Chapter 2
Smooth Oscillating Processes

This chapter gives an overview of selected analytical methods for smooth oscillating processes. Most of such methods are quasi-linear by nature. The corresponding technical implementations usually employ the harmonic oscillator as a generating model. The description focuses on such ideas and technical details that are further combined with nonsmooth methods. The related procedures of the asymptotic integration usually include the stage of a preliminary transformation of the original system to the form that admits a straightforward solution. In particular, the averaging algorithm based on the Hausdorff equation for operators Lie is reproduced.

2.1 Linear and Weakly Nonlinear Approaches

By both practical and theoretical reasons, the quantitative methods of dynamics were developed first for smooth processes. As a rule, smooth oscillations can be directly observed under no special conditions. For instance, projection of any fixed point of a body rotating with constant angular speed makes a perfect impression about harmonic oscillations. Interestingly, in 1693, Leibniz derived the differential equation for sine geometrically by considering a circle. Much later, original analytical ideas of nonlinear vibrations emerged from the celestial mechanics considering perturbations of circular orbits of rigid-body motions rather than any mass-spring oscillators. Robert Hooke (1635–1703) was probably first who suggested the basic elastic mass-spring model, whereas Galileo and Huygens were investigating the pendulum. Later, d'Alembert, Daniel Bernoulli, and Euler considered a one-dimensional continual model of a string. It was found that the vibrating string represents the infinity of harmonic oscillators corresponding to different mode shapes of the string. It is well-known that a serious discussion occurred about whether or not the sum of smooth functions, such as sines, can

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represent a nonsmooth shape of the string. These discussions were finalized by the Fourier theorem.

Let us reproduce the result for a periodic function of time $f(t)$ of the period T in the complex form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\Omega t) \tag{2.1}$$

$$c_k = \frac{1}{T}\int_0^T f(t)\exp(-ik\Omega t)dt, \quad \Omega = \frac{2\pi}{T}$$

This relation generates a one-to-one mapping between the function $f(t)$ and its Fourier coefficients

$$f(t) \longleftrightarrow \{\ldots c_{-2}, c_{-1}, c_1, c_2, \ldots\} \tag{2.2}$$

Note that mathematical expressions (2.1) do not necessarily imply that the periodic signal $f(t)$ must be produced by a linear system even though the right-hand side of (2.1) combines free vibrations of linear oscillators.[1] Therefore, the Fourier analysis with its associates should be viewed as a *linear language* for *nonlinear systems* regardless of specifics of analytical algorithms. Most quantitative methods for weakly nonlinear periodic motions, one way or another, recover Fourier coefficients of the corresponding solutions. On one hand, such tools possess a high level of generality. On the other hand, even elementary strongly nonlinear phenomena, as defined in Chap. 1, may become quite difficult to describe in the linear language. Nevertheless, the quantitative theory of nonlinear vibration has been advanced by new asymptotic techniques developed originally in a formal way for solving nonlinear differential equations. Most traditional methods are essentially based on the ideas of perturbation or averaging [69]. Similar results can be obtained within the theory of Poincaré normal forms [153], which retains resonance terms, whereas all non-resonance terms are eliminated by means of a coordinate transformation. Such a normal form is qualified as the simplest possible form of the equations of motion.

2.2 A Brief Overview of Smooth Methods

2.2.1 Periodic Motions of Quasi-linear Systems

Consider a weakly nonlinear oscillator of the form

$$\ddot{x} + \Omega_0^2 x = \varepsilon f(x, \dot{x}) \tag{2.3}$$

[1] Recall rigid-body analogies in Sect. 1.4.3.

where ε is a small parameter from the interval $0 < \varepsilon \ll 1$, and $f(x, \dot{x})$ is a smooth enough function of both arguments.

Periodic solutions of Eq. (2.3) can be found by splitting the nonlinear system into a sequence of linear oscillators by means of the power series of the small parameter ε

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \tag{2.4}$$

The perturbation on the right-hand side of Eq. (2.3) changes the fundamental frequency of the oscillator as

$$\Omega^2 = \Omega_0^2(1 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \ldots) \tag{2.5}$$

The new frequency, Ω, is introduced explicitly into the differential equation of motion by switching to the phase argument

$$\varphi = \Omega t \tag{2.6}$$

As a result, series (2.4) appears to be composed of trigonometric functions of multiple phases φ, 2φ, 3φ,.... The numbers γ_1, γ_2,... are used for canceling the so-called secular terms from the solution.

A similar idea was implemented by Lyapunov for systems of first-order equations, such as

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + f_1(x_1, x_2) \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + f_2(x_1, x_2) \end{aligned} \tag{2.7}$$

where f_1 and f_2 are nonlinear functions, it is assumed that system (2.7) admits first analytical integral, and the corresponding linearized system has only periodic solutions. Then periodic solutions of (2.7) admit power series expansions with respect to the amplitude.

There exist at least two extensions of Lyapunov theory, such as local and global approaches to nonlinear normal modes, see, for instance, [136, 155, 241].

2.2.2 *One-Phase Averaging*

Averaging with Van der Pol Amplitude-Phase Variables

Let us illustrate different implementations of the averaging by reproducing some technical details. The description focuses on such tools that remain applicable to nonconservative systems.

To illustrate van der Pol's averaging procedure, let us represent Eq. (2.3) as a system of two first-order equations by introducing the velocity variable, v, as

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\Omega_0^2 x + \varepsilon f(x, v) \end{aligned} \tag{2.8}$$

The next step includes transition to the amplitude-phase variables on the system phase plane as $\{x,\ v\} \to \{a, \varphi\}$:

$$x = a\cos\varphi, \quad v = -a\Omega_0 \sin\varphi \tag{2.9}$$

Now substituting (2.9) in (2.8) and considering the result as a system of two algebraic equations with respect to the derivatives $\dot{a}$ and $\dot{\varphi}$ give

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{\Omega_0} f(a\ \cos\varphi, -a\Omega_0 \sin\varphi) \sin\varphi \\ \dot{\varphi} &= \Omega_0 - \frac{\varepsilon}{\Omega_0 a} f(a\ \cos\varphi, -a\Omega_0 \sin\varphi) \cos\varphi \end{aligned} \tag{2.10}$$

This system is still an exact equivalent of the original equation (2.3). Despite the formal complexity, system (2.10) has the essential advantage due to different time scales of the variables, a and φ. Noticing that the new system is 2π-periodic with respect to the fast phase φ suggests its elimination from the right-hand side of the system by applying the operator of averaging

$$< \cdots >_\varphi \equiv \frac{1}{2\pi} \int_0^{2\pi} \cdots d\varphi$$

as follows:

$$\dot{a} = -\frac{\varepsilon}{\Omega_0} < f(a\ \cos\varphi, -a\Omega_0 \sin\varphi) \sin\varphi >_\varphi \tag{2.11}$$

$$\dot{\varphi} = \Omega_0 - \frac{\varepsilon}{\Omega_0 a} < f(a\ \cos\varphi, -a\Omega_0 \sin\varphi) \cos\varphi >_\varphi \tag{2.12}$$

Solutions of system (2.12) are considered as approximate leading-order averaged solutions of the original system (2.10). The main achievement from the above manipulations is due to independence of the amplitude equation from the phase φ.

Example of Rayleigh-Duffing Oscillator

Very often, amplitude equation (2.11) can be solved exactly by separation of variables as illustrated below based on Rayleigh-Duffing oscillator:

$$\ddot{x} + \Omega_0^2 x = \varepsilon \left[\left(1 - \frac{1}{3}\dot{x}^2 \right) \dot{x} - \alpha x^3 \right] \tag{2.13}$$

In this case, substituting (2.9) in $f(x, v) = \left(1 - v^2/3\right) v - \alpha x^3$ and then conducting the averaging in (2.12) give

$$\begin{aligned} \dot{a} &= \frac{1}{2}\varepsilon \left(a - \frac{1}{4}\Omega_0^2 a^3 \right) \\ \dot{\varphi} &= \Omega_0 + \varepsilon \frac{3\alpha}{8\Omega_0} a^2 \end{aligned} \tag{2.14}$$

where the amplitude a is described by a separable equation, which is independent on the phase variable, φ.

Assuming the initial condition $a(0) = a_0$ and separating the variables give explicit solution:

$$a = \frac{2a_0}{\sqrt{a_0^2 \Omega_0^2 \left[1 - \exp(-\varepsilon t)\right] + 4\exp(-\varepsilon t)}} \tag{2.15}$$

Now the phase φ can be obtained from the second equation in (2.14) by the direct integration as

$$\varphi = \Omega_0 t + \frac{3\alpha}{2\Omega_0^3} \ln \left\{ 1 + \frac{1}{4} a_0^2 \Omega_0^2 \left[\exp(\varepsilon t) - 1 \right] \right\} + \varphi_0 \tag{2.16}$$

where $\varphi_0 = \varphi(0)$.

Solution (2.15) and (2.16) should be considered as solutions in the leading asymptotic order of the averaging procedure. Such types of approximations are often sufficient for practical estimations as follows from Fig. 2.1. Comparing the graphs shows a sufficient agreement of the result of direct numerical integration of Eq. (2.13) with the analytical approximation given by (2.15) and (2.16).

Krylov-Bogolyubov Generalization

This above averaging procedure was essentially generalized in the 1930s of the last century [34] by combining the Lindstedt-Poincaré and van der Pol's ideas to obtain high asymptotic orders of the averaging. Let us outline the corresponding formalism based on the general one-phase system

$$\begin{aligned} \dot{x} &= \varepsilon X(x, y) \\ \dot{y} &= \Omega(x) + \varepsilon Y(x, y) \end{aligned} \tag{2.17}$$

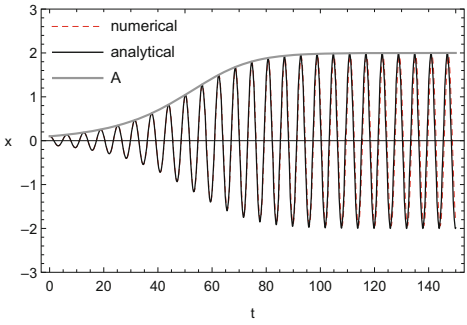

Fig. 2.1 Time history response of Rayleigh-Duffing oscillator under the following parameters and initial conditions: $\varepsilon = 0.1$, $\alpha = 0.3$, $\Omega_0 = 1.0$, $A_0 = 0.1$, $\varphi_0 = 0$

where y and x are scalar and vector variables, respectively, and the right-hand side is assumed to be 2π-periodic with respect to y.

In contrast to (2.10), the frequency Ω in (2.17) depends on the slow varying vector-function x, which may represent a set of amplitudes of an $n-$degrees-of-freedom vibrating system, $x = (a_1, \ldots, a_n)$. Sometimes, such kind of systems is called essentially nonlinear since the condition $\varepsilon = 0$ does not make the frequency state independent. Note that, if $\varepsilon = 0$, the system has no fast phase on the right-hand side. Thus the problem is to find a close to identical transformation

$$\begin{aligned} x &= q + \varepsilon u_1(q, \psi) + \varepsilon^2 u_2(q, \psi) + \ldots \\ y &= \psi + \varepsilon v_1(q, \psi) + \varepsilon^2 v_2(q, \psi) + \ldots \end{aligned} \tag{2.18}$$

which eliminates the fast phase entirely from the system and brings it to the form

$$\begin{aligned} \dot{q} &= \varepsilon A_1(q) + \varepsilon^2 A_2(q) + \ldots \\ \dot{\psi} &= \Omega_0(q) + \varepsilon \Omega_1(q) + \varepsilon^2 \Omega_2(q) + \ldots \end{aligned} \tag{2.19}$$

This problem is solved by substituting expansions (2.18) in Eqs. (2.17), enforcing Eqs. (2.19) to eliminate the derivatives $\dot{q}$ and $\dot{\psi}$, and then separating different orders of ε. Then the resultant system is solved iteratively. In zero-order of ε, the second equation in (2.17) gives $\Omega_0(q) = \Omega(q)$. As a result, the problem of order ε takes the form

$$\Omega(q)\frac{\partial u_1}{\partial \psi} = X(q, \psi) - A_1(q) \tag{2.20}$$

$$\Omega(q)\frac{\partial v_1}{\partial \psi} = Y(q, \psi) + \Omega'(q)u_1 - \Omega_1(q) \tag{2.21}$$

These partial differential equations are solved for u_1 and v_1 under the condition that solutions must be bounded with respect to the fast phase ψ. For that reason, the average of the right-hand side of both equations must be zero, which is achieved by setting

$$A_1(q) = < X(q,\psi) >_\psi \tag{2.22}$$

$$\Omega_1(q) = < Y(q,\psi) >_\psi + \Omega'(q) < u_1 >_\psi \tag{2.23}$$

Then, integration gives

$$u_1 = \frac{1}{\Omega(q)} \int_0^\psi [X(q,\psi) - < X(q,\psi) >_\psi] d\psi \tag{2.24}$$

$$v_1 = \frac{1}{\Omega(q)} \int_0^\psi [Y(q,\psi) - < Y(q,\psi) >_\psi] d\psi \tag{2.25}$$

$$+ \frac{\Omega'(q)}{\Omega(q)} \int_0^\psi [u_1 - < u_1 >_\psi] d\psi$$

Strictly speaking, both these solutions should include arbitrary functions of q as summands. These arbitrary functions are chosen to be zero assuming that both the old and new variables, $\{x, y\}$ and $\{q, \psi\}$ in (2.18), satisfy the same initial conditions. On the next step of asymptotic integration, values (2.22) through (2.25) are substituted in equations

$$\Omega(q) \frac{\partial u_2}{\partial \psi} = u_1 \frac{\partial}{\partial q} X(q,\psi) + v_1 \frac{\partial}{\partial \psi} X(q,\psi) \tag{2.26}$$

$$- A_1(q) \frac{\partial u_1}{\partial q} - \Omega_1(q) \frac{\partial u_1}{\partial \psi} - A_2(q)$$

and

$$\Omega(q) \frac{\partial v_2}{\partial \psi} = u_1 \frac{\partial}{\partial q} Y(q,\psi) + v_1 \frac{\partial}{\partial \psi} Y(q,\psi) \tag{2.27}$$

$$- A_1(q) \frac{\partial v_1}{\partial q} - \Omega_1(q) \frac{\partial v_1}{\partial \psi} + \Omega'(q) u_2 + \frac{1}{2} \Omega''(q) u_1^2 - \Omega_2(q)$$

Despite a more complicated form, these equations have the same structure as Eqs. (2.20) and (2.21). Moreover, it is easy to see that this structure will be maintained for any order with an obvious increase of the technical complexity though. Practically, high-order approximations can be obtained by means of automatic systems of symbolic manipulations.

Example of Rayleigh Equation

Let us represent Rayleigh's equation, $\ddot{z} + \Omega^2 z = \varepsilon\left(1 - \dot{z}^2/3\right)\dot{z}$, as a set of two first-order equations

$$\begin{aligned} \dot{z} &= v \\ \dot{v} &= -\Omega^2 z + \varepsilon\left(1 - \frac{1}{3}v^2\right)v \end{aligned} \tag{2.28}$$

The coordinate transformation, $z = x\cos y$ and $v = -x\Omega\sin y$, brings system (2.28) to the form of system (2.17), where

$$X(x, y) = x\left(1 - \frac{1}{3}x^2\Omega^2\sin^2 y\right)\sin^2 y \tag{2.29}$$

$$Y(x, y) = \frac{1}{6}\left(3 - x^2\Omega^2\sin^2 y\right)\sin 2y \tag{2.30}$$

and Ω is constant.

Substituting (2.29) and (2.30) in (2.22) through (2.25), conducting integration, and substituting the result in (2.19) and (2.18) give, respectively,

$$\dot{q} = \frac{1}{2}\varepsilon\left(q - \frac{1}{4}\Omega^2 q^3\right) + O(\varepsilon^3) \tag{2.31}$$

$$\dot{\psi} = \Omega - \frac{1}{256\Omega}\varepsilon^2\left(32 - 24\Omega^2 q^2 + 5\Omega^4 q^4\right) + O(\varepsilon^3) \tag{2.32}$$

and

$$x = q - \frac{1}{48\Omega}\varepsilon q[12 - (4 - \cos 2\psi)q^2\Omega^2]\sin 2\psi + O(\varepsilon^2) \tag{2.33}$$

$$y = \psi + \frac{1}{12\Omega}\varepsilon\left(6 - q^2\Omega^2\sin^2\psi\right)\sin^2\psi + O(\varepsilon^2) \tag{2.34}$$

The original variable is given by $z = x\cos y$. Taking into account first-order approximation and setting the right-hand side of Eqs. (2.26) and (2.27) to zero give $A_2 = 0$ whereas $\Omega_2 \neq 0$ as seen from Eqs. (2.31) and (2.32).

Solutions of Rayleigh's equation in first and second asymptotic orders are compared in Fig. 2.2. Both solutions are in a sufficient agreement with the result of direct numerical integration even for the parameter ε, which is not very small as compared to unity. The effect of improvement in the order ε^2 still can be observed after multiple cycles of oscillation as follows from Fig. 2.2b.

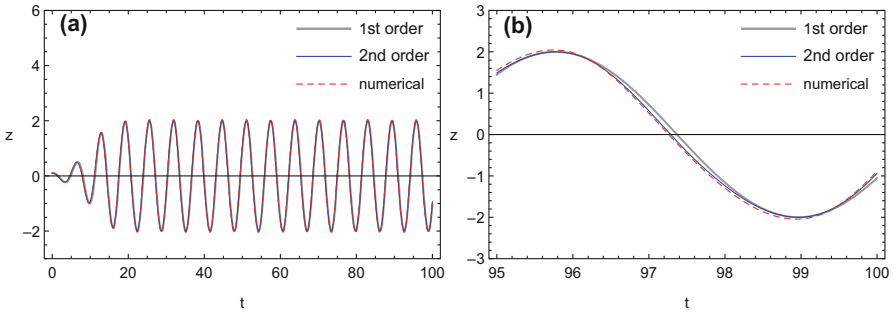

Fig. 2.2 Solutions of Rayleigh's equation in first and second asymptotic orders for the parameters $\varepsilon = 0.5$, $\Omega = 1.0$, and the initial conditions $q(0) = 0.1$, $\psi(0) = 0.0$: (**a**) developing steady state and (**b**) comparison of temporal mode shapes after multiple cycles

2.2.3 Two-Phase Averaging for Mathew Equation

In a multiple frequency case, the averaging procedure may require an extra step of the system adaptation. For illustrating purpose, consider Mathew's equation with damping

$$\ddot{x} + 2\zeta\Omega_0\dot{x} + \Omega_0^2(1 + \varepsilon\cos\Omega t)x = 0 \tag{2.35}$$

where the damping ratio ζ is assumed to be a small parameter of order ε, and $\Omega = 2$ as required by the standard form of Mathew's equation. Therefore, in addition to the natural frequency Ω_0, there is one more frequency, Ω, associated with the parametric loading term.

It will be shown below that the dissipative term $2\zeta\Omega_0\dot{x}$ can be eliminated from the equation by means of a substitution as soon as the equation remains linear. However, keeping in mind possible generalizations on nonlinear cases, this term will be maintained through the manipulations. Let us introduce the phase variable $\psi = \Omega t$ and represent Eq. (2.35) in the form of two first-order equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\Omega_0^2 x + F(x, v) \end{aligned} \tag{2.36}$$

where

$$F = -2\zeta\Omega_0 v - \varepsilon\Omega_0^2 x\cos\psi \quad \sim \varepsilon \tag{2.37}$$

Applying transformation (2.9) to (2.36) gives the system with two fast phases, φ and ψ, as

$$\begin{aligned}
\dot{a} &= -\zeta a\Omega_0(1-\cos 2\varphi) + \frac{1}{4}\varepsilon a\Omega_0\,[\sin(2\varphi-\psi)+\sin(2\varphi+\psi)] \\
\dot{\varphi} &= \Omega_0(1-\zeta\sin 2\varphi) + \frac{1}{4}\varepsilon\Omega_0\,[2\cos\psi+\cos(2\varphi-\psi)+\cos(2\varphi+\psi)] \\
\dot{\psi} &= \Omega
\end{aligned} \tag{2.38}$$

Averaging the right-hand side of system (2.38) separately over φ and ψ would lead to the system $\dot{a} = -\zeta a\Omega_0$, $\dot{\varphi} = \Omega_0$, and $\dot{\psi} = \Omega$, in which the effect of parametric loading is vanished. As seen from the right-hand side of system (2.38), such an averaging becomes inadequate when $2\Omega_0 \sim \Omega$, and therefore $2\dot{\varphi} - \dot{\psi} \sim \varepsilon$. This means that, in addition to the amplitude a, another slow variable, $2\varphi - \psi = \theta$, occurs in the system. As a result, both terms $\sin(2\varphi-\psi) = \sin\theta$ and $\cos(2\varphi-\psi) = \cos\theta$ must be interpreted as frozen when averaging with respect to either φ or ψ. The corresponding formalization is conducted by excluding one of the fast phases, say ψ by means of relationship

$$\psi = 2\varphi - \theta \tag{2.39}$$

Substituting (2.39) in (2.38), taking into account that $\Omega = 2$, and applying the averaging with respect to φ give

$$\begin{aligned}
\dot{a} &= -\zeta\,\Omega_0 a + \frac{1}{4}\varepsilon\Omega_0 a\sin\theta \\
\dot{\theta} &= 2(\Omega_0 - 1) + \frac{1}{2}\varepsilon\Omega_0\cos\theta \\
\dot{\varphi} &= \Omega_0 + \frac{1}{4}\varepsilon\Omega_0\cos\theta
\end{aligned} \tag{2.40}$$

The second equation of this system shows that the phase θ is a slow varying quantity if $\Omega_0 - 1 \sim \varepsilon$. In this case, system (2.40) describes the two slow variables, a and θ, and one fast phase, φ. In other words, compared to the non-resonance case of Sect. 2.2.2, the dimension of the so-called slow manifold is increased by one due to the internal resonance condition.

The original variable is given by the relationship $x = a\cos\varphi$, where the slow phase θ does not explicitly show up but affects the amplitude a and phase φ through system (2.40). The stationary case, $\dot{a} = \dot{\theta} = 0$, determines a family of periodic solutions whenever the system parameters satisfy condition

$$\zeta^2 + \left(1 - \frac{1}{\Omega_0}\right)^2 = \frac{\varepsilon^2}{16} \tag{2.41}$$

or

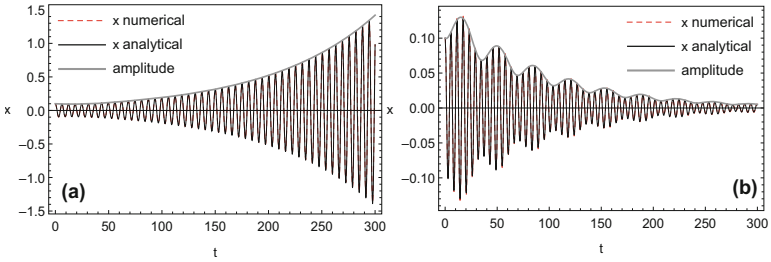

Fig. 2.3 Comparison of the analytical approximation and numerical solution of equation (2.35) for parameter values $\zeta = 0.01$ and $\varepsilon = 0.08$: (**a**) $\Omega = 1$-inside instability zone, (**b**) $\Omega = 1.1$-outside instability zone

$$\Omega_0 = \left[1 \pm \sqrt{\frac{\varepsilon^2}{16} - \zeta^2}\right]^{-1} \tag{2.42}$$

Condition (2.41) is obtained by setting the right-hand side of the first two equations of system (2.40) to zero and then eliminating the phase angle θ. From the geometrical viewpoint, Eq. (2.41) represents boundaries of the main instability zone in the parameter plane $\Omega_0 - \varepsilon$. Figure 2.3 illustrates what happens to the dynamics when crossing the boundary and confirms that the result of averaging and numerical solutions remain in a reasonable agreement. When $\zeta = 0$, Eq. (2.42) gives $\Omega_0 = 1 \pm \varepsilon/4 + O(\varepsilon^2)$. Note that a complete set of boundaries is often given for Mathew's equation of the form $\ddot{y} + (\delta + 2\epsilon \cos 2t)y = 0$, which can be obtained from Eq. (2.35) by means of the substitution $x(t) = \exp(-\zeta \Omega_0 t)y(t)$. Such a substitution leads to the above Mathew's equation, if $\delta = \Omega_0^2(1 - \zeta)$ and $2\epsilon = \Omega_0^2 \varepsilon$.

2.2.4 *Averaging in Complex Variables*

In physical literature, vibration problems are usually considered in terms of complex variables [120]. The idea of averaging can be implemented as follows. If $\varepsilon = 0$ then general solution of Eq. (2.3) is represented in the complex form

$$x = \frac{1}{2}[A \exp(i\Omega_0 t) + \bar{A} \exp(-i\Omega_0 t)] \tag{2.43}$$

where A and $\bar{A}$ are arbitrary complex conjugate constants, and the numerical factor $1/2$ is introduced for further convenience of calculation, although it is not strictly necessary.

Following the idea of parameter variations, let us assume that A and $\bar{A}$ are time-dependent quantities hopefully to satisfy Eq. (2.3) with non-zero right-hand side, $\varepsilon \neq 0$. Under such an assumption, differentiating (2.43) gives

$$\dot{x} = \frac{1}{2}\left[\frac{dA}{dt}\exp(i\Omega_0 t) + \frac{d\bar{A}}{dt}\exp(-i\Omega_0 t)\right] \tag{2.44}$$
$$+\frac{1}{2}i\Omega_0[A\exp(i\Omega_0 t) - \bar{A}\exp(-i\Omega_0 t)]$$

Further, the following condition is imposed on the arbitrary functions, $A(t)$ and $\bar{A}(t)$, with the intent to eliminate the derivatives of amplitudes from (2.44)

$$\frac{dA}{dt}\exp(i\Omega_0 t) + \frac{d\bar{A}}{dt}\exp(-i\Omega_0 t) = 0 \tag{2.45}$$

This condition brings derivative (2.44) to the form

$$\dot{x} = \frac{1}{2}i\Omega_0[A\exp(i\Omega_0 t) - \bar{A}\exp(-i\Omega_0 t)] \tag{2.46}$$

Differentiating (2.46) and taking into account (2.43) give

$$\begin{aligned}\ddot{x} &= \frac{1}{2}i\Omega_0\left[\frac{dA}{dt}\exp(i\Omega_0 t) - \frac{d\bar{A}}{dt}\exp(-i\Omega_0 t)\right] \\ &\quad -\frac{1}{2}\Omega_0^2[A\exp(i\Omega_0 t) + \bar{A}\exp(-i\Omega_0 t)] \\ &= \frac{1}{2}i\Omega_0\left[\frac{dA}{dt}\exp(i\Omega_0 t) - \frac{d\bar{A}}{dt}\exp(-i\Omega_0 t)\right] - \Omega_0^2 x\end{aligned} \tag{2.47}$$

Now, substituting (2.47) in the original equation (2.3) gives second equation for $A(t)$ and $\bar{A}(t)$ in the form

$$\frac{1}{2}i\Omega_0\left[\frac{dA}{dt}\exp(i\Omega_0 t) - \frac{d\bar{A}}{dt}\exp(-i\Omega_0 t)\right] = \varepsilon f \tag{2.48}$$

where the function $f = f(x, \dot{x})$ must be expressed through $A(t)$ and $\bar{A}(t)$ by means of relationships (2.43) and (2.46).

Solving the linear system (2.45) and (2.48) for the derivatives of complex amplitudes gives

$$\frac{dA}{dt} = \frac{\varepsilon}{i\Omega_0}\exp(-i\Omega_0 t) f \tag{2.49}$$

$$\frac{d\bar{A}}{dt} = \frac{\varepsilon}{-i\Omega_0}\exp(i\Omega_0 t) f \tag{2.50}$$

where $f = f(x, \dot{x})$ must be expressed through (2.43) and (2.46).

System (2.49) and (2.50) is still exact equivalent of the original equation (2.3) and represents the result of changing the state variables

$$\{x, \dot{x}\} \to \{A, \bar{A}\} \tag{2.51}$$

The advantage of using the complex variables is that it is sufficient to consider only one amplitude equation, for instance, (2.49) since the other one is simply its complex conjugate. Besides, solving Eqs. (2.43) and (2.46) for A gives the so-called complex amplitude, which is often used in both physics and nonlinear mechanics,

$$A = \frac{1}{i\Omega_0} \exp(-i\Omega_0 t)(\dot{x} + i\Omega_0 x) \tag{2.52}$$

Note that similar formal manipulations remain valid in degenerated cases of multiple degrees of freedom systems. For instance, Eq. (2.3) can be interpreted as a vector equation with the scalar factor Ω_0^2.

Finally, if the parameter ε is small, then the amplitude A is slow; hence the averaging can be applied as

$$\frac{dA}{dt} = \frac{1}{2\pi i}\varepsilon \int_0^{2\pi/\Omega_0} \exp(-i\Omega_0 t) f dt \tag{2.53}$$

Complex Form Solution for Van der Pol Oscillator

For example, let us consider oscillator

$$\ddot{x} + x = \varepsilon f(x, \dot{x}) \tag{2.54}$$

where

$$f(x, \dot{x}) = -(x^2 - 1)\dot{x} \tag{2.55}$$

Substituting (2.55) in Eq. (2.53) gives the following equation for the complex amplitude

$$\dot{A} = \frac{\varepsilon}{2\pi i} \int_0^{2\pi} \exp(-it)(1 - x^2)\dot{x} dt \tag{2.56}$$

where x and $\dot{x}$ are given by (2.43) and (2.46), respectively, after setting $\Omega_0 = 1$.

Conducting the corresponding algebraic manipulations and then integration with respect to time over the period 2π gives

$$\dot{A} = \frac{1}{8}\varepsilon(4 - |A|^2)A \tag{2.57}$$

To solve this equation, let us switch to the exponential form of the complex amplitude

$$A = \rho \exp(i\theta) \tag{2.58}$$

where $\rho = \rho(t)$ and $\theta = \theta(t)$ are the modulus and argument, respectively. Substituting (2.58) in (2.57) and separating imaginary and real parts give

$$\dot{\theta} = 0, \qquad \dot{\rho} = \frac{1}{8}\varepsilon(4 - \rho^2) \tag{2.59}$$

Therefore, θ is a constant phase, while ρ is determined by separating the variables as

$$\rho = \frac{2}{\sqrt{1 + (4\rho_0^{-2} - 1)\exp(-\varepsilon t)}} \tag{2.60}$$

where $\rho_0 = \rho(0)$.

The reverse transition to the original variable $x(t)$ through (2.43) and (2.58) finally gives a general solution of van der Pol's equation in the leading asymptotic order

$$x = \frac{2\cos(t + \theta)}{\sqrt{1 + (4\rho_0^{-2} - 1)\exp(-\varepsilon t)}} \tag{2.61}$$

In the particular case of zero initial velocity, one can set $\theta = 0$ and $\rho_0 = x(0)$ within the same asymptotic order.

2.3 Lie Groups Formalism

The one-parameter Lie[2] group approaches are motivated by the idea of matching the tool and the object of study as explained in references [253] and [256]. Briefly, it is suggested to seek transformation (2.18) among solutions of some dynamical systems rather than the class of the arbitrary nonlinear transformations. Original materials and overviews of the mathematical structure of Lie groups, Lie algebras, and Lie transforms with applications to nonlinear differential equations can be

[2] Marius Sophus Lie (1842–1899), Norwegian mathematician; different mathematical objects are named after him, for instance, groups, operators, algebras, and series.

found in [29, 46, 50, 81, 127]. An essential ingredient of this version of asymptotic integration is the Hausdorff formula, which relates Lie group operators of the original and new systems and the operator of coordinate transformation. According to [254] and [256], most of the averaging techniques just reproduce this formula every time implicitly, during the transformation process. Hence it is more reasonable to start with Hausdorff's relationship rather than recover it during the transformation procedure. This enables one of optimizing the number of manipulations for high-order approximations of asymptotic integration [109].

2.3.1 Hausdorff Equation

The theory of Lie groups deals with a set of transformations. In other words, some dynamical system

$$\dot{x} = f(x) \tag{2.62}$$

is transformed into another system

$$\dot{q} = g(q, \varepsilon), \qquad 0 < \varepsilon \ll 1 \tag{2.63}$$

by means of a near identical coordinate transformation $x \to q$, which is produced by a solution of the initial value problem for a third dynamical system

$$\frac{dx}{d\varepsilon} = s\,(x)\,, \qquad x|_{\varepsilon=0} = q \tag{2.64}$$

Here, the choice for the right-hand side, $s\,(x)$, is dictated by the desired properties of the transformed system. The parameter of group, ε, is interpreted as an independent variable. As follows from the initial condition in (2.64), the transformation is assumed to be identical, $x = q$, when $\varepsilon = 0$, and therefore

$$g(q, 0) = f(q) \tag{2.65}$$

Hausdorff's formula determines the right-hand side of Eq. (2.63) for nonzero ε. For illustrating purpose, consider the one-dimensional case, $x, q \in R^1$ by noticing that the final relationship can be applied to a multidimensional case too. The idea is to conduct the transformation, (2.62)→(2.64)→(2.63), in terms of the operators Lie as

$$F = f(x)\frac{\partial}{\partial x} \longrightarrow S = s(x)\frac{\partial}{\partial x} \longrightarrow G = g(q, \varepsilon)\frac{\partial}{\partial q} \tag{2.66}$$

Using operators (2.66) opens a formal way to representing the above dynamical systems in the linear form as

$$\dot{x} = Fx \longrightarrow \left\{ \frac{dx}{d\varepsilon} = Sx, \qquad x|_{\varepsilon=0} = q \right\} \longrightarrow \dot{q} = Gq \tag{2.67}$$

Now the problem is to find the new operator, G, produced by the operator of transformation, S, from the old operator, F. As follows from (2.67), if the operator S is known, then the related transformation, $x \longrightarrow q$, takes the form of Lie series

$$x = \exp(\varepsilon S)q = (1 + \varepsilon S + \ldots)q = q + \varepsilon s(q) + \ldots \tag{2.68}$$

One of the advantages of this approach is that relationship (2.68) easily generates the inverse transformation, $\exp(-\varepsilon S)x = q$, where the variable q is replaced with x in the operator S.

To derive Hausdorff formula, let us take first time derivative of both sides of transformation (2.68) and enforce Eqs. (2.62) and (2.63) to exclude $\dot{x}$ and $\dot{q}$. This gives

$$f(q + \varepsilon s + \ldots) = (1 + \varepsilon \frac{\partial s}{\partial q} + \ldots)g(q, \varepsilon) \tag{2.69}$$

Using power series expansions for the functions f and g, taking into account relationship (2.65), and rearranging the terms bring (2.69) to the form

$$\frac{\partial}{\partial \varepsilon} g = s \frac{\partial}{\partial q} g - g \frac{\partial}{\partial q} s \qquad \text{as } \varepsilon \longrightarrow 0 \tag{2.70}$$

Multiplying both sides of Eq. (2.70) by the differential operator $\partial/\partial q$ on the right on both sides and using notations (2.66) give Hausdorff equation

$$\frac{\partial}{\partial \varepsilon} G = SG - GS \equiv [S, G] \tag{2.71}$$

where the operator of transformed system, G, satisfies the initial condition

$$G|_{\varepsilon=0} = F = f(q) \frac{\partial}{\partial q} \tag{2.72}$$

according to the assumption that $x = q$ when $\varepsilon = 0$.

A power series solution of the initial value problem (2.71)–(2.72) for operators Lie is given by Hausdorff formula [29]:

$$G = F + \varepsilon [S, F] + \frac{1}{2!} \varepsilon^2 [S, [S, F]] + \ldots \tag{2.73}$$

This formula relates operators F, S, and G according to (2.66) and can be used for the asymptotic integration of the original system (2.62) as follows.

2.3.2 Asymptotic Integration in Terms of Operators Lie

Let the right-hand side of system (2.62) depend upon a small parameter ε as $f = f_0(x) + \varepsilon f_1(x)$, assuming that the system $\dot{x} = f_0(x)$ admits a straightforward integration. Respectively, let us represent the operator of original system, F, the operator of transformation, S, and the operator of transformed system, G, in the power series form

$$\begin{aligned} F &= F_0 + \varepsilon F_1 \\ S &= S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \ldots \\ G &= G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots \end{aligned} \tag{2.74}$$

The problem is to iteratively obtain the operator of transformation, such that operator of transformed system possesses the same useful property as the operator of original system at $\varepsilon = 0$. If this problem is solved then the new system, $\dot{q} = Gq$, can be integrated the same way as the system $\dot{x} = f_0(x)$. Such an asymptotic procedure is formalized by substituting (2.74) in Hausdorff formula (2.73) and matching terms of the same power of ε as

$$G_0 = F_0 \tag{2.75}$$

$$G_1 = F_1 + [S_0, F_0] \tag{2.76}$$

$$G_2 = F_2 + [S_0, F_1] + [S_1, F_0] + \frac{1}{2!}[S_0, [S_0, F_0]] \tag{2.77}$$

$$\ldots$$

Note that all the operator relationships remain applicable to multidimensional cases in the same form.

In order to illustrate the averaging procedure in a two-dimensional case, let us consider system (2.17) assuming that the frequency Ω is fixed:

$$\begin{aligned} \dot{x} &= \varepsilon X(x, y) \\ \dot{y} &= \Omega + \varepsilon Y(x, y) \end{aligned} \tag{2.78}$$

The operator Lie of this system is given by

$$F = F_0 + \varepsilon F_1 \tag{2.79}$$

$$F_0 = \Omega \frac{\partial}{\partial y}, \qquad F_1 = X(x, y)\frac{\partial}{\partial x} + Y(x, y)\frac{\partial}{\partial y} \tag{2.80}$$

When $\varepsilon = 0$, system (2.78) does not have the fast variable y on the right-hand side. The problem is to find the transformation $\{x, y\} \longrightarrow \{q, \psi\}$, such that the

transformed system possesses the same property for nonzero ε, namely, its right-hand side does not have the fast variable (phase) ψ. Thus the problem is formulated in the same way to Krylov-Bogolyubov, which is formalized by Eqs. (2.17), (2.18), and (2.19). In contrast, the present approach does not require the substitution (2.18) in (2.17). The operator of transformed system, G, is given by (2.75) through (2.77) after the replacement of variables $\{x, y\}$ with $\{q, \psi\}$, for instance, as

$$G_0 = \Omega \frac{\partial}{\partial \psi} \tag{2.81}$$

$$G_1 = F_1 - \Omega \frac{\partial}{\partial \psi} S_0 \tag{2.82}$$

Applying the operator G_0 to $\{q, \psi\}$ eliminates the fast phase ψ on the right-hand side of transformed system

$$\{\dot{q}, \dot{\psi}\} = G_0\{q, \psi\} = \{0, \Omega\}$$

The same property is assigned to the operator G_1 by averaging the right-hand side of (2.82) as

$$G_1 = < F_1 >_\psi = < X(q, \psi) >_\psi \frac{\partial}{\partial q} + < Y(q, \psi) >_\psi \frac{\partial}{\partial \psi} \tag{2.83}$$

Note that the integration with respect to the variable ψ does not affect the differential operator $\partial/\partial\psi$. This relationship should be also viewed as a definition for averaging of operators Lie. Since condition (2.83) is imposed on the left-hand side of Eq. (2.82), the equality can be achieved by a proper choice for the operator S_0, which is still unknown. Thus, substituting (2.83) in (2.82) and integrating with respect to ψ give the corresponding approximation for the operator of transformation:

$$\begin{aligned} S_0 &= \frac{1}{\Omega} \int_0^\psi (F_1 - < F_1 >_\psi) d\psi = \frac{1}{\Omega} \int_0^\psi (F_1 - G_1) d\psi \\ &= \frac{1}{\Omega} \int_0^\psi (X - < X >_\psi) d\psi \frac{\partial}{\partial q} + \frac{1}{\Omega} \int_0^\psi (Y - < Y >_\psi) d\psi \frac{\partial}{\partial \psi} \end{aligned} \tag{2.84}$$

Taking into account that $F_2 = 0$ and calculating the commutator $[S_1, F_0]$ bring Eq. (2.77) to the form

$$G_2 = [S_0, F_1] + \frac{1}{2!}[S_0, [S_0, F_0]] - \Omega \frac{\partial}{\partial \psi} S_1 \tag{2.85}$$

Now reiterating manipulations of the previous step gives

$$G_2 = < [S_0, F_1] >_{\psi} + \frac{1}{2!} < [S_0, [S_0, F_0]] > \psi \tag{2.86}$$

and

$$S_1 = \frac{1}{\Omega} \int_0^{\psi} \left([S_0, F_1] + \frac{1}{2!} [S_0, [S_0, F_0]] - G_2 \right) d\psi \tag{2.87}$$

The operators of transformation, S_0 and S_1, and the operators of averaged system, G_1 and G_2, generate solution of system (2.17) in the second asymptotic order and the corresponding averaged system as, respectively,

$$x = \exp(\varepsilon S) q = (1 + \varepsilon S_0 + \varepsilon^2 S_1) q + O(\varepsilon^3) \tag{2.88}$$

$$y = \exp(\varepsilon S) \psi = (1 + \varepsilon S_0 + \varepsilon^2 S_1) \psi + O(\varepsilon^3) \tag{2.89}$$

and

$$\dot{q} = (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) q + O(\varepsilon^3) \tag{2.90}$$

$$\dot{\psi} = (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) \psi + O(\varepsilon^3) \tag{2.91}$$

Solution of Rayleigh Equation in Terms of Operators Lie

For comparison reason, let us consider the example of Rayleigh equation (2.28). According to (2.29) and (2.30), in terms of van der Pol's variables, the corresponding operator Lie is

$$\begin{aligned} F_0 = \Omega \frac{\partial}{\partial \psi}, \qquad F_1 &= X(q, \psi) \frac{\partial}{\partial q} + Y(q, \psi) \frac{\partial}{\partial \psi} \\ &= q \left(1 - \frac{1}{3} q^2 \Omega^2 \sin^2 \psi \right) \sin^2 \psi \frac{\partial}{\partial q} \\ &+ \frac{1}{6} \left(3 - q^2 \Omega^2 \sin^2 \psi \right) \sin 2\psi \frac{\partial}{\partial \psi} \end{aligned} \tag{2.92}$$

where the replacement $\{x, y\} \longrightarrow \{q, \psi\}$ has been made according to the above-described algorithm.

Conducting the averaging in (2.83), (2.84), (2.86), and (2.87) and then substituting the result in (2.88) and (2.89) give

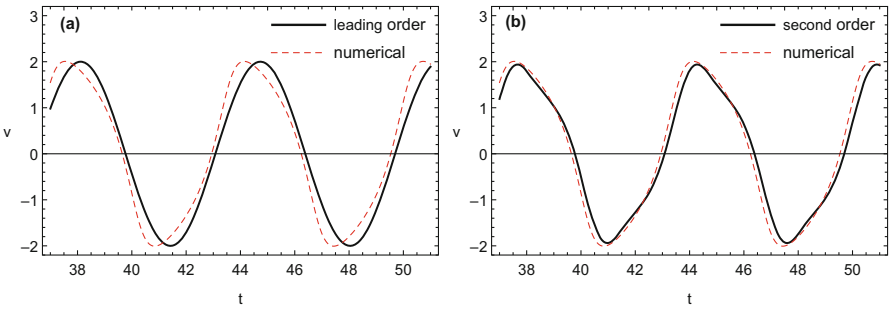

Fig. 2.4 Comparison of numerical solution of Rayleigh equation (2.28) to the leading order (**a**) and second order (**b**) approximations for $v = \dot{z}$ obtained by means of Lie group operators; $\varepsilon = 0.9$, $\Omega = 1$, $q(0) = 0.1$, and $\psi(0) = 0$

$$
\begin{aligned}
x &= q - \frac{\varepsilon}{48\Omega} q[12 - (4 - \cos 2\psi) q^2\Omega^2] \sin 2\psi - \frac{\varepsilon^2}{192\Omega^2} \\
&\quad \times q[48 - \left(12 - q^2\Omega^2\right)\left(6 - q^2\Omega^2 \sin^2\psi\right)] \sin^2\psi + O(\varepsilon^3) \\
y &= \psi + \frac{\varepsilon}{12\Omega}\left(6 - q^2\Omega^2 \sin^2\psi\right) \sin^2\psi + \frac{\varepsilon^2}{1536\Omega^2} \\
&\quad \times [96 - q^2\Omega^2\left(24 - 5q^2\Omega^2\right)(4 - \cos 2\psi)] \sin 2\psi + O(\varepsilon^3)
\end{aligned}
\tag{2.93}
$$

where the functions $q = q(t)$ and $\psi = \psi(t)$ are given by the averaged system (2.90) and (2.91) leading to (2.31) and (2.32). Note that the terms of order ε coincide with those in (2.33) and (2.34).

The effectiveness of second-order approximation is illustrated by Fig. 2.4, where the parameter ε was intentionally chosen to be close to unity since the terms of order ε^2 appeared to have quite small numerical factors in solutions (2.93). The graphs represent temporal shapes of the velocity, $v = \dot{z}$, in order to better observe the effect of anharmonicity.

2.3.3 Linearization Near Equilibrium Manifold

Methods considered in the previous sections of this chapter essentially employ solutions of linearized systems. The linearization procedure assumes the system to remain near a single (stationary) equilibrium point. This condition cannot be guaranteed if the total energy of the system is above the potential barrier on the way to another equilibrium point. As a result, the problem becomes nonlocal and usually multidimensional since the path connecting both points is not necessarily

straight. Following reference [170], let us illustrate this situation based on a two-mode approximation for a simply supported cylindrical panel of thickness h with a sinusoidal initial imperfection of the amplitude α

$$\begin{aligned} \ddot{q}_1 + \varepsilon^2 (q_1 - 1) + \frac{1}{4}\left({q_1}^2 + 4{q_2}^2 - 1\right) q_1 = 0 \\ \ddot{q}_2 + 16\varepsilon^2 q_2 + \left({q_1}^2 + 4{q_2}^2 - 1\right) q_2 = 0 \end{aligned} \tag{2.94}$$

where q_1 and q_2 are time-dependent amplitudes of the first and second sine wave modes, respectively, and ε is a small parameter characterizing the panel flexibility as

$$\varepsilon^2 = \frac{1}{12}\left(\frac{h}{\alpha}\right)^2 \ll 1$$

Model (2.94) can be represented by its Lagrangian as

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1, q_2) \tag{2.95}$$

with the potential energy of elastic deformations given by

$$V(q_1, q_2) = \frac{1}{2} f(q_1, q_2)^2 + \varepsilon^2 \Phi(q_1, q_2) \tag{2.96}$$

$$f(q_1, q_2) = \frac{\sqrt{2}}{4}\left(q_1^2 + 4q_2^2 - 1\right) \tag{2.97}$$

$$\Phi(q_1, q_2) = \frac{1}{2}[(q_1 - 1)^2 + 16q_2^2] \tag{2.98}$$

where the functions f and Φ associate with the tension-compression and bending deformations, respectively. A typical shape of the potential energy with a sample trajectory inside the potential well is shown in Fig. 2.5. If $\varepsilon = 0$, system (2.94) has one equilibrium point $P_0(0, 0)$ on the plane $q_1 q_2$ corresponding to a horizontal configuration of the panel, which is obviously unstable by Lyapunov due to the extreme compression. Also, there is a continuous manifold of the elliptic shape, ${q_1}^2 + 4{q_2}^2 = 1$, along which the panel can move with zero strain of its center line. If $\varepsilon \neq 0$, this manifold disappears by generating four equilibrium points, of which two are located on the straight line $q_2 = 0$ and correspond to the original and inverted positions of the panel: $P_1(1, 0)$ and $P_2(-(1 + \sqrt{1 - 16\varepsilon^2})/2, 0)$, respectively. The other two points are unstable equilibria involving the second (sine wave) spatial

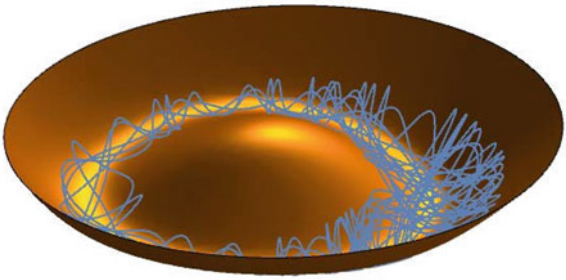

Fig. 2.5 Top-side qualitative view of the potential well V with a sample system trajectory inside the well under the flexibility $\varepsilon = 0.1$, and other parameters as: $a_1 = 1, a_2 = 0, q_1(0) = 1, q_2(0) = 0, \dot{q}_1(0) = 1, \dot{q}_2(0) = 0.25$

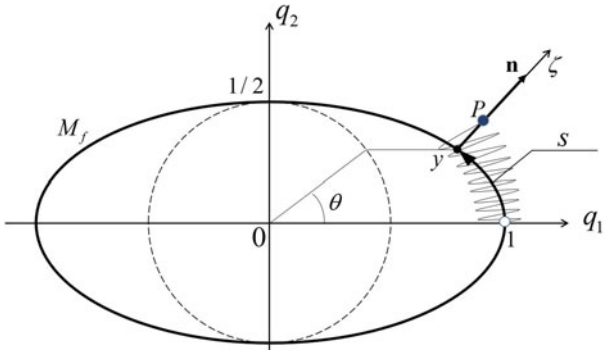

Fig. 2.6 Planar geometry of the coordinate transformation $\{q_1, q_2\} \to \{\zeta, s\}$ with interpretation of the arclength parametrization $s = s(\theta)$

mode: $P_{3,4}(-1/3, \pm\sqrt{2 - 36\varepsilon^2}/3)$. The point P_0 remains on the line $q_2 = 0$ with some shift from zero: $P_0((\sqrt{1 - 16\varepsilon^2} - 1)/2, 0)$. All of the above equilibria exist under the condition $\varepsilon < \sqrt{2}/6$. The minimum potential barrier on the path between two stable equilibrium points, P_1 and P_2, is calculated by substituting the coordinates of one of the two symmetric saddle points, $P_{3,4}$, in (2.96) as

$$V_* = \frac{8}{3}\varepsilon^2 - 16\varepsilon^4 \tag{2.99}$$

The methodology is based on a global linearization of Eqs. (2.94) near the elliptic manifold of equilibrium positions of the perfectly flexible panel, $\varepsilon = 0$. In the present case of just two modes, such a linearization is conducted by means of two new generalized coordinates, such as the deviation ζ from the ellipse and the angular coordinate θ (Fig. 2.6)

$$\begin{aligned} q_1 &= y_1(\theta) + \varepsilon n_1(\theta)\zeta \\ q_2 &= y_2(\theta) + \varepsilon n_2(\theta)\zeta \end{aligned} \tag{2.100}$$

where $\{y_1, y_2\}$ is a projection of the point $P(q_1, q_2)$ onto the ellipse

$$y_1(\theta) = \cos\theta, \quad y_2(\theta) = \frac{1}{2}\sin\theta \tag{2.101}$$

and $\{n_1, n_2\}$ is the corresponding unit vector

$$n_1 = \frac{1}{\omega}\frac{\partial f(y_1, y_2)}{\partial y_1}, \quad n_2 = \frac{1}{\omega}\frac{\partial f(y_1, y_2)}{\partial y_2} \tag{2.102}$$

$$\omega = \sqrt{\left(\frac{\partial f}{\partial y_1}\right)^2 + \left(\frac{\partial f}{\partial y_2}\right)^2} = \frac{\sqrt{2}}{2}\sqrt{1 + 3\sin^2\theta}$$

The nonlinear coordinate transformation $\{q_1, q_2\} \longrightarrow \{\zeta, \theta\}$ (2.100) is conducted by means of Routh function combining the Lagrangian and Hamiltonian formulations for the normal and tangential to the ellipse motion components, respectively [150]. Another approach was using a local Cartesian frame following the point P along the ellipse [136]. Although the nonlinear coordinate transformation technically complicates the differential equations of motion as compared to system (2.94), the new coordinates become closer to the system physical meaning: the fast coordinate ζ associates with the tension-compression of the panel, whereas the slow coordinate θ describes the bending deformations. As a result, the system is reduced based on the idea of separation of motions followed by the averaging procedure. The leading order asymptotic integration gives finally

$$\ddot{\zeta} + \omega^2\zeta = O(\varepsilon) \tag{2.103}$$

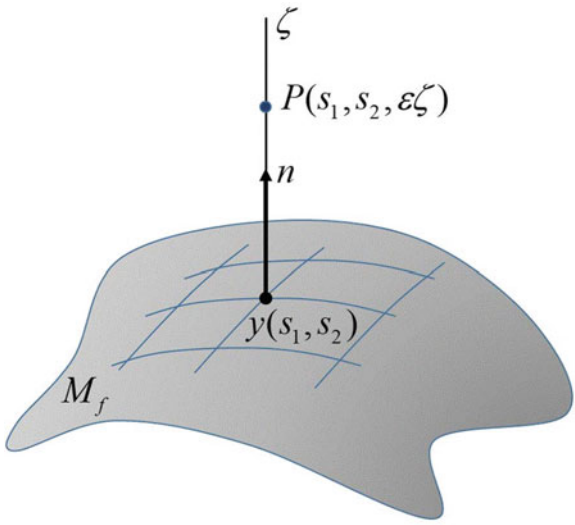

Fig. 2.7 Three-dimensional illustration of zero strain manifold, $M_f = \{q : f(q) = 0\}$, and the corresponding generalized Lagrangian coordinates: $\{\zeta, s_1, s_2\}$

and

$$\frac{1}{2}\left(\omega\frac{d\theta}{dt}\right)^2+\varepsilon^2\left[2(1-\cos\theta)+3\sin^2\theta\right]=E=\text{const.} \tag{2.104}$$

According to (2.99), the least energy level of system (2.94), at which the global motion may become possible, is of order ε^2. Analyzing Eqs. (2.103) and (2.104) for the vanishing ε recalls the discussion on rigid-body motions from the standpoint of linearity and nonlinearity concepts. In the present case, Eqs. (2.103) and (2.104) associate with rotations and translations, respectively, as discussed in Chap. 1.

Figure 2.7 gives an outline for increasing the number of flexural modes of the panel. Although the dimension of essentially nonlinear component can be reduced by one, a parametrization of the equilibrium manifold would lead to significant technical complexities.