega}\right)$ is analytic in $\breve{B}_{\frac{1}{R}}(0)$ and expands there into a Laurent series

$$\varphi(\omega) = \sum_{n=-\infty}^{+\infty} b_n\,\omega^n = \underbrace{\sum_{n=-\infty}^{-1} b_n\,\omega^n}_{\textit{the principal part}} + \underbrace{\sum_{n=0}^{+\infty} b_n\,\omega^n}_{\textit{the regular part}}, \quad \omega \in \breve{B}_{\frac{1}{R}}(0).$$

Returning to the variable $z = \frac{1}{\omega}$, we get the Laurent series for f around ∞ :

$$\begin{aligned} f(z) = \varphi\left(\frac{1}{z}\right) &= \underbrace{\sum_{n=-\infty}^{-1} b_n\,\frac{1}{z^n}}_{\textit{the principal part}} + \underbrace{\sum_{n=0}^{+\infty} b_n\,\frac{1}{z^n}}_{\textit{the regular part}} \\ &= \underbrace{\sum_{n=1}^{+\infty} b_{-n}\,z^n}_{\textit{the principal part}} + \underbrace{\sum_{n=-\infty}^{0} b_{-n}\,z^n}_{\textit{the regular part}} = \sum_{n=-\infty}^{+\infty} c_n z^n, \end{aligned} \tag{6.24}$$

where $c_n = b_{-n}$. But now, the series $\sum_{n=1}^{+\infty} c_n z^n$ with positive powers of z is principal part of the Laurent series of f in $\breve{B}_R(\infty)$, and $\sum_{n=-\infty}^{0} c_n z^n$ is the regular one.

Summing up, we can state that the principal part of a Laurent series contains terms that become unbounded when approaching the isolated singularity (it can be both finite and the point at infinity).

Therefore, the Theorems 6.3, 6.4 and 6.5 are reformulated with precision up to the definition of the principal part of a Laurent series around ∞. For example, *the point ∞ is a pole of an analytic function f if and only if the principal part of its Laurent series around ∞ contains a finite number of nonzero terms.*

Example 6.6 Let us clarify the type of isolated singular points of the function $e^{\frac{1}{z}}$. It is easy to see that these are the points 0 and ∞. In Example 6.5 it was shown that 0 is an essential singularity for $e^{\frac{1}{z}}$. The series (6.23) is also the Laurent series of $e^{\frac{1}{z}}$ around ∞. But now the principal part of the Laurent series (6.23) around ∞ is absent, so ∞ is removable for the function $e^{\frac{1}{z}}$. It is also easy to check that $\lim_{z\to\infty} e^{\frac{1}{z}} = 1$.

6.4 Behavior of an Analytic Function Near Its Essential Singularity

The behavior of $f(z)$ is clear as z approaches a removable singularity or a pole of f. But the behavior of f near its essential singularity is “horrible” and needs further study.

Theorem 6.6 (Casorati–Sokhotskyi–Weierstrass Theorem) *Let $f \in \mathcal{A}(\breve{B}_R(z_0))$ and $z_0 \in \mathbb{C}$. If z_0 is essential for f, then for every $\alpha \in \overline{\mathbb{C}}$ there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to+\infty} z_n = z_0 \quad \textit{and} \quad \lim_{n\to+\infty} f(z_n) = \alpha,$$

i.e., the function f approaches any complex number, including ∞, in any neighborhood of z_0.

Proof Let first $\alpha = \infty$. Since the function f cannot be bounded in an arbitrary punctured disk centered at z_0 (see Corollary 6.2), then there exists a point

$$z_1 \in \breve{B}_R(z_0) \quad \text{such that} \quad |f(z_1)| > 1.$$

We use the same arguments to justify the existence of a point

$$z_2 \in \breve{B}_{\frac{R}{2}}(z_0) \quad \text{at which} \quad |f(z_2)| > 2.$$

Continuing in the same way, we find $z_n \in \breve{B}_{\frac{R}{n}}(z_0)$ at which $|f(z_n)| > n$, $n \in \mathbb{N}$. Thus,

$$\lim_{n\to+\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to+\infty} f(z_n) = \infty. \tag{6.25}$$

Now let $\alpha \neq \infty$. Then two cases are possible, namely

1. for any $n \in \mathbb{N}$ there is a point $z_n \in \breve{B}_{\frac{1}{n}}(z_0)$ such that $f(z_n) = \alpha$, but this is what needs to be proven;
2. otherwise, there is a number $r \in (0, R)$ such that $f(z) \neq \alpha$ for all $z \in \breve{B}_r(z_0)$. Thus, we can determine the analytic function

$$\varphi(z) := \frac{1}{f(z) - \alpha}, \quad z \in \breve{B}_r(z_0),$$

for which the point z_0 is also an essential singularity. Then, according to the first item of the proof, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that the limits (6.25) hold (the second one for the function φ). But then also

$$\lim_{n\to+\infty} f(z_n) = \alpha + \lim_{n\to+\infty} \frac{1}{\varphi(z_n)} = \alpha.$$

The theorem is proved. □

Exercise 6.1 Prove the Casorati–Sokhotskyi–Weierstrass theorem in the case when $z_0 = \infty$.

Exercise 6.2 Prove the Casorati–Sokhotskyi–Weierstrass theorem in the case when z_0 is a limit point of the poles, i.e., z_0 is a non-isolated singular point (see, e.g., the function (6.15)).

Let us present, without proof, a theorem that additionally characterizes the sophisticated behavior of an analytic function in a punctured neighborhood of an essential singular point (in fact, the first case from the proof of Theorem 6.6 takes place).

Theorem 6.7 (Great Picard's Theorem) *Let $z_0 \in \overline{\mathbb{C}}$ be an essential singular point of an analytic function f. Then, in an arbitrary punctured neighborhood of z_0, the function f takes an infinitely times arbitrary complex number, except perhaps one.*

We demonstrate the Great Picard Theorem using the following example.

Example 6.7 The point ∞ is essential for the function e^z (see Example 6.2). It is easy to verify that for every complex number $A \neq 0$

$$e^z = A \quad \Longleftrightarrow \quad z_k = \ln|A| + i(\arg A + 2\pi k), \quad k \in \mathbb{Z}.$$

Thus, $\lim_{k\to+\infty} z_k = \infty$ and $e^{z_k} = A$ for all $k \in \mathbb{Z}$.

6.5 Classification of Analytic Functions with Respect to Their Isolated Singularities: Theorem on a Meromorphic Function

Obviously, the point at infinity is an isolated singular point for any entire function. It turns out that by knowing the type of singularity at ∞ for an entire function, one can determine its form and many other properties.

Lemma 6.1 *If ∞ is removable for an entire function f, then f is constant.*

Proof By Theorem 5.3, the function f can be represented as the sum of the power series centered at zero:

$$f(z) = \sum_{n=0}^{+\infty} c_n\, z^n \quad \text{for all} \ \ z \in \mathbb{C}. \tag{6.26}$$

Due to the uniqueness of the Laurent series expansion (Theorem 6.2), the series (6.26) is also the Laurent series of f around ∞ and $\sum_{n=1}^{+\infty} c_n\, z^n$ is its principal part. Since ∞ is removable for f, the principal part must be absent, i.e., $c_n = 0$ for all $n \in \mathbb{N}$. Thus, $f \equiv c_0$. □

Lemma 6.2 *If ∞ is a pole for an entire function f, then f is a polynomial.*

Proof Since ∞ is a pole for f, the principal part of the Laurent series (6.26) of f around ∞ must contain a finite number of nonzero terms, i.e.,

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m, \quad c_m \neq 0 \quad \text{and} \quad c_n = 0 \quad \text{for all} \ \ n > m.$$

The lemma is proved. □

Definition 6.6 An entire function is called a transcendental entire function if ∞ is its essential singular point.

Example 6.8 Obviously, the following functions are entire transcendental functions: e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$.

From Liouville's Theorem 5.4 it follows that the image of an entire non-constant function must be unbounded. Based on the Great Picard Theorem 6.7 and the fundamental theorem of algebra (Theorem 5.5), the following more stronger statement becomes obvious.

Theorem 6.8 (Little Picard's Theorem) *The domain of values of a non-constant entire function is the whole complex plane, except perhaps for only one point.*

Exercise 6.3 Give a reasoned answer to the question:
Is there a non-constant analytic function f defined on $\mathbb{C} \setminus \{i\}$ which maps $\mathbb{C} \setminus \{i\}$ into $\{z\colon\ 0 < |z| < 1\}$?

Now consider an interesting class of analytic functions whose only isolated singularities are poles.

Definition 6.7 A function f is said to be meromorphic in a domain Ω if f is analytic in Ω except for the poles.

Clearly the class of meromorphic functions includes both analytic functions (the set of poles is empty) and rational functions (a ratio of two polynomials).

Remark 6.2 Taking into account that poles are isolated singularities, it follows from Definition 6.7 that

- the set of poles of a meromorphic function is at most countable;
- if the set of poles of a meromorphic function in Ω is countable, then the limit points of this set lie on the boundary of Ω;
- the limit point of poles of a meromorphic function in $\mathbb{C}$ can only be ∞;
- a meromorphic function in $\overline{\mathbb{C}}$ has a finite number of poles, and ∞ is either its removable point or a pole.

Example 6.9 Obviously, the functions

$$\tan(z) \quad \text{and} \quad \cot(z)$$

are meromorphic in the complex plane $\mathbb{C}$, each of which has a countable set of poles. The function $f(z) = \left(\sin \frac{\pi}{z}\right)^{-1}$ is not meromorphic in $\mathbb{C}$, since the origin is its non-isolated singularity (see (6.15)).

Similar as for entire functions, one can sometimes establish the form of a meromorphic function, knowing the structure of the set of its poles.

Theorem 6.9 *If f is a meromorphic function in $\overline{\mathbb{C}}$, then f is rational.*

Proof It follows from the theorem condition and the fourth point of Remark 6.2 that f has a finite number of poles $\{a_1, \ldots, a_n\}$ and ∞ is either a removable point or a pole for it.

By Theorem 6.4, the principal part of the Laurent series of f around the pole a_k has a finite number of nonzero terms; we denote this principal part as follows

$$p_k(z) := \frac{c^{(k)}_{-N_k}}{(z-a_k)^{N_k}} + \ldots + \frac{c^{(k)}_{-1}}{(z-a_k)}, \qquad c^{(k)}_{-N_k} \neq 0.$$

If ∞ is a pole, then we introduce the following notation for of the principal part of the Laurent series of f around ∞:

$$p_0(z) := c_1 z + c_2 z^2 + \ldots + c_N z^N, \qquad c_N \neq 0.$$

Now consider the function $\varphi := f - \sum_{k=0}^{n} p_n$. Since the principal part of the Laurent series of φ around each point of the set $\{a_1, \ldots, a_n, \infty\}$ is absent, Theorem 6.3 implies that the points $a_1, \ldots, a_n, \infty$ are removable for φ. Extending the function φ by continuity at these points (see Corollary 6.2), we obtain an analytic function in $\overline{\mathbb{C}}$. But then it follows from Lemma 6.1 that $\varphi \equiv c_0$, whence

$$f(z) = c_0 + c_1 z + \ldots + c_N z^N + \sum_{k=1}^{n} \left(\frac{c^{(k)}_{-N_k}}{(z-a_k)^{N_k}} + \ldots + \frac{c^{(k)}_{-1}}{(z-a_k)} \right), \qquad (6.27)$$

i.e., f is rational. □

Remark 6.3 The formula (6.27) obtained in the proof shows that an arbitrary rational function can be decomposed into its integer part (polynomial) and the sum of simple fractions.

Now we complete the picture of the behavior of a meromorphic function with a theorem, which is given here without proof.

Theorem 6.10 (Little Picard's Theorem for Meromorphic Functions) *A non-constant meromorphic function f in $\mathbb{C}$ attains every complex number except maybe one or two.*

A number that is not assumed by a meromorphic function is called a *Picard exceptional value* for that function.

Example 6.10 It is easy to check that the Picard exceptional values of the meromorphic function $\tan z$ are the numbers $\pm i$.

Zero is just one Picard exceptional value of the meromorphic function $\frac{1}{1-z}$.

In Sect. 7.6 we will show that every meromorphic function in $\mathbb{C}$ is a ratio of two entire functions.

Residue Calculus

7

Abstract

Just as a person's character is manifested in extreme situations, so the properties of analytic functions are determined by their behavior in isolated singularities. In this chapter, we will illustrate this claim with examples of integral calculations. It turns out that in order to calculate the integral of an analytic function along a curve, it is necessary to determine some values, called *residues*, of that function at its singularities. The reader can appreciate both the power and the simplicity of the residue theory developed by Cauchy for calculating complicated integrals, including integrals of real-valued functions. This theory helps to deduce amazing formulas both for the decomposition of a meromorphic function, e.g. $\cot z$, into an infinite sum of simple fractions that are responsible for its poles, and for the factorization of an entire function, e.g. $\sin z$, into an infinite product of factors that are responsible for its zeros. The theory also supplies a ready-made framework for counting zeros and poles of a given meromorphic function or zeros of an analytic function, in particular, we prove the argument principle and Rouché's theorem.

7.1 Cauchy's Residue Theorem

One of the main theorems of complex analysis is proved here, which opens the door to the calculation of integrals of various kinds.

Definition 7.1 Let f be an analytic function in a punctured disk $\breve{B}_R(a)$, where $a \in \mathbb{C}$. The residue of f at the point a is the number

$$\operatorname*{Res}_{z=a} f(z) := \frac{1}{2\pi i} \int\limits_{\{|z-a|=\rho\}^+} f(z)\, dz, \tag{7.1}$$

T. Mel'nyk, *Complex Analysis*, https://doi.org/10.1007/978-3-031-39615-1_7

where ρ is an arbitrary number from the interval $(0, R)$.

Definition 7.2 Let $f \in \mathcal{A}(\breve{B}_R(\infty))$. The residue of f at ∞ is the number

$$\operatorname*{Res}_{z=\infty} f(z) := -\frac{1}{2\pi i} \int\limits_{\{|z|=\rho\}^+} f(z)\, dz, \tag{7.2}$$

where ρ is an arbitrary number from the interval $(R, +\infty)$.

Remark 7.1 The integrals (7.1) and (7.2) are independent of ρ due to Cauchy's integral Theorem 4.5. By this theorem, $\operatorname{Res}_{z=a} g(z) = 0$ if g is an analytic function in a. Since integration is a linear operation, finding the residue is also a linear operation:

$$\operatorname*{Res}_{z=a} \big(\lambda f(z) + \mu\, g(z)\big) = \lambda \operatorname*{Res}_{z=a} f(z) + \mu \operatorname*{Res}_{z=a} g(z) \quad \text{for all } \lambda,\ \mu \in \mathbb{C}.$$

Theorem 7.1 (Cauchy's Residue Theorem) *Let*

- *the conditions of Theorem 4.8 be satisfied for a bounded domain Ω,*
- *a function f be analytic in the closure of Ω except for a finite number of points $a_1, \ldots, a_p$ from Ω (we denote this as follows: $f \in \mathcal{A}(\overline{\Omega} \setminus \{a_1, \ldots, a_p\})$).*

Then

$$\int_{\partial^+\Omega} f(z)\, dz = 2\pi i \sum_{k=1}^{p} \operatorname*{Res}_{z=a_k} f(z).$$

Proof There is a positive number r such that for all $k \in \{1, \ldots, p\}$

$$\overline{B_r(a_k)} \subset \Omega, \quad \text{and} \quad \overline{B_r(a_k)} \cap \overline{B_r(a_m)} = \emptyset \quad \text{for } k \neq m.$$

Denote by

$$\Omega_r := \Omega \setminus \bigcup_{k=1}^{p} \overline{B_r(a_k)}.$$

By virtue of the theorem conditions, the function $f \in \mathcal{A}(\overline{\Omega}_r)$. Then it follows from Corollary 4.5 that $\int_{\partial^+\Omega_r} f(z)\, dz = 0$ or, taking into account that

$$\int_{\partial^+\Omega_r} f(z)\, dz = \int_{\partial^+\Omega} f(z)\, dz - \sum_{k=1}^{p} \int_{\partial^+ B_r(a_k)} f(z)\, dz,$$

we get

$$\int_{\partial^+\Omega} f(z)\,dz = \sum_{k=1}^{p} \int_{\partial^+ B_r(a_k)} f(z)\,dz \overset{by\,(7.1)}{=} 2\pi i \sum_{k=1}^{p} \operatorname*{Res}_{z=a_k} f(z).$$

The theorem is proved. □

Theorem 7.2 (On the Full Sum of Residues) *Let a function f be analytic in $\mathbb{C}$ except for a finite number of points $\{a_1, a_2, \ldots, a_p\}$, i.e., $f \in \mathcal{A}(\mathbb{C}\backslash\{a_1, a_2, \ldots, a_p\})$. Then*

$$\sum_{k=1}^{p} \mathop{Res}_{z=a_k} f(z) + \mathop{Res}_{z=\infty} f(z) = 0.$$

Proof It is clear that one can pick a positive number R such that $a_k \in B_R(0)$ for all $k \in \{1, \ldots, p\}$. According to the previous theorem,

$$\int_{\partial^+ B_R(0)} f(z)\,dz = 2\pi i \sum_{k=1}^{p} \operatorname*{Res}_{z=a_k} f(z),$$

or

$$\sum_{k=1}^{p} \operatorname*{Res}_{z=a_k} f(z) - \frac{1}{2\pi i} \int_{\partial^+ B_R(0)} f(z)\,dz = 0,$$

from where, remembering Definition 7.2, we have

$$\sum_{k=1}^{p} \operatorname*{Res}_{z=a_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

□

7.2 Formulas for Calculating Residues

To apply the theorems from the previous section, we need to be able to calculate residues. Below we derive the main formulas.

1. Let $f \in \mathcal{A}(\breve{B}_R(a))$ and $a \in \mathbb{C}$. According to Theorem 6.1, the function f expands into a Laurent series in $\breve{B}_R(a)$, whose coefficients $\{c_n\}_{n\in\mathbb{Z}}$ are determined by the formulas (6.5). Recalling Definition 7.1, we conclude that

$$\operatorname*{Res}_{z=a} f(z) = c_{-1}. \tag{7.3}$$

2. Let $f \in \mathcal{A}(\breve{B}_R(\infty))$. The coefficients of the Laurent series of f in $\breve{B}_R(\infty)$ are determined as follows

$$c_n = \frac{1}{2\pi i} \int_{\{|z|=\rho\}^+} \frac{f(z)}{z^{n+1}} dz, \quad n \in \mathbb{Z}, \quad \forall \rho \in (R, +\infty).$$

Based on Definition 7.2, we conclude that

$$\operatorname*{Res}_{z=\infty} f(z) = -c_{-1}. \tag{7.4}$$

3. Let $f \in \mathcal{A}(\breve{B}_R(a))$ and $a \in \mathbb{C}$. If a is removable for the function f, then the principal part of the Laurent series of f around a is absent, i.e., $c_n = 0$ for all negative integer n. Thus, due to (7.3) we have $\text{Res}_{z=a} f(z) = 0$.

Important
If ∞ is a removable singular point of f, then it cannot be asserted that $\text{Res}_{z=\infty} f(z) = 0$, because the coefficient c_{-1} is not included in the principal part of the Laurent series of f around ∞ (see Sect. 6.3).
But if ∞ is a zero of order m for f (see Theorem 5.15) and $m \geq 2$, then $\text{Res}_{z=\infty} f(z) = 0$.

4. Let $f \in \mathcal{A}(\breve{B}_R(a))$, $a \in \mathbb{C}$, and the point a be a pole of order m for f; $m \geq 2$. In this case, in virtue of Theorem 6.4 the Laurent series of f around a has the form

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \ldots + \frac{c_{-1}}{z-a} + \sum_{n=0}^{+\infty} c_n (z-a)^n, \quad z \in \breve{B}_R(a).$$

Multiplying this equality by $(z-a)^m$, we get

$$(z-a)^m f(z) = c_{-m} + c_{-m}(z-a) + \ldots + c_{-1}(z-a)^{m-1} + \sum_{n=0}^{+\infty} c_n (z-a)^{n+m}.$$

Then differentiating the previous equality $(m-1)$ times, we obtain

$$\frac{d^{m-1}}{dz^{m-1}}\Big((z-a)^m f(z)\Big) = (m-1)!\, c_{-1} + \sum_{n=0}^{+\infty} \frac{(n+m)!}{(n+1)!}\, c_n\, (z-a)^{n+1}, \quad z \in \breve{B}_R(a).$$

Passing here to the limit as $z \to a$, we find the coefficient c_{-1}. Taking (7.3) into account, we conclude that the residue of f at the mth order pole a is calculated by the formula

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \Big((z-a)^m f(z) \Big). \tag{7.5}$$

Example 7.1 By Corollary 6.3, the point 1 is a pole of order 2 for the function

$$f(z) = \frac{\cos 2z}{(z-1)^2}.$$

Using (7.5) with $m = 2$, we obtain

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{d}{dz} \Big((z-1)^2 f(z) \Big) = \lim_{z \to 1} \frac{d}{dz} \Big(\cos 2z \Big) = -2 \sin 2.$$

5. If $a \in \mathbb{C}$ is a simple pole of f, then the Laurent series of f is as follows

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{+\infty} c_n (z-a)^n, \quad z \in \breve{B}_R(a).$$

Finding the coefficient c_{-1} from this equality, we have

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z). \tag{7.6}$$

It should be noted that the formula (7.6) coincides with (7.5) for $m = 1$.

6. Let

$$f(z) = \frac{\psi(z)}{\varphi(z)}, \quad z \in \breve{B}_R(a),$$

where the functions φ and ψ are analytic in the disk $B_R(a)$, the point $a \in \mathbb{C}$ is a simple zero of φ and $\psi(a) \neq 0$. Then, obviously, the point a is a simple pole of the function f. Applying the formula (7.6), we find

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) \frac{\psi(z)}{\varphi(z)} = \lim_{z \to a} \frac{\psi(z)}{\frac{\varphi(z)-\varphi(a)}{z-a}} = \frac{\psi(a)}{\varphi'(a)}. \tag{7.7}$$

Example 7.2 The function $\cot z = \frac{\cos z}{\sin z}$ has isolated singularities at the points $z_k = \pi k$, $k \in \mathbb{Z}$. By Corollary 6.3 these points are simple poles since for each $k \in \mathbb{Z}$ the representation

$$\cot z = \frac{\Psi(z)}{z - \pi k}, \qquad z \in \breve{B}_{\frac{\pi}{2}}(\pi k),$$

holds, where the function

$$\Psi(z) = \begin{cases} (-1)^k \cos z \cdot \frac{z-\pi k}{\sin(z-\pi k)}, & \text{if } \ z \in \breve{B}_{\frac{\pi}{2}}(\pi k); \\ 1, & \text{if } \ z = \pi k, \end{cases}$$

is analytic in the disk $B_{\frac{\pi}{2}}(\pi k)$ and is not equal to zero there (see Example 6.3).

By using (7.7), we have

$$\operatorname*{Res}_{z=\pi k} \cot z = \frac{\cos \pi k}{(\sin z)'|_{z=\pi k}} = 1 \quad \text{for all } \ k \in \mathbb{Z}. \tag{7.8}$$

7. Let $f \in \mathcal{A}(\breve{B}_R(\infty))$ and ∞ be a pole of order m for f. Then the Laurent series of the function f around ∞ is given by

$$f(z) = \sum_{n=-\infty}^{0} c_n z^n + \underbrace{c_1 z + \ldots + c_m z^m}_{\textit{the principal part}}, \qquad z \in \breve{B}_R(\infty)$$

(see Sect. 6.3). Consider the function

$$\varphi(z) = f\left(\tfrac{1}{z}\right) = \sum_{n=-\infty}^{-2} c_n \, z^{-n} + c_{-1}\, z + c_0 + \frac{c_1}{z} + \ldots + \frac{c_m}{z^m}, \qquad z \in \breve{B}_{\frac{1}{R}}(0).$$

Similarly, as in the fourth item above, we find the coefficient c_{-1}:

$$\lim_{z\to 0} \frac{d^{m+1}}{dz^{m+1}} \big(z^m \varphi(z)\big) = (m+1)!\, c_{-1}.$$

Considering the formula (7.4), we get

$$\operatorname*{Res}_{z=\infty} f(z) = -\frac{1}{(m+1)!} \lim_{z\to 0} \frac{d^{m+1}}{dz^{m+1}} \Big(z^m f\big(\tfrac{1}{z}\big)\Big). \tag{7.9}$$

Example 7.3 Consider the function

$$f(z) = \frac{2}{z} + 1 + z^2 \tag{7.10}$$

for which ∞ is a pole of order 2.

Due to Theorem 6.2 (on the uniqueness of a Laurent expansion), the right-hand side of (7.10) is the Laurent series of f around ∞, whose principal part equals z^2, and the coefficient $c_{-1} = 2$. Therefore, by using (7.4), we get

$$\operatorname*{Res}_{z=\infty} f(z) = -2.$$

The formula (7.9) gives the same result:

$$\begin{aligned}\operatorname*{Res}_{z=\infty} f(z) &= -\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}\left(z^2\big(2z + 1 + \frac{1}{z^2}\big)\right)\\ &= -\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}\left(2z^3 + z^2 + 1\right) = -2.\end{aligned}$$

8. If an analytic function f is even and the points 0 and ∞ are its isolated singular points, then

$$\operatorname*{Res}_{z=0} f(z) = 0 \quad \text{and} \quad \operatorname*{Res}_{z=\infty} f(z) = 0.$$

Indeed, if we consider, for example, the point 0, then for the function f the following representations hold:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n\, z^n \quad \text{and} \quad f(-z) = \sum_{n=-\infty}^{+\infty} c_n\,(-z)^n, \qquad z \in \breve{B}_R(0),$$

wherefrom, taking into account the evenness of f, we get

$$0 = \sum_{k=-\infty}^{+\infty} c_{2k+1}\, z^{2k+1}, \qquad z \in \breve{B}_R(0). \tag{7.11}$$

By Theorem 6.2, we get from (7.11) that $c_{2k+1} = 0$ for all $k \in \mathbb{Z}$; so $c_{-1} = 0$.

7.3 Methods for Calculating Integrals

The purpose of this section is to present a collection of methods and examples that can be used to calculate various types of integrals, including improper real integrals that often cannot be calculated using methods of real analysis.

Integrals Over Closed Curves

When calculating integrals over closed curves, Cauchy's residue Theorems 7.1 and 7.2 (on the full sum of residues) are used.

Example 7.4 Compute $\displaystyle\int_{\{|z|=2\}^+} \frac{\cos z}{z^3}\, dz =: I.$

Solution For the function

$$f(z) := \frac{\cos z}{z^3}$$

there is only one isolated singularity $z = 0$, which is inside the integration contour—the circle $\{|z| = 2\}$. By Theorem 7.1, the integral $I = 2\pi i \operatorname{Res}_{z=0} f(z)$.

There are a number of ways to find the residue of f at 0. It is ease to see that the point 0 is a third order pole for f, so we can use the formula (7.5) with $m = 3$. However, if it is easy to expand a function into a Laurent series, then it leads to the result faster. In our case

$$\frac{\cos z}{z^3} = \frac{1}{z^3}\sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{(2n)!} = z^{-3} - \frac{z^{-1}}{2} + \sum_{n=2}^{+\infty} \frac{(-1)^n}{(2n)!} z^{2n-3} \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.$$

So, the coefficient $c_{-1} = -\frac{1}{2}$. By (7.3) we have

$$I = 2\pi i \operatorname*{Res}_{z=0} f(z) = -\pi i.$$

Example 7.5 Compute $\displaystyle\int_{\{|z|=2\}^+} \frac{dz}{(z^7+1)^3} =: I.$

Solution The function

$$f(z) := \frac{1}{(z^7+1)^3}$$

has 7 poles, namely

$$a_k = e^{i(\frac{\pi}{7} + \frac{2\pi k}{7})}, \quad k \in \{0, 1, \ldots, 6\},$$

inside the circle $\{|z| = 2\}$, and all these poles have order 3. By Theorem 7.1

$$I = 2\pi i \sum_{k=0}^{6} \operatorname*{Res}_{z=a_k} f(z).$$

In this case, to avoid tedious calculations, it is better to use Theorem 7.2, which gives

$$\sum_{k=0}^{6} \operatorname*{Res}_{z=a_k} f(z) = - \operatorname*{Res}_{z=\infty} f(z).$$

Since

$$f(z) = \frac{1}{(z^7+1)^3} \sim \frac{1}{z^{21}} \quad \text{as } z \to \infty,$$

the point at infinity is a zero of order 21 for f. Therefore, the coefficient c_{-1} of the Laurent series of f around ∞ is equal to 0 (see Important in Sect. 7.2). This means that $I = 0$.

Trigonometric Integrals

Here we show how to calculate definite trigonometric integrals of the form

$$I := \int_0^{2\pi} R\big(\cos(n\varphi), \sin(m\varphi)\big)\, d\varphi, \tag{7.12}$$

where $R(u, v)$ is a rational function of two real variables u and v, i.e., it can be written as a ratio of two polynomials

$$R(u, v) = \frac{Q(u, v)}{P(u, v)} \quad \text{for all } (u, v) \in \mathbb{R}^2,$$

and $P(u, v) \neq 0$ for all (u, v) such that $u^2 + v^2 = 1$.

The trick is to use the substitution $z = e^{i\varphi}$, $\varphi \in [0, 2\pi]$, to transform the integral (7.12) into a complex integral over the unit circle, to which one can use Cauchy's residue theorem.

It is easy to see that the parametrization $z = e^{i\varphi}$, $\varphi \in [0, 2\pi]$, give us the positively oriented circle $\{z\colon |z| = 1\}$. In addition,

$$dz = i\, e^{i\varphi} d\varphi \implies d\varphi = -iz^{-1}\, dz,$$

$$\cos(n\varphi) = \frac{e^{in\varphi} + e^{-in\varphi}}{2} = \frac{z^n + z^{-n}}{2}, \quad \sin(m\varphi) = \frac{z^m - z^{-m}}{2i}.$$

As a result, we get the integral

$$I = \int_{\{|z|=1\}^+} R_1(z)dz,$$

where

$$R_1(z) = \frac{1}{iz} R\left(\tfrac{1}{2}(z^n + z^{-n}), \tfrac{1}{2i}(z^m - z^{-m})\right)$$

is a new rational function of a complex variable, which can be evaluated using the residue theorem.

Example 7.6 Compute the integral

$$I := \int_0^{2\pi} \frac{d\varphi}{1 - 2a\cos\varphi + a^2},$$

where $a \in \mathbb{R}$ and $|a| \neq 1$.

Solution For $a = 0$, the answer is obvious. Therefore, we further assume that $a \neq 0$. After the substitution $z = e^{i\varphi}$, $\varphi \in [0, 2\pi]$, we get

$$I = \int\limits_{\{|z|=1\}^+} \frac{-iz^{-1}dz}{1 - a(z + z^{-1}) + a^2} = -i \int\limits_{\{|z|=1\}^+} \frac{dz}{(a-z)(az-1)}$$

$$= \frac{i}{a} \int\limits_{\{|z|=1\}^+} \frac{dz}{(z-a)(z-\frac{1}{a})}.$$

We see that the integrand

$$f(z) = \frac{1}{(z-a)(z-\frac{1}{a})}$$

has two simple poles at $z = a$ and $z = \frac{1}{a}$, and only one of which is inside of the circle $\{|z| = 1\}$ (it depends on a). Thus, by Theorem 7.1, we have

- if $|a| < 1$, then

$$I = \frac{i}{a} 2\pi i \operatorname*{Res}_{z=a} f(z) = -\frac{2\pi}{a} \lim_{z\to a}(z-a)f(z) = -\frac{2\pi}{a} \lim_{z\to a} \frac{1}{z-\frac{1}{a}} = \frac{2\pi}{1-a^2};$$

- if $|a| > 1$, then

$$I = -\frac{2\pi}{a} \operatorname*{Res}_{z=\frac{1}{a}} f(z) = -\frac{2\pi}{a} \lim_{z\to\frac{1}{a}} \big(z - \tfrac{1}{a}\big) f(z) = \frac{2\pi}{a^2 - 1}.$$

Integrals Along the Real Line

First, let us prove the lemma.

Lemma 7.1 *Let the following conditions be satisfied:*

1. $f \in \mathcal{A}\big(\{z : \operatorname{Im} z \geq 0\}\backslash\{a_1, \ldots, a_m\}\big)$, *where* $\operatorname{Im} a_k > 0$, $k \in \{1, 2, \ldots, m\}$;
2. *there exist positive numbers* M, R_0 *and* $\delta > 0$ *such that for all* z *from the set* $\{\xi : |\xi| > R_0, \ \operatorname{Im} \xi \geq 0\}$

$$|f(z)| \leq \frac{M}{|z|^{1+\delta}}. \tag{7.13}$$

Then

$$\lim_{r\to+\infty} \int_{\gamma_r} f(z)\,dz = 0, \quad \textit{where} \quad \gamma_r = re^{it}, \ t \in [0, \pi].$$

Proof It can be considered that $r > \max\{R_0, |a_1|, \ldots, |a_m|\}$. Then, taking (4.5) and (7.13) into account, we deduce

$$\left|\int_{\gamma_r} f(z)\,dz\right| \leq \int_{\gamma_r} \frac{M}{|z|^{1+\delta}}\,dl = \frac{M}{r^{1+\delta}} \int_{\gamma_r} dl = \frac{M\pi}{r^\delta} \to 0 \quad \text{as } r \to +\infty.$$

The lemma is proved. □

Theorem 7.3 *Let the conditions of Lemma 7.1 be satisfied. Then*

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z). \tag{7.14}$$

Proof First, we note that, based on the second condition of Lemma 7.1, the improper integral $\int_{-\infty}^{+\infty} f(x)\,dx$ converges.

Now let us take $r > \max\{R_0, |a_1|, \ldots, |a_m|\}$ and consider the positively oriented closed contour $[-r, r] \cup \gamma_r$, where $\gamma_r = re^{it}$, $t \in [0, \pi]$. The interior of this contour contains all the isolated singularities $\{a_1, \dots, a_m\}$ of f that are in the upper half-plane. Under the Cauchy residue Theorem 7.1 we have

$$\int_{-r}^{r} f(x)\,dx + \int_{\gamma_r} f(z)\,dz = \int_{[-r,r]\cup\gamma_r} f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z).$$

Letting r tend to infinity in this equality and using Lemma 7.1, we arrive at the formula (7.14). □

Example 7.7 Compute the integral $\displaystyle\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,dx =: I.$

Solution Since the integrand is even,

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,dx.$$

Let us verify the conditions of Lemma 7.1 for the function

$$f(z) = \frac{z^2+1}{z^4+1} = \frac{\psi(z)}{\varphi(z)}, \quad \text{where} \quad \psi(z) = z^2+1, \;\; \varphi(z) = z^4+1.$$

The function f is rational and has poles at the points where the function φ vanishes; they are

$$a_1 = e^{i\frac{\pi}{4}}, \quad a_2 = e^{i\frac{3\pi}{4}}, \quad a_3 = e^{i\frac{5\pi}{4}}, \quad a_4 = e^{i\frac{7\pi}{4}}.$$

Only a_1 and a_2 lie in the upper half-plane $\{z\colon \operatorname{Im} z > 0\}$. They are simple poles because

$$\psi(a_k) \neq 0 \quad \text{and} \quad \varphi'(a_k) = 4\,a_k^3 \neq 0, \qquad k \in \{1, 2, 3, 4\}.$$

If $|z| > 2$, then

$$|f(z)| = \frac{1}{|z|^2}\left|\frac{1+\frac{1}{z^2}}{1+\frac{1}{z^4}}\right| \le \frac{1}{|z|^2}\,\frac{1+\frac{1}{|z|^2}}{1-\frac{1}{|z|^4}} \le \frac{4}{|z|^2}.$$

Hence the second condition of Lemma 7.1 is satisfied ($M = 4$, $R_0 = 2$, $\delta = 1$).

By using (7.14) and (7.7), we find

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,dx = \pi i\Big(\operatorname*{Res}_{z=a_1} f(z) + \operatorname*{Res}_{z=a_2} f(z)\Big)$$

$$= \pi i \Big(\frac{\psi(a_1)}{\varphi'(a_1)} + \frac{\psi(a_2)}{\varphi'(a_2)} \Big) = \pi i \Big(\frac{e^{\frac{i\pi}{2}} + 1}{4e^{\frac{i3\pi}{4}}} + \frac{e^{\frac{i3\pi}{2}} + 1}{4e^{\frac{i9\pi}{4}}} \Big)$$
$$= -\frac{\pi i}{2} \Big(e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}} \Big) = \pi \sin \tfrac{\pi}{4} = \pi \tfrac{\sqrt{2}}{2}.$$

Fourier Transform Type Integrals $\int_{-\infty}^{+\infty} f(x)\, e^{i\lambda x} dx \ (\lambda > 0)$

Such integrals occur in physical and engineering applications, and Jordan's Lemma plays a fundamental role in their computation.

Lemma 7.2 (Jordan's Lemma) *Let the following conditions be satisfied:*

- $f \in \mathcal{A}\big(\{z\colon \operatorname{Im} z \geq 0\}\setminus\big(\{a_1, \ldots, a_m\} \cup \{x_1, \ldots, x_n\}\big)\big)$, *where* $\operatorname{Im} a_k > 0$ *for* $k \in \{1, \ldots, m\}$, *and* $\{x_1, \ldots, x_n\} \subset \mathbb{R}$;
-

$$\max_{z \in E_{\gamma_r}} |f(z)| =: M_r \to 0 \quad \textit{as} \ \ r \to +\infty, \tag{7.15}$$

where $\gamma_r(t) = re^{it}, \ \ t \in [0, \pi]$.

Then

$$\lim_{r \to +\infty} \int_{\gamma_r} f(z)\, e^{i\lambda z}\, dz = 0 \qquad (\lambda > 0). \tag{7.16}$$

Proof Take $r > \max\{|a_1|, \ldots, |a_m|, |x_1|, \ldots, |x_n|\}$. Then the statement of this lemma follows from the following considerations:

$$\left| \int_{\gamma_r} f(z) e^{i\lambda z}\, dz \right| \leq \int_{\gamma_r} |f(z)| \left| e^{i\lambda z} \right| dl \leq M_r \int_0^{\pi} \left| e^{i\lambda r(\cos t + i \sin t)} \right| r\, dt$$
$$= M_r\, r \int_0^{\pi} e^{-\lambda r \sin t} dt = 2M_r r \int_0^{\frac{\pi}{2}} e^{-\lambda r \sin t}\, dt$$
$$\leq 2M_r r \int_0^{\frac{\pi}{2}} e^{-\lambda r \frac{2}{\pi} t}\, dt = \frac{M_r \pi}{\lambda}(1 - e^{-\lambda r}) \to 0 \ \text{ as } \ r \to +\infty.$$

In the last line, we used the obvious inequality $\frac{2}{\pi}\, t \leq \sin t$ for $t \in [0, \frac{\pi}{2}]$. □

Remark 7.2 Comparing the second condition of the Jordan lemma with the second condition of Lemma 7.1, we see that f can tend to zero at infinity more slowly. This is due to the presence of the factor $e^{i\lambda z}$ near f.

Theorem 7.4 *Let the conditions of Lemma 7.2 be satisfied and* $\{x_1, \ldots, x_n\}$ *be simple poles of* f *and* $x_1 < x_2 < \ldots < x_n$. *Then*

$$p.v. \int\limits_{-\infty}^{+\infty} f(x)e^{i\lambda x}\,dx = 2\pi i \sum_{k=1}^{m} \underset{z=a_k}{Res}\, f(z)e^{i\lambda z} + \pi i \sum_{k=1}^{n} \underset{z=x_k}{Res}\, f(z)e^{i\lambda z}. \qquad (7.17)$$

Before proving, we recall that the principal value of an improper integral, which is divergent, is the way in which we assign a finite value to it. For example, $\int_{-\infty}^{+\infty} \frac{dx}{x}$ is divergent, but

$$\lim_{r\to+\infty,\ \varepsilon\to 0} \left(\int_{-r}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{r} \frac{dx}{x} \right) = 0.$$

So, $p.v. \int_{-\infty}^{+\infty} \frac{dx}{x} = 0$. In (7.17)

$$p.v. \int\limits_{-\infty}^{+\infty} f(x)e^{i\lambda x}\,dx := \lim_{\substack{r\to+\infty \\ \varepsilon\to 0}} \int\limits_{J_{r,\varepsilon}} f(x)e^{i\lambda x}\,dx,$$

where $J_{r,\varepsilon} := [-r, r] \setminus \left(\bigcup_{k=1}^{n} (x_k - \varepsilon, x_k + \varepsilon) \right)$.

Proof We choose r as in Jordan's lemma. Let

$$\varepsilon_0 := \min_{\substack{k\in\{1,\ldots,n\} \\ l\in\{1,\ldots,m\}}} |x_k - a_l|, \quad \varepsilon_1 := \min_{k\in\{1,\ldots,n-1\}} |x_{k+1} - x_k|.$$

Then for any $\varepsilon \in (0, \varepsilon_2)$, where $\varepsilon_2 = \min\{\varepsilon_0, \varepsilon_1/2\}$, we have

$$E_{\gamma_k^\varepsilon} \cap E_{\gamma_j^\varepsilon} = \emptyset \quad \text{for } k \neq j,$$

where $\gamma_k^\varepsilon(t) = x_k - \varepsilon e^{-it},\ t \in [0, \pi],\ k \in \{1, \ldots, n\}$.

Now consider the closed positively oriented contour

$$C_{r,\varepsilon} := \gamma_r \bigcup J_{r,\varepsilon} \bigcup \left(\bigcup_{k=1}^{n} \gamma_k^\varepsilon \right),$$

where the upper semicircle γ_r is defined in Jordan's lemma. The interior of $C_{r,\varepsilon}$ contains only the isolated singularities $\{a_1, \dots, a_m\}$ of the function f; the poles $\{x_1, \dots, x_n\}$ are in the exterior.

The Cauchy residue Theorem 7.1 says that

$$\int_{C_{r,\varepsilon}} f(z)e^{i\lambda z}\, dz = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z)e^{i\lambda z}, \tag{7.18}$$

Considering (7.16) and passing to the limit in (7.18) as $r \to +\infty$ and $\varepsilon \to 0$, we find

$$p.v. \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}\, dx = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} f(z)e^{i\lambda z} - \sum_{k=1}^{n} \lim_{\varepsilon\to 0} \int_{\gamma_k^\varepsilon} f(z)e^{i\lambda z}\, dz. \tag{7.19}$$

It remains to find the limits in the right-hand side of (7.19). Since for any $k \in \{1, \dots, n\}$ the point x_k is a simple pole of the function $f(z)e^{i\lambda z}$, its Laurent series is as follows

$$f(z)e^{i\lambda z} = \frac{c_{-1}^{(k)}}{z - x_k} + \sum_{j=0}^{+\infty} c_j^{(k)}(z - x_k)^j, \quad z \in \breve{B}_{\delta_k}(x_k),$$

where δ_k is a positive number. Denote by g_k the sum of the power series $\sum_{j=0}^{+\infty} c_j^{(k)}(z - x_k)^j$. Due to Theorem 5.6 the function g_k can be considered analytic in the closed disk $\overline{B_{\delta_k}(x_k)}$.

In what follows, we assume that $\varepsilon < \min\{\delta_1, \dots, \delta_n\}$. Using (7.3), we get

$$\int_{\gamma_k^\varepsilon} f(z)e^{i\lambda z}\, dz = \int_{\gamma_k^\varepsilon} \frac{c_{-1}^{(k)}}{z - x_k} dz + \int_{\gamma_k^\varepsilon} g_k(z)\, dz = \int_0^\pi \frac{c_{-1}^{(k)}}{-\varepsilon e^{-it}}\, \varepsilon i e^{-it}\, dt + \int_{\gamma_k^\varepsilon} g_k(z)\, dz$$

$$= -\pi i\, c_{-1}^{(k)} + \int_{\gamma_k^\varepsilon} g_k(z)\, dz = -\pi i \operatorname*{Res}_{z=x_k} f(z)e^{i\lambda z} + \int_{\gamma_k^\varepsilon} g_k(z)\, dz. \tag{7.20}$$

Since for any $k \in \{1, \dots, n\}$ the function $g_k \in \mathcal{A}(\overline{B_{\delta_k}(x_k)})$, there exists a positive constant M_k such that for all $z \in B_{\delta_k}(x_k)$ the inequality $|g_k(z)| \le M_k$ holds. Then

$$\left| \int_{\gamma_k^\varepsilon} g_k(z)\, dz \right| \le \int_{\gamma_k^\varepsilon} |g_k(z)|\, dl \le M_k\, \pi\, \varepsilon \ \to\ 0 \quad \text{as} \quad \varepsilon \to 0. \tag{7.21}$$

Based on (7.20) and (7.21), from (7.19) the equality (7.17) follows. □

Example 7.8 Compute the integral

$$p.v.\int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{x}\,dx =: I \qquad (\lambda > 0).$$

Solution It is easy to check that the function $f(z) = \frac{1}{z}$ satisfies the conditions of Jordan's lemma and the point 0 is its only simple pole. Therefore, from (7.17) and (7.6), we have

$$I = \pi i \operatorname*{Res}_{z=0} \frac{e^{i\lambda z}}{z} = \pi i \lim_{z\to 0} e^{i\lambda z} = \pi i.$$

Using this result, we simply evaluate the Dirichlet integral

$$\int_0^{+\infty} \frac{\sin\lambda x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin\lambda x}{x}\,dx = \frac{1}{2}\operatorname{Im}\left(p.v.\int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{x}\,dx\right) = \frac{\pi}{2}.$$

A key point here is the replacement $\sin\lambda x$ by $e^{i\lambda x}$ and the identity

$$\sin\lambda x = \frac{1}{2}\operatorname{Im} e^{i\lambda x}.$$

Remark 7.3 It is known from mathematical analysis that an antiderivative of the integrand in the Dirichlet integral is not an elementary function. Nevertheless, the value of the integral can be obtained in various (not always short and simple) ways, including double integration and differentiation under the integral sign. In Example 7.8, the calculation of the Dirichlet integral takes one line!

Example 7.9 Compute $\displaystyle\int_{-\infty}^{+\infty} \frac{(x-1)\cos 5x}{x^2-2x+5}\,dx =: I.$

Solution As before, using the Euler formula, we get

$$I = \operatorname{Re}\left(p.v.\int_{-\infty}^{+\infty} \frac{(x-1)}{x^2-2x+5} e^{5ix}\,dx\right).$$

The function

$$f(z) = \frac{z-1}{z^2-2z+5} \sim \frac{1}{z} \quad \text{as} \quad z \to \infty,$$

it has two singular points $a_1 = 1 + 2i$ and $a_2 = 1 - 2i$. They are simple poles and only a_1 lies in the upper half-plane. So, all conditions of Theorem 7.4 are satisfied and it implies that

$$I = \operatorname{Re}\left(2\pi i \operatorname*{Res}_{z=a_1} \frac{z-1}{(z-a_1)(z-a_2)} e^{5iz}\right)$$
$$= 2\pi \operatorname{Re}\left(i\,\frac{a_1-1}{a_1-a_2} e^{5ia_1}\right) = -e^{-10}\pi \sin 5.$$

7.4 Argument Principle: Rouché's Theorem and Its Applications

In this section, we consider applications that help to count the zeros and poles of a given meromorphic function. Let f be meromorphic in a domain D.

Definition 7.3 The function $\frac{f'}{f}$, where it is defined, is called the logarithmic derivative of f.

Remark 7.4 This definition is justified by the fact that

$$\frac{d}{dz}\big(\log_k f(z)\big) = \frac{f'(z)}{f(z)},$$

where $\log_k$ is any branch of the multi-valued function Log (see (3.9)).

Let a be a zero of order n for the function f. According to Theorem 5.13 (on the zero of an analytic function), there is a positive number δ and a unique function $\varphi \in \mathcal{A}(B_\delta(a))$ such that

$$f(z) = (z-a)^n \varphi(z) \quad \text{and} \quad \varphi(z) \neq 0 \quad \text{for all } z \in B_\delta(a).$$

Using this representation, we find that for all $z \in \breve{B}_\delta(a)$

$$\frac{f'(z)}{f(z)} = \frac{n(z-a)^{n-1}\varphi(z) + (z-a)^n \varphi'(z)}{(z-a)^n \varphi(z)} = \frac{n}{z-a} + \frac{\varphi'(z)}{\varphi(z)},$$

whence, remembering Remark 7.1, we get

$$\operatorname*{Res}_{z=a} \frac{f'(z)}{f(z)} = n. \tag{7.22}$$

Let b be a pole of order p for f. Corollary 6.3 (on the pole of an analytic function) says there is a positive number δ_1 and a unique analytic nonzero function ψ in $B_{\delta_1}(b)$ such that

$$f(z) = \frac{\psi(z)}{(z-b)^p} \quad \text{for all } z \in \breve{B}_{\delta_1}(b).$$

As before, we find for all $z \in \breve{B}_{\delta_1}(b)$

$$\frac{f'(z)}{f(z)} = \frac{-\frac{p}{(z-b)^{p+1}}\psi(z) + \frac{1}{(z-b)^p}\psi'(z)}{\frac{1}{(z-b)^p}\psi(z)} = -\frac{p}{z-b} + \frac{\psi'(z)}{\psi(z)},$$

from where

$$\operatorname*{Res}_{z=b} \frac{f'(z)}{f(z)} = -p. \tag{7.23}$$

Let us now denote by $\{a_k\}$ the set of zeros, and by $\{b_k\}$ the set of poles of the function f. Consider a bounded domain Ω, whose boundary is the union of a finite number of pairwise disjoint Jordan curves, such that

$$\overline{\Omega} \subset D,$$
$$\partial\Omega \cap \{a_k\} = \emptyset, \qquad \partial\Omega \cap \{b_k\} = \emptyset,$$
$$\Omega \cap \{a_k\} = \{a_1, \dots, a_q\}, \qquad \Omega \cap \{b_k\} = \{b_1, \dots, b_m\}.$$

Under these assumptions,

$$\frac{f'}{f} \in \mathcal{A}\big(\overline{\Omega} \setminus (\{a_1, \dots, a_q\} \cup \{b_1, \dots, b_m\})\big). \tag{7.24}$$

Hereinafter we consider that there are neither zeros nor poles of f on $\partial\Omega$.

Definition 7.4 Let the above assumptions hold. The logarithmic residue of f with respect to the positively oriented boundary of Ω is the integral

$$\frac{1}{2\pi i}\int_{\partial^+\Omega} \frac{f'(z)}{f(z)}\,dz.$$

Theorem 7.5 (On a Logarithmic Residue) *Let the above assumptions be satisfied. Then*

$$\frac{1}{2\pi i}\int_{\partial^+\Omega} \frac{f'(z)}{f(z)}\,dz = Z - P, \tag{7.25}$$

where $Z := n_1 + n_2 + \ldots + n_q$, $P := p_1 + p_2 + \ldots + p_m$, *and* n_i *is the order of the zero* a_i *and* p_i *is the order of the pole* b_i.

Proof Since the inclusion (7.24) holds, the Cauchy residue Theorem 7.1 and formulas (7.22) and (7.23) yield

$$\frac{1}{2\pi i}\int\limits_{\partial^+\Omega}\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{q}\operatorname*{Res}_{z=a_k}\frac{f'(z)}{f(z)} + \sum_{k=1}^{m}\operatorname*{Res}_{z=b_k}\frac{f'(z)}{f(z)} = \sum_{k=1}^{q} n_k - \sum_{k=1}^{m} p_k = Z - P.$$

□

Example 7.10 The formula (7.25) can be used for calculation of integrals, for example,

$$\int_{\partial^+ B_3(0)}\frac{dz}{\sin 2z} = \frac{1}{2}\int_{\partial^+ B_3(0)}\frac{(\tan z)'}{\tan z}\,dz = \pi i(1-2) = -\pi i.$$

Here it was easy to see that tan has one zero and two poles in the disk $B_3(0)$.

Theorem 7.6 (Argument Principle) *Let the conditions of Theorem 7.5 for a meromorphic function* f *be satisfied and* Ω *be a simply connected domain.*

Then the difference between the number of its zeros Z *and the number of its poles* P *inside* Ω *is equal to the increment of the argument of* $f(z)$ *when* z *passes once along the positively oriented boundary* $\partial^+\Omega$, *divided by* 2π :

$$Z - P = \frac{1}{2\pi}\,\Delta_{\partial^+\Omega}\operatorname{Arg} f(z). \tag{7.26}$$

Before the proof, we explain what the increment of the argument along a curve is. Let $z = \gamma(t)$, $t \in [\alpha, \beta]$, be a curve whose trace does not contain the origin, i.e., $E_\gamma \cap \{0\} = \emptyset$. The angle of rotation of the vector z when the point z moves along the trace of γ from its initial point to the end point is called *the increment of the argument* z *along* γ and is denoted by $\Delta_\gamma \operatorname{Arg} z$ (see Fig. 7.1).

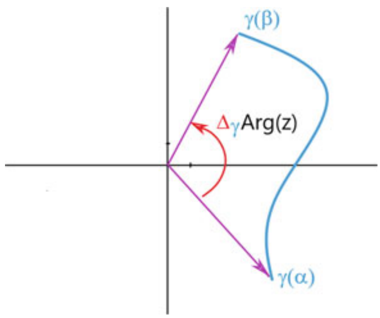

Fig. 7.1 The increment of the argument along a curve γ

Example 7.11 It is easy to verify that

- if $z = \gamma(t) = 1 + it$, $t \in [-1, 1]$, then $\Delta_\gamma \operatorname{Arg} z = \frac{\pi}{2}$;
- if $z = \gamma(t) = e^{-it}$, $t \in [0, \pi]$, then $\Delta_\gamma \operatorname{Arg} z = -\pi$;
- if $z = \gamma(t) = 2e^{it}$, $t \in [-\pi, \pi]$, then $\Delta_\gamma \operatorname{Arg} z = 2\pi$;
- If $z = \gamma(t) = 2 + e^{it}$, $t \in [-\pi, \pi]$, then $\Delta_\gamma \operatorname{Arg} z = 0$.

This example shows that the increment of the argument along a curve does not depend on the continuous branch of the argument that we choose at the beginning of the movement of the point z. Therefore, in this example, in the equality (7.26) and further in this section, any continuous branch of the multi-valued function Arg is assumed.

Proof Since the inclusion (7.24) holds, there is a positive number $\delta > 0$ such that $\frac{f'}{f} \in \mathcal{A}(U_\delta(\partial\Omega))$, where $U_\delta(\partial\Omega) = \{z : \operatorname{dist}(z, \partial\Omega) < \delta\}$.

Because the domain Ω is simply connected, its boundary coincides with the trace of a positively oriented Jordan curve $z = \gamma(t)$, $t \in [\alpha, \beta]$, i.e., $E_\gamma = \partial\Omega$. By Theorem 4.3 (on an antiderivative along a curve), there exists an antiderivative Ψ of the function $\frac{f'}{f}$ along γ. It follows from Theorem 4.3 and Remark 7.4 that

$$\Psi(t) = \ln|f(\gamma(t))| + i \operatorname{Arg}(f(\gamma(t)) + C, \qquad t \in [\alpha, \beta].$$

Then by the Newton-Leibnitz formula (see Theorem 4.4), we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial^+\Omega} \frac{f'(z)}{f(z)}\, dz &= \frac{1}{2\pi i}\big(\Psi(\beta) - \Psi(\alpha)\big) \\ &= \frac{1}{2\pi}\Big(\lim_{t\to\beta-0} \operatorname{Arg} f(\gamma(t)) - \lim_{t\to\alpha+0} \operatorname{Arg} f(\gamma(t))\Big) \\ &=: \frac{1}{2\pi} \Delta_{\partial^+\Omega} \operatorname{Arg} f(z). \end{aligned} \tag{7.27}$$

From (7.25) and (7.27) it follows (7.26). □

Example 7.12 Let's check the formula (7.26) for the function $f(z) = z^3$ in the domain $\Omega := B_1(0)$; $\gamma = e^{it}, t \in [0, 2\pi]$, is a positively oriented Jordan curve whose trace coincides with $\partial\Omega$.

On one hand it is easy to see that $Z = 3$ and $P = 0$ for f, and on the other hand

$$\frac{1}{2\pi} \Delta_{\partial^+\Omega} \operatorname{Arg} f(z) = \frac{1}{2\pi} \left(\operatorname{Arg} e^{3it}\right)\Big|_0^{2\pi} = \frac{1}{2\pi}(6\pi - 0) = 3.$$

Here we have fixed such a continuous branch that $\operatorname{Arg} e^{3it}|_{t=0} = 0$.

Important
If f is analytic, i.e., $P = 0$, then the formula (7.26) counts zeros (with their multiplicities) of f in Ω, i.e.,

$$Z = \frac{1}{2\pi} \Delta_{\partial^+\Omega} \operatorname{Arg} f(z). \tag{7.28}$$

Informally, this formula can be explained as follows: an analytic function f has as many zeros in a simply connected domain Ω as many times the radius vector $f(z)$ rotates around the origin when the point z passes once the boundary of Ω counterclockwise.

In the same way, we can count how many times an analytic function f takes the value w_0 (such points are called w_0*-points of* f).

Definition 7.5 An analytic function f is said to take the value w_0 at a point $z_0 \in \Omega$ with multiplicity $n \in \mathbb{N}$ if

$$f(z) = w_0 + (z - z_0)^n g(z), \quad z \in B_\delta(z_0), \tag{7.29}$$

where $g \in \mathcal{A}(B_\delta(z_0))$ and $g(z_0) \neq 0$. ▲

Similarly to how (7.25) was proved, we obtain

$$\frac{1}{2\pi i} \int_{\partial^+\Omega} \frac{f'(z)}{f(z) - w_0}\, dz = Z_f(w_0), \tag{7.30}$$

where $Z_f(w_0)$ is the number of w_0-points of f in Ω counting multiplicity. If Ω is simply connected, then

$$Z_f(w_0) = \frac{1}{2\pi} \Delta_{\partial^+\Omega} \operatorname{Arg}(f(z) - w_0).$$

The argument principle is used indirectly through Rouche's theorem, which has many important applications, some of which are proved in this subsection and others in Chap. 9.

Theorem 7.7 (Rouché's Theorem) *Let* Ω *be a simply connected domain and* $f, g \in \mathcal{A}(\overline{\Omega})$. *If*

$$|g(z)| < |f(z)| \quad \textit{for all} \;\; z \in \partial\Omega, \tag{7.31}$$

then f and $f+g$ have the same number of zeros (counting multiplicity) in the domain Ω, i.e.,

$$Z_f = Z_{f+g} \quad \textit{in} \ \ \Omega,$$

where Z_f is the number of zeros (counting multiplicity) of f.

Proof From the condition (7.31) it follows that

$$|f(z)| \neq 0 \quad \text{and} \quad |f(z)+g(z)| \neq 0 \quad \text{for all} \ \ z \in \partial\Omega.$$

To argue the last relation we also used the inequality

$$|f(z)+g(z)| \geq |f(z)| - |g(z)| > 0 \quad \text{for all} \ \ z \in \partial\Omega.$$

Then, recalling the properties of the argument (see Sect. 1.1), we obtain

$$\operatorname{Arg}(f(z)+g(z)) = \operatorname{Arg} f(z) + \operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right) \quad \text{for all} \ \ z \in \partial\Omega. \tag{7.32}$$

Here, as in Theorem 7.6, $\operatorname{Arg}(f+g)$ is any continuous branch of the multi-valued function Arg. Again using (7.31), we get

$$\left|\left(1+\frac{g(z)}{f(z)}\right) - 1\right| = \frac{|g(z)|}{|f(z)|} < 1 \quad \text{for all} \ \ z \in \partial\Omega.$$

This means that $1+\dfrac{g(z)}{f(z)}$ remains in the disk $B_1(1)$ for all $z \in \partial\Omega$. Therefore,

$$\Delta_{\partial^+\Omega} \operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right) = 0.$$

Considering this and (7.32), Theorem 7.6 implies

$$Z_{f+g} = \frac{1}{2\pi}\,\Delta_{\partial^+\Omega}\operatorname{Arg}(f+g) = \frac{1}{2\pi}\,\Delta_{\partial^+\Omega} f = Z_f.$$

The theorem is proved. □

This theorem is often used to find the number of roots of an equation in a given domain.

Example 7.13 Find the number of roots of the equation

$$z^9 - 6z^4 + 3z - 1 = 0$$

in the unit disk $B_1(0)$.

Solution Let $f(z) = -6z^4$ and $g(z) = z^9 + 3z - 1$. It is easy to see that

$$|g(z)| \le |z|^9 + 3|z| + 1 = 5 < 6 = |f(z)| \quad \text{for all } z \in \partial B_1(0).$$

Thus, all conditions of the Rouché theorem are satisfied for the functions f and g in $B_1(0)$. Therefore, $Z_{f+g} = Z_f = 4$ in $B_1(0)$.

Recall that zeros are counted taking into account their multiplicities. Clearly, the function f has one zero in $B_1(0)$ and this is a zero of order 4.

Exercise 7.1 Prove the fundamental theorem of algebra using Rouché's theorem.

Exercise 7.2 Find the number of roots of the equation $z^4 + 10z + 1 = 0$ in the annulus $\{z\colon 1 < |z| < 2\}$.

Theorem 7.8 (Sufficient Conditions of Conformality) *If a function f is analytic and univalent in a domain Ω, then f is conformal in Ω.*

Proof Let us prove the theorem by contradiction. Suppose there exists a point $z_0 \in \Omega$ such that $f'(z_0) = 0$. Then z_0 is an isolated and finite multiple zero of the derivative f'. If this is not the case, then, according to Corollaries 5.6 and 5.8, $f \equiv \text{const}$, and this contradicts the univalence of f. Thus, there is a positive number δ, such that

$$f'(z) \neq 0 \quad \text{for all } z \in \overline{B_\delta(z_0)} \setminus \{z_0\}. \tag{7.33}$$

By Theorem 5.3, we have

$$f(z) = c_0 + \sum_{k=n}^{+\infty} c_k (z - z_0)^k \quad \text{for all } z \in \overline{B_\delta(z_0)}, \tag{7.34}$$

where $n \ge 2$ and $c_n = \frac{f^{(n)}(z_0)}{n!} \neq 0$. It is easy to understand that $(n-1)$ is the order of the zero z_0 of the derivative f'. Since $c_n \neq 0$, there exists $\delta_1 \in (0, \delta)$ such that

$$\sum_{k=n}^{+\infty} c_k (z - z_0)^{k-n} \neq 0 \quad \text{for all } z \in \overline{B_{\delta_1}(z_0)}. \tag{7.35}$$

Define the function

$$\varphi(z) := \sum_{k=n}^{+\infty} c_k (z - z_0)^k = (z - z_0)^n \sum_{k=n}^{+\infty} c_k (z - z_0)^{k-n}, \qquad z \in \overline{B_{\delta_1}(z_0)}.$$

Obviously, $\varphi \in \mathcal{A}\left(\overline{B_{\delta_1}(z_0)}\right)$ as the sum of a power series, and z_0 is a zero of multiplicity n of the function φ. Due to (7.35)

$$m_0 := \min_{z \in \partial B_{\delta_1}(z_0)} |\varphi(z)| = \delta_1^n \min_{z \in \partial B_{\delta_1}(z_0)} \left| \sum_{k=n}^{+\infty} c_k (z - z_0)^{k-n} \right| \neq 0.$$

Let $\psi(z) = -\alpha$ for all $z \in \overline{B_{\delta_1}(z_0)}$, where α is a fixed positive number less than m_0. Since $|\psi(z)| = \alpha < m_0 \leq |\varphi(z)|$ for all $z \in \partial B_{\delta_1}(z_0)$, Rouché's theorem yields $Z_{\varphi+\psi} = Z_\varphi = n$ in $B_{\delta_1}(z_0)$. But

$$\varphi(z) + \psi(z) = f(z) - c_0 - \alpha \quad \text{for all } z \in \overline{B_{\delta_1}(z_0)}.$$

Thus,

$$Z_{f-c_0-\alpha} = n \quad \text{in } B_{\delta_1}(z_0),$$

or, more precisely, in $\breve{B}_{\delta_1}(z_0)$, because $\big(f(z) - c_0 - \alpha\big)|_{z=z_0} = -\alpha$.

Since

$$\frac{d}{dz}\big(f(z) - c_0 - \alpha\big) = f'(z) \neq 0 \quad \text{for all } z \in \overline{B_{\delta_1}(z_0)} \setminus \{z_0\}$$

(see (7.33)), the function $f - c_0 - \alpha$ has n $(n \geq 2)$ distinct simple zeros in $\breve{B}_{\delta_1}(z_0)$. This means that there are two points $z_1 \neq z_2,\ z_1, z_2 \in \breve{B}_{\delta_1}(z_0)$ such that $f(z_1) = f(z_2)$. But this contradicts to the univalence of the function f.

Hence, $f'(z) \neq 0$ for all $z \in \Omega$, i.e., the function f is conformal in Ω. □

Remark 7.5 The conditions of Theorem 7.8 are not necessary for conformality. Indeed, the function $f(z) = e^z$ is conformal in $\mathbb{C}$, but is not univalent in $\mathbb{C}$ (see Sect. 3.7).

The next statement shows how the zeros of analytic functions that form a uniformly convergent sequence are related to the zero of the limit function.

Theorem 7.9 (Hurwitz's Theorem) *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of analytic functions in a domain Ω, which converges uniformly on any compact set $\mathcal{K} \subset \Omega$ to a function f that is not identically equal to a constant.*

If $z_0 \in \Omega$ is a zero of f, then for any disk $B_r(z_0) \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that

$$Z_{f_n} > 0 \quad \textit{in } B_r(z_0) \quad \textit{for any } n \geq n_0.$$

Proof The function f is continuous in Ω as the limit of uniformly convergent continuous functions. Take any triangle $\triangle$ which, together with its closure, belongs to Ω. Then

$$\int_{\partial^+\triangle} f(z)\,dz = \lim_{n\to+\infty} \int_{\partial^+\triangle} f_n(z)\,dz = 0,$$

and due to Morera's Theorem 5.9 the function $f \in \mathcal{A}(\Omega)$.

Since $f \neq \text{const}$, its zeros are isolated and finitely multiple (see Corollaries 5.6 and 5.8). If $z_0 \in \Omega$ is a zero of f, then for any $r > 0$ such that $B_r(z_0) \subset \Omega$ there exists a number $\delta \in (0, r)$ that

$$f(z) \neq 0 \quad \text{for all } \ z \in \overline{B_\delta(z_0)} \setminus \{z_0\}.$$

Denote by $\mu := \min\limits_{z\in\partial B_\delta(z_0)} |f(z)|$. It is clear that $\mu > 0$. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\overline{B_\delta(z_0)}$ to f, there exists $n_0 \in \mathbb{N}$ such that for all integer $n \geq n_0$:

$$|f_n(z) - f(z)| < \mu \leq |f(z)| \quad \text{for all } \ z \in \partial B_\delta(z_0).$$

This means that the Rouché theorem can be applied to the functions $f_n - f$ and f. As a result, $Z_{f_n} = Z_{(f_n - f)+f} = Z_f > 0$ in $B_\delta(z_0)$. □

Example 7.14 (Counterexample from Mathematical Analysis) For real functions, the statement of Hurwitz's theorem is incorrect. Indeed, consider the function sequence

$$f_n(x) = x^2 + \frac{1}{n}, \quad x \in (-1, 1), \quad n \in \mathbb{N}.$$

Then, it is easy to verify that

$$f_n \overset{(-1,1)}{\rightrightarrows} f = x^2 \quad \text{as} \quad n \to +\infty,$$

and $f(0) = 0$. However, for all $n \in \mathbb{N}$

$$f_n(x) \neq 0 \quad \text{for all } \ x \in (-1, 1).$$

Corollary 7.1 *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic and univalent functions in a domain Ω, which converges uniformly on any compact set $\mathcal{K} \subset \Omega$ to an analytic function f, then either f is univalent or $f \equiv \text{const}$ in Ω.*

Proof As in Theorem 7.9, we show that $f \in \mathcal{A}(\Omega)$. Next, we go on to prove the corollary by contradicting it.

Suppose $f \not\equiv$ const and there are two points $z_1 \neq z_2$ in Ω such that $f(z_1) = f(z_2)$. Consider the function sequence

$$g_n(z) := f_n(z) - f_n(z_1), \quad z \in \overline{B_r(z_2)}, \quad n \in \mathbb{N},$$

where $r < |z_1 - z_2|$ and $\overline{B_r(z_2)} \subset \Omega$. It is obvious that $g_n \in \mathcal{A}(\overline{B_r(z_2)})$,

$$g_n(z) \overset{\overline{B_r(z_2)}}{\rightrightarrows} g(z) = f(z) - f(z_1) \quad \text{as} \quad n \to +\infty,$$

$g(z_2) = 0$, and $g \not\equiv$ const. Using Hurwitz's theorem for $\{g_n\}_{n\in\mathbb{N}}$, we get

$$\exists\, n_0 \in \mathbb{N} \;\; \forall\, n \geq n_0 : \quad Z_{g_n} > 0 \quad \text{in} \;\; B_r(z_2).$$

This means that for any integer $n \geq n_0$ there is a point $z_n^* \in B_r(z_2)$ such that

$$g_n(z_n^*) = f_n(z_n^*) - f_n(z_1) = 0.$$

This contradicts the univalence condition of f_n, because $z_n^* \neq z_1$. □

Example 7.15 (Counterexample from Mathematical Analysis) Consider the function sequence

$$f_n(x) = \begin{cases} x^3, & x \in [0, 1), \\ \frac{1}{n}x^3, & x \in (-1, 0), \end{cases} \qquad n \in \mathbb{N}.$$

It is easy to check that $f_n \in C^1\big((-1, 1)\big)$, f_n is an injection and

$$f_n(x) \overset{(-1,1)}{\rightrightarrows} f(x) := \begin{cases} x^3, & x \in [0, 1), \\ 0, & x \in (-1, 0), \end{cases}$$

But, $f \not\equiv$ const and f is not univalent in $(-1, 1)$.

Exercise 7.3 Prove that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions in a domain Ω, which converges uniformly on any compact $\mathcal{K} \subset \Omega$ to an analytic function f, and each f_n is nonzero everywhere in Ω, then either $f \equiv 0$ or f is also nowhere zero in Ω.

7.5 Partial Fraction Decomposition of a Meromorphic Function

For a meromorphic function with a finite number of poles, the formula (6.27) about its decomposition into the sum of a polynomial and the sum of simple fractions was proved. In this section, we consider the case when a meromorphic function in $\mathbb{C}$ has a countable number of simple poles. As was noted in Remark 6.2 the point at infinity is the limit point of its poles in this case.

Definition 7.6 A sequence of positively oriented Jordan curves $\{\gamma_n\}_{n\in\mathbb{N}}$ is said to be regular if the following conditions are satisfied:

(1) for any $n \in \mathbb{N}$: $\{0\} \subset \operatorname{int}(\gamma_n)$ and $\overline{\operatorname{int}(\gamma_n)} \subset \operatorname{int}(\gamma_{n+1})$;
(2) $d_n := \min\limits_{z\in E_{\gamma_n}} |z| \to +\infty$ as $n \to +\infty$;
(3) there exists a positive constant C such that $\frac{\ell_{\gamma_n}}{d_n} \le C$ for all $n \in \mathbb{N}$.

Remark 7.6 The first condition in Definition 7.6 means that the interior of the curve γ_n with its closure belongs to the interior of γ_{n+1}, and the origin is inside all these Jordan curves. The second condition says that the traces of these curves expand to infinity in any direction as $n \to +\infty$; and the last one means that the traces expand uniformly in all directions.

Example 7.16 The sequence of the circles $\big\{\gamma_n(t) = ne^{it},\ t \in [0, 2\pi]\big\}_{n\in\mathbb{N}}$ is regular because for it: $d_n = n,\ \ell_{\gamma_n} = 2\pi\, n,\ C = 2\pi$.

Example 7.17 If the trace of γ_n coincides with the boundary of the rectangle $[-n^2, n^2] \times [-ni, ni]$, then the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ is not regular. Indeed, for this sequence $d_n = n$ and $\ell_{\gamma_n} = 2n + 2n^2$, but the value $\frac{\ell_{\gamma_n}}{d_n} = 2(1 + n)$ is unbounded as $n \to +\infty$.

Theorem 7.10 *Let*

- *f be a meromorphic function in $\mathbb{C}$ and its poles $\{a_n\}_{n\in\mathbb{N}}$ be simple and*

$$0 < |a_1| \le |a_2| \le \ldots \le |a_n| \le \ldots \to +\infty \quad \text{as } n \to +\infty;$$

- *there be a regular sequence of positively oriented Jordan curves $\{\gamma_n\}_{n\in\mathbb{N}}$ and a positive constant M such that*

$$|f(z)| \le M \quad \text{for all } z \in E_{\gamma_n} \text{ and for all } n \in \mathbb{N}. \tag{7.36}$$

Then for all $z \in \mathbb{C} \setminus \{a_n\}_{n\in\mathbb{N}}$

$$f(z) = f(0) + \sum_{k=1}^{+\infty} A_k\Big(\frac{1}{z - a_k} + \frac{1}{a_k}\Big), \tag{7.37}$$

where $A_k := Res_{z=a_k} f(z)$, *and moreover, for an arbitrary bounded domain* Ω, *the series (7.37) converges uniformly on* $\Omega\setminus\{a_n\}_{n\in\mathbb{N}}$.

Proof For each $n \in \mathbb{N}$ consider the integral

$$I_n(z) := \frac{1}{2\pi i}\int_{\gamma_n} \frac{zf(\xi)}{\xi(\xi - z)}\,d\xi, \quad z \in \mathrm{int}(\gamma_n)\setminus\{a_k\}_{k\in\mathbb{N}}.$$

Due to the theorem conditions, the integrand $F(\xi) := \dfrac{zf(\xi)}{\xi(\xi - z)}$ has only simple poles $0, z, a_1, \ldots, a_{m_n}$ in the interior of the curve γ_n.

Then, by Cauchy's residue Theorem 7.1 we have

$$\begin{aligned} I_n(z) &= \operatorname*{Res}_{\xi=0} F(\xi) + \operatorname*{Res}_{\xi=z} F(\xi) + \sum_{k=1}^{m_n} \operatorname*{Res}_{\xi=a_k} F(\xi) \\ &= -f(0) + f(z) + \sum_{k=1}^{m_n} \frac{z}{a_k(a_k - z)} A_k. \end{aligned} \tag{7.38}$$

Since

$$\frac{z}{a_k(a_k - z)} = -\Big(\frac{1}{z - a_k} + \frac{1}{a_k}\Big),$$

we get from (7.38) that

$$f(z) = f(0) + \sum_{k=1}^{m_n} A_k\Big(\frac{1}{z - a_k} + \frac{1}{a_k}\Big) + I_n(z) \tag{7.39}$$

for all $z \in \mathrm{int}(\gamma_n)\setminus\{a_k\}_{k=1}^{m_n}$.

Let us estimate $I_n(z)$. Take an arbitrary bounded domain Ω. Since $\{\gamma_n\}_{n\in\mathbb{N}}$ is a regular sequence of Jordan curves, there exist numbers $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\overline{\Omega} \subset B_R(0) \quad \text{and} \quad \overline{B_R(0)} \subset \mathrm{int}(\gamma_n) \quad \text{for all } n \geq n_0.$$

Then, for all $z \in \mathrm{int}(\gamma_n)\setminus\{a_k\}_{k=1}^{m_n}$

$$|I_n(z)| \leq \frac{1}{2\pi}\int_{\gamma_n} \frac{|z||f(\xi)|}{|\xi||\xi - z|}\,dl \leq \frac{M\,R\,l_{\gamma_n}}{2\pi\, d_n(d_n - R)} \leq \frac{M\,R\,C}{2\pi(d_n - R)} \longrightarrow 0 \tag{7.40}$$

as $n \to +\infty$. Taking (7.40) into account and passing to the limit in (7.39) as $n \to +\infty$, we get (7.37). □

Remark 7.7 The summation in the formula (7.37) proceeds as follows: first, we sum the terms related to the poles from the interior of $\mathrm{int}(\gamma_1)$, then from $\mathrm{int}(\gamma_2) \setminus \mathrm{int}(\gamma_1)$ and so on.

Remark 7.8 The condition (7.41) can be weaken and replaced by

$$|f(z)| \leq M\,|z|^p \quad \text{for all } z \in E_{\gamma_n} \text{ and for all } n \in \mathbb{N}, \tag{7.41}$$

where $p \in \mathbb{N}$. Then the following partial fraction decomposition will be valid:

$$f(z) = \sum_{m=0}^{p} \frac{f^{(m)}(0)}{m!} z^m + \sum_{k=1}^{+\infty} A_k \left(\frac{1}{z - a_k} + \sum_{l=0}^{p} \frac{z^l}{a_k^{l+1}} \right). \tag{7.42}$$

The decomposition (7.42) coincides with (7.37) if $p = 0$.

Example 7.18 Perform the partial fraction decomposition of the cotangent.

Solution We cannot directly apply the formula (7.37) since 0 is a pole of $\cot z$. Therefore we first expand the following meromorphic function:

$$f(z) = \begin{cases} \cot z - \dfrac{1}{z}, & z \neq \{\pi\, n\}_{n \in \mathbb{Z}}, \\ 0, & z = 0. \end{cases}$$

Using L'Hopital's rule twice, we find

$$\lim_{z \to 0} \left(\cot z - \frac{1}{z} \right) = \lim_{z \to 0} \frac{z \cos z - \sin z}{z \sin z} = 0.$$

Thus, poles of f are $\{\pi\, n\}_{n \in \mathbb{Z} \setminus \{0\}}$ and all of them are simple (Example 7.2).

Consider a sequence of positively oriented Jordan curves $\{\gamma_n\}_{n \in \mathbb{N}}$, whose traces coincide with the boundaries of squares $\{A_n B_n C_n D_n\}_{n \in \mathbb{N}}$ (see Fig. 7.2), respectively, where $\alpha_n = \frac{\pi}{2} + \pi\, n$. It is easy to verify that this sequence is regular.

Now we show that the inequality (7.41) holds for the function f. First take any $z \in [D_n, C_n]$, i.e., $z = x + i\alpha_n$, $x \in [-\alpha_n, \alpha_n]$. Using (3.16), we obtain

$$|\cot z| = \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} \right| = \left| \frac{e^{-2\alpha_n} e^{2ix} + 1}{e^{-2\alpha_n} e^{2ix} - 1} \right| \leq \frac{1 + e^{-2\alpha_n}}{1 - e^{-2\alpha_n}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}}.$$

Let us now consider any $z \in [B_n, C_n]$, i.e., $z = \alpha_n + iy$, $y \in [-\alpha_n, \alpha_n]$. Then

$$|\cot z| = \left| \cot \left(\frac{\pi}{2} + \pi n + iy \right) \right| = |\tan (iy)| = |\tanh y| \leq \frac{e^{|y|} - e^{-|y|}}{e^{|y|} + e^{-|y|}} \leq 1.$$

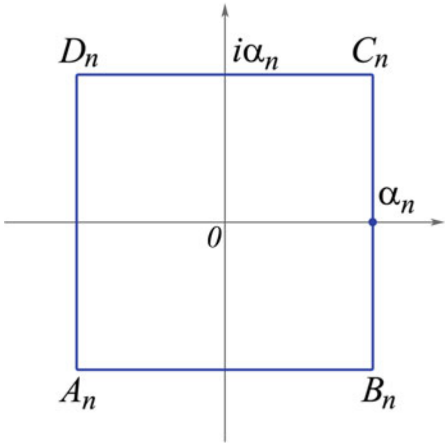

Fig. 7.2 The regular sequence of Jordan curves $\{\gamma_n\}$

Because $|\cot(-z)| = |\cot(z)|$, then

$$|f(z)| \le \frac{1+e^{-\pi}}{1-e^{-\pi}} + 1 \quad \text{for all } \; z \in E_{\gamma_n} \; \text{ and for all } \; n \in \mathbb{N}.$$

In virtue of Remark 7.1 and (7.8) we have that

$$A_k = \operatorname*{Res}_{z=\pi k} f(z) = 1.$$

In $\operatorname{int}(\gamma_k) \setminus \operatorname{int}(\gamma_{k-1})$ there are two poles $\pi\, k$ and $-\pi\, k$ of the function f. Therefore, according to the formula (7.37) and Remark 7.7, we get

$$\begin{aligned} \cot z &= \frac{1}{z} + \sum_{k=1}^{+\infty} \left(\frac{1}{z + k\pi} - \frac{1}{k\pi} + \frac{1}{z - k\pi} + \frac{1}{k\pi} \right) \\ &= \frac{1}{z} + \sum_{k=1}^{+\infty} \frac{2z}{z^2 - k^2\pi^2} \quad \text{for all } \; z \in \mathbb{C} \setminus \{\pi\, k\}_{k \in \mathbb{Z}}. \end{aligned} \tag{7.43}$$

Example 7.19 Find the sum $\sum_{k=1}^{+\infty} \dfrac{1}{k^2 + a^2}$, $\; a \in \mathbb{R} \setminus \{0\}$.

Solution In the formula (7.43) we put $z = i\, a\, \pi$. Then

$$\cot(ia\pi) = -\frac{i}{a\pi} - \frac{2ai}{\pi} \sum_{k=1}^{+\infty} \frac{1}{k^2 + a^2},$$

or

$$\sum_{k=1}^{+\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{2a} \coth(\pi\, a) - \frac{1}{2a^2}.$$

Exercise 7.4 Perform the partial fraction decomposition of the following functions: $\tan z$, $\frac{1}{\sin^2 z}$, $\frac{1}{e^z - 1}$.

Exercise 7.5 Find the sum $\sum_{k=1}^{+\infty} \dfrac{1}{(k^2 + a^2)^2}$, $a \in \mathbb{R} \setminus \{0\}$.

7.6 Factorization of an Entire Function Into an Infinite Product

It is known (see Theorem 5.14) that every polynomial can be factorized into a product of elementary factors. In this section, we will show that, under some additional assumptions, every entire function can also be decomposed into a product (possibly infinite) of elementary factors.

Let an entire function f have a finite number of zeros $a_1, \dots, a_m$ of order $n_1, \dots, n_m$, respectively. Then the function

$$\Phi(z) = \frac{f(z)}{(z - a_1)^{n_1} \cdot \ldots \cdot (z - a_m)^{n_m}} \tag{7.44}$$

has isolated singularities at the points $a_1, \dots, a_m$. By Theorem 5.13, they are removable, and therefore the function Φ can be extended by continuity at these points (see Corollary 6.2). As a result, Φ is an entire function without zeros. Hence $F(z) = \log\big(\Phi(z)\big)$ is also an entire function, where log is the principal branch of the logarithm. From the last relation and (7.44), we obtain

$$f(z) = e^{F(z)} (z - a_1)^{n_1} \cdot \ldots \cdot (z - a_m)^{n_m}, \quad z \in \mathbb{C}. \tag{7.45}$$

A natural question is: what decomposition does an entire function have if it has a countable number of zeros, e.g. $\sin z$? To answer this question, we recall some definitions from mathematical analysis and rewrite them in terms of complex values.

Definition 7.7 An infinite product $\prod_{k=1}^{+\infty} \big(1 + f_k(z)\big)$ is said to converge to a function f in a domain Ω if $1 + f_k(z) \neq 0$ for all $k \in \mathbb{N}$ and for all $z \in \Omega$, and

$$\lim_{n \to +\infty} \prod_{k=1}^{n} \big(1 + f_k(z)\big) = f(z). \tag{7.46}$$

If the limit (7.46) is uniform in $z \in \Omega$, then the product $\prod_{k=1}^{+\infty}\big(1+f_k(z)\big)$ is called uniformly convergent in Ω to the function f. ▲

If the product $\prod_{k=1}^{+\infty}\big(1+f_k(z)\big)$ uniformly in Ω converges to f and $f_k \in \mathcal{A}(\Omega)$, then, based on Corollary 5.5, the function f is analytic in Ω.

Theorem 7.11 (Weierstrass's Factorization Theorem) *Let*

- *f be an entire function, $f(0) \neq 0$, and moduli of its zeros $\{a_k\}_{k\in\mathbb{N}}$ form a non-decreasing sequence:*

$$0 < |a_1| \le |a_2| \le \ldots \le \ldots \le |a_k| \le \ldots, \quad \textit{and} \quad \lim_{k\to+\infty} |a_k| = +\infty;$$

- *the function $\dfrac{f'}{f}$ be uniformly bounded on a regular sequence of positively oriented Jordan curves $\{\gamma_k\}_{k\in\mathbb{N}}$.*

Then for all $z \in \mathbb{C}$ we have

$$f(z) = f(0)\exp\big(\tfrac{f'(0)}{f(0)}z\big)\prod_{k=1}^{+\infty}\Big(1-\frac{z}{a_k}\Big)^{n_k}\exp\big(\tfrac{n_k}{a_k}z\big), \tag{7.47}$$

where n_k is the order of zero a_k. Moreover, for any bounded domain Ω the product (7.47) is uniformly convergent in Ω.

Proof It is easy to see that the function $\frac{f'}{f}$ is meromorphic in $\mathbb{C}$ and has poles only at the points $\{a_k\}_{k\in\mathbb{N}}$; in addition, they are simple poles and due to the formula (7.22) $\operatorname{Res}_{z=a_k}\frac{f'}{f} = n_k$. Considering the theorem conditions, we can apply the formula (7.37) to $\frac{f'}{f}$. As a result,

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{k=1}^{+\infty} n_k\Big(\frac{1}{z-a_k} + \frac{1}{a_k}\Big) \quad \text{for all } z \in \mathbb{C}\setminus\{a_k\}_{k\in\mathbb{N}}. \tag{7.48}$$

Furthermore, the series (7.48) is uniformly convergent in $\Omega\setminus\{a_k\}_{k\in\mathbb{N}}$, where Ω is an arbitrary bounded domain.

Fix any point $\xi \in \mathbb{C}\setminus\{a_k\}_{k\in\mathbb{N}}$ and take a curve $z = \gamma(t),\ t \in [\alpha,\beta]$, whose trace does not pass through the zeros of f, i.e., $E_\gamma \cap \{a_k\}_{k\in\mathbb{N}} = \emptyset$, and whose endpoints are $0 = \gamma(\alpha)$ and $\xi = \gamma(\beta)$. Integrating (7.48) along the curve γ, we obtain

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{f'(0)}{f(0)}\int_\gamma dz + \sum_{k=1}^{+\infty} n_k \int_\gamma \Big(\frac{1}{z-a_k} + \frac{1}{a_k}\Big)dz.$$

With the help of the Newton-Leibnitz formula (see (4.17)) we get

$$\log \frac{f(\xi)}{f(0)} = \frac{f'(0)}{f(0)}\xi + \sum_{k=1}^{+\infty} n_k \left(\log\left(1 - \frac{\xi}{a_k}\right) + \frac{\xi}{a_k}\right), \tag{7.49}$$

whence

$$\begin{aligned} f(\xi) &= f(0)\,\exp\left(\frac{f'(0)}{f(0)}\xi + \sum_{k=1}^{+\infty} n_k \left(\log\left(1 - \frac{\xi}{a_k}\right) + \frac{\xi}{a_k}\right)\right) \\ &= f(0)\,\exp\left(\frac{f'(0)}{f(0)}\xi\right) \prod_{k=1}^{+\infty} \left(1 - \frac{\xi}{a_k}\right)^{n_k} \exp\left(\tfrac{n_k}{a_k}\xi\right). \end{aligned} \tag{7.50}$$

Obviously, we have the identity $0 = 0$ for $\xi = a_k$ in the product (7.50). Therefore, (7.50) holds for all $\xi \in \mathbb{C}$. The uniform convergence of (7.50) in any bounded domain Ω is equivalent to the uniform convergence of the series

$$\sum_{k=1}^{+\infty} n_k \left(\log\left(1 - \frac{z}{a_k}\right) + \frac{z}{a_k}\right)$$

in $\Omega \setminus \{a_k\}_{k\in\mathbb{N}}$. But this is a consequence of the uniform convergence of the series (7.48). □

Remark 7.9 By using the formula (7.47), one can construct entire functions that have zeros of a given multiplicity at given points.

Exercise 7.6 Prove that a meromorphic function in $\mathbb{C}$ can be represented as a ratio of two entire functions.

Example 7.20 Factorize the function $\sin z$ into an infinite product.

Solution Obviously, the function

$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

is an entire function, $f(0) \neq 0$ and $\{\pi k\}_{k\in\mathbb{Z}\setminus\{0\}}$ are simple poles of f. In addition, the function

$$\frac{f'(z)}{f(z)} = \frac{\frac{z\cos z - \sin z}{z^2}}{\frac{\sin z}{z}} = \cot z - \frac{1}{z}$$

is uniformly bounded on the regular sequence of Jordan curves from Example 7.18, and $f'(0) = 0$.

Thus, all conditions of Theorem 7.11 are satisfied for the function f and the formula (7.47) gives

$$\sin z = z \prod_{k=1}^{+\infty} \left(1 + \frac{z}{\pi k}\right) \exp\left(-\frac{z}{\pi k}\right) \left(1 - \frac{z}{\pi k}\right) \exp\left(\frac{z}{\pi k}\right)$$

$$= z \prod_{k=1}^{+\infty} \left(1 - \frac{z^2}{\pi^2 k^2}\right) \quad \text{for all } z \in \mathbb{C}. \tag{7.51}$$

Taking $z = \frac{\pi}{2}$ in (7.51), we get the Wallis formula

$$1 = \frac{\pi}{2} \prod_{k=1}^{+\infty} \left(1 - \frac{1}{(2k)^2}\right) \iff \frac{\pi}{2} = \prod_{k=1}^{+\infty} \frac{(2k)^2}{(2k-1)(2k+1)}$$

or

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots$$

Exercise 7.7 Factorize the function $e^z - 1$ into an infinite product.

8 Analytic Continuations

Abstract

Analytic functions have surprised us in previous chapters with their properties. In this chapter we will learn about another interesting property of analytic functions, namely the possibility of continuing an analytic function from the domain in which it is defined to a larger one. An important property of the analytic continuation procedure is that it is unique. In essence, this means that knowing the values of an analytic function in a small domain, or even on a curve, uniquely determines the value of the function at any other point where that function can be continued. This is a rather surprising and extremely strong assertion. Moreover, for some analytic functions, the analytic continuation can lead to a new concept of a function, which we have already experienced empirically in Sects. 3.6 and 3.7. Here we will find out under which conditions the analytic continuation leads to a multi-valued function, and under which conditions the newly extended function is single-valued. Along the way we will be introduced to various continuation techniques and other fundamental concepts of complex analysis such as monodromy, global analytic functions, their singularities and Riemann surfaces.

8.1 Analytic Function Elements

Thanks to the uniqueness theorem (see Theorem 5.12), the usual way to define analytic functions is to first specify the function only in a small domain, and then extend it by analytic continuation to the largest possible domain. Therefore, we now give some basic definitions of continuation.

Definition 8.1 Let Ω be a domain in $\mathbb{C}$ and E be a subset of Ω. An analytic function $F : \Omega \mapsto \mathbb{C}$ is called an analytic continuation of a function $f : E \mapsto \mathbb{C}$ from the set E into the domain Ω if $F(z) = f(z)$ for all $z \in E$.

T. Mel'nyk, *Complex Analysis*, https://doi.org/10.1007/978-3-031-39615-1_8

In other words, the restriction of F to the set E is the function f.

Definition 8.2 Let G be a domain in $\mathbb{C}$ and $f \in \mathcal{A}(G)$. An ordered pair (f, G) is called an analytic function element.

Definition 8.3 An analytic function element $\big(f, B_r(a)\big)$ is said to be a canonic analytic function element if the radius of the disk $B_r(a)$ is

$$r = \left(\limsup_{n\to+\infty} \sqrt[n]{\frac{|f^{(n)}(a)|}{n!}}\right)^{-1}, \tag{8.1}$$

i.e., $B_r(a)$ is the largest disk in which the function f can be expanded as the sum of a power series centered at a.

Definition 8.4 Two analytic function elements (f_1, G_1) and (f_2, G_2) are called direct analytic continuations of each other if $G_1 \cap G_2 =: D \neq \emptyset$, D is a domain and $f_1(z) = f_2(z)$ for all $z \in D$ (Fig. 8.1).

Lemma 8.1 *Let (f_2, G_2) and (f_3, G_2) be two direct analytic continuations of an analytic function element (f_1, G_1). Then $(f_2, G_2) = (f_3, G_2)$, i.e., no more than one function can be analytic in G_2 and coincides with f_1 in $D = G_1 \cap G_2$.*

Proof By Definition 8.4, $f_1(z) = f_2(z) = f_3(z)$ for all $z \in D$. Then, due to Theorem 5.12 we have that $f_2 \equiv f_3$ in G_2. □

Example 8.1 Consider two canonic analytic function elements (f_1, G_1) and (f_2, G_2), where

$$f_1(z) = \sum_{n=0}^{+\infty} z^n, \quad z \in G_1 := \{z \in \mathbb{C}\colon |z| < 1\},$$

$$f_2(z) = \sum_{n=0}^{+\infty} \frac{1}{1-i}\left(\frac{z-i}{1-i}\right)^n, \quad z \in G_2 := \{z \in \mathbb{C}\colon |z-i| < \sqrt{2}\}.$$

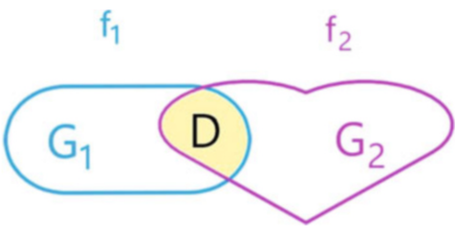

Fig. 8.1 Analytic function elements (f_1, G_1) and (f_2, G_2) that are direct analytic continuations of each other

Since the sums of these series are

$$f_1(z) = \frac{1}{1-z}, \quad z \in G_1, \quad \text{and} \quad f_2(z) = \frac{1}{1-i}\,\frac{1}{1-\frac{z-i}{1-i}} = \frac{1}{1-z}, \quad z \in G_2,$$

the function elements (f_1, G_1) and (f_2, G_2) are direct analytic continuations of each other. Obviously, each of these analytic function elements is the direct analytic continuation of the analytic function element (f_3, G_3), where

$$f_3(z) = \frac{1}{1-z}, \quad z \in G_3 := \mathbb{C} \setminus \{1\}.$$

The question naturally arises: how to construct an analytic continuation of a given analytic function, defined on some domain, into a larger domain? The mathematicians Weierstrass and Riemann proposed a method of analytic continuation called the method of re-expansion of power series. Next, we have a look at how it works.

Let $\big(f,\ B_r(a)\big)$ be an analytic function element, where

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n, \quad z \in B_r(a).$$

Take any point $b \in B_r(a)$ and rewrite this representation as follows

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} c_n (z-a)^n = \sum_{n=0}^{+\infty} c_n \big((z-b) + (b-a)\big)^n \\ &= \sum_{n=0}^{+\infty} c_n \sum_{k=0}^{n} \binom{k}{n} (z-b)^k (b-a)^{n-k} = \sum_{n=0}^{+\infty} d_n (z-b)^n, \end{aligned} \tag{8.2}$$

where the coefficients $d_n = \frac{f^{(n)}(b)}{n!}$, $n \in \mathbb{N}_0$, are uniquely determined by the theorem on the uniqueness of the expansion of an analytic function in a power series (see Theorem 5.7).

Next we find the convergence radius of the power series $\sum_{n=0}^{+\infty} d_n (z-b)^n$

$$r_1 = \left(\limsup_{n \to +\infty} \sqrt[n]{\frac{|f^{(n)}(b)|}{n!}} \right)^{-1}$$

and determine the function

$$f_1(z) = \sum_{n=0}^{+\infty} d_n (z-b)^n, \quad z \in B_{r_1}(b).$$

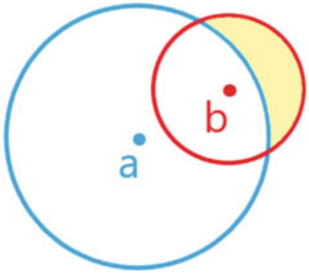

Fig. 8.2 The case $r_1 > r - |b - a|$

Then $\left(f_1,\ B_{r_1}(b)\right)$ is a new canonic analytic function element.

Due Theorem 5.3, the radius $r_1 \geq r - |b-a|$. Two cases are possible, namely

- if $r_1 > r - |b - a|$ (see Fig. 8.2), then $\left(f_1,\ B_{r_1}(b)\right)$ is the direct analytic continuation of the analytic function element $\left(f,\ B_r(a)\right)$;
- if $r_1 = r - |b - a|$, then it is said that through the point $z_0 = \partial B_r(a) \cap \partial B_{r_1}(b)$ it is not possible to analytically extend the function element $\left(f,\ B_r(a)\right)$ and the point z_0 is singular for $\left(f,\ B_r(a)\right)$.

It is easy to see that the point $z_0 = 1$ is singular for the analytic function element $(f_1,\ G_1)$ from Example 8.1. We may apply this approach to any domain on which f is analytic.

Definition 8.5 Let $(f,\ \Omega)$ be an analytic function element and $z_0 \in \partial\Omega$.

We say that the analytic function element $(f,\ \Omega)$ is continued through the point z_0 if there is an analytic function element $\left(F,\ B_r(z_0)\right)$ which is a direct analytic continuation of $(f,\ \Omega)$.

If such a continuation does not exist, then the point z_0 is called a singular point of the analytic function element $(f,\ \Omega)$.

Definition 8.6 A domain Ω is called a domain of analyticity (domain of holomorphy or natural domain) of an analytic function f if every point on $\partial\Omega$ is a singular point of the analytic function element $(f,\ \Omega)$.

Example 8.2 Consider the function

$$f(z) = \sum_{k=1}^{+\infty} z^{2^k}$$

By the formula (5.4), the convergence radius of this power series is 1, so f is analytic in the unit circle $B_1(0)$. If z from $B_1(0)$ approaches z_0, where z_0 is a root of one of the equations $z^{2^k} = 1,\ k \in \mathbb{N}$, then $f(z) \to \infty$. Therefore, f has a singularity at every 2^kth root of 1. Since these roots are dense on $\partial B_1(0)$, the disk $B_1(0)$ is a domain of analyticity of f.

Remark 8.1 Every domain in $\mathbb{C}$ is a domain of analyticity of some analytic function. Indeed, it possible to define an analytic function with isolated zeros accumulating everywhere on the boundary of a given domain, which must then be its domain of analyticity.

Theorem 8.1 (On a Singularity of a Canonic Analytic Element) *Let $\big(f,\ B_r(a)\big)$ be a canonic analytic function element. Then it has a singular point $z_0 \in \partial B_r(a)$.*

Proof Since $\big(f,\ B_r(a)\big)$ is a canonic analytic function element, the function f can be represented as the sum of the power series

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n \quad \text{in } B_r(a), \qquad \text{where } r = \Big(\limsup_{n\to+\infty} \sqrt[n]{|c_n|}\Big)^{-1}.$$

Assume that the canonic analytic function element $\big(f,\ B_r(a)\big)$ has no singular points on $\partial B_r(a)$. Then for any point $z_0 \in \partial B_r(a)$ there is an analytic function element $\big(f_{z_0},\ B_{\delta(z_0)}(z_0)\big)$ which is the direct analytic continuation of $\big(f,\ B_r(a)\big)$.

As $\partial B_r(a)$ is compact, it is possible to choose a finite subcover (see Fig. 8.3)

$$\{B_{r_n}(z_n)\}_{n=1}^{N} \quad \text{of the cover} \quad \{B_{\delta(z_0)}(z_0)\}_{z_0\in\partial B_r(a)}$$

such that $\partial B_r(a) \subset \bigcup_{n=1}^{N} B_{r_n}(z_n)$. Denote by

$$G := \Big(\bigcup_{n=1}^{N} B_{r_n}(z_n)\Big) \cup B_r(a).$$

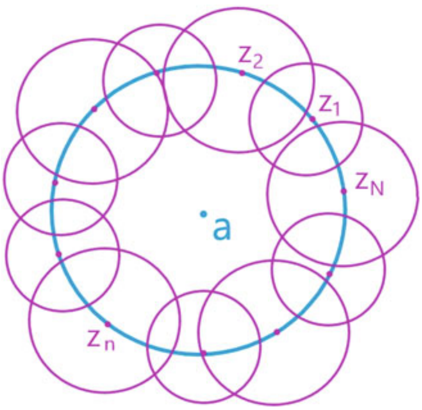

Fig. 8.3 The subcover $\{B_{r_n}(z_n)\}_{n=1}^{N}\}$

In the domain G we define the function

$$F(z) := \begin{cases} f(z), & z \in B_r(a), \\ f_n(z), & z \in B_{r_n}(z_n), \quad n \in \{1, \dots, N\}. \end{cases} \tag{8.3}$$

Since $\{(f_n,\ B_{r_n}(z_n))\}_{n=1}^N$ are direct analytic continuations of the analytic function element $(f,\ B_r(a))$, we have

$$f_n(z) = f_{n+1}(z) = f(z) \quad \text{for all } z \in B_{r_n}(z_n) \cap B_{r_{n+1}}(z_{n+1}) \cap B_r(a)$$

and for all $n \in \{1, \dots, N-1\}$, and

$$f_N(z) = f_1(z) = f(z) \quad \text{for all } z \in B_{r_N}(z_N) \cap B_{r_1}(z_1) \cap B_r(a).$$

Therefore, according to Theorem 5.12, for any $n \in \{1, \dots, N-1\}$

$$f_n(z) = f_{n+1}(z) \quad \text{for all } z \in B_{r_n}(z_n) \cap B_{r_{n+1}}(z_{n+1}),$$
$$f_N(z) = f_1(z) \quad \text{for all } z \in B_{r_N}(z_N) \cap B_{r_1}(z_1).$$

Consequently, the formula (8.3) correctly defines the analytic function F in the domain G.

It is easy to recognise (Fig. 8.3) that the distance from ∂G to $B_r(a)$ is positive; let us denote it by $\alpha := \text{dist}(\partial G, B_r(a)) > 0$. In virtue of the analyticity of F in the disk $B_{r+\alpha}(a)$, it can be expanded in the power series

$$F(z) = \sum_{n=0}^{+\infty} d_n (z-a)^n \quad \text{for all } z \in B_{r+\alpha}(a),$$

and it follows from (5.4) and (8.3) that

$$\frac{1}{r+\alpha} \geq \limsup_{n\to+\infty} \sqrt[n]{|d_n|} = \limsup_{n\to+\infty} \sqrt[n]{\frac{|F^{(n)}(a)|}{n!}} = \limsup_{n\to+\infty} \sqrt[n]{\frac{|f^{(n)}(a)|}{n!}} = \frac{1}{r}.$$

Since $\alpha > 0$, this inequality is a contradiction, which completes the proof. □

Remark 8.2 Theorem 8.1 can be used to find radii of convergence of power series. For example, the radius of convergence of the power series

$$\tan z = \sum_{n=0}^{+\infty} c_n z^n, \quad z \in B_r(0),$$

is equal to the distance from the point 0 to the nearest singular point of the function $\tan z$, i.e. $r = \frac{\pi}{2}$.

It is clear that the method of re-expansion of power series is inefficient for constructing analytic continuations. In the next section, we will get to know more effective methods of analytic continuation, which can be used to construct new important non-elementary functions.

8.2 Methods of Analytic Continuation: Schwarz's Reflection Principle

To feel the power statement of the following theorem, let us first consider an example. We define two real-valued functions

$$g_1(x_1, x_2) = -x_1, \quad (x_1, x_2) \in (-1, 0) \times (0, 1),$$

and

$$g_2(x_1, x_2) = x_1, \quad (x_1, x_2) \in (0, 1) \times (0, 1).$$

These functions are smooth in their domains and

$$\lim_{x_1 \to 0-} g_1(x_1, x_2) = \lim_{x_1 \to 0+} g_2(x_1, x_2) \quad \text{for all } \; x_2 \in (0, 1).$$

But, the function

$$g(x_1, x_2) = \begin{cases} g_1(x_1, x_2), & (x_1, x_2) \in (-1, 0] \times (0, 1), \\ g_2(x_1, x_2), & (x_1, x_2) \in (0, 1) \times (0, 1), \end{cases}$$

is only continuous in the rectangle $(-1, 1) \times (0, 1)$.

For complex-valued analytic functions, the situation is quite different. The following theorem is true.

Theorem 8.2 (Analytic Continuation by Continuity) *Assume that the following conditions are satisfied:*

(1) domains Ω_1 and Ω_2 do not intersect, however the intersection of their closures is $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$, and it is the trace of some smooth curve;
(2) in each domain Ω_k, an analytic function f_k is given, which is also defined on Γ and is continuous on $\Omega_k \cup \Gamma$, i.e., $f_k \in \mathcal{A}(\Omega_k) \cap C(\Omega_k \cup \Gamma)$, $\; k \in \{1, 2\}$;
(3) for all $\; z \in \Gamma$

$$f_1(z) = f_2(z).$$

Then the function

$$f(z) := \begin{cases} f_1(z), & z \in \Omega_1 \cup \Gamma, \\ f_2(z), & z \in \Omega_2, \end{cases}$$

is analytic in the domain D, where $\overline{D} = \overline{\Omega}_1 \cup \overline{\Omega}_2$.

Proof From the theorem's conditions follows that $f \in C(D)$. Consider an arbitrary triangle $\triangle$ which, together with its closure, belongs to the domain D. Then two cases are possible: either $\overline{\triangle} \cap \Gamma = \emptyset$, or $\overline{\triangle} \cap \Gamma \neq \emptyset$.

If $\overline{\triangle} \cap \Gamma = \emptyset$, then based on the Cauchy-Goursat theorem for triangles (Theorem 4.1)

$$\int_{\partial^+\triangle} f\, dz = 0.$$

Consider the case of $\overline{\triangle} \cap \Gamma \neq \emptyset$. We assume that Γ divides the triangle $\triangle$ into two open sets Ξ_1 and Ξ_2 (see Fig. 8.4); other cases are either treated similarly or are obviously simplified, for example, when Γ intersects only a side or a vertex of $\overline{\triangle}$. Then

$$\int_{\partial^+\triangle} f\, dz = \int_{\partial^+\Xi_1} f_1\, dz + \int_{\partial^+\Xi_2} f_2\, dz. \tag{8.4}$$

Since $f_k \in \mathcal{A}(\Xi_k) \cap C(\overline{\Xi_k})$ for $k \in \{1, 2\}$ and taking into account Remark 4.10, each of the integrals on the right-hand side of (8.4) is equal to zero.

Thus, for an arbitrary triangle $\triangle$ which, together with its closure, belongs to the domain D, $\int_{\partial^+\triangle} f\, dz = 0$. Then Morera's theorem (Theorem 5.9) says that $f \in \mathcal{A}(D)$. □

Remark 8.3 The theorem remains valid when Γ is the union of at most a countable number of smooth curves.

Exercise 8.1 Let $f \in \mathcal{A}\big(B_1(0)\big) \cap C(B_1(0) \cup \Gamma_0)$, where Γ_0 is an arc of the circle $\{z \in \mathbb{C} \colon |z| = 1\}$, and $f(z) = 0$ for all $z \in \Gamma_0$. Prove that $f = 0$ in $B_1(0)$.

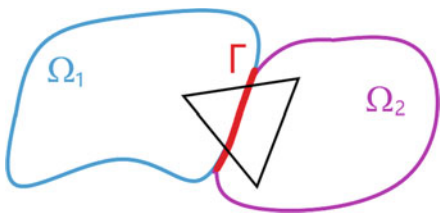

Fig. 8.4 Intersection of Γ with a triangle $\triangle$

Example 8.3 Example 5.8 shows that the gamma function

$$\Gamma(z) = \int_0^{+\infty} e^{-t}\, t^{z-1}\, dt, \quad z \in \Omega_r = \{\xi : \ \operatorname{Re} \xi > 0\}, \tag{8.5}$$

is analytic in the right half-plane Ω_r. Let us show that this function can be continued in the complex plane with the exception of simple poles.

Solution Integration by parts in the integral (8.5) gives

$$\Gamma(z+1) = z\,\Gamma(z) \iff \Gamma(z) = \frac{\Gamma(z+1)}{z} \quad \text{for all } \ z \in \Omega_r. \tag{8.6}$$

The last identity in (8.6) makes it possible to continue the gamma function in the half plane $\{z : \ \operatorname{Re} z > -1\}$, except at the origin. Obviously, this extension is analytic in the vertical strip $\{z : \ -1 < \operatorname{Re} z < 0\}$. It is continuous at the points of the imaginary axis, except at the origin. Indeed,

$$\lim_{z \to iy} \Gamma(z) = \lim_{z \to iy} \frac{\Gamma(z+1)}{z} = \frac{\Gamma(iy+1)}{iy} = \Gamma(iy) \ \text{ for all } \ y \in \mathbb{R} \setminus \{0\}.$$

Considering that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$, we get

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \sim \frac{\Gamma(1)}{z} = \frac{1}{z} \quad \text{as } \ z \to 0. \tag{8.7}$$

Thus, Theorem 8.2 says that Γ is analytic in $\{z : \ \operatorname{Re} z > -1\} \setminus \{0\}$, and the asymptotic relation (8.7) indicates that Γ has a simple pole at the origin and

$$\operatorname{Res}_{z=0} \Gamma(z) = 1.$$

Continuing in the same way, we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} \quad \text{for all } \ z \in \{\xi : \ \operatorname{Re} \xi > -2\} \setminus \{0, -1\},$$

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1) \cdot \ldots \cdot (z+k)} \tag{8.8}$$

for all $z \in \{\xi : \ \operatorname{Re} \xi > -k-1\} \setminus \{0, -1, \ldots, -k\}$. It follows from (8.8) that

$$\Gamma(z) \sim \frac{(-1)^k}{k!\,(z+k)} \quad \text{as } \ z \to -k. \tag{8.9}$$

Thus, the analytic continuation of the gamma function from the right half-plane gives us a meromorphic function with simple poles at non-positive integers

$\{-k\}_{k\in\mathbb{N}_0}$ at which the residue of Γ is equal to $\frac{(-1)^k}{k!}$, respectively. In addition, its restriction on the positive real axis coincides with the real-valued gamma function (5.21), where it is positive, and due to (8.8) it is also positive on the intervals $(-k-1, -k)$, $k \in \mathbb{N}_0$.

Using the uniqueness Theorem 5.12, as in Sect. 5.5, one can prove functional identities for the gamma function in the complex plane that are valid for real numbers. For example, it is well known that

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{for all } \ x \in (0,1).$$

Consequently,

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{for all } \ z \in \mathbb{C} \setminus \mathbb{Z}. \tag{8.10}$$

An immediate consequence of the identity (8.10) is the absence of zeros of the gamma function. This means that $\frac{1}{\Gamma}$ is an entire function that has simple zeros at non-positive integers.

Example 8.4 Example 5.7 shows that the Riemann zeta function is analytic in the half-plane $\{z\colon \ \operatorname{Re} z > 1\}$. From the functional Eq. (5.24) and the fact that $\frac{1}{\Gamma}$ is entire, it follows that

$$\zeta(z) = \frac{1}{\Gamma(z)}\left(\int_0^1 \frac{t^{z-1}}{e^t - 1}\,dt + g(z)\right), \tag{8.11}$$

where the entire function g is defined in (5.19). Taking into account the power expansion for the function $z/(e^z - 1)$ around $z = 0$, namely

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + \sum_{k=2}^{+\infty} c_k\, z^k \quad \text{for all } \ z \in B_{2\pi}(0)$$

(to find the radius of convergence, Remark 8.2 is used), and the third statement of Theorem 5.2, we deduce

$$\begin{aligned}\int_0^1 \frac{t^{z-1}}{e^t - 1}\,dt &= \int_0^1 t^{z-1}\Big(\frac{1}{t} - \frac{1}{2} + \sum_{k=2}^{+\infty} c_k\, t^{k-1}\Big)\,dt \\ &= \frac{1}{z-1} - \frac{1}{2z} + \sum_{k=1}^{+\infty} \frac{c_{k+1}}{z+k}\end{aligned}$$

According to (8.11), we get the formula

$$\zeta(z) = \frac{1}{\Gamma(z)} \cdot \frac{1}{z-1} + \frac{1}{\Gamma(z)}\left(-\frac{1}{2z} + \sum_{k=1}^{+\infty} \frac{c_{k+1}}{z+k} + g(z)\right) \tag{8.12}$$

which provides the analytic continuation of ζ in $\mathbb{C} \setminus \{1\}$. Indeed, despite the fact that the bracketed expression in (8.12) has a simple pole at each non-positive integer, all these poles are cancelled by the zeros of $\frac{1}{\Gamma}$ based on the formula (8.9); moreover, $\zeta(0) = -\frac{1}{2}$, and ζ has a simple pole at $z = 1$ with residue 1. ▲

Remark 8.4 By modifying the contour integration, Riemann deduced from the identity (5.24) the surprisingly mysterious functional equation

$$\zeta(z) = \Gamma(1-z)\,\zeta(1-z)\,2^z \pi^{z-1} \sin\frac{\pi z}{2} \tag{8.13}$$

that relates values of the zeta function at the points z and $1-z$. In particular, it follows from (8.13) that the zeta function has a simple zero at the points $\{z_n = -2n\}_{n\in\mathbb{N}}$, known as the *trivial zeros* of ζ. When $z = 2m$, $m \in \mathbb{N}$, the limit of the product $\Gamma(1-z)\,\sin\frac{\pi z}{2}$ is equal to $(-1)^m \pi/2(2m-1)!$ because of (8.9); due to (8.12) the limit $\zeta(1-z)\,\sin\frac{\pi z}{2}$ is $-\frac{\pi}{2}$ as z tends to zero.

In 1859 Riemann famously conjectured that there are infinitely many non-trivial zeros of the zeta function and that all of these zeros lie on the line $\operatorname{Re} z = \frac{1}{2}$. The Riemann hypothesis has been confirmed by many theoretical and numerical studies, but still remains unproven. The Clay Mathematical Institute in 2000 announced a US\$1 million prize for the first correct solution of this conjecture.

Theorem 8.3 (Schwarz's Reflection Principle) *Let Ω be a domain in $\mathbb{C}$ which is symmetric with respect to the real axis. We set $\Omega^+ = \{z \in \Omega : \operatorname{Im} z > 0\}$, $\Omega^- = \{z \in \Omega : \operatorname{Im} z < 0\}$ and $J = \Omega \cap \mathbb{R}$ (see Fig. 8.5).*

If f is an analytic function in Ω^+, continuous in $\Omega^+ \cup J$ and takes real values on J, then f has an analytic extension $\widetilde{f}$ to Ω which determined by the formula

$$\widetilde{f}(z) = \begin{cases} f(z), & z \in \Omega^+ \cup J, \\ \overline{f(\overline{z})}, & z \in \Omega^-. \end{cases}$$

Proof Take any point $z_0 \in \Omega^-$. Then

$$\frac{\widetilde{f}(z) - \widetilde{f}(z_0)}{z - z_0} = \frac{\overline{f(\overline{z})} - \overline{f(\overline{z}_0)}}{z - z_0} = \overline{\left(\frac{f(\overline{z}) - f(\overline{z}_0)}{\overline{z} - \overline{z}_0}\right)} \longrightarrow \overline{f'(\overline{z}_0)} \text{ as } z \to z_0.$$

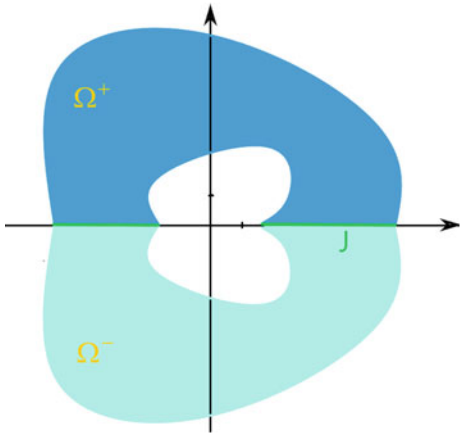

Fig. 8.5 Domain $\Omega = \Omega^+ \cup \Omega^- \cup J$

This means that there is a derivative of $\widetilde{f}$ at z_0 and

$$\frac{d\widetilde{f}}{dz}(z_0) = \overline{f'(\overline{z}_0)}.$$

Hence, $\widetilde{f} \in \mathcal{A}(\Omega^-)$.

Since f is continuous in $\Omega^+ \cap J$ and $f(x) \in \mathbb{R}$ for all $x \in J$,

$$\lim_{\Omega^+ \ni z \to x \in J} \widetilde{f}(z) = \lim_{\Omega^+ \ni z \to x \in J} f(z) = f(x)$$

and

$$\lim_{\Omega^- \ni z \to x \in J} \widetilde{f}(z) = \lim_{\Omega^- \ni z \to x \in J} \overline{f(\overline{z})} = \overline{f(x)} = f(x).$$

Then $\widetilde{f} \in \mathcal{A}(\Omega)$ according to Theorem 8.2. □

Corollary 8.1 *Let conditions of Theorem 8.3 be satisfied and, moreover,*

- *let the function f univalently map the domain Ω^+ onto a domain G^+ that is located in the upper half-plane $\{w \in \mathbb{C} \colon \operatorname{Im} w > 0\}$ and*
- *the mapping $f : J \mapsto I$ be a bijection, where $I := \partial G^+ \cap \mathbb{R}$.*

Then the function

$$\widetilde{f}(z) = \begin{cases} f(z), & z \in \Omega^+ \cup J, \\ \overline{f(\overline{z})}, & z \in \Omega^-, \end{cases}$$

is a univalent and conformal mapping of the domain Ω onto the domain G, where $G = G^+ \cup I \cup G^-$ and G^- is the symmetric image of G^+ with respect to the real axis.

Proof Based on Theorem 8.3, we have that $\widetilde{f} \in \mathcal{A}(\Omega)$. It follows from the corollary conditions that the mapping $\widetilde{f}\colon \Omega \mapsto G$ is a bijection. Then, from Theorem 7.8 (on sufficient conditions of conformality), we have that $\widetilde{f}$ is conformal in Ω. □

Example 8.5 Find a univalent and conformal mapping of the domain $\Omega := \mathbb{C} \setminus \big([-1, 1] \cup [-i, i]\big)$ onto the domain $G := \mathbb{C} \setminus [-1, 1]$.

Solution The domains Ω and G are symmetric with respect to the real axis and

$$\Omega = \Omega^+ \cup \Omega^- \cup J, \qquad G = G^+ \cup G^- \cup I,$$

where $J := \Omega \cap \mathbb{R} = (-\infty, -1) \cup (1, +\infty)$, $I := G \cap \mathbb{R} = (-\infty, -1) \cup (1, +\infty)$,

$$\Omega^+ := \{z \in \mathbb{C}\colon \ \mathrm{Im}\, z > 0\} \setminus [0, i], \quad \Omega^- := \{z \in \mathbb{C}\colon \ \mathrm{Im}\, z < 0\} \setminus [-i, 0],$$
$$G^+ := \{w \in \mathbb{C}\colon \ \mathrm{Im}\, w > 0\}, \quad G^- := \{w \in \mathbb{C}\colon \ \mathrm{Im}\, w < 0\}.$$

To apply Corollary 8.1, we must find a univalent and conformal mapping of Ω^+ onto G^+. It is easy to see that this is the function

$$f(z) := \frac{1}{\sqrt{2}} \sqrt[0]{z^2 + 1}, \quad z \in \Omega^+,$$

where $\sqrt[0]{\cdot}$ is the 0th-branch of the square root (see Sect. 3.6); in addition, it is a bijection of the interval $(1, +\infty)$ onto $(1, +\infty)$. Since

$$\lim_{\Omega^+ \ni z \to x < -1} \frac{1}{\sqrt{2}} \sqrt[0]{z^2 + 1} = \frac{\sqrt{x^2 + 1}}{\sqrt{2}}\, e^{i\pi} = -\frac{\sqrt{x^2 + 1}}{\sqrt{2}},$$

f is also a bijection of $(-\infty, -1)$ onto $(-\infty, -1)$.

Thus, all the conditions of Corollary 8.1 are satisfied, which means that the function

$$\widetilde{f}(z) = \begin{cases} \frac{1}{\sqrt{2}} \sqrt[0]{z^2 + 1}, & z \in \Omega^+ \cup (-\infty, -1) \cup (1, +\infty), \\[2ex] \frac{1}{\sqrt{2}} \overline{\sqrt[0]{(\overline{z})^2 + 1}}, & z \in \Omega^-, \end{cases}$$

is the desired mapping.

8.3 Analytic Continuation Along a Curve: The Monodromy Theorem

In the previous section we introduced two specific methods of analytic continuation. Here we will study all the possible analytic extensions of a canonic analytic function element.

Definition 8.7 A finite set of analytic function elements $\left\{(f_k, D_k)\right\}_{k=0}^{n}$ is called an analytic chain if consecutive pairs are analytic continuations of each other, i.e., (f_{k-1}, D_{k-1}) and (f_k, D_k) are direct analytic continuations of each other for all $k \in \{1, \dots, n\}$. Hereby (f_n, D_n) is referred to as an analytic continuation of the analytic function element (f_0, D_0) along the given chain.

According to Lemma 8.1 the analytic continuation along an analytic chain is unique.

Example 8.6 Let (f_0, B_0), (f_1, B_1), (f_2, B_2) be analytic function elements, where $B_0 := B_1(0)$, $B_1 := B_1(i)$, $B_2 := B_1(-1)$,

$$f_0(z) = \sqrt{|z|}\,\exp(i\tfrac{\varphi}{2}),\quad z \in B_0,\quad \varphi \in \operatorname{Arg}(z),\quad \varphi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2});$$

$$f_1(z) = \sqrt{|z|}\,\exp(i\tfrac{\varphi}{2}),\quad z \in B_1,\quad \varphi \in \operatorname{Arg}(z),\quad \varphi \in (0, \pi);$$

$$f_2(z) = \sqrt{|z|}\,\exp(i\tfrac{\varphi}{2}),\quad z \in B_2,\quad \varphi \in \operatorname{Arg}(z),\quad \varphi \in (\tfrac{\pi}{2}, \tfrac{3\pi}{2}).$$

It is easy to verify that $\{(f_0, B_0), (f_1, B_1), (f_2, B_2)\}$ is an analytic chain, and (f_2, B_2) is the analytic continuation of the analytic function element (f_0, B_0) along this chain.

Similarly, we check that $\{(f_0, B_0), (f_{-1}, \mathcal{B}_{-1}), (f_{-2}, \mathcal{B}_{-2})\}$ is also an analytic chain, where $\mathcal{B}_{-1} := B_1(-i)$, $\mathcal{B}_{-2} := B_1(-1)$,

$$f_{-1}(z) = \sqrt{|z|}\,\exp(i\tfrac{\varphi}{2}),\quad z \in \mathcal{B}_{-1},\quad \varphi \in \operatorname{Arg}(z),\quad \varphi \in (-\pi, 0);$$

$$f_{-2}(z) = \sqrt{|z|}\,\exp(i\tfrac{\varphi}{2}),\quad z \in \mathcal{B}_{-2},\quad \varphi \in \operatorname{Arg}(z),\quad \varphi \in (-\tfrac{3\pi}{2}, -\tfrac{\pi}{2}).$$

However, $f_2(z) \neq f_{-2}(z)$ for all $z \in B_2 = \mathcal{B}_{-2}$. Thus, the results obtained by continuing (f_0, B_0) along these two chains are different. And this is an illustration of how repeated continuation of a single-valued analytic function element can produce a multi-valued function or a single-value function given on the corresponding Riemann surface (see Sects. 3.6 and 3.7). ▲

From this example, a natural question arises: *under what conditions can it be guaranteed that the result of an analytic continuation along different analytic chains will be the same?*

Before answering this question, let us give a somewhat similar but more convenient definition of analytic continuation. As in Sect. 4.4, we assume that curves are given on the segment $I = [0, 1]$.

Definition 8.8 Let $\big(f_0, B_{r_0}(a)\big)$ be a canonic analytic function element, and let $\gamma\colon I \mapsto \mathbb{C}$ be a curve with the origin $a = \gamma(0)$ and the endpoint $b = \gamma(1)$.

We say that $\big(f_0, B_{r_0}(a)\big)$ is analytically continued along γ if there exists a partition $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ of the segment I and an analytic chain $\big\{\big(f_k, B_{r_k}(\gamma(t_k))\big)\big\}_{k=0}^{n}$ such that

(1) $\gamma\big([t_{k-1}, t_k]\big) \subset B_{r_{k-1}}\big(\gamma(t_{k-1})\big)$ for all $k \in \{1, \ldots, n\}$,
(2) and $\big(f_n, B_{r_n}(b)\big)$ is a canonic analytic function element.

Hereby, $\big(f_n, B_{r_n}(b)\big)$ is called an analytic continuation of $\big(f_0, B_{r_0}(a)\big)$ along the curve γ ; this is denoted as follows

$$\big(f_0, B_{r_0}(a)\big) \ \underset{\gamma}{\overset{\text{a.c.}}{\longrightarrow}} \ \big(f_n, B_{r_n}(b)\big).$$

Exercise 8.2 Prove that the analytic continuation of $\big(f_0, B_{r_0}(a)\big)$ along γ does not depend on a partition of the segment I.

Remark 8.5 The reader may wonder why, in this definition, we need the canonic analytic function elements at the beginning and the end of the analytic continuation along γ. There is no loss generality, since an analytic function element $\big(f, B_r(a)\big)$ can easily be made a canonic one. To do this, we need to use Theorem 5.3 (on the expansion of an analytic function in a power series) and find the convergence radius using the formula (8.1). Also, without losing generality, we can replace the domains in an analytic chain with disks.

Example 8.7 The analytic chain $\{(f_0, B_0), (f_1, B_1), (f_2, B_2)\}$ from Example 8.6 can be replaced with the analytic continuation of (f_0, B_0) along the curve $\gamma(t) = e^{i\pi t}$, $t \in [0, 1]$. The corresponding analytic chain is as follows $\big\{\big(f_k, B_1(\gamma(t_k))\big)\big\}_{k=0}^{4}$, where the partition

$$\{0 = t_0 < t_1 = \tfrac{1}{4} < t_2 = \tfrac{1}{2} < t_3 = \tfrac{3}{4} < t_4 = 1\}$$

and

$$f_k(z) = \sqrt{|z|}\, \exp(i\tfrac{\varphi}{2}), \ \ z \in B_1(\gamma(t_k)), \ \ \varphi \in \operatorname{Arg}(z), \ \ \varphi \in (-\tfrac{\pi}{2} + \tfrac{\pi k}{4}, \tfrac{\pi}{2} + \tfrac{\pi k}{4}).$$

Thus,

$$(f_0, B_0) \ \underset{\gamma}{\overset{\text{a.c.}}{\longrightarrow}} \ (f_2, B_2). \tag{8.14}$$

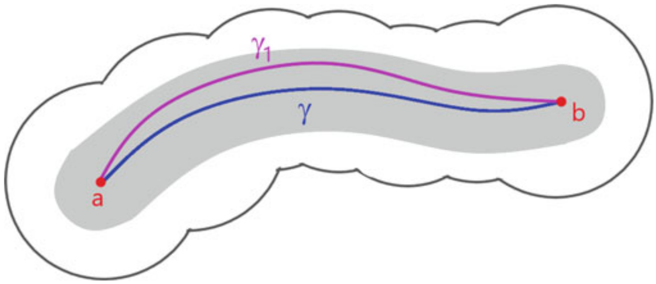

Fig. 8.6 Curvilinear strip $\mathcal{U}_\varepsilon(\gamma)$

Clearly (see Example 8.6),

$$(f_0, B_0) \xrightarrow[\gamma_1]{\text{a.c.}} (f_{-2}, \mathcal{B}_{-2}), \tag{8.15}$$

where $\gamma_1(t) = e^{-i\pi t}, \ t \in [0, 1]$. ▲

Remark 8.6 For the analytical chain $\{(f_k, B_{r_k}(\gamma(t_k)))\}_{k=0}^n$ from Definition 8.8, the following properties hold:

(1) there is a positive number ε such that $r_k \geq \varepsilon$ for all $k \in \{0, 1, \dots, n\}$;
(2) for any curve $\gamma_1 : I \mapsto \mathbb{C}$ such that $\gamma_1(0) = a$, $\gamma_1(1) = b$ and whose trace E_{γ_1} belongs to the curvilinear strip

$$\mathcal{U}_\varepsilon(\gamma) := \{z \in \mathbb{C} : \ \text{dist}(z; \ E_\gamma) < \varepsilon\}, \tag{8.16}$$

(see Fig. 8.6) there exists an analytic continuation of $(f_0, B_{r_0}(a))$ along γ_1 and

$$(f_0, B_{r_0}(a)) \xrightarrow[\gamma_1]{\text{a.c.}} (f_n, B_{r_n}(b)).$$

The same result of the analytic continuation of $(f_0, B_{r_0}(a))$ along γ_1 is guaranteed by Theorem 5.12. ▲

It turns out that if an analytic function element can be analytically continued along homotopic curves (see Definition 4.4), then the results of the extension coincide. The following theorem proves this.

Theorem 8.4 (Monodromy Theorem) *Let $\varphi\colon I \times I \mapsto \Omega$ be a homotopy between two curve*

$$\gamma_0 \colon I \mapsto \Omega \quad \textit{and} \quad \gamma_1 \colon I \mapsto \Omega,$$

i.e., $\gamma_0 \approx \gamma_1$ *in* Ω, *and let* $a = \gamma_0(0) = \gamma_1(0)$ *and* $b = \gamma_0(1) = \gamma_1(1)$. *For each* $s \in I$, *denote by*

$$\gamma_s(t) := \varphi(s,t), \quad t \in I.$$

If for every $s \in I$ *a given canonic analytic function element* $\big(f_0, B_{r_0}(a)\big)$ *is analytically continued along* γ_s *and*

$$\big(f_0, B_{r_0}(a)\big) \xrightarrow[a.c.]{\gamma_s} \big(f_{n_s}^{(s)}, B_{r_{n_s}^{(s)}}(b)\big),$$

then

$$\big(f_{n_s}^{(s)}, B_{r_{n_s}^{(s)}}(b)\big) = \big(f_{n_0}^{(0)}, B_{r_{n_0}^{(0)}}(b)\big) \quad \text{for all } s \in I. \tag{8.17}$$

Proof Thanks to Definition 8.8, for every $s \in I$ there is an analytic chain

$$\Big\{\big(f_0, B_{r_0}(b)\big), \big(f_{r_1^{(s)}}^{(s)}, B_{r_1^{(s)}}(\gamma_s(t_1^{(s)}))\big), \ldots, \big(f_{n_s}^{(s)}, B_{r_{n_s}^{(s)}}(b)\big)\Big\}$$

which leads to the canonic analytic function element $\big(f_{n_s}^{(s)}, B_{r_{n_s}^{(s)}}(b)\big)$ as a result of the analytic continuation of $\big(f_0, B_{r_0}(a)\big)$ along γ_s.

From the first point of Remark 8.6 we see that there exists $\varepsilon_s > 0$ such that $r_k^{(s)} \geq \varepsilon_s$ for all $k \in \{0, 1, \ldots, n_s\}$.

Due to the uniform continuity of φ, there is a positive number δ_s such that for all $\mu \in \Upsilon_{\delta_s} := (s - \delta_s,\ s + \delta_s) \cap I$

$$\max_{t\in[0,1]} |\gamma_s(t) - \gamma_\mu(t)| = \max_{t\in[0,1]} |\varphi(s,t) - \varphi(\mu,t)| < \varepsilon_s.$$

This means that the trace E_{γ_μ} belongs to $\mathcal{U}_\varepsilon(\gamma_s)$ (see (8.16)) for all $\mu \in \Upsilon_{\delta_s}$.

By the second point of Remark 8.6,

$$\big(f_{n_\mu}^{(\mu)}, B_{r_{n_\mu}^{(\mu)}}(b)\big) = \big(f_{n_s}^{(s)}, B_{r_{n_s}^{(s)}}(b)\big) \quad \text{for all } \mu \in \Upsilon_{\delta_s}. \tag{8.18}$$

Since $I = [0,1]$ is compact, it can be covered by a finite number of intervals $\{\Upsilon_{\delta_{s_0}}, \ldots, \Upsilon_{\delta_{s_p}}\}$, where $s_0 = 0,\ s_p = 1$, on each of which equality (8.18) holds for any $\mu \in \Upsilon_{\delta_k}$, $k \in \{1, \ldots, p\}$. Then, starting from $s_0 = 0$ and taking a finite number of steps, we arrive at $s_p = 1$, which means that the identity (8.17) is satisfied. □

The following corollary follows from the monodromy theorem and from the fact that arbitrary curves with a common origin and end are homotopic in a simply connected domain.

Corollary 8.2 *Let* Ω *be a simply connected domain and* $a \in \Omega$.

If for any curve γ starting at a and such that $E_\gamma \subset \Omega$ there exists an analytic continuation of a given canonic analytic function element $\big(f, B_r(a)\big)$ along γ, then the result of the analytic continuation does not depend on γ but is uniquely determined by its end.

Remark 8.7 If the conditions of Corollary 8.2 hold, then a single-valued analytic function F is determined in Ω such that the analytic function element (F, Ω) is a direct analytic continuation of $\big(f, B_r(a)\big)$.

Example 8.8 The curves γ and γ_1 from Example 8.7 are homotopic in $\mathbb{C}$. However, the analytic continuations of the canonic analytic function element (f_0, B_0) along these curves yields different results (see (8.14) and (8.15)). The reason is that along a curve whose trace contains the origin and connects the points 1 and -1, it is impossible to analytically continue (f_0, B_0) (see Sect. 3.6). This means that the condition of Theorem 8.4 is not satisfied.

8.4 Global Analytic Functions

Example 8.8 shows that all possible analytic continuations of a canonic analytic function element can lead to a new object, which is not necessarily a single-valued function. Such an object is called a global analytic function which appears as a collection of canonic analytic function elements related to each other in a prescribed way.

Definition 8.9 A set of canonic analytic function elements

$$\mathcal{F} := \big\{(f_\alpha, B_\alpha)\big\}_{\alpha\in\Xi}$$

is called a global analytic function if the following conditions are met:

(1) for any $(f_{\alpha_1}, B_{\alpha_1}) \in \mathcal{F}$ and $(f_{\alpha_2}, B_{\alpha_2}) \in \mathcal{F}$ there is a curve γ such that

$$(f_{\alpha_1}, B_{\alpha_1}) \xrightarrow[\gamma]{\text{a.c.}} (f_{\alpha_2}, B_{\alpha_2});$$

(2) if for some canonic analytic function element (g, B) there exists an element $(f_{\alpha_3}, B_{\alpha_3}) \in \mathcal{F}$ and a curve γ_0 such that

$$(f_{\alpha_3}, B_{\alpha_3}) \xrightarrow[\gamma_0]{\text{a.c.}} (g, B),$$

then necessarily $(g, B) \in \mathcal{F}$. ▲

It follows from Definition 8.9 that two global analytic functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are equal if they have at least one element in common.

Exercise 8.3 Check that the relation " $\xrightarrow[\gamma]{\text{a.c.}}$ " between canonic analytic function elements is an equivalence relation, i.e., it is reflexive, symmetric and transitive. ▲

Therefore, the set of all canonic analytic function elements is partitioned into equivalence classes by the equivalence relation " $\xrightarrow[\gamma]{\text{a.c.}}$ ", and these equivalence classes are different global analytic functions.

Working with such bulky objects is not convenient. Next, a new definition is introduced which characterises the concept of a global analytic function in more detail.

For each global analytic function $\mathcal{F} = \{(f_\alpha, B_\alpha)\}_{\alpha\in\Xi}$ we define the set

$$\mho := \bigcup_{\alpha\in\Xi} B_\alpha. \tag{8.19}$$

It is easy to see that for each point $a \in \mho$ there exists a canonic analytic function element $(f, B_r(a))$ which belongs to the global analytic function $\mathcal{F}$. It also means that $B_r(a) \in \mho$, and therefore $\mho$ is an open set.

Consider two arbitrary points a and b from $\mho$. Then there are canonic analytic function elements $(f_1, B_{r_1}(a))$ and $(f_2, B_{r_2}(b))$ from $\mathcal{F}$. By Definition 8.9, there exists a curve γ such that

$$(f_1, B_{r_1}(a)) \xrightarrow[\gamma]{\text{a.c.}} (f_2, B_{r_2}(b)),$$

and this means (see Definition 8.8) that the trace of γ belongs to $\mho$. Thus, $\mho$ is a domain.

If there is an analytic function element $(g, \mho)$ which is the direct analytic continuation of each $(f_\alpha, B_\alpha) \in \mathcal{F}$ (see examples below), then the global analytic function $\mathcal{F}$ bijectively specifies the single-valued analytic function $g\colon \mho \mapsto \mathbb{C}$ for which $\mho$ is its domain of analyticity.

Therefore, we also call the domain $\mho$ defined by (8.19) *a domain of analyticity of the global analytic function* $\mathcal{F}$.

Definition 8.10 Let $\mathcal{F} = \{(f_\alpha, B_\alpha)\}_{\alpha\in\Xi}$ be a global analytic function, and D be a subdomain of $\mho = \bigcup_{\alpha\in\Xi} B_\alpha$.

A single-valued continuous function g defined in D is called a branch of the global analytic function $\mathcal{F}$ if for every $z_0 \in D$ there exists a canonic analytic function element $(f_{\alpha_1}, B_{\alpha_1}) \in \mathcal{F}$ such that $z_0 \in B_{\alpha_1}$ and

$$g(z) = f_{\alpha_1}(z) \quad \text{for all } z \in D \cap B_{\alpha_1}.$$

▲

It follows from this definition that g is an analytic function in D. It is also obvious that for any index $\alpha \in \Xi$, the function f_α defined in B_α is a branch of the global analytic function $\mathcal{F}$ in the disk B_α. Depending on the domain $\mho$, the following cases are possible.

1. If $\mho$ is a simply connected domain, then by Corollary 8.2 a unique branch is defined in $\mho$ and the domain $\mho$ is its domain of analyticity. In this case, the global analytic function is a single-valued analytic function. For example, for the global analytic function $\mathcal{F} = \{(\sin z, \mathbb{C})\}$ the domain $\mho = \mathbb{C}$ and $g(z) = \sin z$, $z \in \mathbb{C}$.
2. If $\mho$ is multiply connected, then two cases are possible:

- In the domain $\mho$ one can still pick out a unique branch of the global analytic function $\mathcal{F}$, and then $\mho$ is a domain of analyticity for this branch. For example, for the global analytic function

$$\mathcal{F} = \left\{\left(\tfrac{1}{z}, B_{|a|}(a)\right)\right\}_{a \in \mathbb{C} \setminus \{0\}},$$

  the domain $\mho = \mathbb{C} \setminus \{0\}$ and $g(z) = \frac{1}{z}$, $z \in \mathbb{C} \setminus \{0\}$.
- A branch of $\mathcal{F}$ cannot be selected in $\mho$ (see Example 8.9 below).

Example 8.9 Consider the global analytic function

$$\{\sqrt{z}\} = \left\{\left(f, B_{|a|}(a)\right)\right\}_{a \in \mathbb{C} \setminus \{0\}}, \tag{8.20}$$

where $f(z) = \sqrt{|z|}\,\exp(i\frac{\operatorname{Arg} z}{2})$, and $\operatorname{Arg} z$ is a continuous in $B_{|a|}(a)$ branch of the multi-valued function Arg. From Example 8.7 it can be seen that for this global analytic function it is not possible to select a single-valued branch in the domain $\mho = \mathbb{C} \setminus \{0\}$. For any point $a \in \mathbb{C} \setminus \{0\}$, the global analytic function $\{\sqrt{z}\}$ has two different canonic analytic function elements, namely $(f, B_{|a|}(a))$ and $(-f, B_{|a|}(a))$.

Remark 8.8 The French mathematician Henri Poincaré (1954–1912) and the Italian mathematician Vito Volterra (1860–1940) independently proved that a global analytic function can have at most a countable number of different canonic analytic function elements centred at the same point. ▲

If it is impossible to select a single-valued branch for a global analytic function $\mathcal{F}$ in its domain $\mho$ of analyticity, then we cut $\mho$ by connecting the components of its boundary to obtain a simply connected domain $\widetilde{\mho}$. Next, we take an arbitrary canonic analytic function element of $\mathcal{F}$ and, based on Corollary 8.2, select a branch of $\mathcal{F}$ in $\widetilde{\mho}$. According to Remark 8.8 there can be at most a countable number of such branches in $\widetilde{\mho}$. Thus, instead of a global analytic function $\mathcal{F}$, one can consider the set of all its branches in the domain $\widetilde{\mho}$.

Example 8.10 Consider the global analytic function $\{\sqrt{z}\}$ (see Example 8.9). For this function, $\mho = \mathbb{C} \setminus \{0\}$. If we connect the points 0 and ∞, the simplest and most convenient way to do this is to use a real positive semi-axis, then we get a simply connected domain

$$\widetilde{\mho} := \mathbb{C} \setminus \{z\colon \ \operatorname{Im} z = 0, \ \ \operatorname{Re} z \geq 0\}.$$

In $\widetilde{\mho}$ it is possible to select two different branches for the global analytic function $\{\sqrt{z}\}$, namely

$$g_k(z) = \sqrt{|z|}\, e^{i(\frac{\arg z}{2} + \pi k)}, \quad k \in \{0, 1\};$$

here $0 < \arg z < 2\pi$. For every canonic analytic function element of $\{\sqrt{z}\}$, there exists a branch that is its direct analytic continuation onto some larger domain. For example, for $\big(f, B_1(1)\big)$, where $f(z) = \sqrt{|z|}\, \exp(i\frac{\varphi}{2})$ and $\varphi \in \operatorname{Arg}(z)$, $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the direct analytic continuation onto the upper plane $D_0 := \{z\colon \ \operatorname{Im} z > 0\}$ is the analytic function element (g_0, D_0).

Therefore, the specification of the global analytic function $\{\sqrt{z}\}$ is equivalent to the specification of its two branches: g_0 and g_1 in $\widetilde{\mho}$ (see Sect. 3.6).

Example 8.11 Consider the following global analytic function

$$\{\operatorname{Log}\} = \big\{\big(f, B_{|a|}(a)\big)\big\}_{a \in \mathbb{C}\setminus\{0\}},$$

where $f(z) = \log|z| + i \operatorname{Arg} z$, and $\operatorname{Arg} z$ is any continuous in $B_{|a|}(a)$ branch of the multi-valued function Arg. For this function, $\mho = \mathbb{C} \setminus \{0\}$, and there it is impossible to select a single-valued branch of {Log}. However, by making a cut along the real negative semi-axis, we obtain the simply connected domain $\widetilde{\mho} = \mathbb{C} \setminus \{z\colon \ \operatorname{Im} z = 0, \ \operatorname{Re} z \leq 0\}$, in which it is possible to pick out the following branches:

$$g_k(z) = \log|z| + i \arg z + i2\pi k, \quad k \in \mathbb{Z}; \tag{8.21}$$

here $-\pi < \arg z < \pi$ (see Sect. 3.7). For every canonic analytic function element of {Log}, there exists a branch which is the direct analytic continuation of it on a somewhat larger domain. For example, for $\big(f, B_1(-1)\big)$, where $f(z) = \log|z| + i\varphi$ and $\varphi \in \operatorname{Arg}(z)$, $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{2})$, the direct analytic continuation onto the upper plane D_0 is the analytic function element (g_0, D_0).

Therefore, specifying the global analytic function {Log} is equivalent to specifying its branches (8.21) in $\widetilde{\mho}$ (see Sect. 3.7). ▲

The following operations can be performed on global analytic functions:

(1) multiplication by an arbitrary complex number or by an arbitrary single-valued analytic function (of course, it must be defined in the corresponding domain $\mho$); for example:

$$\sin z \cdot \mathcal{F} = \big\{\big(\sin z \cdot f_\alpha,\, B_\alpha\big)\big\}_{\alpha\in\Xi},$$

where $\mathcal{F} = \big\{(f_\alpha, B_\alpha)\big\}_{\alpha\in\Xi}$ is a global analytic function;

(2) superposition with any single-valued analytic function; for example:

$$\exp\big(\mathcal{F}\big) = \big\{\big(\exp(f_\alpha),\, B_\alpha\big)\big\}_{\alpha\in\Xi};$$

(3) when integrating (differentiating) a global analytic function, one selects its branches in the corresponding domain $\widetilde{\mho}$, and then integrates (differentiates) over the required branch.

But, it is impossible to correctly determine the addition or multiplication of global analytic functions. For example, if we assume that the sum $\{\sqrt{z}\} + \{\sqrt{z}\}$ (see (8.20)) is uniquely determined as the sum of all possible canonic analytic function elements of these functions, then, on the one hand, the sum of the elements $(f, B_{|a|}(a))$ and $(-f, B_{|a|}(a))$ defines a function that is identically equal to zero; on the other hand, the sum of the elements $(f, B_{|a|}(a))$ and $(f, B_{|a|}(a))$ defines the global analytic function $\{2\sqrt{z}\}$.

8.5 Riemann Surfaces of Global Analytic Functions

In Sects. 3.6 and 3.7 we showed how to construct Riemann surfaces of the multi-valued functions Log and $\sqrt[n]{\cdot}$ in a simple way. The aim of such constructions is to make a single-valued function out of a multi-valued one. The purpose of this section is to present a unified topological approach to the construction of Riemann surfaces for global analytic functions, in which the above examples are special cases and which shows that Riemann surfaces can be considered as mathematical objects worthy of independent study.

In what follows, we will use the abbreviation $\widetilde{f_a}$ for a given canonic analytic function element $(f, B_r(a))$ indicating the function f itself and, as an index, the point a (the center of this element).

Next, we construct a set $\mathfrak{R}$, in which elements (points) are ordered pairs $A := (a, \widetilde{f_a})$, where a is an arbitrary point from $\overline{\mathbb{C}}$, and $\widetilde{f_a}$ is an arbitrary canonic analytic function element centered at a.

On $\mathfrak{R}$, we introduce a topology τ[1] generated by a set of ε-neighbourhoods of each point $A = \big(a, (f, B_r(a))\big) \in \mathfrak{R}$, where ε is any positive number from $(0, r)$. An ε-neighbourhood of a point $A \in \mathfrak{R}$ is defined as follows

$$U_\varepsilon(A) := \Big\{B = (b, \widetilde{g}_b) \in \mathfrak{R}\colon\ |a - b| < \varepsilon \\ \text{and}\quad \widetilde{g}_b \ \text{ is a direct analytic continuation of } \ \widetilde{f}_a\Big\}. \tag{8.22}$$

It is easy to verify that

- for every $A \in \mathfrak{R}$, there is an ε-neighbourhood such that $A \in U_\varepsilon(A)$;
- if $C = (c, \widetilde{h}_c) \in U_{\varepsilon_1}(A) \cap U_{\varepsilon_2}(B)$, then there is $U_{\varepsilon_3}(C)$ such that

$$C \in U_{\varepsilon_3}(C) \subset U_{\varepsilon_1}(A) \cap U_{\varepsilon_2}(B).$$

To do this, we need to take ε_3 such that the disk $B_{\varepsilon_3}(c) \subset B_{\varepsilon_1}(a) \cap B_{\varepsilon_2}(b)$.

This means that the set of ε-neighbourhoods of each point $A \in \mathfrak{R}$ is a basis of a topology, and a subset in $\mathfrak{R}$ is defined to be open if it is a union of ε-neighbourhoods. Thus, the collection τ of all open sets, including $\emptyset$ by definition, is a topology on $\mathfrak{R}$, and $(\mathfrak{R}, \tau)$ is a *topological space*.

Lemma 8.2 *The space* $(\mathfrak{R}, \tau)$ *is a Hausdorff space (for each pair of distinct points, there are their non-intersecting neighborhoods), i.e., for all* $A \neq B \in \mathfrak{R}$ *there exists a number* $\varepsilon > 0$ *such that* $U_\varepsilon(A) \cap U_\varepsilon(B) = \emptyset$.

Proof Consider two distinct points $A = (a, \widetilde{f}_a)$ and $B = (b, \widetilde{g}_b)$ lying in $\mathfrak{R}$. There are two possible cases. The first one is $a \neq b$. In this case, we set $\varepsilon = \frac{|a-b|}{4}$. Then, based on the definition of the ε-neighbourhood of a point in the space $\mathfrak{R}$ (see (8.22)), it is clear that $U_\varepsilon(A) \bigcap U_\varepsilon(B) = \emptyset$.

In the second case, we have $a = b$ and $\widetilde{f}_a \neq \widetilde{g}_a$, i.e.,

$$(f, B_{r_1}(a)) \neq (g, B_{r_2}(a)).$$

The last relation means that $f \neq g$ at the intersection $B_{r_1}(a) \cap B_{r_2}(a)$. Taking $\varepsilon < \min\{r_1, r_2\}$, we get that $U_\varepsilon(A) \bigcap U_\varepsilon(B) = \emptyset$. □

Recall that a topological space Y is *connected* if there are no nonempty open sets $U, V \subset Y$ such that

$$U \cap V = \emptyset \quad \text{and} \quad U \cup V = Y.$$

[1] A topology τ on a set Y is a collection of subsets of Y such that $\emptyset$ and Y lie in τ, the finite intersection of subsets of τ is in τ, and the union of arbitrarily many subsets of τ is also in τ. An element of τ is called an open set, and Y with τ is called a topological space.

Lemma 8.3 *The topological space* $(\mathfrak{R}, \tau)$ *is not connected.*

Proof Consider the set $D = \{(a, (1, \overline{\mathbb{C}})) : a \in \overline{\mathbb{C}}\}$. It is an open set, since every point of D lies in D together with its ε-neighbourhood

$$U_\varepsilon\big((a, (1, \overline{\mathbb{C}}))\big) \subset D,$$

where $\varepsilon > 0$.

Now let us show that D is a closed set. Let $C = (c, \widetilde{h_c})$ be a limit point of D, i.e., for every $\varepsilon > 0$ there is a point $(b, (1, \overline{\mathbb{C}}))$ which belongs to $U_\varepsilon(C)$. This means that $|c - b| < \varepsilon$ and the canonic analytic function elements $(h, B_r(c))$ and $(1, \overline{\mathbb{C}})$ are direct analytic continuations of each other. This implies that $h \equiv 1$ in $\mathbb{C}$. Thus, $C \in D$.

Therefore, $\mathfrak{R}$ can be represented as a union of two open sets D and $\mathfrak{R} \setminus D$, which do not intersect. □

Let $\mathcal{F} = \{\widetilde{(f_\alpha)}_{a_\alpha}\}_{\alpha \in \Xi}$ be a global analytic function, where

$$\widetilde{(f_\alpha)}_{a_\alpha} := \big(f_\alpha, B_{r_\alpha}(a_\alpha)\big).$$

The set

$$\mathfrak{R}_{\mathcal{F}} := \big\{A_\alpha = \big(a_\alpha, \widetilde{(f_\alpha)}_{a_\alpha}\big)\big\}_{\alpha \in \Xi}$$

is said to be a *Riemann surface* of $\mathcal{F}$.

Lemma 8.4 *The set* $\mathfrak{R}_{\mathcal{F}}$ *is a domain in the topological space* $(\mathfrak{R}, \tau)$.

Proof Let $\mathcal{F} = \{\widetilde{(f_\alpha)}_{a_\alpha}\}_{\alpha \in \Xi}$ be a global analytic function. Consider any point

$$A_\alpha = \big(a_\alpha, \widetilde{(f_\alpha)}_{a_\alpha}\big) \in \mathfrak{R}_{\mathcal{F}}.$$

Then $U_\varepsilon(A_\alpha) \subset \mathfrak{R}_{\mathcal{F}}$, where $\varepsilon \in (0, r_\alpha)$. Indeed, if a point

$$B = (b, \widetilde{g_b}) \in U_\varepsilon(A_\alpha),$$

then $|b - a_\alpha| < \varepsilon$ and the canonic analytic function elements $\widetilde{g_b}$ and $\widetilde{(f_\alpha)}_{a_\alpha}$ are direct analytic continuations of each other. According to Definition 8.9, the element $\widetilde{g_b} \in \mathcal{F}$, and therefore $B \in \mathfrak{R}_{\mathcal{F}}$. Thus, $U_\varepsilon(A_\alpha) \subset \mathfrak{R}_{\mathcal{F}}$, i.e., $\mathfrak{R}_{\mathcal{F}}$ is an open set.

Now let us show that $\mathfrak{R}_{\mathcal{F}}$ is path-connected. Consider two arbitrary points

$$A_{\alpha_1} = \big(a_{\alpha_1}, \widetilde{(f_{\alpha_1})}_{a_{\alpha_1}}\big) \in \mathfrak{R}_{\mathcal{F}} \quad \text{and} \quad A_{\alpha_2} = \big(a_{\alpha_2}, \widetilde{(f_{\alpha_2})}_{a_{\alpha_2}}\big) \in \mathfrak{R}_{\mathcal{F}}.$$

Since $\widetilde{(f_{\alpha_1})}_{a_{\alpha_1}} \in \mathcal{F}$ and $\widetilde{(f_{\alpha_2})}_{a_{\alpha_2}} \in \mathcal{F}$, based on Definition 8.9, there exists a curve $\gamma\colon I \mapsto \mathbb{C}$ such that $\gamma(0) = a_{\alpha_1}$, $\gamma(1) = a_{\alpha_2}$ and

$$\widetilde{(f_{\alpha_1})}_{a_{\alpha_1}} \xrightarrow[\gamma]{\text{a.c.}} \widetilde{(f_{\alpha_2})}_{a_{\alpha_2}}.$$

This means (see Definition 8.8) that there exists an analytic chain

$$\left\{\widetilde{(f_{\alpha_1})}_{a_{\alpha_1}}, \widetilde{(f_1)}_{\gamma(t_1)}, \dots, \widetilde{(f_{n-1})}_{\gamma(t_{n-1})}, \widetilde{(f_{\alpha_2})}_{a_{\alpha_2}}\right\}$$

in which each analytic function element $\widetilde{(f_k)}_{\gamma(t_k)} = \big(f_k, B_{r_k}(\gamma(t_k))\big)$ can be considered canonic.

According to the second part of Definition 8.9, the point

$$\big(\gamma(t_k), \widetilde{(f_k)}_{\gamma(t_k)}\big)$$

belongs to $\mathfrak{R}_{\mathcal{F}}$, where $k \in \{1, \dots, n-1\}$. In addition, for every $t \in I = [0, 1]$ there is a number $k \in \{1, \dots, n\}$ and a unique canonic analytic function element $\widetilde{(f)}_{\gamma(t)}$ such that $t \in [t_{k-1}, t_k]$ and the elements $\widetilde{(f)}_{\gamma(t)}$ and $\widetilde{(f_k)}_{\gamma(t_k)}$ are direct analytic continuations of each other. Then, thanks to Definitions 8.9 and 8.11, we have

$$\big(\gamma(t), \widetilde{(f)}_{\gamma(t)}\big) \in \mathfrak{R}_{\mathcal{F}} \quad \text{for all } \; t \in [0, 1]. \tag{8.23}$$

Since $\gamma\colon I \mapsto \mathbb{C}$ is a curve, based on (8.23) and the definition of an ε-neighbourhood of a point $A \in \mathfrak{R}$, we can state that the function

$$\Gamma(t) := \big(\gamma(t), \widetilde{(f)}_{\gamma(t)}\big), \quad t \in [0, 1],$$

is continuous as a function acting from the segment I into $\mathfrak{R}_{\mathcal{F}}$. This means that Γ is a curve whose trace lies in $\mathfrak{R}_{\mathcal{F}}$ with the starting point $\Gamma(0) = A_{\alpha_1}$ and ending point $\Gamma(1) = A_{\alpha_2}$. □

The definition of a Riemann surface given before Lemma 8.4 was slightly simplified in the notation in order to make it easier to understand the proof of the lemma and not to clutter it up with additional notation. By Remark 8.8, in general, a global analytic function can have a countable number of canonic analytic function elements centered at the same point, i.e.

$$\widetilde{(f_{k,\alpha})}_{a_\alpha} = \big(f_{k,\alpha}, B_{r(\alpha,k)}(a_\alpha)\big), \quad k \in \mathbb{N};$$

if the number of such elements is finite, we show how the index k changes. So, the next definition is proposed.

Definition 8.11 Let

$$\mathcal{F} = \left\{\widetilde{(f_{k,\alpha})_{a_\alpha}}\right\}_{\alpha\in\Xi,\, k\in\mathbb{N}}$$

be a global analytic function. The set

$$\mathfrak{R}_{\mathcal{F}} := \left\{A_\alpha^{(k)} = \left(a_\alpha, \widetilde{\left(f_{k,\alpha}\right)_{a_\alpha}}\right)\right\}_{\alpha\in\Xi\ k\in\mathbb{N}}$$

is called a *Riemann surface* of $\mathcal{F}$.

Important
Summarizing the results proved in this section and recalling Definitions 8.9 and 8.11, we can make the following conclusions:

(1) for every global analytic function $\mathcal{F}$ one can uniquely determine its Riemann surface $\mathfrak{R}_{\mathcal{F}}$, which is a domain in the topological space $(\mathfrak{R}, \tau)$;
(2) if $\mathcal{F}_1 \neq \mathcal{F}_2$, then $\mathfrak{R}_{\mathcal{F}_1} \cap \mathfrak{R}_{\mathcal{F}_2} = \emptyset$, i.e., the Riemann surfaces of two different global analytic functions cannot intersect;
(3) each domain in the topological space $(\mathfrak{R}, \tau)$ corresponds to a unique global analytic function;
(4) every global analytic function $\mathcal{F}$ can be associated with a single-valued function on its Riemann surface, namely the function $\mathcal{F} : \mathfrak{R}_{\mathcal{F}} \mapsto \mathbb{C}$ that maps each point

$$A_\alpha^{(k)} = \left(a_\alpha, \widetilde{\left(f_{k,\alpha}\right)_{a_\alpha}}\right) \in \mathfrak{R}_{\mathcal{F}}$$

to the complex number $f_{k,\alpha}(a_\alpha)$.

Example 8.12 The global analytic function from Example 8.11 can be rewritten as follows

$$\{\text{Log}\} = \left\{\left(f_k, B_{|a|}(a)\right)\right\}_{k\in\mathbb{Z},\, a\in\mathbb{C}\setminus\{0\}},$$

where $f_k(z) = \log|z| + i \arg z + i2\pi k$, and we assume that either $\arg z \in (-\pi, \pi]$ or $\arg z \in [0, 2\pi]$ (it depends on a disk $B_{|a|}(a)$).

Its Riemann surface (see Definition 8.11) is the set

$$\mathfrak{R}_{\text{Log}} = \left\{\left(a, \left(f_k, B_{|a|}(a)\right)\right)\right\}_{k\in\mathbb{Z},\, a\in\mathbb{C}\setminus\{0\}};$$

and if $k \neq m$, then $\big(a, \big(f_k, B_{|a|}(a)\big)\big)$ and $\big(a, \big(f_m, B_{|a|}(a)\big)\big)$ are two distinct points of the Riemann surface $\mathfrak{R}_{\mathrm{Log}}$.

The associated function $\mathrm{Log}\colon \mathfrak{R}_{\mathrm{Log}} \mapsto \mathbb{C}$ is single-valued because for each fixed $k \in \mathbb{Z}$ and $a \in \mathbb{C} \setminus \{0\}$ the point

$$\big(a, \big(f_k, B_{|a|}(a)\big)\big) \in \mathfrak{R}_{\mathrm{Log}}$$

is uniquely mapped to the complex number $f_k(a)$ (see also Sect. 3.7).

Exercise 8.4 For the global analytic function from Example 8.10, write down its Riemann surface and the single-valued function associated with it.

8.6 Singularities of Global Analytic Functions

As shown in Sects. 3.6 and 3.7, the multi-valued functions $\sqrt[n]{\cdot}$ and Log have special isolated singularities called branch points (now we can talk about points around which a multiple-valued function has nontrivial monodromy). So in this section, we give a classification of singularities of global analytic functions using the abstract approach introduced in the previous sections.

Definition 8.12 Let $\mathcal{F} = \big\{\widetilde{\big(f_{k,\alpha}\big)}_{a_\alpha}\big\}_{\alpha\in\Xi,\, k\in\mathbb{N}}$ be a global analytic function.

A point $b \in \overline{\mathbb{C}}$ is called an isolated singular point of $\mathcal{F}$ if there exists a positive number r such that, for any point $a_\alpha \in \breve{B}_r(b)$, every canonic analytic function element $\widetilde{(f_{k,\alpha})}_{a_\alpha}$ is analytically continued along every curve γ that starts at the point a_α and whose trace belongs to $\breve{B}_r(b)$. ▲

Let us prove a lemma which allows to classify singularities of global analytic functions.

Lemma 8.5 *Let $\mathcal{F} = \big\{\widetilde{\big(f_{k,\alpha}\big)}_{a_\alpha}\big\}_{\alpha\in\Xi,\, k\in\mathbb{N}}$ be a global analytic function, let a point b be an isolated singular point of $\mathcal{F}$, and let the following conditions be satisfied:*

(1) there exists a canonic analytic function element $\widetilde{(f_{k,\alpha_1})}_{a_{\alpha_1}} \in \mathcal{F}$, where the point $a_{\alpha_1} \in \breve{B}_r(b)$, and a closed curve γ_0, whose trace $E_{\gamma_0} \subset \breve{B}_r(b)$ and $\gamma_0(0) = \gamma_0(1) = a_{\alpha_1}$ (see Fig. 8.7), such that

$$\widetilde{(f_{k,\alpha_1})}_{a_{\alpha_1}} \xrightarrow[\gamma_0]{a.c.} \widetilde{(f_{k,\alpha_1})}_{a_{\alpha_1}};$$

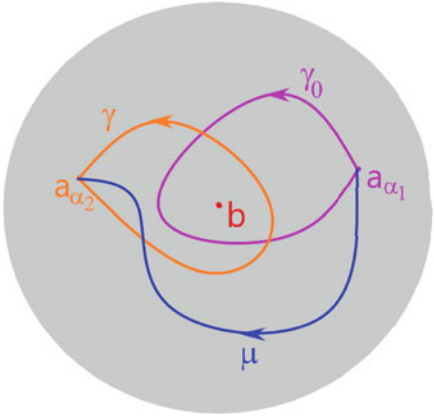

Fig. 8.7 Isolated singular point b of $\mathcal{F}$

(2) there exists a canonic analytic function element $\widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}} \in \mathcal{F}$, *where the point* $a_{\alpha_2} \in \breve{B}_r(b)$, *and a curve* μ, *whose trace* $E_\mu \subset \breve{B}_r(b)$ *and* $\mu(0) = a_{\alpha_1}$ *and* $\mu(1) = a_{\alpha_2}$, *such that*

$$\widetilde{(f_{k,\alpha_1})}_{a_{\alpha_1}} \xrightarrow[\mu]{a.c.} \widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}}.$$

Then for any closed curve γ *for which* $\gamma(0) = \gamma(1) = a_{\alpha_2}$ *and which is homotopic to the curve* γ_0 *in* $\breve{B}_r(b)$ $(\gamma \approx \gamma_0)$, *we have*

$$\widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}} \xrightarrow[\gamma]{a.c.} \widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}}. \tag{8.24}$$

Proof Consider the curve $\widetilde{\gamma} := \mu^{-1} \cup \gamma_0 \cup \mu$ (see (4.3)). It is easy to see (Fig. 8.7) that $\widetilde{\gamma} \approx \gamma_0$ in $\breve{B}_r(b)$ and

$$\widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}} \xrightarrow[\widetilde{\gamma}]{\text{a.c.}} \widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}}.$$

Since by the condition $\gamma \approx \gamma_0$ in $\breve{B}_r(b)$, the curve $\gamma \approx \widetilde{\gamma}$ in $\breve{B}_r(b)$. According to Definition 8.12, the canonic analytic function element $\widetilde{(f_{m,\alpha_2})}_{a_{\alpha_2}}$ is analytically continued along any curve whose trace belongs to $\breve{B}_r(b)$. Therefore, by Theorem 8.4, we have (8.24). □

Remark 8.9 If a closed curve $\gamma \approx 0$ in $\breve{B}_r(b)$ and it starts at $a_{\alpha_3} \in \breve{B}_r(b)$, then the analytic continuation of any canonic global function element $\widetilde{(f_{k,\alpha_3})}_{a_{\alpha_3}} \in \mathcal{F}$ along γ results the same function element. In fact, the curve γ is continuously deformed into a curve belonging to the disk of this canonic global function element, and the continuation along such a curve obviously does not change it. Next we have to apply Lemma 8.5.

Definition 8.13 Let $\mathcal{F}$ be a global analytic function and let a point b be its isolated singularity.

The point b is called an isolated single-valued singular point of $\mathcal{F}$ if there exists a Jordan curve γ whose trace E_γ belongs to $\breve{B}_r(b)$ and the point b lies in the interior of γ so that the analytic continuation along γ of any canonic analytic function element of $\mathcal{F}$ centered at $\gamma(0)$ does not change it. ▲

Remark 8.10 If b is a single-valued singular point of a global analytic function $\mathcal{F}$, then, based on Lemma 8.5, the analytic continuation along any closed curve γ, whose trace belongs to $\breve{B}_r(b)$, of any canonic analytic function element of $\mathcal{F}$ centered at a point in $\breve{B}_r(b)$ will result in the same element. Thus, single-valued branches of $\mathcal{F}$ are distinguished in $\breve{B}_r(b)$, and the point b for them can be either a pole, or removable, or essential (Definition 6.4).

Definition 8.14 Let $\mathcal{F} = \big\{\widetilde{\big(f_{k,\alpha}\big)_{a_\alpha}}\big\}_{\alpha\in\Xi,\, k\in\mathbb{N}}$ be a global analytic function and let a point b be its isolated singularity.

The point b is called a branch point of $\mathcal{F}$ if there exists a Jordan curve γ, $E_\gamma \subset \breve{B}_r(b)$ and $b \in \operatorname{int}(\gamma)$, and two different canonic analytic function elements $\widetilde{(f_k)}_{\gamma(0)} \in \mathcal{F}$ and $\widetilde{(f_m)}_{\gamma(0)} \in \mathcal{F}$ such that

$$\widetilde{(f_k)}_{\gamma(0)} \xrightarrow[\gamma]{\text{a.c.}} \widetilde{(f_m)}_{\gamma(0)}.$$

Moreover,

(1) if there exists an integer $n \geq 2$ (the smallest selected) such that

$$\widetilde{(f_k)}_{\gamma(0)} \xrightarrow[\gamma^n]{\text{a.c.}} \widetilde{(f_k)}_{\gamma(0)},$$

where $\gamma^n := \underbrace{\gamma \cup \ldots \cup \gamma}_{n}$, the point b is said to be a branch point of finite order, namely of order $n-1$;

(2) if such a number does not exist, then the point b is called a logarithmic branch point (or branch point of infinite order). ▲

Remark 8.11 Thus, a branch point of a global analytic function is its isolated singular point such that the analytic continuation of a global analytic function element of that function along a Jordan curve enclosing this point leads to a new function element. Branch points are divided into two categories: If, after the analytic continuation along the specified Jordan curve (based on Lemma 8.5 it suffices to take one such curve) n times, we obtain the original function element again, then this point is a branch point of finite order; if this does not happen, then the point is a logarithmic branch point (branch point of infinite order).

Example 8.13 Find isolated singular points of the global analytic function $\{\sqrt{1+\sqrt{z}}\}$ and characterize them.

Solution It known that the point 0 is a first-order branch point for the global analytic function $\{\sqrt{z}\}$ (see Sect. 3.6) and two branches of $\{\sqrt{z}\}$ are selected (see Example 8.10). Let us denote $\sqrt[0]{z} := \sqrt{|z|}\,e^{i\,\frac{\arg z}{2}}$ the zeroth branch, where $\arg z \in (-\pi, \pi]$. Then in the disk $B_{\frac{1}{2}}(\frac{1}{2})$, according to Corollary 8.2, four branches can be selected for the global analytic function $\{\sqrt{1+\sqrt{z}}\}$, namely

$$f_0(z) := \sqrt[0]{1+\sqrt[0]{z}}, \qquad f_1(z) := \sqrt[0]{1-\sqrt[0]{z}},$$
$$f_2(z) := -\sqrt[0]{1-\sqrt[0]{z}}, \qquad f_3(z) := -\sqrt[0]{1+\sqrt[0]{z}}.$$

It is clear that the points 0, 1 and ∞ are suspected to be branching points.

Let $\gamma_1(t) = \frac{1}{2}\exp(it),\ t \in [0, 2\pi]$. Then the values of the outer roots in these branches are not changed by analytic continuations along γ_1. They only change the values of the inner root. Therefore,

$$\widetilde{(f_0)}_{\frac{1}{2}} \xrightarrow[\gamma_1]{\text{a.c.}} \widetilde{(f_1)}_{\frac{1}{2}} \xrightarrow[\gamma_1]{\text{a.c.}} \widetilde{(f_0)}_{\frac{1}{2}}, \tag{8.25}$$

and

$$\widetilde{(f_2)}_{\frac{1}{2}} \xrightarrow[\gamma_1]{\text{a.c.}} \widetilde{(f_3)}_{\frac{1}{2}} \xrightarrow[\gamma_1]{\text{a.c.}} \widetilde{(f_2)}_{\frac{1}{2}}. \tag{8.26}$$

By Definition 8.14, the relations (8.25) and (8.25) mean that the global analytic function $\{\sqrt{1+\sqrt{z}}\}$ has two first-order branch points at the origin. This is shown schematically in Fig. 8.8.

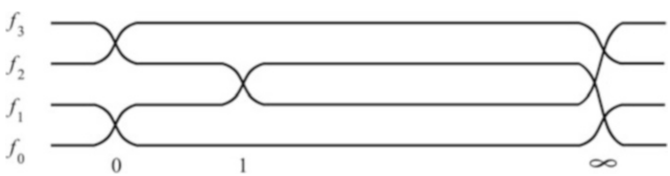

Fig. 8.8 Diagram of branch points of the global analytic function $\{\sqrt{1+\sqrt{z}}\}$

Now consider the Jordan curve $\gamma_2(t) = 1 - \frac{1}{2}\exp(it)$, $t \in [0, 2\pi]$. Then the analytic continuation along γ_2 doesn't change the canonic analytic function elements $\widetilde{(f_0)}_{\frac{1}{2}}$ and $\widetilde{(f_3)}_{\frac{1}{2}}$. But,

$$\widetilde{(f_1)}_{\frac{1}{2}} \xrightarrow[\gamma_2]{\text{a.c.}} \widetilde{(f_2)}_{\frac{1}{2}} \xrightarrow[\gamma_2]{\text{a.c.}} \widetilde{(f_1)}_{\frac{1}{2}}.$$

Thus, at the point 1, the global analytic function $\{\sqrt{1+\sqrt{z}}\}$ has only one first-order branch point (Fig. 8.8). For the branches f_0 and f_3 the point 1 is a removable singular point.

To study the point ∞, consider the curve $\gamma_3(t) = 9\exp(it)$, $t \in [0, 2\pi]$. Then $\sqrt[0]{\gamma_3(t)} = 3\exp(i\frac{t}{2})$, $t \in [0, 2\pi]$. For each value $k \in \{0, 1, 2, 3\}$ we consider the restriction $f_k(\gamma_3(t))$, $t \in [0, 2\pi]$, and find its values at the points $t = 0$ and $t = 2\pi$. Considering these values, we conclude that

$$\widetilde{(f_0)}_9 \xrightarrow[\gamma_3]{\text{a.c.}} \widetilde{(f_1)}_9 \xrightarrow[\gamma_3]{\text{a.c.}} \widetilde{(f_3)}_9 \xrightarrow[\gamma_3]{\text{a.c.}} \widetilde{(f_2)}_9 \xrightarrow[\gamma_3]{\text{a.c.}} \widetilde{(f_0)}_9.$$

Thus, the point ∞ is a branch point of order 3 for the global analytic function $\{\sqrt{1+\sqrt{z}}\}$ (see Fig. 8.8).

Exercise 8.5 Let $f \in \mathcal{A}(\breve{B}_r(b))$ and b is a simple zero or pole of f.
Prove that

- b is a branch point of order $n-1$ of the global analytic function $\{\sqrt[n]{f}\}$;
- b is a branch point of infinite order of the global analytic function $\{\text{Log}(f)\}$.

Exercise 8.6 Find isolated singular points of the global analytic function $\{\sqrt{z}\,\sqrt{z-1}\}$ and characterize them.

Qualitative Properties of Analytic Functions

9

Abstract

Inspired by the properties of analytic functions proved in the previous sections, in the last section we are ready to explore new, no less amazing properties of such functions. In Sect. 9.1 we show that analyticity is sufficient for a nonconstant function being an open map. This property indicates that the modulus of a non-constant analytic function cannot have a strict local maximum. A direct application of the maximum modulus principle is Schwarz's Lemma, established by the German mathematician K. A. Schwarz (1943–1921) in 1869, which is important in the theory of bounded analytic functions, where it is fundamental to most estimates. Sect. 9.2 shows how methods of complex analysis can be used to efficiently find inverse functions and expand them into Lagrange series (for single-valued inverse functions) and Puiseux series (for multi-valued inverse functions). Sections 9.3 and 9.4 are a preparation for the proof of Riemann's theorem, namely here we are interested in the conformal classification of domains of the complex plane and the finding of a sufficient condition for the precompactness of a family of analytic functions (Montel's theorem). In the last section there is a proof of the Riemann mapping theorem, which is undoubtedly one of the most beautiful theorems in mathematics.

9.1 Open Mapping Theorem, Maximum Modulus Principle, Schwarz Lemma

In real analysis, there are many examples of differential functions that are not open mappings, for example, the function $f(x) = x^2$ maps the open interval $(-2, 2)$ onto the half-open interval $[0, 4)$. The following open mapping theorem once again points to the essential difference between the properties of analytic functions in complex analysis and smooth functions in real analysis.

T. Mel'nyk, *Complex Analysis*, https://doi.org/10.1007/978-3-031-39615-1_9

Theorem 9.1 (Open Mapping Theorem) *Let f be a non-constant analytic function in a domain Ω. Then the image of Ω under the mapping f is also a domain in $\mathbb{C}$.*

Proof Denote by $\Omega^* := f(\Omega)$. Clearly, Ω^* is path-connected as a continuous image of a connected set. Indeed, for any two distinct points w_1, w_2 from Ω^* there are two points $z_1 \neq z_2$ from Ω such that

$$f(z_1) = w_1 \quad \text{and} \quad f(z_2) = w_2.$$

Since Ω is a domain, there is a curve $z = \gamma(t),\ t \in [0, 1]$, such that $\gamma(0) = z_1$, $\gamma(1) = z_2$, and its trace $E_\gamma \subset \Omega$. Then $w = f\big(\gamma(t)\big),\ t \in [0, 1]$, is a curve that connects w_1 and w_2, whose trace belongs to Ω^*.

Let us show that Ω^* is open. For any point $w_0 \in \Omega^*$ there is a point $z_0 \in \Omega$ such that $f(z_0) = w_0$. It follows from the theorem conditions that the w_0-point of the function f is isolated, i.e., there is an $r > 0$ such that

$$\overline{B_r(z_0)} \subset \Omega \quad \text{and} \quad f(z) \neq w_0 \quad \text{for all } z \in \overline{B_r(z_0)} \setminus \{z_0\}. \tag{9.1}$$

If this is not the case, then, by Theorem 5.12, the function $f \equiv w_0$ in Ω.

Denote by

$$\mu := \min_{z \in \partial B_r(z_0)} |f(z) - w_0| > 0. \tag{9.2}$$

Since $|f(z) - w_0|$ is a positive (see (9.1)) and continuous function on $\partial B_r(z_0)$, the extreme value theorem guarantees the existence of its positive minimum.

Consider any w_1 in the disk $B_\mu(w_0)$ and two functions

$$F(z) := f(z) - w_0 \quad \text{and} \quad G(z) := w_0 - w_1 \quad \text{in } \overline{B_r(z_0)}.$$

It is easy to see that $F, G \in \mathcal{A}(\overline{B_r(z_0)})$ and

$$|F(z)| = |f(z) - w_0| \geq \mu > |w_1 - w_0| = |G(z)| \quad \text{for all } z \in \partial B_r(z_0).$$

Then by Rouché's Theorem 7.7, these functions have the same number of zeros (counted with multiplicity), i.e., $Z_F = Z_{F+G}$ in $B_r(z_0)$ or

$$Z_{f(z)-w_1} = Z_{f(z)-w_0} > 0 \quad \text{in } B_r(z_0), \tag{9.3}$$

because $Z_{f(z)-w_0} > 0$ and $F(z) + G(z) = f(z) - w_1$.

This means that there is a point $z_1 \in B_r(z_0)$ such that $f(z_1) = w_1$. Hence, $w_1 \in \Omega^*$. But w_1 was arbitrary from $B_\mu(w_0)$. Therefore, the disk $B_\mu(w_0)$ is contained in Ω^*. As w_0 was also any point in Ω^*, Definition 1.16 says that Ω^* is an open set. □

We can now gracefully prove a very interesting property of analytic functions, namely that the modulus of an analytic non-constant function f cannot have a strict local maximum within the domain of f.

Theorem 9.2 (Maximum Modulus Principle) *Let f be an analytic function in a domain Ω and let there be a disk $B_R(a)$ in Ω such that*

$$\max_{z\in\overline{B_R(a)}} |f(z)| = |f(a)|. \tag{9.4}$$

Then $f \equiv \text{const}$ in Ω.

Proof Let $w_0 := f(a)$ and assume that $f \not\equiv \text{const}$ in Ω. Since in this case the w_0-point of the function f is isolated, there exists a number $r \in (0, R)$ such that

$$f(z) \neq w_0 \quad \text{for all } \; z \in \overline{B_r(a)} \setminus \{a\}.$$

Then by the previous theorem $f(\Omega) =: \Omega^*$ is a domain in $\mathbb{C}$ and the disk $B_\mu(w_0)$ belongs to Ω^*, where μ is determined by the formula (9.2).

Obviously, there is a point $w_1 \in B_\mu(w_0)$ such that $|w_0| < |w_1|$. And this means that there exists a point $z_1 \in B_r(a) \subset B_R(a)$ such that $f(z_1) = w_1$. But this contradicts to (9.4) because

$$|f(a)| = |w_0| < |w_1| = |f(z_1)|.$$

Thus, f is constant throughout Ω. □

Corollary 9.1 *Let $f \in \mathcal{A}(\Omega) \cap C(\overline{\Omega})$ and Ω be a bounded domain in $\mathbb{C}$. Then*

$$\max_{z\in\overline{\Omega}} |f(z)| = \max_{z\in\partial\Omega} |f(z)|.$$

Proof If $f \equiv \text{const}$, then the statement is obvious. Therefore, assume that $f \not\equiv \text{const}$. Since $|f| \in C(\overline{\Omega})$, the extreme value theorem assures the existence of $z_0 \in \overline{\Omega}$ such that

$$|f(z_0)| = \max_{z\in\overline{\Omega}} |f(z)|.$$

However, if $z_0 \in \Omega$, then by Theorem 9.2 the function f is constant in Ω, which contradicts the assumption. Therefore, $z_0 \in \partial\Omega$. □

Example 9.1 It is easy to see that the function $f(z) = z,\ z \in \overline{B_1(0)}$, is analytic and non-constant in the closed disk $\overline{B_1(0)}$ and

$$\min_{z\in\overline{B_1(0)}} |f(z)| = |f(0)|.$$

This example shows that both the assertion of Theorem 9.2 and the assertion of Corollary 9.1 fail for the minimum modulus of an analytic function. However, the following statements hold.

Exercise 9.1 If a function f is analytic, non-constant, and non-zero in a domain Ω, then its modulus cannot have a strict local minimum in Ω.

Exercise 9.2 Let $f \in \mathcal{A}(\Omega) \cap C(\overline{\Omega})$, Ω be an bounded domain and $f(z) \neq 0$ for all $z \in \overline{\Omega}$. Prove that

$$\min_{z\in\overline{\Omega}} |f(z)| = \min_{z\in\partial\Omega} |f(z)|.$$

Exercise 9.3 Let $f \in \mathcal{A}(\Omega)$, $f = u + iv$, and

$$\sup_{(x,y)\in\Omega} u(x, y) = u(x_1, y_1) \quad \big(\text{or} \quad \inf_{(x,y)\in\Omega} u(x, y) = u(x_1, y_1)\big).$$

Prove that $u \equiv \text{const}$ in the domain Ω.

A direct application of the maximum modulus principle is Schwarz's lemma which helps to prove the Riemann mapping theorem (see Sect. 9.5).

Lemma 9.1 (Schwarz's Lemma) *Let the following conditions be satisfied:*

1. *a function f is analytic in the unit disk $B_1(0)$;*
2. *$|f(z)| \leq 1$ for all $z \in B_1(0)$;*
3. *$f(0) = 0$.*

Then

$$|f(z)| \leq |z| \quad \textit{for all} \;\; z \in B_1(0), \qquad \textit{and} \;\; |f'(0)| \leq 1.$$

Moreover, if there is a point $z_1 \in B_1(0) \setminus \{0\}$ such that $|f(z_1)| = |z_1|$, then f is a rotation, i.e., $f(z) = e^{i\alpha}z$, where α is some real number. The same statement holds if $|f'(0)| = 1$.

Proof Consider the function $\varphi(z) = \frac{f(z)}{z}$, $z \in B_1(0) \setminus \{0\}$. By virtue of the first and third conditions, the point 0 is removable for φ and

$$\lim_{z\to 0} \varphi(z) = f'(0).$$

By Corollary 6.2, φ can be extended by continuity at 0, i.e., $\varphi(0) = f'(0)$, and as a result, $\varphi \in \mathcal{A}\big(B_1(0)\big)$.

Fix any point $z_0 \in B_1(0)$. Applying the maximum modulus principle to the function φ in the disk $B_r(0)$, where $r \in (|z_0|,\ 1)$, we get

$$\max_{z\in\overline{B_r(0)}} |\varphi(z)| = \max_{z\in\partial B_r(0)} |\varphi(z)| = \max_{z\in\partial B_r(0)} \frac{|f(z)|}{r} \le \frac{1}{r},$$

where the second condition is used. By letting $r \to 1$, we find that $|\varphi(z_0)| \le 1$ for all $z_0 \in B_1(0)$. Thus, $|f(z_0)| \le |z_0|$ for all $z_0 \in B_1(0)$, and $|f'(0)| = |\varphi(0)| \le 1$.

We now prove the second assertion of this lemma. Let there be a point $z_1 \in B_1(0) \setminus \{0\}$ such that $|f(z_1)| = |z_1|$. This means that $|\varphi(z_1)| = 1$, i.e., the modulus of φ has a local maximum within the disk $B_1(0)$, and therefore by Theorem 9.2

$$\varphi \equiv \text{const} \quad \text{in } B_1(0), \qquad \text{and} \qquad |\varphi| \equiv 1.$$

Consequently, $\varphi(z) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$, or $f(z) = e^{i\alpha} z$. The same reasoning can be applied to the case $|f'(0)| = 1$. □

Exercise 9.4 Instead of the third condition of the Schwarz lemma, let the following condition be satisfied:

$$f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0.$$

Then prove that $|f(z)| \le |z|^n$ for all $z \in B_1(0)$.

Exercise 9.5 (Schwarz–Pick Theorem) Let f be an analytic function in $B_1(0)$ and $|f(z)| \le 1$ for all $z \in B_1(0)$. Prove that $|f'(a)| \le \dfrac{1-|b|^2}{1-|a|^2}$ for all $a \in B_1(0)$, where $b = f(a)$.

9.2 Inverse Function Theorem: Puiseux Series

Assume that a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain Ω and $f'(z_0) \neq 0$, where $z_0 = x_0 + iy_0 \in \Omega$. By Theorem 9.1, the image $\Omega^* := f(\Omega)$ is a domain in $\mathbb{C}$. In addition, based on (2.15), the Jacobian of the vector mapping $\binom{u}{v} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ at the point (x_0, y_0) is equal to

$$J \begin{pmatrix} u \\ v \end{pmatrix}\Bigg|_{(x_0,y_0)} = |f'(z_0)|^2 \neq 0.$$

It is well known from multivariable calculus that this relation is a sufficient condition for the existence of an inverse vector function in some neighborhood of the point $w_0 = f(z_0)$. However, only methods of complex analysis make it possible to effectively determine this neighborhood and find the inverse function. We will now show how this can be done.

By the same considerations as in the proof of Theorem 9.1, we obtain that there exists a number $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$ and

$$f(z) \neq w_0 \quad \forall\, z \in \overline{B_r(z_0)} \setminus \{z_0\}, \quad \text{and} \quad f'(z) \neq 0 \quad \forall\, z \in \overline{B_r(z_0)}. \tag{9.5}$$

Take any $w_1 \in B_\mu(w_0)$, where the radius μ is determined by the formula (9.2). According to (9.3) and (9.5),

$$Z_{f-w_1} = Z_{f-w_0} = 1 \quad \text{in } B_r(z_0), \tag{9.6}$$

Thus, there exists a unique point $z_1 \in B_r(z_0)$ such that $f(z_1) = w_1$, i.e., the inverse mapping $f^{-1}\colon B_\mu(w_0) \mapsto B_r(z_0)$ is defined, for which, due to (2.16) and the right relation in (9.5), we have

$$\big(f^{-1}(w)\big)' = \frac{1}{\big(f'(z)\big)}\bigg|_{z=f^{-1}(w)} \quad \text{for all } \ w \in B_\mu(w_0).$$

Since $f^{-1} \in \mathcal{A}\big(B_\mu(w_0)\big)$, it can be represented as the sum of the power series

$$f^{-1}(w) = \sum_{n=0}^{+\infty} d_n\,(w - w_0)^n \quad \text{for all } \ w \in B_\mu(w_0). \tag{9.7}$$

The series (9.7) is called *Bürmann-Lagrange series.*

To find the coefficients d_n in (9.7), we fix a point $w \in B_\mu(w_0)$ and define the function

$$h(\xi) = \frac{\xi\, f'(\xi)}{f(\xi) - w}, \qquad \xi \in B_r(z_0).$$

It is easy to see that $h \in \mathcal{A}\big(B_r(z_0) \setminus \{f^{-1}(w)\}\big)$, and the point $z = f^{-1}(w)$ is a simple pole of the function h, because $f'(z) \neq 0$ (see (9.5)). Then applying Cauchy's residue theorem, we get

$$\frac{1}{2\pi i} \int\limits_{\partial^+ B_r(z_0)} h(\xi)\, d\xi = \operatorname{Res}_{\xi=z} h(\xi) = \lim_{\xi \to z} \frac{\xi\, f'(\xi)}{f(\xi) - w}\,(\xi - z) = z$$

or

$$f^{-1}(w) = \frac{1}{2\pi i} \int\limits_{\partial^+ B_r(z_0)} \frac{\xi\, f'(\xi)}{\big(f(\xi) - w_0\big)\big(1 - \frac{w-w_0}{f(\xi)-w_0}\big)}\, d\xi. \tag{9.8}$$

Since

$$\frac{|w - w_0|}{|f(\xi) - w_0|} \le \frac{|w - w_0|}{\mu} =: q < 1 \quad \text{for all } \xi \in \partial B_r(z_0),$$

we obtain from (9.8) by the infinite geometric progression formula that

$$f^{-1}(w) = \sum_{n=0}^{+\infty} d_n \, (w - w_0)^n,$$

where

$$d_n = \frac{1}{2\pi i} \int_{\partial^+ B_r(z_0)} \frac{\xi \; f'(\xi)}{\big(f(\xi) - w_0\big)^{n+1}} \, d\xi, \qquad n \ge 0. \tag{9.9}$$

Clearly, $d_0 = f^{-1}(w_0) = z_0$. For $n \ge 1$ we integrate by parts in (9.9) and use the Cauchy residue theorem, namely

$$\begin{aligned} d_n &= -\frac{1}{2\pi i \, n} \int_{\partial^+ B_r(z_0)} \xi \, d\left(\frac{1}{\big(f(\xi) - w_0\big)^n}\right) \\ &= \frac{1}{2\pi i \, n} \int_{\partial^+ B_r(z_0)} \frac{d\xi}{\big(f(\xi) - w_0\big)^n} \\ &= \frac{1}{n!} \, \frac{d^{n-1}}{dz^{n-1}} \left[\left(\frac{z - z_0}{f(z) - w_0} \right)^n \right] \Bigg|_{z=z_0}. \end{aligned} \tag{9.10}$$

Thus, the following theorem has been proved.

Theorem 9.3 (Inverse Function Theorem) *Let $f \in \mathcal{A}(\Omega)$, and let $f'(z_0) \ne 0$, where $z_0 \in \Omega$, and $w_0 = f(z_0)$.*

Then, there are numbers $r > 0$ and μ, where μ is defined by (9.2), and an inverse function $f^{-1}\colon B_\mu(w_0) \mapsto B_r(z_0)$, which is the sum of the Bürmann-Lagrange series (9.7) whose coefficients are determined by formulas (9.10).

From Theorem 9.3 and the theorem on sufficient conditions for conformality (see Theorem 7.8) we get immediately necessary and sufficient conditions for local univalence.

Lemma 9.2 *Let $f \in \mathcal{A}(\Omega)$ and $z_0 \in \Omega$. The function f is univalent in a neighborhood of z_0 if and only if $f'(z_0) \ne 0$.*

Example 9.2 (Counterexample From Mathematical Analysis) For differentiated mappings in real analysis, the condition $J\binom{u}{v}|_{(x_0,y_0)} \ne 0$ is not necessary for the local injectivity of the mapping $\binom{u}{v}$. Indeed, for injective mapping

$$\begin{pmatrix} x^3 \\ y \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2,$$

we have

$$J \begin{pmatrix} x^3 \\ y \end{pmatrix}\Bigg|_{(0,0)} = \begin{vmatrix} 3x^2 & 0 \\ 0 & 1 \end{vmatrix}_{(0,0)} = 0.$$

Example 9.3 Find the inverse function of the function $f(z) = ze^{-az}$ in a neighborhood of the point $0 = f(0)$, where a is a positive number.

Solution Since $f'(0) = 1 \neq 0$, then according to the formula (9.10) we have

$$d_0 = 0, \quad d_n = \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\left(\left(\frac{z}{ze^{-az}}\right)^n\right)\Bigg|_{z=0} = \frac{1}{n!}\, a^{n-1}n^{n-1} \quad \text{for } n \in \mathbb{N}.$$

Thus,

$$z = f^{-1}(w) = \sum_{n=1}^{+\infty} \frac{a^{n-1}n^{n-1}}{n!}\, w^n \quad \text{for all } w \in B_R(0),$$

where $R = \lim_{n\to+\infty} \frac{d_n}{d_{n+1}} = \frac{1}{a\,e}$.

Bürmann-Lagrange series are often used in the search for the asymptotics of solutions to various transcendental equations.

Example 9.4 Consider the transcendental equation

$$\tan x = \frac{1}{x}, \qquad x \in (0, +\infty). \tag{9.11}$$

It can be seen from Fig. 9.1 that each interval $(\pi p,\ \pi p + \frac{\pi}{2})$ contains only one root of the Eq. (9.11), i.e.

$$\forall\, p \in \mathbb{N} \cup \{0\} \quad \exists\,!\, x_p \in (\pi p,\ \pi p + \tfrac{\pi}{2}) : \quad \tan x_p = \frac{1}{x_p},$$

and, moreover, $x_p - \pi p \to 0$ as $p \to +\infty$.

After the substitution $t = x_p - \pi p$, the equation becomes as follows

$$\tan t = \frac{1}{t + \pi p} \quad \Longleftrightarrow \quad \frac{1}{\pi p} = \frac{\sin t}{\cos t - t \sin t}. \tag{9.12}$$

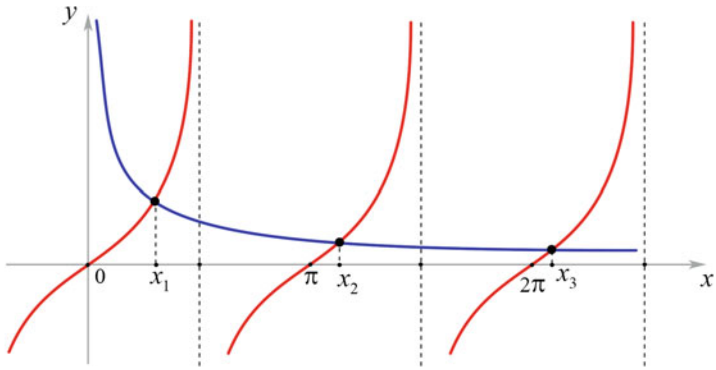

Fig. 9.1 Roots of the transcendental Eq. (9.11)

Now consider the complex-valued function

$$w = f(z) := \frac{\sin z}{\cos z - z\sin z}.$$

It is easy to verify that $f(0) = 0$ and $f'(0) = 1$. Then, Theorem 9.3 says that the inverse function f^{-1} exists in a neighborhood of the point $0 = f(0)$ and

$$z = f^{-1}(w) = \sum_{n=1}^{+\infty} d_n\, w^n, \tag{9.13}$$

where

$$d_n = \frac{1}{n!}\,\frac{d^{n-1}}{dz^{n-1}}\left(\left(\frac{z(\cos z - z\ \sin z)}{\sin z}\right)^n\right)\Bigg|_{z=0}, \qquad n \in \mathbb{N}.$$

The function in these equalities is even. Therefore its odd derivative is odd, meaning $d_n = 0$ for all even $n \geq 2$. For $n = 1$, we directly calculate $d_1 = 1$.

The last equation in (9.12) can be rewritten with the help of the function f as follows

$$f(t) = \frac{1}{\pi p}, \quad \text{or} \quad t = f^{-1}\Big(\frac{1}{\pi p}\Big).$$

Keeping in mind that $t = x_p - \pi p$ and using (9.13), we find from the last equation that

$$x_p - \pi p = \frac{1}{\pi p} + d_3\frac{1}{(\pi p)^3} + d_5\frac{1}{(\pi p)^5} + \dots,$$

whence we obtain the following asymptotics of the roots of the transcendental Eq. (9.11):

$$x_p = \pi p + \frac{1}{\pi p} + \mathcal{O}\Big(\frac{1}{p^3}\Big) \quad \text{as} \quad p \to +\infty.$$

▲

Theorem 9.3 states that an analytic function f is invertible in some neighborhood of a given point z_0 if $f'(z_0) \neq 0$. But what can be said if this condition is not satisfied and we have the following: there exists a unique number $p \in \mathbb{N}$ ($p \geq 2$) such that

$$f'(z_0) = \ldots = f^{(p-1)}(z_0) = 0 \quad \text{and} \quad f^{(p)}(z_0) \neq 0.$$

Then, as in (9.5), we can conclude that there exists a number $r > 0$ such that

$$f(z) \neq w_0 \quad \text{and} \quad f'(z) \neq 0 \quad \text{for all } z \in \overline{B_r(z_0)} \setminus \{z_0\}.$$

But now the Eq. (9.6) becomes

$$Z_{f-w_1} = Z_{f-w_0} = p \quad \text{in } B_r(z_0). \tag{9.14}$$

This means there exists a set of distinct points $\{z_1 \neq \ldots \neq z_p\} \subset B_r(z_0) \setminus \{z_0\}$ such that

$$f(z_k) = w_1 \quad \text{for all } k \in \{1, \ldots, p\}. \tag{9.15}$$

So we can speak of a p-valued inverse function f^{-1} in the disk $B_\mu(w_0)$, where μ is defined by (9.2).

To find its representation, we write the Taylor series of the function f around the point z_0

$$f(z) = f(z_0) + (z - z_0)^p \sum_{n=p}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-p}, \quad z \in B_r(z_0).$$

Denoting

$$\varphi(z) := \sum_{n=p}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-p}, \quad z \in B_r(z_0),$$

we obtain from the previous equation that $f(z) = w_0 + (z - z_0)^p \varphi(z)$. Obviously, $\varphi \in \mathcal{A}\big(B_r(z_0)\big)$ and can regard that $\varphi(z) \neq 0$ for all $z \in B_r(z_0)$, since $f^{(p)}(z_0) \neq 0$. Thus, for all $z \in B_r(z_0)$

$$w - w_0 = (z - z_0)^p \varphi(z) \implies \sqrt[p|0]{w - w_0}\, e^{-i\frac{2\pi k}{p}} = \psi(z) \tag{9.16}$$

for $k \in \{0, 1, \ldots, p-1\}$, where $\psi(z) := (z - z_0)\, \sqrt[p|0]{\varphi(z)}$ and

$$\sqrt[p|0]{z} := \sqrt[p]{|z|}\, e^{i\frac{\arg z}{p}}$$

is the 0th-branch of the global analytic function $\{\sqrt[p]{z}\}$.

Since $\varphi \neq 0$ in $B_r(z_0)$, the function ψ is single-valued and analytic in $B_r(z_0)$, and, moreover, $\psi(z_0) = 0$ and $\psi'(z_0) \neq 0$. Therefore, according to Theorem 9.3, the inverse function $\psi^{-1}\colon B_{\mu_1}(0) \mapsto B_r(z_0)$ exists and

$$\psi^{-1}(\xi) = \sum_{n=0}^{+\infty} d_n\, \xi^n \quad \text{for all } \xi \in B_{\mu_1}(0),$$

where $d_0 = z_0$ and

$$d_n = \lim_{z \to z_0} \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{(z - z_0)^n}{\psi^n(z)} \right), \quad n \in \mathbb{N}.$$

Then, it follows from (9.16) that

$$z_k = \psi^{-1}\Big(\sqrt[p|0]{w - w_0}\, e^{-i\frac{2\pi k}{p}} \Big) = \sum_{n=0}^{+\infty} d_n \left(\sqrt[p|0]{w - w_0}\, e^{-i\frac{2\pi k}{p}} \right)^n,$$

for all $k \in \{0, 1, \ldots, p-1\}$. Combining these branches into a global analytic function, the previous equalities can be rewritten as a series

$$z = \sum_{n=0}^{+\infty} d_n\, (w - w_0)^{\frac{n}{p}}, \quad w \in B_{\mu_2}(w_0),$$

which is called *Puiseux series*. Here, $\mu_2 := \mu_1^p < \mu$.

9.3 Conformal Isomorphisms and Automorphisms

Section 3.4 studied fractional-linear isomorphisms and automorphisms. Here we generalize and continue these studies.

Definition 9.1 Domains Ω_1 and Ω_2 in $\overline{\mathbb{C}}$ are said to be conformally equivalent if there exists a conformal univalent mapping f that maps Ω_1 onto Ω_2. In this case, the mapping $f: \Omega_1 \mapsto \Omega_2$ is called a conformal isomorphism of Ω_1 onto Ω_2. ▲

It is clear that if $f: \Omega_1 \mapsto \Omega_2$ is a conformal isomorphism of Ω_1 onto Ω_2, then there exists an inverse mapping that is a conformal isomorphism of Ω_2 onto Ω_1.

Definition 9.2 A conformal isomorphism of a domain Ω onto itself is called a conformal automorphism of Ω. ▲

The set of all conformal automorphisms of a domain Ω is a group concerning the composition of mappings. It is denoted by $\mathrm{Aut}(\Omega)$.

Lemma 9.3 (On the Set of Conformal Isomorphisms) *Let f_0 be a conformal isomorphism of Ω_1 onto Ω_2. Then for any conformal isomorphism $f : \Omega_1 \mapsto \Omega_2$ there is a conformal automorphism $\varphi \in \mathrm{Aut}(\Omega_2)$ such that*

$$f = \varphi \circ f_0.$$

Proof For any conformal isomorphism $f : \Omega_1 \mapsto \Omega_2$, we define the following mapping $\varphi := f \circ f_0^{-1}: \Omega_2 \mapsto \Omega_2$, which obviously belongs to $\mathrm{Aut}(\Omega_2)$. Then

$$\varphi \circ f_0 = (f \circ f_0^{-1}) \circ f_0 = f \circ (f_0^{-1} \circ f_0) = f$$

that had to be proved. □

Exercise 9.6 Let Ω_1 and Ω_2 be conformally equivalent domains. Prove that the groups $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_2)$ are isomorphic (i.e., there is a one-to-one correspondence between the elements of the groups that preserves the given group operations).

The domains $\overline{\mathbb{C}}$, $\mathbb{C}$, and $B_1 := B_1(0)$ are called *canonical domains*. Next, for each canonical domain, we find the group of its conformal automorphisms.

It follows from Theorems 3.2 and 3.3 that the set of all fractional-linear mappings

$$\Lambda = \left\{ \mathfrak{F}(z) = \frac{az+b}{cz+d} : \ a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0 \right\}$$

is a subgroup of $\mathrm{Aut}\big(\overline{\mathbb{C}}\big)$, and from Proposition 3.2 that the set

$$\left\{ \mathfrak{F}(z) = e^{i\beta} \frac{z-b}{1-\overline{b}\, z} : \ \ b \in B_1, \ \beta \in \mathbb{R} \right\}$$

is a subgroup of $\mathrm{Aut}\big(B_1\big)$. It is also clear that the set of all linear mappings

$$\left\{ w = az + b : \ \ a, b \in \mathbb{C}, \ a \neq 0 \right\}$$

is a subgroup of $\text{Aut}(\mathbb{C})$. In fact, it is possible to put equal signs in these inclusions, as the following theorem shows.

Theorem 9.4 (On Conformal Automorphisms of Canonical Domains) *Every conformal automorphism of a canonical domain is its fractional-linear automorphism.*

Proof

1. Let $\varphi \in \text{Aut}(\overline{\mathbb{C}})$. Then, there is a unique point $z_0 \in \overline{\mathbb{C}}$ such that $\varphi(z_0) = \infty$. It can be assumed that $z_0 \in \mathbb{C}$. In the opposite case, it is necessary to consider the function $\varphi(\frac{1}{z})$.

Thus φ is a meromorphic function and z_0 is its unique pole. Suppose that it is of order n and $n \geq 2$. Then z_0 is a zero of multiplicity n for the function $f := \frac{1}{\varphi}$. Therefore,

$$f'(z)|_{z=z_0} = \ldots = f^{(n-1)}(z)|_{z=z_0} = 0, \quad f^{(n)}(z)|_{z=z_0} \neq 0,$$

and this means (see (9.15)) that f is not univalent in a neighborhood of z_0, which contradicts the univalence of the function φ. Thus, z_0 is a simple pole of φ. Then, by Theorem 6.9 (or more exactly by the formula (6.27)),

$$\varphi(z) = \frac{A}{z - z_0} + B, \quad \text{if } z_0 \neq \infty \quad (A \neq 0),$$
$$\varphi(z) = Az + B, \quad \text{if } z_0 = \infty \quad (A \neq 0).$$

Hence, φ is a fractional-linear mapping.

2. Let $\varphi \in \text{Aut}(\mathbb{C})$. Then, the point at infinity is an isolated point of the function φ. If ∞ is removable, then due to Lemma 6.1, $\varphi \equiv$ const. If ∞ is essential for φ, then by Picard's great theorem (Theorem 6.7) the function φ is not univalent, which cannot be the case. Thus, ∞ is a pole. Similar to the first point of the proof, we show that ∞ is a simple pole of φ and

$$\varphi(z) = Az + B \qquad (A \neq 0).$$

3. Let $\varphi \in \text{Aut}(B_1)$. Then $\varphi(0) = w_0 \in B_1$.

Consider the function $\zeta = f(z) := \mathfrak{F}(\varphi(z)),\ z \in B_1$, where

$$\mathfrak{F}(w) = \frac{w - w_0}{1 - \overline{w}_0\, w} \tag{9.17}$$

is a fractional-linear automorphism of B_1, which maps the point w_0 to the origin. Clearly, $f(0) = 0$. Thus, the function $f \in \text{Aut}(B_1)$, and it satisfies all the conditions of Schwarz's Lemma 9.1. Therefore, $|f(z)| \leq |z|$ for all $z \in B_1$.

Obviously, the inverse function

$$z = f^{-1}(\zeta) = \varphi^{-1}\big(\mathfrak{F}^{-1}(\zeta)\big), \quad \zeta \in B_1,$$

also satisfies the conditions of Schwartz's lemma, and hence $|f^{-1}(\zeta)| \le |\zeta|$ for all $\zeta \in B_1$. It follows from this inequality that $|z| \le |f(z)|$ for all $z \in B_1$. Given this inequality and considering the inverse inequality proved in the previous paragraph, we obtain that $|f(z)| = |z|$ for all $z \in B_1$.

Then, according to the second statement of the Schwarz lemma, we get

$$f(z) = e^{i\beta} z \quad \text{for all } \; z \in B_1,$$

where $\beta \in \mathbb{R}$, or $\varphi(z) = \mathfrak{F}^{-1}(e^{i\beta} z)$ for all $z \in B_1$. Thus, φ is a fractional-linear automorphism of B_1.

□

Now we want to find out: Can canonical domains be conformally equivalent?

If there is a conformal isomorphism $f\colon \mathbb{C} \mapsto B_1$, then $f \in \mathcal{A}(\mathbb{C})$ and $|f(z)| < 1$ for all $z \in \mathbb{C}$. According to Liouville's theorem (Theorem 5.4), the function f is constant, which cannot be the case. For the same reasons, there is no conformal isomorphism $f\colon \overline{\mathbb{C}} \mapsto B_1$. Now let $f\colon \overline{\mathbb{C}} \to \mathbb{C}$ be a conformal isomorphism. Then $f \in \mathcal{A}(\mathbb{C})$ and $|f(\infty)| < +\infty$, from which again $f \equiv$ const. This is why the canonical domains cannot be conformally isomorphic to each other.

Let's find out when a domain Ω is conformally equivalent to one of the canonical domains. If $\partial\Omega = \emptyset$, then $\Omega = \overline{\mathbb{C}}$. If $\partial\Omega = \{z_0\}$, then $\Omega = \overline{\mathbb{C}} \setminus \{z_0\}$ and the fractional-linear function $\frac{1}{z-z_0}$ maps it onto $\mathbb{C}$. If Ω is a simply connected domain whose boundary contains more than one point, then according to Riemann's theorem (see Theorem 2.5), which we are preparing to prove in Sect. 9.5, the domain Ω is conformally equivalent to B_1.

9.4 Montel's Theorem

Corollary 5.5 has introduced to us one of the characteristics of convergent analytic functions. Another fundamental convergence property is Montel's theorem on compactness conditions for a family of analytic functions, obtained by the French mathematician Paul Montel (1876–1975).

Definition 9.3 Let $\mathfrak{M}$ be a set of functions defined on a domain Ω. The set $\mathfrak{M}$ is said to be locally uniformly bounded on Ω if for any bounded domain G, which together with its closure, belongs to Ω, there exists a constant C such that

$$|f(z)| \le C \quad \text{for all} \quad f \in \mathfrak{M} \quad \text{and for all} \; z \in G.$$

Definition 9.4 Let $\mathfrak{M}$ be a set of functions defined on a domain Ω. The set $\mathfrak{M}$ is called locally equicontinuous on Ω if for any bounded domain G, which together with its closure, belongs to Ω and for an arbitrary $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, G) > 0$ such that

$$|f(z_1) - f(z_2)| < \varepsilon \quad \text{for all} \;\; f \in \mathfrak{M} \quad \text{and for all} \;\; z_1, z_2 \in G \;\; \text{with} \;\; |z_1 - z_2| < \delta.$$

Lemma 9.4 *Let $\mathfrak{M}$ be some set of analytic functions in a domain Ω. If $\mathfrak{M}$ is locally uniformly bounded on Ω, then it is locally equicontinuous on Ω.*

Proof Take any bounded domain G, which together with its closure belongs to Ω. We denote the distance from G to the boundary $\partial\Omega$ by 2ρ. Then, the ρ neighborhood $G_\rho := \cup_{z\in G} B_\rho(z)$ of G also belongs to Ω with its closure.

It follows from the theorem's condition and Definition 9.3 that there is a constant C such that

$$|f(z)| \le C \quad \text{for all} \quad f \in \mathfrak{M} \quad \text{and for all} \;\; z \in G_\rho.$$

Then, for all $z_1 \in G$, for all $z \in B_\rho(z_1)$ and for all $f \in \mathfrak{M}$ we have

$$|f(z) - f(z_1)| \le |f(z)| + |f(z_1)| \le 2C. \tag{9.18}$$

For an arbitrary function $f \in \mathfrak{M}$ define the function

$$g(\xi) := \frac{1}{2C}\big(f(\rho\,\xi + z_1) - f(z_1)\big) \quad \text{for all} \;\; \xi \in B_1. \tag{9.19}$$

It is easy to see that the linear function $z = \rho\,\xi + z_1$ maps the unit disk B_1 onto the disk $B_\rho(z_1)$, and thanks to (9.18) the function g satisfies the conditions

$$1)\; g \in \mathcal{A}(B_1), \qquad 2)\; g\colon B_1 \mapsto B_1, \qquad 3)\; g(0) = 0.$$

Therefore, by Schwartz's lemma, we have $|g(\xi)| \le |\xi|$ for all $\xi \in B_1$, or

$$\left|g\big(\tfrac{1}{\rho}(z - z_1)\big)\right| \le \left|\tfrac{1}{\rho}(z - z_1)\right| \quad \text{for all} \;\; z \in B_\rho(z_1).$$

In view of (9.19), the last inequality can be rewritten as

$$|f(z) - f(z_1)| \le \frac{2C}{\rho}\,|z - z_1| \quad \text{for all} \;\; z \in B_\rho(z_1). \tag{9.20}$$

Thus, for any $\varepsilon > 0$ one can choose

$$\delta := \min\left\{\rho,\; \frac{\varepsilon\,\rho}{2\,C}\right\}$$

so that for all $f \in \mathfrak{M}$ and for all $z_1, z_2 \in G$ with $|z_1 - z_2| < \delta$ based on (9.20) we have $|f(z_1) - f(z_2)| < \varepsilon$. This means that the set $\mathfrak{M}$ is locally equicontinuous on Ω. □

Definition 9.5 A set $\mathfrak{M}$ of functions defined on a domain Ω is called locally precompact if, for any compact set (bounded and closed set) from Ω, every sequence from $\mathfrak{M}$ contains a subsequence that converges uniformly on that compact.

Theorem 9.5 (Montel's Theorem) *Let $\mathfrak{M}$ be a set of analytic functions defined on a domain Ω. If $\mathfrak{M}$ is locally uniformly bounded, then $\mathfrak{M}$ is locally precompact.*

Proof

1. Consider the set

$$E := \Omega \cap \{\mathbb{Q} \times \mathbb{Q}\} = \{z_j\}_{j \in \mathbb{N}};$$

it is clear that it is a countable and dense subset of Ω.

Take an arbitrary sequence $\{f_n\}_{n \in \mathbb{N}}$ from $\mathfrak{M}$.

According to the condition of the theorem, the sequence $\{f_n(z_1)\}_{n \in \mathbb{N}}$ is bounded. Therefore, there exists a convergent subsequence $\{f_{n_k}(z_1)\}_{k \in \mathbb{N}}$, which we denote by $\{f_{n,1}(z_1)\}_{n \in \mathbb{N}}$. Now consider the sequence $\{f_{n,1}(z_2)\}_{n \in \mathbb{N}}$ that is also bounded, hence there is a convergent subsequence

$$\{f_{n_k,1}(z_2)\}_{k \in \mathbb{N}} =: \{f_{n,2}(z_2)\}_{n \in \mathbb{N}}.$$

Thus, the sequence $\{f_{n,2}(\cdot)\}_{n \in \mathbb{N}}$ already converges at two points z_1 and z_2.

Continuing this process, we get a subsequence $\{f_{n,k}(\cdot)\}_{n \in \mathbb{N}}$ which is convergent at the points $z_1, \dots, z_k$ from the set E ($k \in \mathbb{N}$).

Now consider the diagonal sequence $\{f_{n,n}\}_{n \in \mathbb{N}}$. It converges at each point $z_p \in E$ since $\{f_{n,n}(z_p)\}_{n \in \mathbb{N}}$ is a subsequence of $\{f_{n,p}(z_p)\}_{n \in \mathbb{N}}$ for $n \geq p$.

2. Now take an arbitrary compact G from the domain Ω and any $\varepsilon > 0$. Thanks to Lemma 9.4, the set $\mathfrak{M}$ is locally equicontinuous on Ω. Therefore, according to Definition 9.3, one can choose $\delta > 0$ such that for all $f \in \mathfrak{M}$ and for all $z_1, z_2 \in G$ with $|z_1 - z_2| < \delta$ we have

$$|f(z_1) - f(z_2)| < \frac{\varepsilon}{3}. \tag{9.21}$$

Obviously, $\Omega \subset \bigcup_{j=1}^{+\infty} B_\delta(z_j)$. The compactness of G implies the existence of a finite subcover of G. We can assume, without loss of generality, that there exists a finite number of points $\{z_1, \dots, z_p\} \subset G$ such that $G \subset \bigcup_{j=1}^{p} B_\delta(z_j)$.

According to what was proved in the first point, the sequences

$$\{f_{n,n}(z_1)\}_{n \in \mathbb{N}}, \dots, \{f_{n,n}(z_p)\}_{n \in \mathbb{N}}$$

are Cauchy sequences. Therefore,

$$\exists n_0 \in \mathbb{N} \ \forall n \geq n_0, \ \forall m \geq n_0 \ \ \forall j \in \{1, 2, \dots, p\} :$$
$$|f_{n,n}(z_j) - f_{m,m}(z_j)| < \frac{\varepsilon}{3}. \tag{9.22}$$

Now take any $z \in G$. Then there exists a $j_0 \in \{1, \dots, p\}$ such that the point $z \in B_\delta(z_{j_0})$, and using (9.21) and (9.22) we deduce that

$$|f_{n,n}(z) - f_{m,m}(z)| \leq |f_{n,n}(z) - f_{n,n}(z_{j_0})| + |f_{n,n}(z_{j_0}) - f_{m,m}(z_{j_0})|$$
$$+ |f_{m,m}(z_{j_0}) - f_{m,m}(z)| < \varepsilon.$$

Thus the Cauchy criterion for uniform convergence of $\{f_{n,n}(\cdot)\}_{n\in\mathbb{N}}$ on G is satisfied. According to Definition 9.5, the set $\mathfrak{M}$ is locally precompact.

□

Definition 9.6 Let $\mathfrak{M}$ be a set of functions defined on a domain Ω.

A functional $J : \mathfrak{M} \longmapsto \mathbb{C}$ is said to be continuous on $\mathfrak{M}$ if for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$, which converges uniformly on any compact $G \subset \Omega$ to a function $f_0 \in \mathfrak{M}$,

$$\lim_{n\to+\infty} J(f_n) = J(f_0).$$

Example 9.5 Let $\mathfrak{M} = \mathcal{A}(\Omega)$. Consider the functional

$$J(f) = \frac{1}{p!} \frac{d^p f(z)}{dz^p}\Big|_{z=a}, \qquad f \in \mathfrak{M},$$

where a is a point from Ω and p is a fixed positive integer. Let us show that J is continuous on $\mathfrak{M}$.

Solution Take any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$, which converges uniformly on any compact $G \subset \Omega$ to a function $f_0 \in \mathfrak{M}$. Then, for a closed disk $\overline{B_r(a)} \subset \Omega$ and for any $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$|f_n(z) - f_0(z)| < \varepsilon \quad \text{for all } \ n \geq n_0 \quad \text{and for all } \ z \in \overline{B_r(a)}.$$

Then, using Cauchy's inequalities for the coefficients of a power series (see Corollary 5.1), we have that for all $n \geq n_0$

$$|J(f_n) - J(f_0)| = \left| \frac{1}{p!} \frac{d^p(f_n - f_0)}{dz^p}\Big|_{z=a} \right| \leq \frac{\varepsilon}{r^p}.$$

This means that $\lim_{n\to+\infty} J(f_n) = J(f_0)$. ▲

Definition 9.7 Let $\mathfrak{M}$ be a locally precompact set of functions defined on a domain Ω. The set $\mathfrak{M}$ is said to be locally compact if, for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ that converges uniformly to f_0 on any compact set $G \subset \Omega$, the function f_0 necessarily belongs to $\mathfrak{M}$.

Lemma 9.5 *Let $\mathfrak{M}$ be a locally compact set of functions defined on a domain Ω, and let $J : \mathfrak{M} \longmapsto \mathbb{C}$ be a continuous functional.*

Then there exists a function $f_0 \in \mathfrak{M}$ such that

$$\sup_{f\in\mathfrak{M}} |J(f)| = |J(f_0)|.$$

Proof Denote by $A := \sup_{f\in\mathfrak{M}} |J(f)|$. Then there exists a maximising sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ such that $\lim_{n\to+\infty} |J(f_n)| = A$. Since the set $\mathfrak{M}$ is locally compact, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}} \subset \{f_n\}_{n\in\mathbb{N}}$ and $f_0 \in \mathfrak{M}$ such that f_{n_k} converges uniformly to f_0 on every compact set $G \subset \Omega$. The continuity of the functional J implies that

$$A = \lim_{k\to+\infty} J(f_{n_k}) = J(f_0),$$

which had to be proved. □

9.5 Riemann Mapping Theorem

We are now in a position to demonstrate the proof of Riemann's mapping theorem (see Theorem 2.5). It was first formulated and proved in 1851 by B. Riemann, using potential theoretical methods that were not yet fully substantiated at that time. For most modern proofs, Montel's theorem is used. This is the fastest way to prove this theorem. We will separate the formulation of Theorem 2.5 and carry out its proof in several steps.

Theorem 9.6 *An arbitrary simply connected domain $\Omega \subset \overline{\mathbb{C}}$ whose boundary contains more than one point is conformally equivalent to the unit disk B_1.*

Proof

1. The first thing we wand to do is to show that there is a univalent analytic function f_1 which maps Ω into B_1.

From the conditions of the theorem it follows that there are two different points $\alpha \neq \beta$ that lie on the boundary $\partial\Omega$. Since Ω is simply connected, two single-valued analytic branches φ_1 and φ_2 can be distinguished in Ω for the global analytic function

$$\left\{\sqrt{\frac{z-\alpha}{z-\beta}}\right\}.$$

Clearly, $\varphi_1(z) = -\varphi_2(z)$ for all $z \in \Omega$. Moreover, they are univalent. Indeed, if there are $z_1, z_2 \in \Omega$ such that $\varphi_j(z_1) = \varphi_j(z_2)$ $(j \in \{1, 2\})$, then

$$\frac{z_1-\alpha}{z_1-\beta} = \frac{z_2-\alpha}{z_2-\beta},$$

from which, based on the univalence of a fractional-linear mapping, we get $z_1 = z_2$.

Theorem 9.1 says that $\Omega_1^* := \varphi_1(\Omega)$ and $\Omega_2^* := \varphi_2(\Omega)$ are domains. Assume that $\Omega_1^* \cap \Omega_2^* \neq \emptyset$. Then there are two points $z_1, z_2 \in \Omega$ such that $\varphi_1(z_1) = \varphi_2(z_2)$. Hence

$$\frac{z_1-\alpha}{z_1-\beta} = \frac{z_2-\alpha}{z_2-\beta},$$

which means $z_1 = z_2$. So $\varphi_1(z_1) = \varphi_2(z_1) = -\varphi_1(z_1)$, whence $\varphi_1(z_1) = 0$. This means that it is necessary $z_1 = \alpha$. As a result, we have a contradiction, because $\alpha \notin \Omega$. Thus, $\Omega_1^* \cap \Omega_2^* = \emptyset$.

Since Ω_2^* is a domain, for any $w_0 \in \Omega_2^*$ there exists a $\rho > 0$ such that $\overline{B_\rho(w_0)} \subset \Omega_2^*$. Define the function

$$f_1(z) := \frac{\rho}{\varphi_1(z) - w_0}, \quad z \in \Omega.$$

Here the denominator is not equal to zero, because $\Omega_1^* \cap \Omega_2^* = \emptyset$. Then it is easy to see that $f_1 \in \mathcal{A}(\Omega)$, $|f_1(z)| < 1$ for all $z \in \Omega$, and f_1 is univalent in Ω as a composition of two univalent functions.

2. Fix any point $a \in \Omega$. By Theorem 7.8 (on sufficient conditions of conformality), $|f_1'(a)| > 0$. Now define the following function set

$$\mathfrak{M} := \Big\{ f \in \mathcal{A}(\Omega) : \ f \ \text{ is univalent in } \ \Omega, \quad |f'(a)| \geq |f_1'(a)|, \ \text{ and } \\ |f(z)| < 1 \ \text{ for all } \ z \in \Omega \Big\}.$$

The set $\mathfrak{M} \neq \emptyset$, since the function f_1 defined in the first item of the proof belongs to $\mathfrak{M}$. It follows from the Montel Theorem 9.5 that $\mathfrak{M}$ is a locally precompact set. Let us show that $\mathfrak{M}$ is locally compact.

Consider any sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ that converges uniformly to a function g on any compact set $G \subset \Omega$. By Corollary 5.5, the function $g \in \mathcal{A}(\Omega)$. In addition, from Example 9.5 it follows that $\lim_{n\to+\infty} g_n'(a) = g'(a)$. Since

$|g_n'(a)| \geq |f_1'(a)| > 0$, then and $|g'(a)| \geq |f_1'(a)| > 0$. By Corollary 7.1 from the Hurwitz theorem, either the function $g \equiv$ const in Ω (which is impossible because $|g'(a)| > 0$) or g is univalent in Ω. Next, since

$$|g_n(z)| < 1 \quad \text{for all } z \in \Omega \quad \text{and for all } n \in \mathbb{N},$$

and $|g(z)| \leq 1$. But the maximum modulus principle gives (see Theorem 9.2) that $|g(z)| < 1$ for all $z \in \Omega$. Thus, $g \in \mathfrak{M}$ and $\mathfrak{M}$ is locally compact.

3. Now define the functional $J(f) := |f'(a)|$ for all $f \in \mathfrak{M}$. According to Example 9.5, the functional J is continuous, and due to Lemma 9.5 there exists a function $f_0 \in \mathfrak{M}$ such that

$$\sup_{f \in \mathfrak{M}} |f'(a)| = |f_0'(a)|. \tag{9.23}$$

The next step will be a demonstration that f_0 is the conformal isomorphism of interest.

First, we prove that $f_0(a) = 0$. Assume that is not the case, i.e., $f_0(a) \neq 0$, and define the function

$$g_0(z) := \frac{f_0(z) - f_0(a)}{1 - \overline{f_0(a)}\, f_0(z)}, \quad z \in \Omega.$$

Since g_0 is a composition of a fractional-linear automorphism of B_1 (see (9.18)) and f_0, the function g_0 is analytic and univalent, and $|g(z)| < 1$ for all $z \in \Omega$. It is easy to calculate that

$$|g_0'(a)| = \left| \frac{f_0'(a)}{1 - |f_0(a)|^2} \right| = \frac{|f_0'(a)|}{1 - |f_0(a)|^2} > |f_0'(a)| \geq |f_1'(a)| > 0.$$

Thus, $g \in \mathfrak{M}$ and $|g'(a)| > |f_0'(a)|$. However, the last inequality is in contradiction with the equality (9.23). Therefore, $f_0(a) = 0$.

Now we show that the function $f_0 \colon \Omega \mapsto B_1$ is surjective. If this is not the case, then there exists a point $b \in B_1$ such that its preimage is empty, i.e., $f_0(z) \neq b$ for all $z \in \Omega$. Since $f_0(a) = 0$, the point $b \neq 0$. We can therefore define the function

$$\psi(z) := \sqrt{\frac{f_0(z) - b}{1 - \overline{b}\, f_0(z)}}, \quad z \in \Omega.$$

Here by $\sqrt{\cdot}$ we mean one of the branches of the global analytic function $\{\sqrt{\cdot}\}$, which is uniquely defined in Ω because $f_0(z) \neq b$ and $f_0(z) \neq \frac{1}{\overline{b}}$, $(\left|\frac{1}{\overline{b}}\right| > 1)$ for all $z \in \Omega$. Thus, ψ is a single-valued, analytic, and univalent function in Ω, and also $|\psi(z)| < 1$ for all $z \in \Omega$ (see the explanation above in this item).

Since $\psi(a) = \sqrt{-b}$, then $|\psi(a)|^2 = |b|$. Next we find

$$\psi'(a) = \frac{1}{2\sqrt{-b}} \frac{f_0'(a) - f_0'(a)\,|b|^2}{1} = f_0'(a)\frac{1-|b|^2}{2\sqrt{-b}}. \tag{9.24}$$

Now define this function

$$h(z) := \frac{\psi(z) - \psi(a)}{1 - \overline{\psi(a)}\,\psi(z)}, \quad z \in \Omega,$$

which is single-valued and analytic in Ω, and $|h(z)| < 1$ for all $z \in \Omega$. In addition, using (9.24), we find

$$|h'(a)| = \frac{|\psi'(a)|}{1 - |\psi(a)|^2} = |f_0'(a)|\,\frac{1-|b|^2}{2\sqrt{|b|}}\,\frac{1}{1-|b|} = |f_0'(a)|\,\frac{1+|b|}{2\sqrt{|b|}}.$$

Since $b \neq 0$, $\frac{1+|b|}{2\sqrt{|b|}} > 1$. And therefore,

$$|h'(a)| > |f_0'(a)| \geq |f_1'(a)| > 0.$$

Thus, $h \in \mathfrak{M}$ and $|h'(a)| > |f_0'(a)|$. However, the last inequality contradicts the equality (9.23). This means that the mapping $f_0 \colon \Omega \mapsto B_1$ is surjective.

On the basis of sufficient conformality conditions (see Theorem 7.8), the function f_0 is conformal in the domain Ω. Thereby, f_0 is conformal isomorphism of Ω onto B_1.

□

From this theorem follows immediately the following statement.

Corollary 9.2 *Any two simply connected domains in* $\overline{\mathbb{C}}$ *whose boundaries contain more than one point are conformally equivalent.*

It is clear that there is no unique conformal isomorphism of the domain Ω onto the unit disk B_1. This follows from the fact that the set of all conformal automorphisms is infinite (see Theorem 9.4 (part 3) and Proposition 3.2). However, taking into account the form of all conformal automorphisms of B_1, the following statement gives additional conditions under which there exists a unique conformal isomorphism Ω onto B_1.

Theorem 9.7 (On Uniqueness of Conformal Isomorphism) *Let Ω be a simply connected domain $\Omega \subset \overline{\mathbb{C}}$ whose boundary contains more than one point. Then for any $z_0 \in \Omega$ and any real number $\alpha \in (-\pi, \pi]$ there exists a unique conformal isomorphism $f\colon \Omega \mapsto B_1$ such that*

$$f(z_0) = 0 \quad \textit{and} \quad \arg f'(z_0) = \alpha. \tag{9.25}$$

Proof Let g be any conformal isomorphism of the domain Ω onto B_1 which exists by Theorem 9.6. Then, by Lemma 9.3 and Theorem 9.4 (part 3), an arbitrary conformal isomorphism $f : \Omega \mapsto B_1$ can be represented as $f = \mathfrak{F} \circ g$, where $\mathfrak{F} \in \operatorname{Aut} B_1$, i.e.,

$$f(z) = e^{i\theta} \frac{g(z) - a}{1 - \overline{a}\, g(z)}, \quad z \in \Omega,$$

where a is an arbitrary point from B_1 and $\theta \in \mathbb{R}$.

Take $a = g(z_0)$ and consider the following conformal isomorphism

$$f_1(z) = e^{i\theta} \frac{g(z) - g(z_0)}{1 - \overline{g(z_0)}\, g(z)}, \quad z \in \Omega.$$

It is easy to verify that $f_1(z_0) = 0$ and

$$f_1'(z)|_{z=z_0} = e^{i\theta} \frac{g'(z_0)}{1 - |g(z_0)|^2}, \qquad \operatorname{Arg} f_1'(z_0) = \theta + \arg g'(z_0).$$

Let us set $\theta = \alpha - \arg g'(z_0)$. Thus there exists a conformal isomorphism Ω onto B_1 which satisfies the relations (9.25). Next, we will show that it is unique.

Suppose there is another conformal isomorphism $f_2 : \Omega \mapsto B_1$ which satisfies the relations (9.25). Then $\varphi := f_1 \circ f_2^{-1} \in \operatorname{Aut} B_1$. Theorem 9.4 (part 3) says that φ is a fractional-linear automorphism of B_1, and moreover, $\varphi(0) = 0$,

$$\varphi'(0) = \frac{f_1'(z_0)}{f_2'(z_0)} \quad \text{and hence} \quad \arg \varphi'(0) = 0.$$

These equalities mean that $\varphi(w) = w$ for all $w \in B_1$, whence we get that $f_1(z) = f_2(z)$ for all $z \in \Omega$. □

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