Modal Analysis of a 2R Flexible Platform Using Screw Theory

Martín Pucheta(✉) and Alejandro Gallardo

Research Center in Informatics for Engineering (CIII) and National Scientific and Technical Research Council (CONICET), Córdoba Regional Faculty, National University of Technology (FRC-UTN), X5016ZAA Cordoba, Argentina

{mpucheta,agallardo}@frc.utn.edu.ar

Abstract. Compliant mechanisms perform precision movements generated by the elastic deformation of their flexible elements. For most applications, resonance frequencies must be bounded by a defined range of values. Therefore, a modal analysis of the mechanisms is required at the initial stage of design. In this work, a novel flexible platform with 2 degrees of rotational freedom restricted by beam-type flexure elements is analyzed. The Screw Theory description of the movement of the mechanism is used for the rigid and the flexible elements. Using this formalism, the stiffness and mass matrices for a two-node beam element are derived and can be assembled using a standard finite-element-like procedure to perform the modal analysis of the mechanism. Starting from the desired movements, the mechanism is synthesized using screws and a hybrid (series/parallel) solution is proposed. The analytical results for the modal analysis obtained by Screw Theory are compared with the results of finite element analysis. Due to the computational efficiency, the analytical equations are chosen to be applied in the optimization of the designs.

Keywords: 3D compliant mechanisms · 2R flexible platform · Beam Theory · Screw Theory · modal analysis

1 Introduction

Compliant mechanisms have wide applications in the field of precision engineering, medical devices, and optical instrumentation, among others [1, 2]. In the initial stage of design and optimization, a modal analysis of the mechanisms must be carried out [2–4] to satisfy requirements on the resonant frequencies.

For parallel and series flexible mechanisms constrained by plate and beam-type elements there are well-established methods for their linear static analysis applying Screw Theory [5]. Using screws, the kinematics of rigid bodies linked by clamped-clamped flexible elements is determined by a 6×6 flexibility matrix (and its inverse, the 6×6 stiffness matrix) [4, 6, 7]. Hopkins et al. [6] derive the stiffness matrix of hybrid topologies considering relative screw displacements starting from the ground of

A. Vizán Idoipe and J. C. García Prada (Eds.): IACME 2022, *Proceedings of the XV Ibero-American Congress of Mechanical Engineering*, pp. 50–56, 2023.
https://doi.org/10.1007/978-3-031-38563-6_8

the structure. Wu et al. [7] firstly determines the potential energy of the structure using the 6×6 flexibility matrices for each flexure element; subsequently, they determine the minimum state of the energy and obtains the stiffness matrix of the complete mechanism.

In this work, the Screw Theory formalism by Ding and Selig [8, 9] is applied to obtain the stiffness and mass matrices of a two-node beam element that is assembled applying the standard techniques of the finite element method [10] to build the stiffness and mass matrices of the complete flexible mechanism required to perform its modal analysis. This enables analysts to consider more than one beam element per flexure element, increasing the accuracy in the modal analysis. As an application example, a flexible platform with a hybrid structure and 2 rotational degrees of freedom (RR) restricted with beam-type flexure elements is analyzed. This kind of platform is mainly used in optical applications for guiding light or laser beams through mirrors of several scales [2]. The analytical results obtained through the Screw Theory are compared with the results of finite element analysis.

2 Beam Element Derivation Using Screw Theory

The beam element that will be developed in this work is a slender beam with a straight centroidal axis, with a constant cross section, and subjected to external loads applied only at its end nodes, see Fig. 1(a).

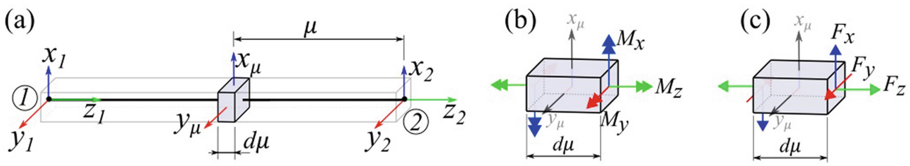

Fig. 1. Notation for a two-node beam element

A differential segment of the beam element is selected, see Fig. 1(a). This segment will be analyzed as a beam of length $d\mu$ with a reference system located in the center of the segment. When a moment is applied at one of the ends of the beam of length $d\mu$, on the other end there must be a moment with equal direction and magnitude, but in the opposite sense to enforce static equilibrium, see Fig. 1(b). Equations (1)–(3) are the constitutive relationships for the moments of a Euler-Bernoulli beam [10]

$$M_z = GI_p \frac{d\theta_z}{d\mu} \rightarrow d\theta_z(\mu) = \frac{d\mu}{GI_p} M_z \quad (1)$$

$$M_y = EI_y \frac{d\theta_y}{d\mu} \rightarrow d\theta_y(\mu) = \frac{d\mu}{EI_y} M_y \quad (2)$$

$$M_x = EI_x \frac{d\theta_x}{d\mu} \rightarrow d\theta_x(\mu) = \frac{d\mu}{EI_x} M_x \quad (3)$$

where E is the longitudinal elasticity modulus, G is the transversal elasticity modulus, I_x (I_y) is the moment of inertia of the section with respect to the $x(y)$ axis, and I_p is the polar moment of inertia of the section. Throughout the differential segment the internal moment forces are constant. Therefore, the variation of rotations between the extreme sections can be obtained from the constitutive relations given by Eqns. (1)–(3). The same differential beam segment is analyzed for the applied forces at its ends, see Fig. 1(c). The forces transversal to the beam axis generate distortions and bending moments. If $d\mu \to 0$, then the moments produced by these forces will tend to zero. In addition, in the Euler-Bernoulli beam hypotheses, the deformations due to distortions are negligible. Therefore, F_z is the only force that generates an appreciable deformation and its constitutive relationship is expressed as

$$F_z = EA\frac{d\delta_z}{d\mu} \to d\delta_z(\mu) = \frac{d\mu}{EA}F_z \tag{4}$$

Equations (1)–(4) can be expressed in compact form

$$\boldsymbol{dt} = \boldsymbol{c}.\boldsymbol{Q}.\boldsymbol{w}$$

$$\begin{bmatrix} d\theta_x \\ d\theta_y \\ d\theta_z \\ d\delta_x \\ d\delta_y \\ d\delta_z \end{bmatrix} = \begin{bmatrix} d\mu/EI_x & 0 & 0 & & & \\ & d\mu/EI_y & 0 & & \mathbf{0}_{3\times3} & \\ & & d\mu/GI_p & & & \\ & & & 0 & 0 & 0 \\ & sym. & & & 0 & 0 \\ & & & & & d\mu/EA \end{bmatrix} \boldsymbol{Q} \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} \tag{5}$$

where d$\boldsymbol{t}$ is the differential deflection screw, $\boldsymbol{c}$ is called the flexibility density matrix, $\boldsymbol{Q}$ is an exchange matrix, and $\boldsymbol{w}$ is the wrench [8, 9].

A possible boundary condition of the beam element shown in Fig. 1(a) is clamped at node 1 and free at node 2, where a wrench is applied. The displacement twist at the free end can be obtained by integrating Eq. (5). To integrate the differential deflection screw d$\boldsymbol{t}$ of each cross section, they must be expressed in a common reference frame [8], for example, in a frame located at node 2 as

$$\begin{aligned} \textstyle\int_{-L}^{0} d\boldsymbol{t}_2 = \left(\textstyle\int_{-L}^{0} \boldsymbol{H}_{2\mu}\boldsymbol{c}\boldsymbol{Q}\boldsymbol{H}_{2\mu}^{-1}\right)\boldsymbol{w}_2 \to \boldsymbol{t}_2 = \boldsymbol{C}_{22}\boldsymbol{w}_2 \\ \boldsymbol{w}_2 = \boldsymbol{k}_{22}\boldsymbol{t}_2 \end{aligned} \tag{6}$$

where L is the beam length, d$\boldsymbol{t}_2$ is the differential deflection screw of the differential element $d\mu$ expressed in the reference system of node 2, $\boldsymbol{w}_2$ is the wrench applied on node 2, and $\boldsymbol{H}_{2\mu}$ is the matrix that changes the coordinates (a passive translation in μ) from the system located at coordinate μ to a system located at the node 2.

The reactions at the clamped end of the beam must ensure equilibrium

$$(\boldsymbol{w}_1)_2 = -\boldsymbol{w}_2 \tag{7}$$

where $(\boldsymbol{w}_1)_2$ is the wrench applied on node 1 expressed in the reference system of node 2. Taking the expression of $\boldsymbol{w}_2$ from Eq. (6), then using Eq. (7) and applying to $(\boldsymbol{w}_1)_2$ a change of coordinates $\boldsymbol{H}_{21}$ from a node 2 to a node 1 frame, the stiffness matrix that relates the twist of node 2 with the wrench applied on node 1 can be computed as

$$\begin{aligned}(\boldsymbol{w}_1)_2 &= -\boldsymbol{k}_{22}\boldsymbol{t}_2 \\ \boldsymbol{w}_1 &= \left(-\boldsymbol{H}_{21}^{-1}\boldsymbol{k}_{22}\right)\cdot\boldsymbol{t}_2 \\ \boldsymbol{w}_1 &= \boldsymbol{k}_{12}\cdot\boldsymbol{t}_2\end{aligned} \tag{8}$$

where $\boldsymbol{w}_1$ is the wrench at node 1 expressed in the reference frame of node 1.

By clamping the node 2 and applying a wrench on node 1, the same procedure as above is used to get the stiffness matrices $\boldsymbol{k}_{11}$ and $\boldsymbol{k}_{21}$. Finally, the wrench – deflection screw relationship for the beam element can be put in compact form as

$$\begin{bmatrix}\boldsymbol{w}_1\\ \boldsymbol{w}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{k}_{11} & \boldsymbol{k}_{12}\\ \boldsymbol{k}_{21} & \boldsymbol{k}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{t}_1\\ \boldsymbol{t}_2\end{bmatrix} = \boldsymbol{K}_e\begin{bmatrix}\boldsymbol{t}_1\\ \boldsymbol{t}_2\end{bmatrix} \tag{9}$$

The stiffness matrix $\boldsymbol{K}_\mathrm{e}$ obtained in Eq. (9) is equal to that obtained by classical methods for structural analysis, like the Direct Stiffness Method and the Finite Element Method, where bending, torsion, and tension are analyzed in a decoupled way [10]. To compute the total stiffness matrix of a flexible mechanism, the stiffness matrix of each flexible element must be assembled in the same way as is done in the finite element method. Thus, before performing the assembly, each local stiffness matrix $\boldsymbol{K}_e$ must be expressed in a single global reference system, usually located in the moving platform.

In this work, a concentrated mass matrix [10] is adopted, where it is not necessary to assume and/or to know the internal displacements of the beam. By assuming the node 1 of the element as clamped and applying forces at node 2, the Second Newton's Law is expressed as $\boldsymbol{w}_2 = \boldsymbol{m}\,\boldsymbol{Q}\,\ddot{\boldsymbol{t}}_2$ where the mass matrix is defined as

$$\boldsymbol{m}_{6\times 6} = \rho\cdot A\cdot L\cdot diag\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\alpha L^2,\alpha L^2,\frac{I_P}{2A}\right) \tag{10}$$

ρ is the density of the beam material, and α is a non-negative parameter between 0 and 1/50; here $\alpha = 1/100$ is adopted. The same procedure as above is developed for a clamped node 2 and a wrench applied on node 1. Then, by combining this result with that of node 2, the local mass matrix of the beam element is obtained

$$\begin{bmatrix}\boldsymbol{w}_1\\ \boldsymbol{w}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{m}\cdot\boldsymbol{Q} & \boldsymbol{0}_{3\times 3}\\ \boldsymbol{0}_{3\times 3} & \boldsymbol{m}\cdot\boldsymbol{Q}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{t}}_1\\ \ddot{\boldsymbol{t}}_2\end{bmatrix} = \boldsymbol{M}_e\begin{bmatrix}\ddot{\boldsymbol{t}}_1\\ \ddot{\boldsymbol{t}}_2\end{bmatrix} \tag{11}$$

3 Modal Analysis

The dynamic equation of undamped flexible mechanisms without external excitation is $\boldsymbol{M}\cdot\ddot{\boldsymbol{T}}+\boldsymbol{K}\cdot\boldsymbol{T} = 0$, where $\boldsymbol{T}$ is a column vector containing the deflection screws of each node of the structure, $\boldsymbol{M}$ and $\boldsymbol{K}$, are respectively, the global mass and stiffness matrices of the entire mechanism. For small oscillations, harmonic motions are assumed [4, 9, 10] and this equation is transformed into a generalized eigenvalue problem $\left(-\omega^2\boldsymbol{M}+\boldsymbol{K}\right)\cdot\boldsymbol{T} = 0$, where the solutions for ω are the natural frequencies.

The 2R flexible mechanism shown in Fig. 2(a) has two rotational degrees of freedom around two concurrent axes of rotation, x and y. It was designed as two parallel sub-mechanisms in series: An intermediate platform is linked to the moving one by the flexure elements shown in Fig. 2 (b) that allow rotation around the x axis. Then, flexure elements that link the foundation and the intermediate platform are added, allowing the rotation around the y axis; see Fig. 2 (c). The material of the mechanism is aluminum with a density of $\rho = 7700$ kg/m^3, a longitudinal elasticity modulus $E = 2.1 \cdot 10^{11}$ N/m^2 and transversal modulus $G = 8 \cdot 10^{10}$ N/m^2. The flexible elements are 30 mm long and have a cross section with 0.2 mm thick and 5 mm wide.

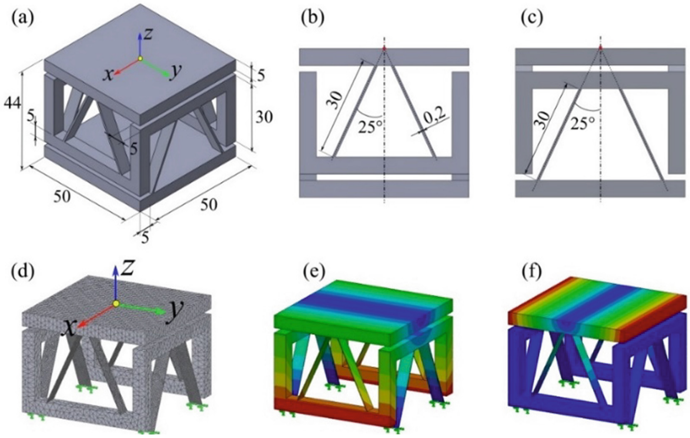

Fig. 2. Dimensions (in mm) of the 2R mechanism (a, b, c) and FEM simulation (d, e, f).

All flexure elements are modeled with the proposed beam for three cases of modal analysis: Considering 1 beam element for each flexure element, (i) including the mass of the beam (ii) and ignoring the mass of the beam, respectively, and (iii) by considering 2 beam elements for each flexure element where the mass of the beam is taken into account, otherwise the mass matrix would be singular.

The values obtained for the first 6 vibration modes are very similar to each other and are shown in Table 1. The results obtained with the proposed method are compared with results that were obtained by the finite element method shown in the 2nd column of Table 1. In Fig. 2(d) the mesh of high-order quadratic tetrahedral elements consisting of 100630 nodes is shown. Figures 2(e) and 2(f) show the results for the first two natural frequencies corresponding to the two desired rotational degrees of freedom. It should be noted that using 2 beam elements per flexure element has less error in the 6th mode compared to using 1 element per flexure element. The execution times to calculate the natural frequencies with the proposed analytical method are 100 orders less if they are compared to results obtained through the finite element method.

Table 1. Natural frequencies (in Hz) of the 2R flexible mechanism

Mode	FEM	(i) With beam mass 1 Element	(ii) W/O beam mass 1 Element	(iii) With beam mass 2 Elements
1st	141.21	140.33	141.82	140.33
2nd	169.52	165.28	165.42	165.28
3rd	2041.2	2101.61	2109.19	2101.61
4th	2422.2	2528.11	2532.47	2528.11
5th	4741.1	5255.80	5279.7	5244.43
6th	7432.3	9742.33	9771.67	7325.57

4 Conclusions

In this work, the modal analysis of flexible platforms with application to precision devices with small displacements and deformations was presented. A stiffness matrix for a Euler-Bernoulli beam was derived by applying the Screw Theory formalism. This matrix allows to be assembled in a simple and direct way to carry out static analysis of flexible mechanisms and, together with a mass matrix of concentrated parameters, it also allows to carry out the modal analysis of flexible mechanisms. Using the proposed beam, the flexure elements of a flexible mechanism with two rotational degrees of freedom were modeled to analyze their vibration modes. The results of the modal analysis obtained with the proposed analytical method were validated with the finite element method and its accuracy was acceptable and had faster execution than the FEM, highlighting that the number of beam elements per flexure element is important to capture the correct physics of the modal analysis.

References

1. Howell, L.L., Magleby, S.P., Olsen, B.M.: Handbook of Compliant Mechanisms. John Wiley & Sons, New York (2013)
2. Song, Y., Panas, R.M., Hopkins, J.B.: A review of micromirror arrays. Precis. Eng. **51**, 729–761 (2018)
3. Hopkins, J.B., Song, Y.P., Wang, S.Y., Behbahani, A.H., Josefson, I.: Optimal actuation of dynamically driven serial and hybrid flexure systems. In: Proceedings of the ASME IDETC/CIE 2014, pp. DETC2014-35181, Buffalo, NY, USA (2014)
4. Li, Z., Kota, S.: Dynamic Analysis of Compliant Mechanisms. Volume 5: 27th Biennial Mechanisms and Robotics Conference (2002)
5. Pucheta, M.A., Gallardo, A.G.: Synthesis of hybrid flexible mechanisms using beams and spatial design constraints. In: Pucheta, M., Cardona, A., Preidikman, S., Hecker, R. (eds.) MuSMe 2021. MMS, vol. 94, pp. 47–56. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-60372-4_6
6. Hopkins, J.B., Lange, K.J., Spadaccini, C.M.: Designing microstructural architectures with thermally actuated properties using freedom, actuation, and constraint topologies. ASME J. Mech. Des. **135**(6), 061004 (2013)

7. Wu, S.L., Shao, Z.X., et al.: An energy-based approach for kinetostatic modeling of general compliant mechanisms. Mech. Mach. Theory **142**, 103588 (2019)
8. Selig, J.M., Ding, X.: A screw theory of static beams. In: IEEE International Conference on Intelligent Robots and Systems, Maui, Hawaii, pp. 2544–2550 (2001)
9. Ding, X.L., Selig, J.M.: Screw theoretic view on dynamics of spatially compliant beam. Appl. Math. Mech. **31**(9), 1173–1188 (2010)
10. Géradin, M., Rixen, J.D.: Mechanical Vibrations: Theory and Application to Structural Dynamics, 3rd edn. John Wiley & Sons, UK (2015)