Chapter 6
NSTT for Linear and Piecewise-Linear Systems

The tool of nonsmooth argument substitutions was introduced first to describe strongly nonlinear vibrations whose temporal mode shapes are asymptotically close to nonsmooth ones. Such cases are known to be most difficult for analyses because different quasi-harmonic methods are already ineffective, whereas the nonsmooth mapping is still inapplicable. It is quite clear however that the nonsmooth arguments can be introduced regardless of the strength of nonlinearity or the form of dynamical systems in general. For instance, it is shown in this chapter that the nonsmooth temporal substitutions can facilitate the analyses of different linear models with nonsmooth or discontinuous inputs.

6.1 Free Harmonic Oscillator: Temporal Quantization of Solutions

Introducing the triangle wave temporal argument into the differential equations of motion may bring some specific features into the corresponding solutions. For illustrating purposes, let us consider the harmonic oscillator

$$\ddot{x} + \Omega_0^2 x = 0 \tag{6.1}$$

First, let us obtain exact general solution of the oscillator (6.1) in terms of the triangle wave temporal argument by using the substitution

$$x = X(\tau) + Y(\tau)\, e \tag{6.2}$$

where $\tau = \tau(t/a)$ and $e = e(t/a)$ are the standard triangle and square wave functions, respectively.

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Substituting (6.2) in (6.1) gives the boundary value problem

$$a^{-2}X''(\tau) + \Omega_0^2 X(\tau) = 0 \tag{6.3}$$

$$a^{-2}Y''(\tau) + \Omega_0^2 Y(\tau) = 0 \tag{6.4}$$

$$X'(\pm 1) = 0, \; Y(\pm 1) = 0 \tag{6.5}$$

By considering the parameter a as an eigen-value of the problem, one obtains the set of eigen-values and the corresponding solutions as, respectively,

$$a_j = \frac{j\pi}{2\Omega_0} \tag{6.6}$$

and

$$X_j = \sin\left(\frac{j\pi\tau}{2} + \varphi_j\right), \quad Y_j = \cos\left(\frac{j\pi\tau}{2} - \varphi_j\right) \tag{6.7}$$

where $\varphi_j = (\pi/4)\left[1 + (-1)^j\right]$, $\tau = \tau(t/a_j)$, and j is any positive real integer.

Therefore, introducing the triangle wave oscillating time produced the discrete family of solutions for harmonic oscillator (6.1). The nature of such kind of quantization is due to the temporal symmetry of periodic motions. In other words, the quantization is associated with a multiple choice for the period

$$T_j = 4a_j = jT \tag{6.8}$$

where $T = 2\pi/\Omega_0$ is the natural period of oscillator (6.1).

In terms of the original temporal variable t, the number j plays no role for the temporal mode shape, given by

$$\begin{aligned} x(t) = A \sin\left[\frac{j\pi}{2}\tau\left(\frac{2\Omega_0 t}{j\pi}\right) + \varphi_j\right] \\ + B \cos\left[\frac{j\pi}{2}\tau\left(\frac{2\Omega_0 t}{j\pi}\right) - \varphi_j\right] e\left(\frac{2\Omega_0 t}{j\pi}\right) \end{aligned} \tag{6.9}$$

where A and B are arbitrary constants, and $x(t)$ is the same harmonic wave regardless of the number j.

In this section, the free linear oscillator was considered for illustrating purposes. There is no other pragmatic reason for introducing the triangle wave time into Eq. (6.1). The situation drastically changes however in non-autonomous cases of nonsmooth or discontinuous inputs. It is shown below that, in such cases, the triangle wave time variable facilitates determining particular solutions. The above-noticed effect of temporal quantization, which is just an identical transformation in the autonomous case, becomes helpful at the presence of external excitations. For instance, according to (6.9), the so-called combination resonances appear to be an inherent property of oscillators.

6.2 Non-autonomous Case

6.2.1 Unipotent Basis

Consider the linear harmonic oscillator under the external forcing described by the linear combination of triangle and square wave functions

$$\ddot{x} + \Omega_0^2 x = F\tau\left(\frac{t}{a}\right) + Ge\left(\frac{t}{a}\right) \tag{6.10}$$

where F and G are constant amplitudes and a is a quarter of the period.

Substituting (6.2) in (6.10) leads to the boundary value problem

$$a^{-2}X''(\tau) + \Omega_0^2 X(\tau) = F\tau \tag{6.11}$$

$$a^{-2}Y''(\tau) + \Omega_0^2 Y(\tau) = G \tag{6.12}$$

under the boundary conditions (6.5).

In contrast to autonomous case (6.1), the parameter a is known. Equations (6.11) and (6.12), are non-homogeneous, and their non-zero solution exists for any a and can be found in few elementary steps. The particular periodic solution of the original Eq. (6.10) takes the form

$$\begin{aligned} x_p(t) = X(\tau) + Y(\tau)e = \frac{F}{\Omega_0^2}\left\{\tau\left(\frac{t}{a}\right) - \frac{\sin[a\Omega_0\tau(t/a)]}{a\Omega_0\cos(a\Omega_0)}\right\} \\ +\frac{G}{\Omega_0^2}\left\{1 - \frac{\cos[a\Omega_0\tau(t/a)]}{\cos(a\Omega_0)}\right\}e\left(\frac{t}{a}\right) \end{aligned} \tag{6.13}$$

The corresponding general solution is $x(t) = A\cos(\Omega_0 t - \varphi) + x_p(t)$, where A and φ are arbitrary amplitude and phase parameters. Note that solution (6.13) immediately shows all possible resonance combinations $a\Omega_0 = (2k+1)\pi/2$ or

$$\frac{\Omega_0}{\Omega} = 2k + 1 \tag{6.14}$$

where $k = 1, 2, 3\ldots$ and $\Omega = 2\pi/T = \pi/(2a)$ is the fundamental frequency of the external forcing.

Let us compare solution (6.13) to solution obtained by using Fourier series

$$\begin{aligned} \tau\left(\frac{t}{a}\right) &= \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin[(2k+1)\,\Omega t] \\ e\left(\frac{t}{a}\right) &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\cos[(2k+1)\,\Omega t] \end{aligned} \tag{6.15}$$

These lead to the particular solution of Eq. (6.10) in the form

$$x_p(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Omega_0^2 - (2k+1)^2 \Omega^2} \times \\ \times \left[\frac{8F}{\pi^2 (2k+1)^2} \sin[(2k+1)\Omega t] + \frac{4G}{\pi (2k+1)} \cos[(2k+1)\Omega t] \right] \tag{6.16}$$

Solution (6.16) reveals the same resonance conditions, (6.14). However, the infinite trigonometric series are less convenient for calculations, especially when dealing with derivatives of solutions since differentiation slows down convergence of the series.

6.2.2 *Idempotent Basis*

Consider the linear oscillator including viscous damping under the square wave external loading

$$\ddot{x} + 2\zeta\Omega_0\dot{x} + \Omega_0^2 x = pe\left(\frac{t}{a}\right) \tag{6.17}$$

The purpose is to obtain periodic steady-state solution with the period of external loading, $T = 4a$. Recall that the idempotent basis is introduced by means of the linear transformation

$$\{1, e\} \longrightarrow \{e_+, e_-\} : \qquad e_\pm = \frac{1}{2}(1 \pm e) \tag{6.18}$$

or, inversely, $1 = e_+ + e_-$ and $e = e_+ - e_-$, where $e_\pm^2 = e_\pm$ and $e_+e_- = 0$; see Chaps. 1 and 4.

Now, the periodic solution and external loading are represented in the new basis as

$$x(t) = U(\tau)e_+ + V(\tau)e_- \tag{6.19}$$
$$pe = p(e_+ - e_-)$$

where $e_\pm = e_\pm(t/a)$ and $U(\tau)$ and $V(\tau)$ are unknown functions of the triangle wave, $\tau = \tau(t/a)$.

Substituting (6.19) in (6.17) and sequentially eliminating derivatives of the square wave, as described in Chap. 4, give equations

$$U'' + 2\zeta\Omega_0 a U' + (\Omega_0 a)^2 U = pa^2$$
$$V'' - 2\zeta\Omega_0 a V' + (\Omega_0 a)^2 V = -pa^2 \tag{6.20}$$

with boundary conditions

$$
\begin{aligned}
(U - V)|_{\tau=\pm 1} &= 0 \\
(U' + V')|_{\tau=\pm 1} &= 0
\end{aligned}
\tag{6.21}
$$

All the coefficients and right-hand sides of both equations in (6.20) are constant, and the equations are decoupled. As a result, solution of boundary value problem, (6.20) and (6.21), is obtained in the closed form

$$
\begin{aligned}
U(\tau) = \frac{p}{\Omega_0^2} - &\frac{2p\exp(-\alpha\tau)}{\beta\Omega_0^2(\cos 2\beta + \cosh 2\alpha)} \\
&\times[\cos\beta\cosh\alpha(\beta\cos\beta\tau + \alpha\sin\beta\tau) + \sin\beta\sinh\alpha(\alpha\cos\beta\tau - \beta\sin\beta\tau)]
\end{aligned}
\tag{6.22}
$$

$$
\begin{aligned}
V(\tau) = -\frac{p}{\Omega_0^2} + &\frac{2p\exp(\alpha\tau)}{\beta\Omega_0^2(\cos 2\beta + \cosh 2\alpha)} \\
&\times[\cos\beta\cosh\alpha(\beta\cos\beta\tau - \alpha\sin\beta\tau) + \sin\beta\sinh\alpha(\alpha\cos\beta\tau + \beta\sin\beta\tau)]
\end{aligned}
\tag{6.23}
$$

where $\alpha = a\zeta\Omega_0$ and $\beta = a\Omega_0\sqrt{1-\zeta^2}$.

Substituting (6.22) and (6.23) in (6.19) gives the closed form particular solution of original Eq. (6.17). Transition to the original temporal variable is given by the functions $\tau(\varphi) = (2/\pi)\arcsin[\sin(\pi\varphi/2)]$ and $e(\varphi) = \mathrm{sgn}[\cos(\pi\varphi/2)]$. Since the system under consideration is linear, the general solution of Eq. (6.17) can be obtained by adding general solution of the corresponding equation with zero right-hand side.

6.3 Systems Under Periodic Pulsed Excitation

Instantaneous impulses acting on a mechanical system can be modeled either by imposing specific matching conditions on the system state vector at pulse times or by introducing Dirac functions into the differential equations of motion. The first approach deals with the differential equations of a free system separately between the impulses; therefore, a sequence of systems under the matching conditions are considered. The second method gives a single set of equations over the whole time interval without any conditions of matching. In latter case, the analysis can be carried out correctly in terms of distributions that requires additional mathematical justifications in nonlinear cases. Both of the above approaches are used for different quantitative and qualitative analyses. The analytical tool, which is described below, eliminates the singular terms from the equations. As a result, solutions are obtained in a closed form of a single analytical expression for the whole time interval.

6.3.1 Regular Periodic Impulses

Introducing the triangle wave temporal argument may significantly simplify solutions whenever loading functions are combined of the triangular wave and its derivatives. For instance, let us seek a particular solution of the first-order differential equation[1]

$$\dot{v} + \lambda v = \mu \sum_{k=-\infty}^{\infty} \left[\delta\left(t + 1 - 4k\right) - \delta\left(t - 1 - 4k\right) \right] \tag{6.24}$$

where λ and μ are constant parameters.

For positive λ, Eq. (6.24) describes the velocity of a particle moving in a viscous media under the periodic impulsive force. The corresponding physical model is shown in Fig. 6.1, where the freely moving massive tank experiences perfectly elastic reflections from the stiff obstacles. By scaling the variables, one can bring the differential equation of motion of the particle to the form (6.24), where $v(t) = \dot{x}(t)$.

First, note that the right-hand side of Eq. (6.24) can be expressed through the generalized derivative of the square wave as follows:

$$\dot{v} + \lambda v = \frac{\mu}{2} \dot{e}\left(t\right) \tag{6.25}$$

Now let us represent the particular solution in the form

$$v\left(t\right) = X\left(\tau\left(t\right)\right) + Y\left(\tau\left(t\right)\right) e\left(t\right) \tag{6.26}$$

Substituting (6.26) in (6.25) gives

$$Y' + \lambda X + \left(X' + \lambda Y\right) e\left(t\right) + \underline{\left(Y - \frac{\mu}{2}\right) \dot{e}\left(t\right)} = 0 \tag{6.27}$$

Apparently, the elements $\{1, e\}$ and $\dot{e}$ on the left-hand side of Eq. (6.27) are linearly independent as functions of different classes of smoothness. Therefore,

$$Y' + \lambda X = 0, \qquad X' + \lambda Y = 0, \qquad Y|_{\tau=\pm 1} = \frac{\mu}{2} \tag{6.28}$$

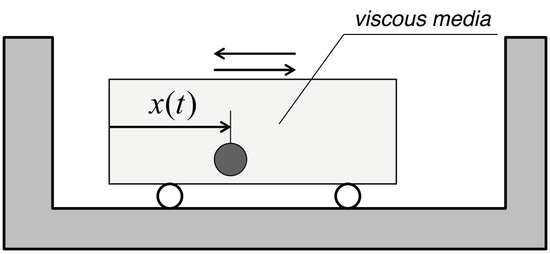

Fig. 6.1 If mass of the particle is very small compared to the total mass of the tank, then the inertia force applied to the particle inside the tank has a periodic pulse-wise character

[1] The case of Dirac comb input was considered in Chap. 1.

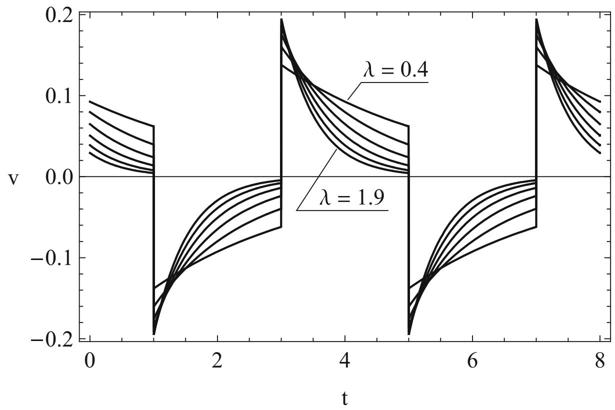

Fig. 6.2 The family of discontinuous periodic solutions for different viscosity parameter of the media inside the tank (Fig. 6.1)

In contrast to Eq. (6.24), this boundary value problem includes no discontinuities, whereas the new independent variable belongs to the standard interval, $-1 \leq \tau \leq 1$. Solving the boundary value problem (6.28) and taking into account substitution (6.26) give the periodic solution of Eq. (6.24) as

$$v = X + Ye = \frac{\mu}{2\cosh\lambda}\left(-\sinh\lambda\tau + e\cosh\lambda\tau\right)$$

or

$$v = \frac{\mu}{2\cosh\lambda}\exp\left[-\lambda\tau\left(t\right)e\left(t\right)\right]e\left(t\right) \tag{6.29}$$

Figure 6.2 illustrates solution (6.29) for $\mu = 0.2$ and different magnitudes of λ.

Note that the discontinuous solution $v\left(t\right)$ is described by the closed form expression (6.29) through the two elementary functions $\tau\left(t\right)$ and $e\left(t\right)$.

6.3.2 Harmonic Oscillators Under the Periodic Impulsive Loading

Resonances in Zero Damping Case

Let us consider the harmonic oscillator subjected to periodic pulses

$$\ddot{x} + \Omega_0^2 x = 2p\sum_{k=-\infty}^{\infty}\left[\delta\left(\omega t + 1 - 4k\right) - \delta\left(\omega t - 1 - 4k\right)\right] \tag{6.30}$$

where p, Ω_0 and ω are constant parameters.

The right-hand side of Eq. (6.30) can be expressed through the first derivative of the square wave as follows:

$$\ddot{x} + \Omega_0^2 x = p\frac{de\,(\omega t)}{d\,(\omega t)} \tag{6.31}$$

Let us seek a periodic solution of the period $T = 4/\omega$ in the form

$$x\,(t) = X\,(\tau\,(\omega t)) + Y\,(\tau\,(\omega t))\,e\,(\omega t) \tag{6.32}$$

Substituting (6.32) in (6.31) under the necessary condition of continuity for the coordinate, $x\,(t)$, gives

$$\omega^2 X'' + \Omega_0^2 X + \left(\omega^2 Y'' + \Omega_0^2 Y\right) e + \underline{\left(\omega^2 X' - p\right)\frac{de\,(\omega t)}{d\,(\omega t)}} = 0 \tag{6.33}$$

Analogously to the previous subsection, Eq. (6.33) gives the boundary value problem

$$X'' + \left(\frac{\Omega_0}{\omega}\right)^2 X = 0, \qquad Y'' + \left(\frac{\Omega_0}{\omega}\right)^2 Y = 0 \tag{6.34}$$

$$X'|_{\tau=\pm 1} = \frac{p}{\omega^2}, \qquad Y|_{\tau=\pm 1} = 0$$

Solving problem (6.34) and taking into account (6.32) give the periodic solution of the original Eq. (6.30) in the form

$$x = X\,(\tau\,(\omega t)) = \frac{p}{\omega \Omega_0}\frac{\sin\left[(\Omega_0/\omega)\,\tau\,(\omega t)\right]}{\cos\,(\Omega_0/\omega)} \tag{6.35}$$

where $Y \equiv 0$.

Solution (6.35) is continuous although nonsmooth at those times t where $\tau\,(\omega t) = \pm 1$. All possible resonances are given by

$$\omega = \frac{2}{\pi}\frac{\Omega_0}{k}; \qquad k = 1, 3, 5, \ldots \tag{6.36}$$

where the factor $2/\pi$ is due to different normalization of the periods for sine and triangle waves.

Viscous Damping Case

Now let us consider the case of standard harmonic oscillator described by the differential equation of motion

$$\ddot{x} + 2\zeta\Omega_0\dot{x} + \Omega_0^2 x = p\frac{de\,(\omega t)}{d\,(\omega t)} \tag{6.37}$$

where ζ is the damping ratio.

In this case, the boundary value problem becomes coupled

$$X'|_{\tau=\pm 1} = \frac{p}{\omega^2}, \qquad Y|_{\tau=\pm 1} = 0$$

$$\begin{aligned} X'' + 2\zeta r Y' + r^2 X &= 0 \\ Y'' + 2\zeta r X' + r^2 Y &= 0 \end{aligned} \tag{6.38}$$

where $r = \Omega_0/\omega$ is the adjusted natural over loading frequency ratio. Recall that the sine wave frequency is given by $\Omega = (\pi/2)\omega$. The principal resonance ratio is therefore $r = \pi/2$, which is obviously equivalent to $\Omega_0 = \Omega$.

As a result, the periodic solution has both X and Y components and is given by

$$\begin{aligned} x = X + Ye = &\frac{p}{\beta\omega^2\left(\cos^2\beta\cosh^2\alpha + \sin^2\beta\sinh^2\alpha\right)} \\ &\times[\cosh\alpha\cos\beta\cosh\alpha\tau\sin\beta\tau - \sinh\alpha\sin\beta\sinh\alpha\tau\cos\beta\tau \\ &+ (\sinh\alpha\cos\beta\tau\cosh\alpha\tau\sin\beta - \sinh\alpha\tau\sin\beta\tau\cosh\alpha\cos\beta)\,e] \end{aligned} \tag{6.39}$$

where $\tau = \tau\,(\omega t)$, $e = e\,(\omega t)$; $\alpha = r\zeta$ and $\beta = r\sqrt{1-\zeta^2}$.

Figure 6.3 illustrates qualitatively different responses of the system when varying the input frequency. In different proportions, the responses combine properties of the harmonic damped motion and the nonsmooth motion due to the impulsive loading. For instance, when $\omega >> \Omega_0$ and $\omega >> \zeta\Omega_0$, the system is near the limit of a free particle under the periodic impulsive force. In this case, the boundary value problem is reduced to

$$X'' = 0, \qquad Y'' = 0; \qquad X'\,|_{\tau=\pm 1} = \frac{p}{\omega^2}, \qquad Y\,|_{\tau=\pm 1} = 0 \tag{6.40}$$

This gives the triangle wave temporal shape, $x = p\tau\,(\omega t)\,/\omega^2$, which is close to the shape in Fig. 6.3d.

Multiple Degrees-of-Freedom Case

Finally, let us consider N-degrees-of-freedom system

$$M\ddot{\mathbf{y}} + K\mathbf{y} = \mathbf{p}\frac{de\,(\omega t)}{d\,(\omega t)} \tag{6.41}$$

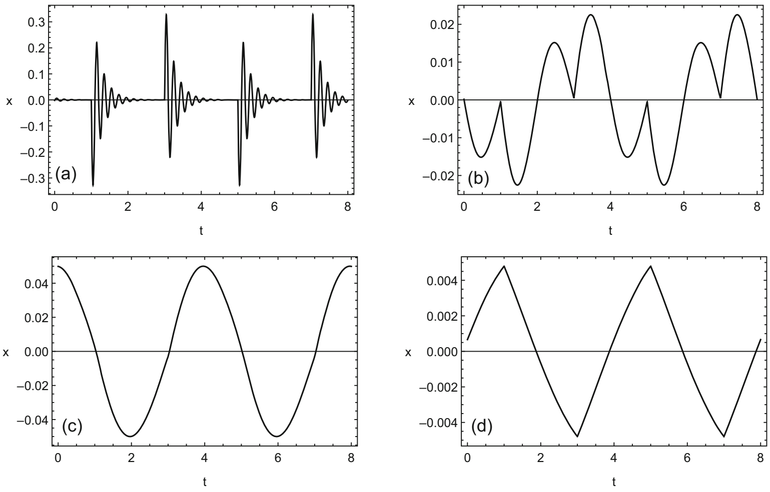

Fig. 6.3 Evolution of the response of the damped harmonic oscillator under the periodic impulsive excitation for $p = 0.1$, $\zeta = 0.125$, $\Omega_0 = 4.0$, and different impulse frequencies $\Omega = (\pi/2)\omega$: (**a**) $\Omega = 0.2$—low-frequency Impulses, (**b**) $\Omega = 2.0$, (**c**) $\Omega = \Omega_0$, and (**d**) $\Omega = 8.0$

where $\mathbf{y}(t)$ is N-dimensional vector-function, $\mathbf{p}$ is a constant vector, and M and K are constant $N \times N$ mass and stiffness matrixes, respectively.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ and $\Omega_1, \ldots, \Omega_N$ be the normal mode basis vectors and the corresponding natural frequencies, respectively, such that

$$K\mathbf{e}_j = \Omega_j^2 M\mathbf{e}_j, \qquad \mathbf{e}_k^T M\mathbf{e}_j = \delta_{kj}$$

for any $k = 1, \ldots, N$ and $j = 1, \ldots, N$.

Introducing the principal coordinates $x^j(t)$,

$$\mathbf{y} = \sum_{j=1}^{N} x^j(t)\,\mathbf{e}_j \tag{6.42}$$

gives a decoupled set of impulsively forced harmonic oscillators of the form (6.31),

$$\ddot{x}^j + \Omega_j^2 x^j = p^j \frac{de(\omega t)}{d(\omega t)} \tag{6.43}$$

where $p^j = \mathbf{e}_j^T \mathbf{p}$.

Therefore, using solution (6.35) for each of the oscillators (6.43) and taking into account (6.42) give

$$\mathbf{y} = \sum_{j=1}^{N} \frac{(\mathbf{e}_j^T \mathbf{p})\mathbf{e}_j}{\omega\Omega_j} \frac{\sin\left[\left(\Omega_j/\omega\right)\tau(\omega t)\right]}{\cos\left(\Omega_j/\omega\right)} \tag{6.44}$$

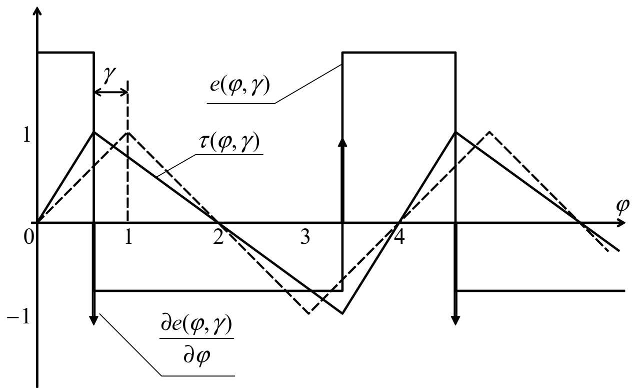

Fig. 6.4 Basic NSTT asymmetric wave functions

The corresponding resonances are determined by the condition

$$\omega = \frac{2}{\pi}\frac{\Omega_j}{k}$$

where $k = 1, 3, 5, \dots$ and $j = 1, \dots, N$.

6.3.3 *Periodic Impulses with a Temporal Dipole Shift*

Let us consider the impulsive excitation with a dipole wise shift of pulse times. In this case, the right-hand side of Eq. (6.25) can be expressed by second derivative of the asymmetric triangle wave with some incline[2] characterized by the parameter γ as shown in Fig. 6.4

$$\begin{aligned} \dot{v} + \lambda v &= p\frac{\partial^2 \tau\,(\omega t, \gamma)}{\partial\,(\omega t)^2} = p\frac{\partial e\,(\omega t, \gamma)}{\partial\,(\omega t)} \\ &= \frac{2p}{1-\gamma^2}\sum_{k=-\infty}^{\infty}\left[\delta\,(\omega t + 1 - \gamma - 4k) - \delta\,(\omega t - 1 + \gamma - 4k)\right] \end{aligned} \tag{6.45}$$

Based on the NSTT identities introduced in Chap. 4, periodic solutions of Eq. (6.45) still can be represented in the form

$$v = X\,(\tau) + Y\,(\tau)\,e \tag{6.46}$$

where $\tau = \tau\,(\omega t, \gamma)$ and $e = e\,(\omega t, \gamma)$; see Fig. 6.4 for graphic illustrations.

[2] Can be viewed as a generalized sawtooth function.

Substituting (6.46) in Eq. (6.45) gives

$$\omega\alpha Y' + \lambda X + \left[\omega\left(X' + \beta Y'\right) + \lambda Y\right]e + (\omega Y - p)\frac{\partial e(\omega t, \gamma)}{\partial(\omega t)} = 0 \tag{6.47}$$

where $\alpha = 1/\left(1-\gamma^2\right)$, $\beta = 2\gamma\alpha$, and the identity $e^2 = \alpha + \beta e$ has been taken into account.

Equation (6.47) is equivalent to the boundary-value problem

$$\begin{aligned} \omega\left(X' + \beta Y'\right) &= -\lambda Y \\ \omega\alpha Y' &= -\lambda X \\ \omega Y|_{\tau=\pm 1} &= p \end{aligned} \tag{6.48}$$

The corresponding solution is

$$\begin{aligned} Y &= \frac{p}{\omega}\left[\cosh\left(\gamma\frac{\lambda}{\omega}\right)\frac{\cosh\left(\frac{\lambda}{\omega}\tau\right)}{\cosh\frac{\lambda}{\omega}} - \sinh\left(\gamma\frac{\lambda}{\omega}\right)\frac{\sinh\left(\frac{\lambda}{\omega}\tau\right)}{\sinh\frac{\lambda}{\omega}}\right]\exp\left(\gamma\frac{\lambda}{\omega}\tau\right) \\ X &= -\frac{\omega\alpha}{\lambda}Y' \end{aligned} \tag{6.49}$$

where the X-component is defined by differentiation due to the second equation in (6.48).

6.4 Parametric Excitation

In this section, two different cases of parametric excitation are considered based on relatively simple linear models. Piecewise-constant and impulsive excitations are described by means of the functions $e(\omega t, \gamma)$ and $\partial e(\omega t, \gamma)/\partial(\omega t)$, respectively. There are at two least reasons for using NSTT as a preliminary analytical step. First, NSTT automatically gives conditions for matching solutions at discontinuity points. Second, due to the automatic matching through the NSTT functions, the corresponding solutions appear to be in the closed form that is important feature when further manipulations with the solutions are required by problem formulations.

6.4.1 Piecewise-Constant Excitation

Let us consider the linear oscillator under the periodic piecewise-constant parametric excitation

$$\ddot{x} + \Omega_0^2[1 + \varepsilon e(\omega t, \gamma)]x = 0 \tag{6.50}$$

where Ω_0, ω, γ, and ε are constant parameters.

We seek periodic solutions with the period of excitation $T = 4/\omega$ in the form

$$x = X(\tau) + Y(\tau)\, e \tag{6.51}$$

where $\tau = \tau(\omega t, \gamma)$ and $e = e(\omega t, \gamma)$.

As follows from the form of Eq. (6.50), the acceleration $\ddot{x}$ may have stepwise discontinuities due to the presence of the function $e(\omega t, \gamma)$, whereas the coordinate $x(t)$ and the velocity $\dot{x}(t)$ must be continuous. Hence neither velocity $\dot{x}(t)$ nor acceleration $\ddot{x}(t)$ can include Dirac δ-functions. Taking first derivative of (6.51) gives

$$\dot{x}(t) = \left[\alpha Y' + (X' + \beta Y')e + \underline{Y\frac{\partial e(\omega t, \gamma)}{\partial(\omega t)}}\right]\omega \tag{6.52}$$

where the last term that consists of the periodic sequence of δ-functions must be excluded by imposing the boundary condition for Y-component

$$Y\,|_{\tau=\pm 1} = 0 \tag{6.53}$$

Under condition (6.53), the second derivative takes the form

$$\begin{aligned}\ddot{x}(t) &= \omega^2[\alpha(X'' + \beta Y'')] + \omega^2[\beta X'' + (\alpha + \beta^2)Y'']e \\ &\underline{+\omega^2(X' + \beta Y')\frac{\partial e(\omega t, \gamma)}{\partial(\omega t)}}\end{aligned} \tag{6.54}$$

In this case, the singular term, which is underlined in (6.54), is eliminated by condition

$$\left(X' + \beta Y'\right)|_{\tau=\pm 1} = 0 \tag{6.55}$$

Substituting (6.51) and (6.54) in the differential equation of motion (6.50) and taking into account the algebraic properties of hyperbolic numbers bring the left-hand side of the equation to the form $\{\cdots\} + \{\cdots\}e$. Then, setting separately each of the two algebraic components to zero gives a set of the differential equations for $X(\tau)$ and $Y(\tau)$ in the following matrix form:

$$\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \alpha + \beta^2 \end{bmatrix} \frac{d^2}{d\tau^2} \begin{bmatrix} X \\ Y \end{bmatrix} + r^2 \begin{bmatrix} 1 & \alpha\varepsilon \\ \varepsilon & 1 + \beta\varepsilon \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \tag{6.56}$$

where $r = \Omega_0/\omega$

Further, any particular solution of linear differential equations with constant coefficients (6.56) is represented in the exponential form

$$\begin{bmatrix} X \\ Y \end{bmatrix} = B \begin{bmatrix} 1 \\ \mu \end{bmatrix} \exp(\lambda\tau) \tag{6.57}$$

where B, μ, and λ are constant parameters.

Substituting (6.57) in (6.56) and using the relationships, $\alpha = 1/\left(1-\gamma^2\right)$ and $\beta = 2\gamma\alpha$, lead to the characteristic equation with two pairs of roots determined by

$$\begin{aligned} \lambda^2 &= \left[-(1-\gamma)\,\varepsilon - (1-\gamma)^2\right] r^2 \equiv \pm k^2 \\ \lambda^2 &= \left[(1+\gamma)\,\varepsilon - (1+\gamma)^2\right] r^2 \equiv \pm l^2 \end{aligned} \tag{6.58}$$

where signs of the notations $\pm k^2$ and $\pm l^2$ depend on the parameters ε and γ.

Let us consider the case of negative signs on the right-hand side of (6.58), when the following condition holds:

$$-(1-\gamma) < \varepsilon < (1+\gamma) \tag{6.59}$$

Due to condition (6.59), the stiffness coefficient in Eq. (6.50) is always positive, whereas (6.58) gives $\lambda = \pm ki$ and $\lambda = \pm li$.As a result, the general solution of Eqs. (6.56) takes the form

$$\begin{aligned} X &= B_1 \sin k\tau + B_2 \cos k\tau + B_3 \sin l\tau + B_4 \cos l\tau \\ Y &= \mu_1 \left(B_1 \sin k\tau + B_2 \cos k\tau\right) + \mu_2 \left(B_3 \sin l\tau + B_4 \cos l\tau\right) \end{aligned} \tag{6.60}$$

where $B_1,\ldots,B_4$ are arbitrary constants, and

$$\mu_1 = -\frac{1}{\alpha}\frac{\alpha k^2 - r^2}{\beta k^2 - \varepsilon r^2} \qquad \text{and} \qquad \mu_2 = -\frac{1}{\alpha}\frac{\alpha l^2 - r^2}{\beta l^2 - \varepsilon r^2}$$

Substituting (6.60) in boundary conditions (6.53) and (6.55) gives the homogeneous set of four linear algebraic equations with respect to the arbitrary constants whose matrix is

$$\begin{bmatrix} \mu_1 \sin k & \mu_1 \cos k & \mu_2 \sin l & \mu_2 \cos l \\ -\mu_1 \sin k & \mu_1 \cos k & -\mu_2 \sin l & \mu_2 \cos l \\ k\left(1+\beta\mu_1\right)\cos k & -k\left(1+\beta\mu_1\right)\sin k & l\left(1+\beta\mu_2\right)\cos l & -l\left(1+\beta\mu_2\right)\sin l \\ k\left(1+\beta\mu_1\right)\cos k & k\left(1+\beta\mu_1\right)\sin k & l\left(1+\beta\mu_2\right)\cos l & l\left(1+\beta\mu_2\right)\sin l \end{bmatrix}$$

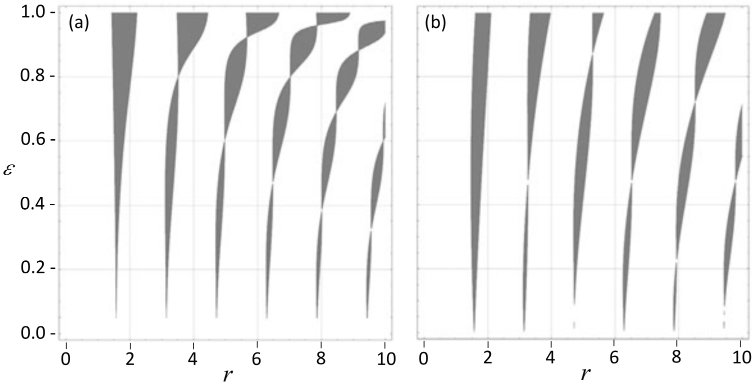

Fig. 6.5 Instability zones of oscillator (6.50) under the piecewise constant parametric excitation for (**a**) $\gamma = 0.0$ and (**b**) $\gamma = 0.7$, where $r = \Omega_0/\omega$

Calculation of the determinant can be eased essentially after a proper summation and subtraction of its rows. Then setting it to zero gives a condition for non-zero solutions in the form

$$\left[\mu_1 (1 + \beta\mu_2)\, l \cos k \sin l - \mu_2 (1 + \beta\mu_1)\, k \cos l \sin k\right] \times \left[\mu_1 (1 + \beta\mu_2)\, l \cos l \sin k - \mu_2 (1 + \beta\mu_1)\, k \cos k \sin l\right] = 0 \quad (6.61)$$

Equation (6.61) describes the family of curves separating stability and instability zones on the plane (r, ε) as shown in Fig. 6.5, where the instability zones are shadowed.

The diagrams in Fig. 6.5 are interpreted in a similar way to Ince-Strutt diagrams showing the transition curves in the parameters' plane. The curves divide the plane into regions corresponding to unbounded/unstable and regions of bounded/stable solutions.

6.4.2 Parametric Impulsive Excitation

Let us consider the case of parametric impulsive excitation whose temporal shape is given by first derivative of the generalized square wave, $e(\omega t, \gamma)$, [177]

$$\ddot{x} + \Omega_0^2 \left[1 + \varepsilon \frac{\partial e(\varphi, \gamma)}{\partial \varphi}\right] x = 0 \quad (6.62)$$

where $\varphi = \omega t$, and

$$\frac{\partial e(\varphi,\gamma)}{\partial \varphi} = \frac{\partial^2 \tau(\varphi,\gamma)}{\partial \varphi^2}$$
$$= \frac{2}{1-\gamma^2}\sum_{k=-\infty}^{\infty}\left[\delta(\varphi+1-\gamma-4k) - \delta(\varphi-1+\gamma-4k)\right]$$

In this case, when substituting (6.51) and (6.54) in Eq. (6.62), the singular term of second derivative (6.54) must be preserved in order to compensate the singularity in Eq. (6.62). The result of such a compensation leads to the boundary conditions (compare to (6.55))

$$\tau = \pm 1: \qquad \omega^2(X' + \beta Y') + \varepsilon \Omega_0^2 X = 0 \tag{6.63}$$

Note that substitution of (6.51) in Eq. (6.62) generates the term $(\partial e/\partial\varphi)eY$, which is generally undefined in the theory of distributions. This term represents a periodic series of δ-functions, $\partial e/\partial\varphi$, "multiplied" by the function e whose stepwise discontinuities coincide with the times of δ-functions, $\{\varphi : \tau(\varphi,\gamma) = \pm 1\}$. Some interpretations of such terms are still possible only within specific contents assuming the common physical nature for both singularities as discussed in the next section. In the present case, the term $(\partial e/\partial\varphi)eY$ is simply removed from the equation since the point-wise singularities at $\{\varphi : \tau(\varphi,\gamma) = \pm 1\}$ are suppressed by continuity condition (6.53) for the coordinate x: $Y\mid_{\tau=\pm 1} = 0$. Then combining separately two group of terms associated with different structural parts of the hyperbolic number gives

$$\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \alpha+\beta^2 \end{bmatrix} \frac{d^2}{d\tau^2}\begin{bmatrix} X \\ Y \end{bmatrix} + r^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = 0 \tag{6.64}$$

where $r = \Omega_0/\omega$.

Further steps follow the previous section. Substituting (6.57) in (6.64) leads to the characteristic equation whose two pairs of roots λ and the corresponding amplitude ratios μ are given by

$$\lambda = \pm ki = \pm r(1+\gamma)i, \qquad \mu_1 = -1+\gamma$$
$$\lambda = \pm li = \pm r(1-\gamma)i, \qquad \mu_2 = 1+\gamma$$

where the notations $\alpha = 1/\left(1-\gamma^2\right)$ and $\beta = 2\gamma\alpha$ were taken into account.

Then, substituting (6.60) in boundary conditions, (6.53) and (6.63), gives a homogeneous linear algebraic system for the constants $B_1, \ldots, B_4$. Setting its determinant to zero gives the condition of existence of the period, $T = 4/\omega$, as

$$\varepsilon^2 = \frac{2\left(1-\gamma^2\right)^2 \sin^2 2r}{r^2(\cos 4r - \cos 4\gamma r)} \tag{6.65}$$

The dependence of ε on r at fixed γ has a branched zone-like structure on the plane (r, ε) which is typical for different cases of parametrically excited oscillators. Interestingly enough, different subsequences of zones may disappear as the parameter γ varies. For instance, the number $\gamma = 1/2$ eliminates every second zone, whereas the number $\gamma = 1/5$ removes every fifth zone. The effect of collapsing instability zones can be explained by considering the regions of definition for condition (6.65), $\cos 4r - \cos 4\gamma r > 0$. This inequality holds inside the white regions on the plane (r, γ) as shown in Fig. 6.6. It is seen that the bottom horizontal line, $\gamma = 1/5$, intersects four white regions before it starts crossing two shadowed areas with no white one in between. It happens because the line intersects the (blue) point at the corners of two shadowed regions. The corresponding vertical straight line, which is intersecting the same point, corresponds to one of the roots of the equation $\sin 2r = 0$. This root, $r = 5\pi/2$, locates the point on the r-axis, from which the missing instability zone would branch out if the line $\gamma = 1/5$ were slightly shifted up or down. Another horizontal line, $\gamma = 1/5$, gives the example, when every second zone is missing.

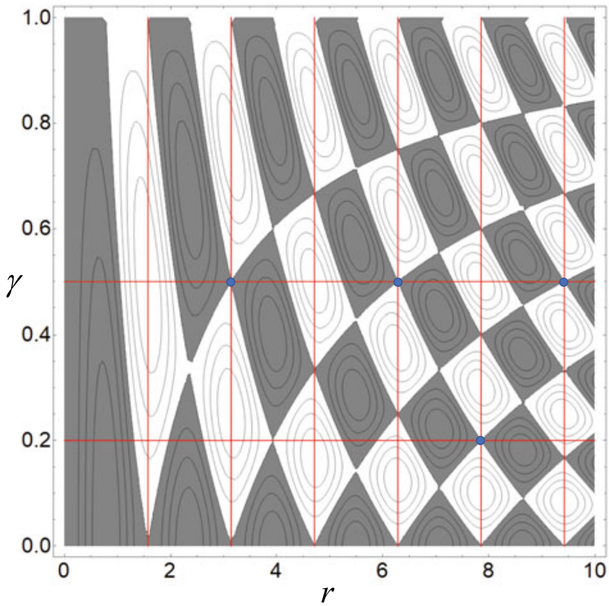

Fig. 6.6 The regions of definition for condition (6.65) are shown in white color; the locations of blue dots explain why the corresponding zones of Ince-Strutt diagrams collapse; see the main text for more details

6.4.3 General Case of Periodic Parametric Excitation

Below, the problem formulation only is discussed for the case of periodic parametric loading with both regular and singular components. It is assumed that there are two discontinuities and two singularities on each period at the same time points. The differential equation of motion is represented in the vector form

$$\ddot{x} + \left[Q(\tau) + P(\tau)\, e + p \frac{\partial e}{\partial \varphi} \right] x = 0 \tag{6.66}$$

where $x(t) \in R^n$ is the coordinates' vector-column, $\tau = \tau(\varphi, \gamma)$, $e = e(\varphi, \gamma)$, $\varphi = \omega t$ is the phase variable, p is a constant $n \times n$ matrix, and $Q(\tau(\varphi, \gamma))$ and $P(\tau(\varphi, \gamma))$ are periodic matrixes of the period $T = 4$ with respect to the phase φ.

In Eq. (6.66), the first two terms of the coefficient can represent any periodic function $q(\varphi)$ with stepwise discontinuities on $\Lambda = \{t : \tau(\varphi, \gamma) = \pm 1\}$. In case the original function $q(\varphi)$ is continuous, one has $P = 0$ on Λ.

Let us represent periodic solutions of the period $T = 4$ in the form (6.51). Substituting (6.51) in Eqs. (6.66), taking into account the equality $e^2 = \alpha + \beta e$, the necessary condition of continuity of the vector function $x(t)$, (6.53), and using (6.52) and (6.54) give equations

$$\begin{aligned} \omega^2 \left(\alpha X'' + \alpha \beta Y'' \right) + QX + \alpha PY = 0 \\ \omega^2 \left[\left(\alpha + \beta^2 \right) Y'' + \beta X'' \right] + PX + QY + \beta PY = 0 \end{aligned} \tag{6.67}$$

and the boundary condition

$$\left[\omega^2 \left(X' + \beta Y' \right) + pX \right] |_{\tau=\pm 1} = 0 \tag{6.68}$$

In the case of fixed sign of impulses, the matrix p should be provided with the factor $\mathrm{sgn}(\tau)$. Together with (6.53), relations (6.67) and (6.68) represent a boundary-value problem for determining the vector functions X and Y and the corresponding conditions for existence of periodic solutions.

Note that substitution (6.51) in Eq. (6.66) generates the specific term $e\partial e/\partial\varphi$. Let us show that, within the theory of distributions, these terms can be interpreted as

$$e \frac{\partial e}{\partial \varphi} = \frac{1}{2} \beta \frac{\partial e}{\partial \varphi} \tag{6.69}$$

The relationship (6.69) is the result of a formal differentiation of both sides of the relation $e^2 = \alpha + \beta e$ with respect to the phase φ. To justify it in terms of distributions, let us assume that $\omega = 1$ so that $\varphi \equiv t$ and consider expression (6.53) locally, near the point $t = 1 - \gamma$, which is a typical point for the entire set of discontinuities at times $\Lambda = \{t : \tau(t) = \pm 1\}$.

Generally speaking, the "product" $f(t)\delta(t)$ requires the function $f(t)$ to be at least continuous at $t = 0$. However, it is possible to provide the left-hand side of (6.69) with a certain meaning due to the fact that both terms of the product are generated by the same family of smooth functions. In order to illustrate this remark and prove equality (6.69), let us consider a family of smooth functions $\{\delta_\epsilon(t)\}$ such that

$$\int_{-\epsilon}^{\epsilon} \delta_\epsilon(t)\,dt = 1 \tag{6.70}$$

for all positive ϵ, and $\delta_\epsilon(t) = 0$ outside the interval $-\epsilon < t < \epsilon$.

Therefore, in terms of weak limits, $\delta_\epsilon(t) \to \delta(t)$ as $\epsilon \to 0$. Now, a family of smooth functions approximating e and $\partial e/\partial t$ in the neighborhood of point $t = 1-\gamma$ within the interval $-1+\gamma < t < 3+\gamma$ can be chosen as, respectively,

$$e_\epsilon = \frac{1}{1-\gamma} - \frac{\beta}{\gamma}\theta_\epsilon(t-1+\gamma) \quad \text{and} \quad \frac{\partial e_\epsilon}{\partial t} = -\frac{\beta}{\gamma}\delta_\epsilon(t-1+\gamma) \tag{6.71}$$

where $\theta_\epsilon(t) = \int_{-\infty}^{t} \delta_\epsilon(\xi)\,d\xi$ is a smoothed version of Heaviside unit-step function associated with $\delta_\epsilon(t)$.

Based on definitions (6.71) for e_ϵ and $\partial e_\epsilon/\partial t$, one has $e_\epsilon \to e$ and $\partial e_\epsilon/\partial t \to \partial e/\partial t$ as $\epsilon \to 0$ in the interval $-1+\gamma < t < 3+\gamma$.

Substituting (6.71) in equality (6.69) instead of e and $\partial e/\partial\varphi$ reduces the problem to the proof of identity

$$\theta_\epsilon(t-1+\gamma)\delta_\epsilon(t-1+\gamma) = \frac{1}{2}\delta_\epsilon(t-1+\gamma) \to \frac{1}{2}\delta(t-1+\gamma) \qquad \text{as}\ \epsilon \to 0 \tag{6.72}$$

For simplicity reason, let us shift the origin to the point $t = 1-\gamma$ and show that the left-hand side of (6.72) gives $\delta(t)/2$ as $\epsilon \to 0$ in the sense of a weak limit. The proof below is based on general properties of the functions $\{\delta_\epsilon\}$ regardless of specifics of their shapes. It is important nonetheless to maintain the relationship $d\theta_\epsilon/dt = \delta_\epsilon$ as shown in Fig. 6.7. First, the area bounded by $\theta_\epsilon\delta_\epsilon$ is

$$\int_{-\epsilon}^{\epsilon} \theta_\epsilon\delta_\epsilon dt = \int_{-\epsilon}^{\epsilon} \theta_\epsilon \frac{d\theta_\epsilon}{dt} dt = \frac{1}{2}\theta_\epsilon^2 \mid_{-\epsilon}^{\epsilon} = \frac{1}{2}$$

Then, let $\phi(t)$ belong to the class of continuous testing functions, which is usually considered in the theory of distributions. By definition, in some ϵ-neighborhood of the point $t = 0$, one has $\mid \phi(t) - \phi(0) \mid < 2\eta$, where η is as small as needed whenever ϵ is sufficiently small. Therefore,

$$\mid \int_{-\epsilon}^{\epsilon} \theta_\epsilon(t)\,\delta_\epsilon(t)\,\phi(t)\,dt - \frac{1}{2}\phi(0) \mid \le \int_{-\epsilon}^{\epsilon} \theta_\epsilon(t)\,\delta_\epsilon(t) \mid \phi(t) - \phi(0) \mid dt \le \eta$$

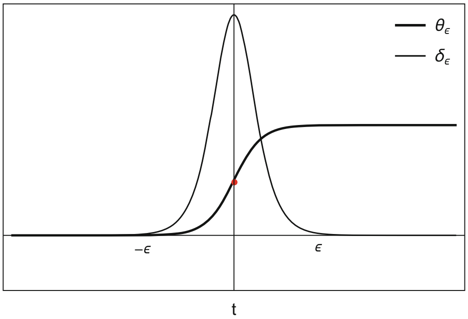

Fig. 6.7 Clarification for the product $\delta(t)\theta(t)$ based on the smooth families of functions $\delta_\epsilon(t)$ and $\theta_\epsilon(t)$

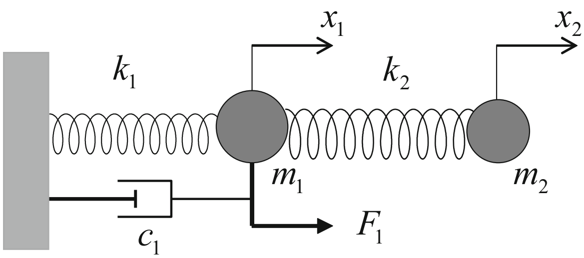

Fig. 6.8 Two mass-spring model

In other words,

$$\int_{-\epsilon}^{\epsilon} \theta_\epsilon(t)\,\delta_\epsilon(t)\,\phi(t)\,dt \to \frac{1}{2}\phi(0)$$

as $\epsilon \to 0$.

This completes the proof.

6.5 Input-Output Systems

The input-output form of dynamical systems may be convenient for different reasons, for instance, when dealing with control problems. In many linear cases, input-output systems are represented in the form of a single high order equation

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \ldots + b_1 \frac{du}{dt} + b_0 u \tag{6.73}$$

where $u = u(t)$ and $y = y(t)$ are input and output, respectively, and $a_n, \ldots, a_1, a_0$, $b_m, \ldots, b_1, b_0$ are constant coefficients.

For illustration purposes, a two-degrees-of-freedom model as shown in Fig. 6.8 is considered, although the general case (6.73) can be handled in the same way.

Eliminating $x_2(t)$ from the system gives a single higher-order equation with respect to the other coordinate, $x_1(t)$, in the form

$$m_1\frac{d^4x_1}{dt^4}+c_1\frac{d^3x_1}{dt^3}+\left(k_1+k_2+\frac{m_1}{m_2}k_2\right)\frac{d^2x_1}{dt^2}+\frac{c_1}{m_2}k_2\frac{dx_1}{dt}+\frac{k_1k_2}{m_2}x_1$$
$$=\frac{d^2F_1}{dt^2}+\frac{k_2}{m_2}F_1 \tag{6.74}$$

System (6.74) is a particular case of (6.73), where $n = 4$ and $m = 2$. Let us consider the stepwise discontinuous periodic function $F_1(t) = u(t) = e(\omega t)$ and represent Eq. (6.74) in the form

$$a_4\frac{d^4y}{dt^4}+\ldots+a_1\frac{dy}{dt}+a_0y=b_2\omega^2e''+b_1\omega e'+b_0e \tag{6.75}$$

where $' \equiv d/d(\omega t)$, and all the coefficients and variables are identified by comparing (6.74)–(6.75).

The right-hand side of Eq. (6.75) contains discontinuous and singular functions; hence, Eq. (6.75) must be treated in terms of distributions. Nonetheless, on the manifold of periodic solutions, Eq. (6.75) is equivalent to some classical boundary-value problem.

To confirm this statement, let us represent the output in the form

$$y(t)=X(\tau)+Y(\tau)e \tag{6.76}$$

where $\tau = \tau(\omega t)$ and $e = e(\omega t)$.

When differentiating expression (6.76) step-by-step, one should eliminate the singular term e' in the first two derivatives by sequentially setting boundary conditions as follows:

$$\frac{dy}{dt}=(Y'+X'e)\omega, \qquad Y|_{\tau=\pm1}=0 \tag{6.77}$$
$$\frac{d^2y}{dt^2}=(X''+Y''e)\omega^2, \qquad X'|_{\tau=\pm1}=0$$

Then, it is dictated by the form of the input in (6.75) that the singular terms e' and e'' must be preserved on the next two steps:

$$\frac{d^3y}{dt^3}=(Y'''+X'''e+Y''e')\omega^3 \tag{6.78}$$
$$\frac{d^4y}{dt^4}=(X^{(4)}+Y^{(4)}e+X'''e'+Y''e'')\omega^4$$

The fourth-order derivative in (6.78) takes into account the equality $ee' = 0$, which easily follows from (6.53) in the symmetric case $\beta = 0$. Substituting (6.77) and (6.78) in (6.75) and considering the elements $\{1, e, e', e''\}$ as a linearly independent give equations

$$a_4\omega^4 X^{IV} + a_3\omega^3 Y''' + a_2\omega^2 X'' + a_1\omega Y' + a_0 X = 0 \tag{6.79}$$
$$a_4\omega^4 Y^{IV} + a_3\omega^3 X''' + a_2\omega^2 Y'' + a_1\omega X' + a_0 Y = b_0$$

under the boundary conditions at $\tau = \pm 1$:

$$Y = 0, \qquad X' = 0 \tag{6.80}$$
$$\omega^2 Y'' = \frac{b_2}{a_4}, \qquad \omega^3 X''' = \frac{1}{a_4}\left(b_1 - \frac{a_3}{a_4} b_2\right)$$

In contrast to Eq. (6.75), the boundary value problem (6.79) and (6.80) does not include discontinuous terms any more. Although the number of equations in (6.79) is doubled as compared to (6.75), such a complication is rather formal due to the symmetry of the equations. Introducing the new variables, $U = X + Y$ and $V = X - Y$, decouples system (6.79) in such a way that the corresponding roots of the characteristic equations differ just by signs. (Besides, this fact reveals the possibility of using the idempotent basis for decoupling the resultant set of equations as discussed in Chap. 4 and will be discussed later in this chapter.) In addition, the type of the symmetry suggests that $X(\tau)$ and $Y(\tau)$ are odd and even functions, respectively. This enables one of reducing the general form of solution to a family of solutions with four arbitrary constants

$$X = \sum_{j=1}^{2}\left[A_j \cosh\left(\frac{\alpha_j}{\omega}\tau\right)\sin\left(\frac{\beta_j}{\omega}\tau\right) + B_j \sinh\left(\frac{\alpha_j}{\omega}\tau\right)\cos\left(\frac{\beta_j}{\omega}\tau\right)\right] \tag{6.81}$$
$$Y = \sum_{j=1}^{2}\left[A_j \sinh\left(\frac{\alpha_j}{\omega}\tau\right)\sin\left(\frac{\beta_j}{\omega}\tau\right) + B_j \cosh\left(\frac{\alpha_j}{\omega}\tau\right)\cos\left(\frac{\beta_j}{\omega}\tau\right)\right] + \frac{b_0}{a_0}$$

where $\alpha_j \pm \beta_j i$ are complex conjugate roots of the characteristic equation

$$a_4 p^4 + \ldots + a_1 p + a_0 = 0 \tag{6.82}$$

The assumption that both of the roots are complex reflects the physical meaning of the example. Finally, substituting (6.81) in (6.80) gives a linear algebraic set of four independent equations with respect to four constants: A_1, A_2, B_1, and B_2. Although the corresponding analytical solution is easy to obtain by using the standard *Mathematica*® commands, the result is somewhat complicated for reproduction. Practically, it may be reasonable to determine the constants by setting the system parameters to their numerical values moreover that only numerical solution is often possible for characteristic equations.

6.6 Piecewise-Linear Oscillators with Asymmetric Characteristics

Piecewise-linear oscillators are often considered as finite degrees-of-freedom models of cracked elastic structures [3, 41, 243], but may occur also due to specific design solutions. In many cases, the corresponded periodic solutions can be combined of different pieces of linear solutions valid for two different subspaces of the configuration space [42, 99, 243]. In this section, it will be shown that the nonsmooth transformation of time results in a closed form analytical solution matching both pieces of the solution automatically by means of elementary functions.

6.6.1 Amplitude-Phase Equations

Let us consider a piecewise linear oscillator of the form

$$m\ddot{q} + k[1 - \varepsilon H(q)]q = 0 \tag{6.83}$$

where $H(q)$ is Heaviside unit-step function; m and k are mass and stiffness parameters, respectively; and $|\varepsilon| \ll 1$; therefore, $k_- = k$ and $k_+ = k(1 - \varepsilon)$ are elastic stiffness of the oscillator for $q < 0$ and $q > 0$, respectively.

The exact general solution of oscillator (6.83) can be obtained by satisfying the continuity conditions for q and $\dot{q}$ at the matching point $q = 0$, where the characteristic has a break. The exact *closed form* solution for a similar oscillator was obtained in Sect. 4.3.4 in terms of NSTT. Such approaches are often facing quite challenging algebraic problems, as the number of degrees of freedom increases or external forces are involved. This is mainly due to the fact that times of crossing the boundary, $q = 0$, are a priori unknown. The problems become even more complicated in the presence of other types of nonlinearities. In this section, it will be shown that applying a combination of asymptotic expansions with respect to ε and NSTT gives a closed form solution for oscillator (6.83) with a possibility of generalization on the normal mode motions of multiple degrees-of-freedom systems. In particular, the nonsmooth temporal transformation:

- Provides an automatic matching of the motions from different subspaces of constant stiffness, and
- justifies quasi-linear asymptotic solutions for the specific nonsmooth case of piecewise linear characteristics.

Let us clarify the above two remarks. Introducing the notation $\Omega^2 = k/m$ brings Eq. (6.83) to the standard form of a weakly nonlinear oscillator

$$\ddot{q} + \Omega^2 q = \varepsilon \Omega^2 H(q) q \tag{6.84}$$

The nonlinear perturbation on the right-hand side of oscillator (6.84) is a continuous but nonsmooth function of the coordinate q. Since the major algorithms of quasi-linear theory assume smoothness of nonlinear perturbations, then such algorithms are not applicable in this case unless appropriate modifications and extensions have been made. Even though deriving first-order asymptotic solutions usually require no differentiation of characteristics, dealing with two pieces of the solution may complicate any further stages.

Let us show that combining quasi-linear methods of asymptotic integration, such as Krylov-Bogolyubov averaging,[3] with nonsmooth temporal transformations results in a closed form analytical solution for piecewise linear oscillator (6.83). Note that oscillator (6.83) plays an illustrative role for the approach developed below. Then a more complicated case will be considered.

Let us introduce the amplitude-phase coordinates $\{A(t), \varphi(t)\}$ on the phase plane of oscillator (6.83) through relationships

$$\begin{aligned} q &= A \cos\varphi \\ \dot{q} &= -\Omega A \sin\varphi \end{aligned} \tag{6.85}$$

The following compatibility condition is imposed on transformation (6.85)

$$\dot{A}\cos\varphi - A\sin\varphi\dot{\varphi} = -\Omega A \sin\varphi \tag{6.86}$$

Substituting (6.85) in (6.84) and taking into account (6.86) give

$$\begin{aligned} \dot{A} &= -\frac{1}{2}\varepsilon\Omega A H(A\cos\varphi)\sin 2\varphi \\ \dot{\varphi} &= \Omega - \varepsilon\Omega H(A\cos\varphi)\cos^2\varphi \end{aligned} \tag{6.87}$$

The right-hand sides of Eqs. (6.87) are 2π-periodic with respect to the phase variable, φ. Therefore, nonsmooth transformation of the phase variable applies through the couple of functions

$$\tau = \tau\left(\frac{2}{\pi}\varphi\right) \qquad \text{and} \qquad e = e\left(\frac{2}{\pi}\varphi\right) \tag{6.88}$$

Assuming that $A \geq 0$ and taking into account identities

[3] See Sect. 2.2.2.

$$\sin\varphi = \sin\left(\frac{\pi}{2}\tau\right)$$
$$\cos\varphi = \cos\left(\frac{\pi}{2}\tau\right)e$$
$$H(A\cos\varphi) = \frac{1}{2}(1+e) \tag{6.89}$$
$$e^2 = 1$$

bring (6.87) to the form

$$\dot{A} = -\frac{1}{4}\varepsilon\Omega(1+e)A\sin\pi\tau \tag{6.90}$$

$$\dot{\varphi} = \Omega - \frac{1}{2}\varepsilon\Omega(1+e)\cos^2\frac{\pi\tau}{2} \tag{6.91}$$

Note that the right-hand sides of (6.90) and (6.91) are nonsmooth but continuous with respect to the phase φ since the stepwise discontinuities of the square wave $e(2\varphi/\pi)$ are suppressed by the factors $\sin\pi\tau$ and $\cos^2(\pi\tau/2)$, respectively.

6.6.2 Amplitude Solution

Let us show that Eq. (6.90) has an exact 2π-periodic solution with respect to the phase variable, φ. According to the algorithm of NSTT, any periodic solution can be represented in the form

$$A = X(\tau) + Y(\tau)e \tag{6.92}$$

where τ and e are defined by (6.88).

Substituting (6.92) in (6.90) and taking into account (6.91) give boundary-value problem

$$(X-Y)' = 0$$
$$\frac{(X+Y)'}{X+Y} = -\frac{\varepsilon\pi}{4}\frac{\sin\pi\tau}{1-\varepsilon\cos^2\frac{\pi\tau}{2}} \tag{6.93}$$

$$Y|_{\tau=\pm 1} = 0 \tag{6.94}$$

where $' \equiv d/d\tau$.

Solution of the boundary value problem, (6.93) and (6.94), is obtained by integration. Then representation (6.92) gives

$$A(\varphi) = \alpha[1+\zeta(\tau)] - \alpha[1-\zeta(\tau)]e \tag{6.95}$$
$$\zeta(\tau) = \left(1-\varepsilon\cos^2\frac{\pi\tau}{2}\right)^{-1/2}$$

where the functions τ and e of the phase φ are defined in (6.88), and α is an arbitrary positive constant.

Note that solution (6.95) exactly captures the amplitude in both subspaces $q < 0$ and $q > 0$. However, the temporal mode shape and the period essentially depend on the phase variable φ described by Eq. (6.91). Generally, this equation admits exact integration, but the result would appear to have implicit form. Alternatively, it is shown below that solution for the phase variable can be approximated by asymptotic series in the explicit form

$$\begin{aligned}\varphi = \phi &- \frac{1}{8}\varepsilon[\pi\tau + (1+e)\sin\pi\tau] \\ &- \frac{1}{128}\varepsilon^2\{4(2-\cos\pi\tau)(\pi\tau+\sin\pi\tau) \\ &- [4\pi\tau(1+\cos\pi\tau) - 8\sin\pi\tau + \sin 2\pi\tau]e\} + O(\varepsilon^3)\end{aligned} \tag{6.96}$$

where the triangle and rectangle waves depend on the new phase variable, $\tau = \tau(2\phi/\pi)$, $e = e(2\phi/\pi)$, and

$$\phi = \Omega\left[1 - \frac{1}{4}\varepsilon - \frac{3}{32}\varepsilon^2 + O(\varepsilon^3)\right]t \tag{6.97}$$

6.6.3 Phase Solution

In this subsection, a second-order asymptotic procedure for phase equations with nonsmooth periodic perturbations is introduced. If applied to Eq. (6.91), the developed algorithm gives solution (6.96).

Let us consider some phase equation of the general form

$$\dot{\varphi} = \Omega[1+\varepsilon f(\varphi)] \tag{6.98}$$

where $f(\varphi)$ is a 2π-periodic, nonsmooth, or even stepwise discontinuous function, and ε is a small parameter, $|\varepsilon| \ll 1$.

Using the basic NSTT identity for $f(\varphi)$ brings Eq. (6.98) to the form

$$\dot{\varphi} = \Omega + \varepsilon\Omega\,[G(\tau) + M(\tau)e] \tag{6.99}$$

where the functions $G(\tau)$ and $M(\tau)$ are expressed through $f(\varphi)$, and the functions τ and e of the phase φ are defined in (6.88).

Note that the class of smoothness of the periodic perturbation in Eq. (6.99) depends on the behavior of functions $G(\tau)$ and $M(\tau)$ and their derivatives at the boundaries $\tau = \pm 1$. If, for instance, $M(\pm 1) \neq 0$, then the perturbation is stepwise discontinuous in φ whenever $\tau = \pm 1$.

Let us introduce the asymptotic procedure for Eq. (6.99) by noticing that, in case $\varepsilon = 0$, the phase φ has a constant temporal rate, $\dot{\varphi} = \Omega$. Hence, following the idea of asymptotic integration, let us find a phase transformation

$$\varphi = \phi + \varepsilon F_1(\phi) + \varepsilon^2 F_2(\phi) + \ldots \tag{6.100}$$

where functions $F_i(\phi)$ are such that the new phase variable, ϕ, also has a constant temporal rate even when $\varepsilon \neq 0$.

In other words, transformation (6.100) should bring Eq. (6.99) to the form

$$\dot{\phi} = \Omega(1 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \ldots) \tag{6.101}$$

where γ_i are constant coefficients to be determined together with $F_i(\phi)$ during the asymptotic procedure.

Note that the procedure, which is described below, has several specific features due to the presence of nonsmooth periodic functions. In particular, high-order approximations require a non-conventional interpretation for the power series expansions as discussed in Remark 6.6.1 at the end of this section. Other modifications occur already in the leading order approximation.

Substituting (6.100) in Eq. (6.99) and then enforcing Eq. (6.101) and collecting the terms of order ε give

$$F_1'(\phi) = G(\tau) + eM(\tau) - \gamma_1 \tag{6.102}$$

where the triangle and square waves depend now on the new phase variable ϕ as $\tau = \tau(2\phi/\pi)$ and $e = e(2\phi/\pi)$, respectively.

According to the conventional averaging procedure, the constant γ_1 is selected to achieve a zero mean on the right-hand side of Eq. (6.102) and thus provide periodicity of the solution, $F_1(\phi)$. In the algorithm below, the periodicity is due to the form of representation for periodic solutions, whereas the operator of averaging occurs automatically from the corresponding conditions of smoothness that is boundary conditions for the solution components. Following this remark, let us seek solution of Eq. (6.102) in the form

$$F_1(\phi) = U_1(\tau) + eV_1(\tau) \tag{6.103}$$

Substituting (6.103) in (6.102) and applying NSTT procedure give the boundary-value problem

$$\begin{aligned} U_1'(\tau) &= \frac{\pi}{2} M(\tau) \\ V_1'(\tau) &= \frac{\pi}{2} [G(\tau) - \gamma_1] \\ V_1(\pm 1) &= 0 \end{aligned} \tag{6.104}$$

There are two conditions on the function $V_1(\tau)$ described by the first-order differential equation in (6.104). There is also a choice for γ_1, which is to satisfy one of the two conditions. As a result, solution of boundary-value problem (6.104) is obtained by integration in the form

$$\begin{aligned} U_1(\tau) &= \frac{\pi}{2} \int_0^\tau M(z) dz \\ V_1(\tau) &= \frac{\pi}{2} \int_{-1}^\tau [G(z) - \gamma_1] dz \\ \gamma_1 &= \frac{1}{2} \int_{-1}^1 G(\tau) d\tau \end{aligned} \tag{6.105}$$

Further, collecting the terms of order ε^2 gives

$$F_2'(\phi) = G_2(\tau) + e M_2(\tau) + P_2(\tau) e' - \gamma_2 \tag{6.106}$$

where $e' \equiv de(2\phi/\pi)/d(2\phi/\pi)$ is a periodic series of δ-functions, and

$$\begin{aligned} M_2(\tau) &= \frac{2}{\pi} [U_1(\tau) G'(\tau) + V_1(\tau) M'(\tau)] - M(\tau) \gamma_1 \\ G_2(\tau) &= \frac{2}{\pi} U_1(\tau) M'(\tau) - G(\tau) \gamma_1 + \gamma_1^2 \\ P_2(\tau) &= \frac{2}{\pi} U_1(\tau) M(\tau) \end{aligned} \tag{6.107}$$

In contrast to first-order Eq. (6.102), the Eq. (6.106) includes the singular term $P_2(\tau)e'$ produced by the power series expansion of the perturbation in Eq. (6.99). If the perturbation is smooth, then $P_2(\pm 1) = 0$ and such singular term disappears. Nonetheless, the second-order approximation makes sense even in discontinuous case, when $P_2(\pm 1) \neq 0$. To clarify the details, let us represent solution of Eq. (6.106) in the form

$$F_2(\phi) = U_2(\tau) + e V_2(\tau) \tag{6.108}$$

Substituting (6.108) in (6.106) gives boundary-value problem

$$
\begin{aligned}
U_2'(\tau) &= \frac{\pi}{2} M_2(\tau) \\
V_2'(\tau) &= \frac{\pi}{2} [G_2(\tau) - \gamma_2] \\
V_2(\pm 1) &= \frac{\pi}{2} P_2(\pm 1)
\end{aligned} \tag{6.109}
$$

In contrast to (6.104), the current boundary-value problem has generally non-homogeneous boundary conditions for V_2. These conditions compensate the singular term e' from differential equation (6.106). As a result Eqs. (6.109) are free of any singularities and admit solution analogously to first-order Eqs. (6.104),

$$
\begin{aligned}
U_2(\tau) &= \frac{\pi}{2} \int_0^{\tau} M_2(z) dz \\
V_2(\tau) &= \frac{\pi}{2} \int_{-1}^{\tau} [G_2(z) - \gamma_2] dz + \frac{\pi}{2} P_2(-1) \\
\gamma_2 &= \frac{1}{2} \int_{-1}^{1} G_2(\tau) d\tau + \frac{1}{2} [P_2(-1) - P_2(1)]
\end{aligned} \tag{6.110}
$$

Example 6.6.1 Now, let us revisit the illustrating model. In particular case (6.91), one has

$$
G(\tau) \equiv M(\tau) \equiv -\frac{1}{2} \cos^2 \frac{\pi \tau}{2} \tag{6.111}
$$

and

$$
\begin{aligned}
G(\pm 1) &= M(\pm 1) = 0 \\
G'(\pm 1) &= M'(\pm 1) = 0 \\
G''(\pm 1) &= M''(\pm 1) = -\pi^2/4
\end{aligned} \tag{6.112}
$$

where $' \equiv d/d\tau$. First two of Eqs. (6.112) provide continuity for the right-hand side of (6.99) and its first derivative at those φ where $\tau = \pm 1$. As follows from (6.107) and (6.111), for this class of smoothness, one has $P_2(\pm 1) = 0$ and hence no singular terms occur in the first two steps of asymptotic procedure. Finally, taking into account (6.111) and (6.112) and conducting integration in (6.105) and (6.110) bring solution (6.100) to the form (6.96) and (6.97). Figure 6.9 compares analytical solution (6.85), (6.95), and (6.96) shown by the solid line and numerical solution shown by the dashed line. As expected, the amplitude shows the perfect match, whereas some phase shift develops after several cycles.

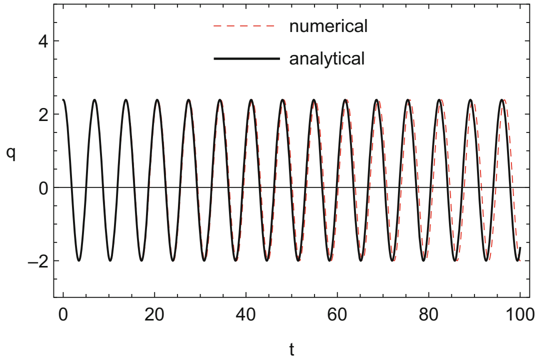

Fig. 6.9 Second-order asymptotic and numerical solutions shown by solid and dashed lines, respectively

6.6.4 The Amplitude-Phase Problem in Idempotent Basis

Recall that the idempotent basis is given by $e_+ = (1+e)/2$ and $e_- = (1-e)/2$ so that $e_+^2 = e_+$, $e_-^2 = e_-$, and $e_+e_- = 0$. Equations (6.90) and (6.91) therefore take the form

$$\dot{A} = -\frac{1}{2}\varepsilon\Omega e_+ A \sin\pi\tau \tag{6.113}$$

$$\dot{\varphi} = \Omega - \varepsilon\Omega e_+ \cos^2\frac{\pi\tau}{2} \tag{6.114}$$

Let us represent the amplitude as a function of φ in the form

$$A(\varphi) = X_+(\tau)e_+ + X_-(\tau)e_- \tag{6.115}$$

where $e_+ = e_+(2\varphi/\pi)$, $e_- = e_-(2\varphi/\pi)$, and $\tau = \tau(2\varphi/\pi)$.

Substituting (6.115) in (6.113) and taking into account (6.114) give

$$\frac{2}{\pi}(X'_+e_+ - X'_-e_-)\left(\Omega - \varepsilon\Omega e_+\cos^2\frac{\pi\tau}{2}\right) = -\frac{1}{2}\varepsilon\Omega e_+(X_+e_+ + X_-e_-)\sin\pi\tau$$

or

$$\left(1 - \varepsilon\cos^2\frac{\pi\tau}{2}\right)X'_+ = -\frac{\pi}{4}\varepsilon X_+\sin\pi\tau$$
$$X'_- = 0 \tag{6.116}$$

under the boundary condition

$$(X_+ - X_-)|_{\tau=\pm 1} = 0 \tag{6.117}$$

The boundary-value problem (6.116) and (6.117) admits exact solution so that (6.115) gives finally

$$A(\varphi) = \alpha\left[\left(1 - \varepsilon\cos^2\frac{\pi\tau}{2}\right)^{-1/2} e_+ + e_-\right] \tag{6.118}$$

where α is an arbitrary positive constant.

Remark 6.6.1 In the classical analysis, nonsmooth functions cannot be represented by Taylor series near their singular points. This can be justified however in terms of distributions as confirmed by the following example. Nonsmoothness of the triangular sine is similar to that function $|t|$ has at zero. Let us consider its formal power series

$$|t+\varepsilon| = |t| + |t|'\varepsilon + \frac{1}{2!}|t|''\varepsilon^2 + \ldots \tag{6.119}$$

where $\varepsilon > 0$ and $-\infty < t < \infty$, and prime indicates Schwartz derivative. It is clear that equality (6.119) has no regular point-wise meaning. For instance, equality (6.119) is obviously not true on the interval $-\varepsilon < t < 0$. In addition, the right-hand side of (6.119) is uncertain at $t = 0$, whereas the left-hand side gives ε. Nevertheless, let us show that equality (6.119) admits a generalized interpretation and holds in terms of distributions. Let $\psi(t)$ be a *test function* in terms of the distribution theory; more precisely, $\psi(t)$ is infinitely differentiable with compact support that is identically zero outside of some bounded interval. Integrating by parts and then shifting the variable of integration give

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(|t| + |t|'\varepsilon + \frac{1}{2!}|t|''\varepsilon^2 + \ldots\right)\psi(t)dt \\
&= \int_{-\infty}^{\infty}|t|\left[\psi(t) - \psi'(t)\varepsilon + \frac{1}{2!}\psi''(t)\varepsilon^2 - \ldots\right]dt \\
&= \int_{-\infty}^{\infty}|t|\psi(t-\varepsilon)dt = \int_{-\infty}^{\infty}|t+\varepsilon|\psi(t)dt
\end{aligned} \tag{6.120}$$

Therefore, equality (6.119) holds in the integral sense of distributions.

6.7 Multiple Degrees-of-Freedom Case

Let us consider a multiple degrees-of-freedom piecewise-linear system of the form

$$M\ddot{x} + Kx = \varepsilon H(Sx)Bx \tag{6.121}$$

where $x(t) \in R^n$ is a vector-function of the system coordinates, M is a mass matrix, H denotes the Heaviside unit-step function, and S is a normal vector to the plane splitting the configuration space into two parts with different elastic properties, so that the stiffness matrix is K when $Sx < 0$ and $K - \varepsilon B$ when $Sx > 0$. It is assumed that the stiffness jump is small, $|\varepsilon| \ll 1$.

The number of possible iterations of the classical perturbation tools usually depends on a class of smoothness of the perturbation. The perturbation term on the right-hand side of (6.121) is continuous but nonsmooth. Therefore, only first-order asymptotic solution can be obtained within the classic theory of differential equations. Also, the piecewise character of the perturbation complicates the form of the solution due to the necessity of matching the different pieces of the solution.

Let us show that NSTT gives a closed-form solution by automatically matching the pieces of solution in two different configuration subspaces of different stiffness properties. We seek a 2π-periodic, with respect to the phase φ, solution of system (6.121) in the form of the following asymptotic expansions:

$$x(\varphi) = A_j \cos\varphi + \varepsilon x^{(1)}(\varphi) + O(\varepsilon^2)$$
$$\varphi = \Omega_j \sqrt{1 + \varepsilon\gamma^{(1)} + O(\varepsilon^2)}t \tag{6.122}$$

where Ω_j and A_j are arbitrary eigen-frequency and eigen vector (normal mode) of the linearized system:

$$\left(-\Omega_j^2 M + K\right) A_j = 0 \qquad (j = 1, \ldots, n) \tag{6.123}$$

Substituting (6.122) in (6.121), taking into account identities (6.89), assuming that algebraic equation (6.123) holds, and collecting terms in the first order of ε give

$$\Omega_j^2 M \frac{d^2 x^{(1)}}{d\varphi^2} + K x^{(1)} = \left[\frac{1}{2} B A_j + \left(\frac{1}{2} B A_j + \gamma^{(1)} K A_j\right) e\right] \cos\frac{\pi\tau}{2} \tag{6.124}$$

where $\tau = \tau(2\varphi/\pi)$, $e = e(2\varphi/\pi)$, and the relationship $(1 + \varepsilon\gamma^{(1)})^{-1} = 1 - \varepsilon\gamma^{(1)} + O(\varepsilon^2)$ was enforced.

Since the function $x^{(1)}(\varphi)$ is sought to be 2π-periodic with respect to φ, it should admit NSTT representation

$$x^{(1)} = X(\tau) + Y(\tau)e \tag{6.125}$$

Substituting (6.125) in (6.124) and conducting NSTT procedure give the boundary-value problem

$$\left(\frac{2\Omega_j}{\pi}\right)^2 MX'' + KX = \frac{1}{2}BA_j \cos\frac{\pi\tau}{2}, \qquad X'|_{\tau=\pm 1} = 0 \tag{6.126}$$

$$\left(\frac{2\Omega_j}{\pi}\right)^2 MY'' + KY = \left(\frac{1}{2}BA_j + \gamma^{(1)}KA_j\right)\cos\frac{\pi\tau}{2} \tag{6.127}$$

$$Y|_{\tau=\pm 1} = 0$$

Representing the corresponding solution in terms of the normal mode coordinates

$$X = \sum_{i=1}^{n} A_i X_i(\tau), \qquad Y = \sum_{i=1}^{n} A_i Y_i(\tau) \tag{6.128}$$

and taking into account M-orthogonality of the set of eigen-vectors give

$$\left(\frac{2\Omega_j}{\pi}\right)^2 X_i'' + \Omega_i^2 X_i = \beta_{ij}\cos\frac{\pi\tau}{2}, \qquad X_i'|_{\tau=\pm 1} = 0 \tag{6.129}$$

$$\left(\frac{2\Omega_j}{\pi}\right)^2 Y_i'' + \Omega_i^2 Y_i = (\beta_{ij} + \gamma^{(1)}\varkappa_{ij})\cos\frac{\pi\tau}{2}, \quad Y_i|_{\tau=\pm 1} = 0 \tag{6.130}$$

where

$$\beta_{ij} = \frac{1}{2}\frac{A_i B A_j}{A_i M A_i}, \qquad \varkappa_{ij} = \frac{A_i K A_j}{A_i M A_i} \tag{6.131}$$

are dimensionless coefficients.

Note that, despite the similar representation for solution (6.122), the current asymptotic procedure differs from the Poincaré-Lindstedt method due to the specific of representation (6.125). According to the Poincaré-Lindstedt method, the frequency correction term, $\gamma^{(1)}$, is to eliminate the so-called secular terms in the asymptotic expansions. In the present case, the secular terms appear to be periodic due to the inherent periodicity of the new temporal argument. Instead solutions are required to satisfy the boundary-value problems, such as (6.129) and (6.130). If $i \neq j$, the term $\gamma^{(1)}$ disappears from (6.130) due to $\varkappa_{ij} = 0$. Then both boundary-value problems, (6.129) and (6.130), admit solutions

$$X_i = \frac{\beta_{ij}}{\Omega_i^2 - \Omega_j^2}\left(\cos\frac{\pi\tau}{2} - \frac{\Omega_j}{\Omega_i}\cos\frac{\pi\Omega_i\tau}{2\Omega_j}\csc\frac{\pi\Omega_i}{2\Omega_j}\right) \tag{6.132}$$

$$Y_i = \frac{\beta_{ij}}{\Omega_i^2 - \Omega_j^2}\cos\frac{\pi\tau}{2} \tag{6.133}$$

When $i = j$, problem (6.129) still has solution, but problem (6.130) generally does not due to the resonance $\Omega_i = \Omega_j$. Fortunately, in this case $\varkappa_{jj} \neq 0$, and hence

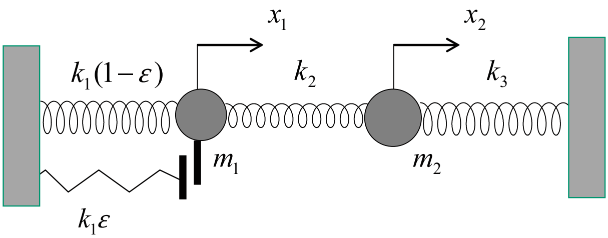

Fig. 6.10 Two degrees-of-freedom piecewise-linear system can be viewed as a model of a rod with a small crack

the right-hand side of the equation can be set to zero by means of the condition on yet undetermined as

$$\gamma^{(1)} = -\frac{\beta_{jj}}{\varkappa_{jj}} \tag{6.134}$$

Due to this condition, problem (6.130) admits zero solution, and thus

$$X_j = \frac{\pi\beta_{jj}}{4\Omega_j^2}\left(\tau \sin\frac{\pi\tau}{2} + \frac{2}{\pi}\cos\frac{\pi\tau}{2}\right) \tag{6.135}$$

$$Y_j = 0 \tag{6.136}$$

Expressions (6.125), (6.128), and (6.132) through (6.136) completely determine the first-order approximation $x^{(1)}(\varphi)$.

Example 6.7.1 Let us consider a two-degrees-of-freedom example of mass-spring model (Fig. 6.10)

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= \varepsilon k_1 H(x_1)x_1 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 &= 0 \end{aligned} \tag{6.137}$$

Equations (6.119) can be represented in the form (6.121), where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}, \qquad B = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In this case, the first-order asymptotic solution for the inphase ($j = 1$) and out-of-phase ($j = 2$) takes the form, respectively,

$$x_1 = e\cos\frac{\pi\tau}{2} + \frac{\varepsilon\pi}{16}\left(\frac{2}{\pi}\cos\frac{\pi\tau}{2} + \tau\sin\frac{\pi\tau}{2}\right) + O(\varepsilon^2)$$

$$x_2 = e\cos\frac{\pi\tau}{2} - \frac{\varepsilon k_1}{8k_2}\left[e\cos\frac{\pi\tau}{2} + \cos\frac{\pi\tau}{2}\right. \tag{6.138}$$

$$\left. - \left(1 + 2\frac{k_2}{k_1}\right)^{-1/2}\cos\left(\sqrt{1+2\frac{k_2}{k_1}}\frac{\pi\tau}{2}\right) / \sin\left(\sqrt{1+2\frac{k_2}{k_1}}\frac{\pi}{2}\right)\right] + O(\varepsilon^2)$$

$$\varphi = \sqrt{\frac{k_1}{m}}\sqrt{1 - \frac{\varepsilon}{4} + O(\varepsilon^2)t}$$

and

$$x_1 = -e\cos\frac{\pi\tau}{2} + \frac{\varepsilon k_1}{8k_2}\left[e\cos\frac{\pi\tau}{2} + \cos\frac{\pi\tau}{2} - \left(1 + 2\frac{k_2}{k_1}\right)\right.$$

$$\left. \times\cos\left(\frac{\pi\tau}{2}/\sqrt{1+2\frac{k_2}{k_1}}\right) / \sin\left(\frac{\pi}{2}/\sqrt{1+2\frac{k_2}{k_1}}\right)\right] + O(\varepsilon^2) \tag{6.139}$$

$$x_2 = e\cos\frac{\pi\tau}{2} + \frac{\varepsilon k_1\pi}{16(k_1+2k_2)}\left(\frac{2}{\pi}\cos\frac{\pi\tau}{2} + \tau\sin\frac{\pi\tau}{2}\right) + O(\varepsilon^2)$$

$$\varphi = \sqrt{\frac{k_1+2k_2}{m}}\sqrt{1 - \frac{\varepsilon k_1}{4(k_1+2k_2)} + O(\varepsilon^2)t}$$

where $m = m_1 = m_2$ is assumed. Solutions (6.138) and (6.102) show that the piecewise linear restoring force may have quite different effect on different modes. In particular, solution (6.138) reveals the possibility of internal resonances, when

$$\sin\left(\frac{\pi\Omega_2}{2\Omega_1}\right) = 0, \qquad \frac{\Omega_2}{\Omega_1} = \sqrt{1+2\frac{k_2}{k_1}} \tag{6.140}$$

If, for instance, the system is close to the frequency ratio $\Omega_2/\Omega_1 = 2$, then the inphase mode may be affected significantly by a crack even under very small magnitudes of the parameter ε. In contrary, solution (6.102) has the denominator $\sin[(\pi/2)\Omega_1/\Omega_2]$, which is never close to zero because $0 < \Omega_1/\Omega_2 < 1$. Therefore, in current asymptotic approximation, the influence of crack on the out-of-phase mode is always of order ε provided that $k_2/k_1 = O(1)$.The influence of the bilinear stiffness on inphase mode trajectories in the closed to internal resonance case is seen from Fig. 6.11, where both analytical and numerical solutions are shown for comparison reasons. The frequency ratio $\Omega_2/\Omega_1 = 2.0025$ is achieved by conditioning the spring stiffness parameters as follows $k_2 = (3/2)k_1 + 0.005$.

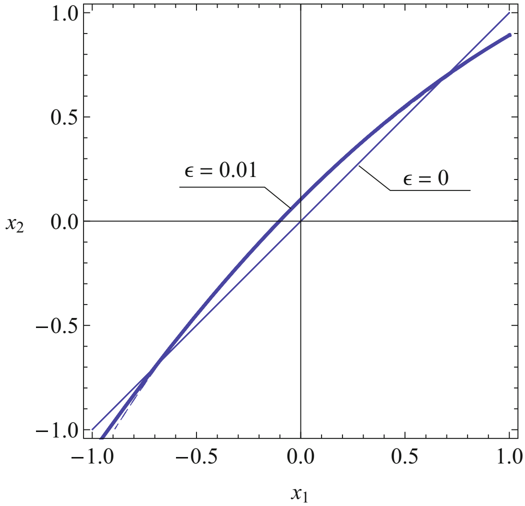

Fig. 6.11 The influence of a small crack, $\varepsilon = 0.01$, on the inphase mode trajectory near the frequency ratio $\Omega_2/\Omega_1 = 2$; the dashed line shows the numerical solution, and the thin solid line corresponds to the linear case, $\varepsilon = 0$