Chapter 14
Spatially Oscillating Structures

This chapter illustrates applications of nonsmooth argument substitutions to modeling spatially oscillating structures such as one-dimensional elastic rods with periodic discrete inclusions and two- or three-dimensional acoustic media with periodic nonsmooth boundary sources of waves. Whenever the corresponding global spatial domains are infinite or cyclical, the related analytical manipulations are similar to those conducted with dynamical systems. The idea of structural homogenization is implemented through the two-variable expansions, where the fast scale is represented by the triangular periodic wave. As a result, closed-form analytical solutions are derived despite of the presence of discrete inclusions or external discontinuous loads. Static and dynamic problems of elasticity dealing with nonsmooth periodic structures are often considered in the literature due to their practical importance; see reviews in [10, 11, 100] for introduction and references. Such problems can also be considered by means of the nonsmooth argument transformations. In this case though, the transformation must be applied to the spatial independent variable related to coordinate along which the structure under consideration is periodic. Such an approach was introduced in [174] for a string on discrete elastic foundation, although the complete description of the tool was given later for the corresponding nonlinear case [199, 200, 213].

14.1 Spatial Triangle Wave Argument

14.1.1 Infinite String on a Discrete Elastic Foundation

For illustrating purpose, let us consider the dynamics of an infinite string on a discrete nonlinearly elastic foundation described by equation

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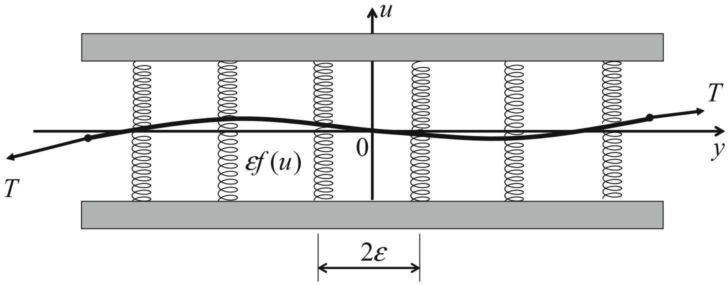

Fig. 14.1 Linear string on the discrete regular set of nonlinearly elastic springs

$$\rho\frac{\partial^2 u}{\partial t^2}-T\frac{\partial^2 u}{\partial y^2}+2f(u)\sum_{k=-\infty}^{\infty}\delta\left(\frac{y}{\varepsilon}-1-2k\right)=q\left(\frac{y}{\varepsilon},y,t\right) \tag{14.1}$$

$$(-\infty<y<\infty)$$

where ρ is the mass density per unit length; T is a constant tension; $q(y/\varepsilon, y, t)$ is the body force or external loading, which is assumed to be periodic in the "fast spatial scale" y/ε ($0<\varepsilon\ll 1$) with the period normalized to four; and a similar assumption is made with respect to the transverse displacement of the string $u = u(y/\varepsilon, y, t)$; see Fig. 14.1 for illustration.

Note that Eq. (14.1) does not allow for a point-wise interpretation due to the presence of Dirac δ-function. Both sides of the equation therefore must be interpreted as distributions producing the same output if applied to the same testing function. Correctness of such type of modeling was intensively discussed in the literature; see, for instance, [65]. Omitting details, the series of δ-functions in Eq. (14.1) has a certain meaning if the function $f(u(y/\varepsilon, y, t))$ is at least continuous in the neighborhoods of points $y=\varepsilon(1+2k)$ with respect to the spatial argument, y, for every k. Note that such a continuity condition is guaranteed by the form of Eq. (14.1). If, for instance, the displacement u was stepwise discontinuous at $y=\varepsilon(1+2k)$, then the derivative $\partial^2 u/\partial y^2$ would produce uncompensated first derivatives of the δ-function. Therefore, the displacement u is at least continuous function of the coordinate y in a match with the physical meaning of the model. Moreover, it will be shown below that introducing the space folding spatial argument, $\tau=\tau(y/\varepsilon)$, eliminates the singularities from Eq. (14.1) and hence takes the problem in the frameworks of classical theory of differential equations. First, the set of localized restoring forces of the elastic foundation are expressed through second derivative of the triangular wave as

$$2f(u)\sum_{k=-\infty}^{\infty}\delta\left(\frac{y}{\varepsilon}-1-2k\right)=-f(u)\mathrm{sgn}(\tau)\tau''\left(\frac{y}{\varepsilon}\right) \tag{14.2}$$

where the derivative τ'' is taken with respect to the entire argument y/ε and $\mathrm{sgn}(\tau)$ is introduced to make all the δ-functions positive, since the foundation reaction forces must be restoring.

Now both the displacement u and the external loading function q are represented as elements of the hyperbolic algebra

$$\begin{aligned} u &= U(\tau, y, t) + V(\tau, y, t)e \\ q &= Q(\tau, y, t) + P(\tau, y, t)e \\ e &= e\,(y/\varepsilon) \equiv d\tau(y/\varepsilon)/d(y/\varepsilon) \end{aligned} \tag{14.3}$$

where the components Q and P are known, whereas X and Y are new unknown functions.

Finally, substituting (14.2) and (14.3) in Eq. (14.1) and using the differential and algebraic rules of nonsmooth argument substitutions (Chap. 4) give the differential equations and boundary conditions as, respectively,

$$\frac{\partial^2 U}{\partial \tau^2} = -2\varepsilon \frac{\partial^2 V}{\partial y \partial \tau} + \varepsilon^2 \left(\frac{\rho}{T} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial y^2} - \frac{Q}{T} \right) \tag{14.4}$$

$$\frac{\partial^2 V}{\partial \tau^2} = -2\varepsilon \frac{\partial^2 U}{\partial y \partial \tau} + \varepsilon^2 \left(\frac{\rho}{T} \frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial y^2} - \frac{P}{T} \right) \tag{14.5}$$

and

$$\tau = \pm 1 : V = 0 \tag{14.6}$$

$$\tau = \pm 1 : \frac{\partial U}{\partial \tau} = \mp \frac{\varepsilon^2}{T} f(U) \tag{14.7}$$

where the form of boundary condition (14.7) was eased by enforcing condition (14.6).

In contrast to Eq. (14.1), the above boundary value problem, (14.4) through (14.7), is free of the δ-function singularities. Therefore, conventional methods of asymptotic integration, such as two-variable expansions, can be applied after a proper modification to account for the problem specifics. Regardless of the types of algorithms, solutions are represented in the form of asymptotic series

$$\begin{aligned} U(\tau, y, t) &= \sum_{k=0}^{\infty} \varepsilon^k U_k(\tau, y, t) \\ V(\tau, y, t) &= \sum_{k=0}^{\infty} \varepsilon^k V_k(\tau, y, t) \end{aligned} \tag{14.8}$$

As just noticed, problem formulations based on Eqs. (14.4) through (14.7) possess certain advantages as compared to Eq. (14.1). For instance, the influence of infinite discontinuities in Eq. (14.1) is captured by the form of substitution (14.3). Also, the structural discreteness of the nonlinear elastic foundation (14.2) is associated with the new spatial variable τ restricted in the range $-1 \le \tau \le 1$. As a result, polynomial expansions with respect to the coordinate τ will not be affecting the regularity of asymptotic expansions (14.8) in terms of the original fast scale y/ε. In other words, the structural periodicity will be maintained in a wide range of asymptotic algorithms due to the inherent periodicity of the coordinate $\tau = \tau(y/\varepsilon)$.

Note that partial differential Eqs. (14.4) and (14.5) are coupled, whereas boundary conditions (14.6) and (14.7) are decoupled with respect to the unknowns U and V. It is usually more convenient to decouple equations by introducing new unknown functions, say $X = U + V$ and $Y = U - V$. Then Eqs. (14.4) and (14.5) take the form

$$\frac{\partial^2 X}{\partial \tau^2} = -2\varepsilon \frac{\partial^2 X}{\partial y \partial \tau} + \varepsilon^2 \left(\frac{\rho}{T} \frac{\partial^2 X}{\partial t^2} - \frac{\partial^2 X}{\partial y^2} - \frac{Q + P}{T} \right) = 0 \tag{14.9}$$

$$\frac{\partial^2 Y}{\partial \tau^2} = 2\varepsilon \frac{\partial^2 Y}{\partial y \partial \tau} + \varepsilon^2 \left(\frac{\rho}{T} \frac{\partial^2 Y}{\partial t^2} - \frac{\partial^2 Y}{\partial y^2} - \frac{Q - P}{T} \right) = 0 \tag{14.10}$$

This is obviously transition to the idempotent basis, $\{1, e\} \longrightarrow \{e_+, e_-\}$, as described in Sect. 4.2.1

$$\begin{aligned} u &= U + Ve = U(e_+ + e_-) + V(e_+ - e_-) \\ &= (U + V)e_+ + (U - V)e_- = Xe_+ + Ye_- \end{aligned}$$

Equations (14.9) and (14.10) have the same structure, except for signs of two terms. Therefore, it is sufficient to solve just one of the equations. Then, solution of another equation can be written by analogy.

14.1.2 Doubling the Array of Springs

The mechanical model, which is shown in Fig. 14.2, was considered in [198] based on the generalized (asymmetric) version of the triangle wave in different notations. In contrast to the model shown in Fig. 14.1, the support sprigs are linearly elastic and shifted in a dipole-wise manner, such that the differential equation of motion with respect to the string deflection $u = u(t, y)$ has the form

$$\rho \frac{\partial^2 u}{\partial t^2} - T \frac{\partial^2 u}{\partial y^2} - \frac{k}{\varepsilon}(1 - \gamma^2) u \, \mathrm{sgn}[\tau(\xi, \gamma)] \frac{\partial^2 \tau(\xi, \gamma)}{\partial \xi^2} = 0 \tag{14.11}$$

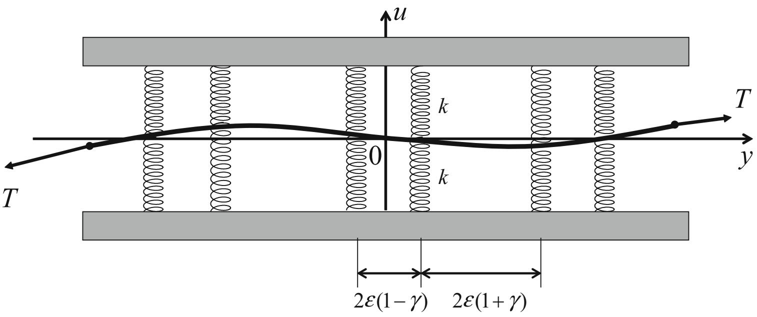

Fig. 14.2 Linear string on the discrete periodic set of linearly elastic dipole-wise shifted springs of the stiffness k

$$\left(-\infty < y < \infty, \qquad \xi = \frac{y}{\varepsilon}\right)$$

where the triangular wave with different positive and negative slopes is given by (Fig. 14.7)

$$\tau = \quad \tau\left(\xi, \gamma\right) = \begin{cases} \xi / \left(1-\gamma\right) & \text{for } -1+\gamma \le \xi \le 1-\gamma \\ \left(-\xi+2\right) / \left(1+\gamma\right) & \text{for } 1-\gamma \le \xi \le 3+\gamma \end{cases} \tag{14.12}$$

$$\forall \xi : \qquad \tau\left(\xi+4, \gamma\right) = \tau\left(4, \gamma\right), \qquad -1 < \gamma < 1$$

Schwartz derivatives of function (14.12), $e = \partial\tau(\xi, \gamma)/\partial\xi$, satisfy the following relationships

$$e^2 = \alpha + \beta e \tag{14.13}$$

$$e\frac{\partial e}{\partial \xi} = \frac{1}{2}\beta\frac{\partial e}{\partial \xi} \tag{14.14}$$

$$\frac{\partial e}{\partial \xi} = 2\alpha \sum_{k=-\infty}^{\infty} \left[\delta\left(\xi+1-\gamma-4k\right) - \delta\left(\xi-1+\gamma-4k\right)\right] \tag{14.15}$$

where $\alpha = 1/\left(1-\gamma^2\right)$ and $\beta = 2\gamma\alpha$.

Let us represent the string deflection in the form

$$u = U(\tau, y, t) + V(\tau, y, t)e \tag{14.16}$$

where $\tau = \tau(\xi, \gamma)$ and $e = \partial\tau(\xi, \gamma)/\partial\xi$.

The components of representation (14.16), U and V, depend on the coordinate y both explicitly and through the triangular wave function τ in such a way that the complete partial derivative $\partial u/\partial y$, including the dependence $\xi = y/\varepsilon$, is equivalent to applying the differential matrix operator

$$D = \varepsilon^{-1} \begin{bmatrix} \varepsilon\partial/\partial y & \alpha\partial/\partial\tau \\ \partial/\partial\tau & \beta\partial/\partial\tau + \varepsilon\partial/\partial y \end{bmatrix} \tag{14.17}$$

to the vector column of the components U and V

$$\frac{\partial u}{\partial y} \Longleftrightarrow D \begin{bmatrix} U \\ V \end{bmatrix} \tag{14.18}$$

under the condition

$$\tau = \pm 1 : V = 0 \tag{14.19}$$

The regular part of the second derivative can be calculated by means of the relationship

$$\frac{\partial^2 u}{\partial y^2} \Longleftrightarrow D^2 \begin{bmatrix} U \\ V \end{bmatrix} \tag{14.20}$$

However, second derivative of the triangular wave function must be preserved in order to eliminate the same kind of singularity from Eq. (14.11). So, substituting (14.16) in (14.11) and collecting separately terms related to different elements of the basis $\{1, e\}$ give

$$\rho \frac{\partial^2 U}{\partial t^2} - T \times \left[\left(\frac{\alpha}{\varepsilon^2} \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial y^2} \right) U + \left(\frac{\alpha\beta}{\varepsilon^2} \frac{\partial^2}{\partial \tau^2} + \frac{2\alpha}{\varepsilon} \frac{\partial^2}{\partial \tau \partial y} \right) V \right] = 0 \tag{14.21}$$

$$\rho \frac{\partial^2 V}{\partial t^2} - T \times \left[\left(\frac{\beta}{\varepsilon^2} \frac{\partial^2}{\partial \tau^2} + \frac{2}{\varepsilon} \frac{\partial^2}{\partial \tau \partial y} \right) U + \left(\frac{\alpha + \beta^2}{\varepsilon^2} \frac{\partial^2}{\partial \tau^2} + \frac{2\beta}{\varepsilon} \frac{\partial^2}{\partial \tau \partial y} + \frac{\partial^2}{\partial y^2} \right) V \right] = 0 \tag{14.22}$$

under the additional to (14.19) condition

$$\tau = \pm 1 : -\frac{T}{\varepsilon^2} \left(\frac{\partial U}{\partial \tau} + \beta \frac{\partial V}{\partial \tau} \right) = \pm \frac{k}{\varepsilon} (1 - \gamma^2) U \tag{14.23}$$

Note that some terms have been eliminated from condition (14.23) by taking into account condition (14.19).

Further analysis of the boundary value problems (14.19) and (14.21) through (14.23) can be implemented by using the asymptotic approach [199] considering ε as a small parameter.

Similar version of the transformation was employed for statics of layered composites in [227].

14.1.3 Elastic Ring Under Periodic Array of Compressive Loads

The buckling of a circular ring loaded by a discrete regular set of concentrated compressive forces was considered in [239]; see Fig. 14.3. Taking into account identity (14.2), the differential equation of equilibrium of such ring can be represented in the form

$$\frac{dy}{dt} = f(y) - \lambda \sum_{k=-\infty}^{\infty} \delta\left(2\frac{t}{\varepsilon} - 1 - 2k\right) \equiv f(y) + \frac{1}{2}\lambda \operatorname{sgn}(\tau)\tau'' \qquad (14.24)$$

where $t = s/R$ $(0 \leq t \leq 2\pi)$ is the arc length of the undeformed ring axis per radius; $y = y(t)$ is a six-component vector-function characterizing elastic states of the ring; $f(y)$ is a vector-function of the elastic states describing both geometrical and physical properties of the ring; $\tau = \tau(2t/\varepsilon)$ is the triangular sine wave of the period $T = 2\varepsilon$ and prime means its Schwartz derivative; ε is a small parameter, as compared to unity, characterizing the distance between loads (Fig. 14.3); λ is

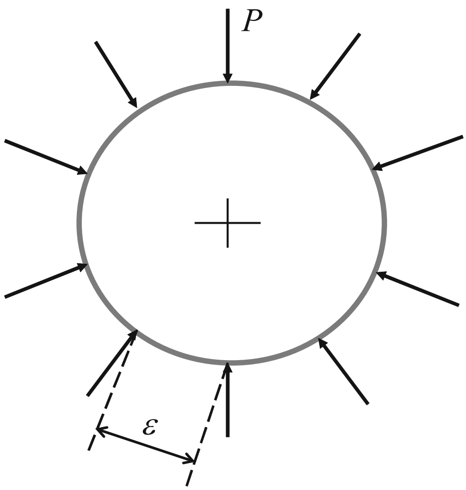

Fig. 14.3 The circular ring, whose radius is scaled to unity, under the discrete regular set of compressive radial forces, where $\varepsilon = 2\pi/N \ll 1$ and N is the number of forces

a dimensionless parameter which is proportional to the load P; and conditions of periodicity are imposed on the vector-function: $y(t + 2\pi) = y(t)$. This periodicity is associated with the formally infinite limits of summation in Eq. (14.24). Shifting the arclength t by 2π is equivalent to shifting the summation index as $k \longrightarrow k + N$. Obviously, the wave length of buckling is not necessarily equal to the spatial period of loading.

Further, the unknown vector-function is represented as an element of the hyperbolic algebra

$$y = X(t, \tau) + Y(t, \tau)e, \qquad e = e(2t/\varepsilon) \tag{14.25}$$

Substituting (14.25) in Eq. (14.24) gives

$$\frac{2}{\varepsilon}\frac{\partial Y}{\partial \tau} + \frac{\partial X}{\partial t} - R(X, Y) + \left[\frac{2}{\varepsilon}\frac{\partial X}{\partial \tau} + \frac{\partial Y}{\partial t} - I(X, Y)\right]e$$
$$+ \left[\frac{2}{\varepsilon}Y - \frac{1}{2}\lambda \mathrm{sgn}(\tau)\right]e' = 0 \tag{14.26}$$

or

$$\frac{\partial Y}{\partial \tau} = \frac{1}{2}\varepsilon\left[R(X, Y) - \frac{\partial X}{\partial t}\right]$$
$$\frac{\partial X}{\partial \tau} = \frac{1}{2}\varepsilon\left[I(X, Y) - \frac{\partial Y}{\partial t}\right] \tag{14.27}$$

and

$$\tau = \pm 1 : Y = \pm\frac{1}{4}\varepsilon\lambda \tag{14.28}$$

where

$$R(X, Y) = \frac{1}{2}\left[f(X + Y) + f(X - Y)\right]$$
$$I(X, Y) = \frac{1}{2}\left[f(X + Y) - f(X - Y)\right]$$

Recall that the role of boundary conditions (14.28) is to eliminate the δ-function singularities, $e' = \tau''$, from Eq. (14.26). The resultant boundary value problem, (14.27)–(14.28), was analyzed by means of regular asymptotic expansions in [239]. Based on the asymptotic solutions, the load discreteness effects on the critical load and postbuckling states of the ring were estimated.

14.2 Homogenization of One-Dimensional Periodic Structures

Let us consider a one-dimensional 4ε-periodic structure whose static elastic states are described by the vector-function $z = z(y) \in R^n$ that depends upon the longitudinal coordinate y attached to the undeformed structure. The number of vector components n can always be increased in such a way that the differential equation of equilibrium takes the form of first-order differential equation, for instance, as follows

$$\frac{dz}{dy} = f(z, y, \xi) + p(y)e'(\xi), \ e(\xi) = \tau'(\xi) \tag{14.29}$$

Here the spatial scale $\xi = y/\varepsilon$ associates with the structural periodicity, the vector-function $f(z, y, \xi) \in R^n$ is continuous with respect to z and y, but it is allowed to be stepwise discontinuous with respect to ξ at the points $\{\xi : \tau(\xi) = \pm 1\}$, and $p(y) \in R^n$ is a continuous vector-function describing the amplitude modulation of the localized loading.

Example 14.2.1 In terms of the matrixes,

$$z = \begin{bmatrix} u(y) \\ v(y) \end{bmatrix}, \ p(y) = \begin{bmatrix} 0 \\ q(y)/(2\varepsilon) \end{bmatrix} \tag{14.30}$$

$$f = \begin{bmatrix} v(y)/\{EF[1 + \alpha e(\xi)]\} \\ 0 \end{bmatrix}$$

Equation (14.29) describes an elastic rod whose cross-sectional area is a piecewise constant periodic function of the longitudinal coordinate as it is shown in Fig. 14.4. Substituting (14.30) in (14.29) and eliminating then $v(y)$ give the second-order differential equation of equilibrium for the rod

$$\frac{d}{dy}\left[EF\,[1 + \alpha e(\xi)]\,\frac{du}{dy}\right] = \frac{1}{2\varepsilon}q(y)e'(\xi) \tag{14.31}$$

Now let us consider the vector-form equation in its general form (14.29). Nonsmooth two-variable expansions will be used by considering the triangular wave

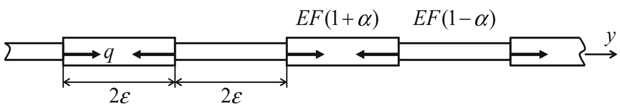

Fig. 14.4 An elastic rod with a periodic nonsmoothly varying cross-sectional area and concentrated loading

oscillating coordinate $\tau = \tau(\xi)$ and the original coordinate $\eta \equiv y$ as fast and slow spatial scales, respectively, provided that the following assumption holds

$$\varepsilon << 1 \tag{14.32}$$

Let us represent solutions of Eq. (14.29) in the form

$$z = X(\tau, \eta) + Y(\tau, \eta)e \tag{14.33}$$

Substituting (14.33) in Eq. (14.29) gives

$$\frac{\partial Y}{\partial \tau} + \varepsilon\left(\frac{\partial X}{\partial \eta} - R_f\right) + \left[\frac{\partial X}{\partial \tau} + \varepsilon\left(\frac{\partial Y}{\partial \eta} - I_f\right)\right]e + [Y - \varepsilon p(\eta)]e' = 0 \tag{14.34}$$

where

$$\left.\begin{array}{l} R_f(X, Y, \tau, \eta) \\ I_f(X, Y, \tau, \eta) \end{array}\right\} = \frac{1}{2}[f(X + Y, \tau, \eta) \pm f(X - Y, 2 - \tau, \eta)] \tag{14.35}$$

Expression (14.34) is equivalent to the following boundary value problem with no discontinuities

$$\frac{\partial X}{\partial \tau} + \varepsilon\left(\frac{\partial Y}{\partial \eta} - I_f\right) = 0 \tag{14.36}$$

$$\frac{\partial Y}{\partial \tau} + \varepsilon\left(\frac{\partial X}{\partial \eta} - R_f\right) = 0$$

$$\tau = \pm 1 : Y = \varepsilon p(\eta) \tag{14.37}$$

Let us represent solutions of the boundary value problem, (14.36) and (14.37), in the form of asymptotic series with respect to ε

$$X = \sum_{i=0}^{\infty} \varepsilon^i X^i(\tau, \eta) \tag{14.38}$$

$$Y = \sum_{i=0}^{\infty} \varepsilon^i Y^i(\tau, \eta)$$

where the functions X^i and Y^i are to be sequentially determined.

Substituting (14.38) in (14.35) generates power series expansions

$$R_f = R_f^0 + \varepsilon R_f^1 + \varepsilon^2 R_f^2 + \ldots \tag{14.39}$$

$$I_f = I_f^0 + \varepsilon I_f^1 + \varepsilon^2 I_f^2 + ...$$

where the following notations are used

$$R_f^i = \frac{1}{i!}\frac{\partial^i R_f}{\partial \varepsilon^i}|_{\varepsilon=0}, \qquad I_f^i = \frac{1}{i!}\frac{\partial^i I_f}{\partial \varepsilon^i}|_{\varepsilon=0}$$

Substituting (14.38) in (14.36) and (14.37), then matching the coefficients of the same powers of ε, gives the corresponding sequence of equations and boundary conditions. In particular, zero-order problem takes the form

$$\frac{\partial X^0}{\partial \tau} = 0, \ \frac{\partial Y^0}{\partial \tau} = 0 \tag{14.40}$$

and

$$\tau = \pm 1 : Y^0 = 0 \tag{14.41}$$

As follows from (14.40), the generating solution is independent on the fast oscillating scale τ. Therefore, taking into account (14.41) gives solution

$$X^0 = A^0(\eta), \qquad Y^0 \equiv 0 \tag{14.42}$$

where A^0 is an arbitrary vector-function of the slow coordinate that will be determined on the next step of the asymptotic procedure.

So, collecting the terms of order ε gives the differential equations and boundary conditions in the form, respectively,

$$\frac{\partial X^1}{\partial \tau} = I_f^0 - \frac{\partial Y^0}{\partial \eta} = I_f(A^0, 0, \tau, \eta) \tag{14.43}$$

$$\frac{\partial Y^1}{\partial \tau} = R_f^0 - \frac{\partial X^0}{\partial \eta} = R_f(A^0, 0, \tau, \eta) - \frac{dA^0}{d\eta} \tag{14.44}$$

and

$$\tau = \pm 1 : Y^1 = p(\eta) \tag{14.45}$$

Integrating Eqs. (14.43) and (14.44) gives first-order terms of the asymptotic solution

$$X^1 = \int_0^\tau I_f^0 d\tau + A^1(\eta) \tag{14.46}$$

$$Y^1 = \int_{-1}^{\tau} R_f^0 d\tau - \frac{dA^0}{d\eta}(\tau + 1) + p(\eta)$$

where A^1 is a new arbitrary vector-function of the slow spatial scale η and the limits of integration for Y^1 are chosen in such a manner that boundary condition (14.45) is satisfied automatically at the point $\tau = -1$, whereas another point, $\tau = 1$, gives equation

$$\frac{dA^0}{d\eta} = \frac{1}{2}\int_{-1}^{1} R_f^0 d\tau \equiv \left\langle R_f^0 \right\rangle \tag{14.47}$$

Note that the "slow scale" Eq. (14.47) was obtained by satisfying the boundary condition in contrast to the conventional scheme of two-variable expansions in which such kind of equations are obtained by eliminating the so-called resonance terms.

Enforcing now Eq. (14.47) brings the component Y^1 to the final form

$$Y^1 = \int_{-1}^{\tau} \left(R_f^0 - \left\langle R_f^0 \right\rangle\right) d\tau + p(y^0) \tag{14.48}$$

At this stage, expressions (14.46) through (14.48) determine the first-order terms of the asymptotic solution; however, the slow scale vector-function $A^1(\eta)$ still remains unknown. The corresponding ordinary differential equation is obtained on the next stage from the boundary condition for Y^2 and can be represented in the form

$$\frac{dA^1}{d\eta} = \left\langle \frac{\partial R_f^0}{\partial A^0} \right\rangle A^1 + F^1(A^0, \eta) \tag{14.49}$$

where $\partial R_f^0 / \partial A^0$ is the Jacobian matrix and the vector-function F^1 is known.

Note that Eq. (14.49) is linear. Moreover, on the next steps, equations for the vector-functions A^2, A^3,... will be of the same linear structure, including the same Jacobian matrix.

14.3 Second-Order Equations

Let us consider now the second-order differential equation with respect to the vector-function $z(y) \in R^n$, however, in the linear form

$$\frac{d^2 z}{dy^2} + [q(\xi, y) + p(y)e'(\xi)]z = g(\xi, y) + r(y)e'(\xi) \tag{14.50}$$

$$\tau = \tau(\xi), \qquad e = \tau'(\xi), \qquad \xi = y/\varepsilon$$

where q and p are $n \times n$-matrixes, g and r are n-dimensional vector-functions, and ξ is the fast spatial scale.

Based on the assumptions of the previous section, the functions q and g and solutions of Eq. (14.50) can be represented in the form, respectively,

$$\begin{aligned} q(\xi, y) &= Q(\tau, \eta) + P(\tau, \eta)e \\ g(\xi, y) &= G(\tau, \eta) + F(\tau, \eta)e \\ \eta &\equiv y \end{aligned} \tag{14.51}$$

and

$$z(y) = X(\tau, \eta) + Y(\tau, \eta)e \tag{14.52}$$

where η and τ represent the slow and fast spatial scales, respectively.

Substituting (14.51) and (14.52) in Eq. (14.50) and conducting differential and algebraic manipulations of NSTT lead to the boundary value problem

$$\frac{\partial^2 X}{\partial \tau^2} = -2\varepsilon \frac{\partial^2 Y}{\partial \tau \partial \eta} - \varepsilon^2 \left(\frac{\partial^2 X}{\partial \eta^2} + QX + PY - G \right) \tag{14.53}$$

$$\frac{\partial^2 Y}{\partial \tau^2} = -2\varepsilon \frac{\partial^2 X}{\partial \tau \partial \eta} - \varepsilon^2 \left(\frac{\partial^2 Y}{\partial \eta^2} + PX + QY - F \right) \tag{14.54}$$

and

$$\tau = \pm 1 : \frac{\partial X}{\partial \tau} = \varepsilon^2 [r(y) - p(y)X], \qquad Y = 0 \tag{14.55}$$

Further, representing the solution of the boundary value problem (14.53), (14.54), and (14.55) in the form of asymptotic series (14.38) gives a sequence of boundary value problems, in which first two steps appear to have quite trivial solutions, such as

$$X^0 = B^0(\eta), \quad Y^0 \equiv 0$$

and

$$X^1 \equiv 0, \quad Y^1 \equiv 0$$

where B^0 is an arbitrary function of the slow argument η.

As a result, first two non-trivial steps of the averaging procedure give

$$z = B^0(\eta) + \varepsilon^2[X^2(\tau,\eta) + Y^2(\tau,\eta)e] + O(\varepsilon^3) \tag{14.56}$$

where, in second order of ε, the solution components are

$$X^2 = \int_{-1}^{\tau} (\tau - s)[G(s,\eta) - \langle G(\tau,\eta)\rangle \tag{14.57}$$

$$-(Q(s,\eta) - \langle Q(\tau,\eta)\rangle)B^0]ds + (r - pB^0)\tau + B^2(\eta)$$

$$Y^2 = \int_{-1}^{\tau} [(\tau - s)(F(s,\eta) - P(s,\eta)B^0) \tag{14.58}$$

$$-\langle(1-\tau)(F(\tau,\eta) - P(\tau,\eta)B^0)\rangle]ds$$

Here, notation $< \bullet >$ means averaging with respect to τ as defined in (14.47); the vector-function $B^0 = B^0(\eta)$ satisfies equation

$$\frac{d^2B^0}{d\eta^2} + \langle Q(\tau,\eta)\rangle B^0 = \langle G(\tau,\eta)\rangle \tag{14.59}$$

The new arbitrary function of the slow coordinate, $B^2(\eta)$, has to be defined on the next step of the procedure.

Note that the δ-function impulses generated by the derivative $e'(\xi)$ are switching their directions twice per one period of the triangular wave. In many practical cases though, the direction of impulses may remain constant. The corresponding reformulation of the problem can be implemented by introducing the factor -sgn(τ) into the differential equation as follows

$$\frac{d^2z}{dy^2} + [q(\xi,y) - p(y)\mathrm{sgn}(\tau)e'(\xi)]z = g(\xi,y) + r(y)\mathrm{sgn}(\tau)e'(\xi) \tag{14.60}$$

Now, in Eq. (14.60), the term $\mathrm{sgn}(\tau)e'(\xi)$ generates δ-functions of the same direction, whereas the boundary condition (14.55) for the X-component takes the form

$$\tau = \pm 1 : \frac{\partial X}{\partial \tau} = \mp\varepsilon^2[r(y) - p(y)X] \tag{14.61}$$

The form of expressions (14.57) and (14.59) is modified as, respectively,

$$X^2 = \int_{-1}^{\tau} (\tau - s)[G(s,\eta) - \langle G(\tau,\eta)\rangle \tag{14.62}$$

$$-(Q(s,\eta) - \langle Q(\tau,\eta)\rangle)B^0]ds - \frac{\tau^2}{2}(r - pB^0) + B^2(\eta)$$

and,

$$\frac{d^2B^0}{d\eta^2} + (\langle Q(\tau,\eta)\rangle + p)B^0 = \langle G(\tau,\eta)\rangle + r \tag{14.63}$$

where $\eta \equiv y$ and the component Y^2 is still described by (14.58).

Example 14.3.1 Let us consider an infinite beam resting on a discrete foundation represented by the periodic set of linearly elastic springs of stiffness c. The corresponding differential equation of equilibrium is

$$EI\frac{d^4w}{dx^4} + \frac{c}{a}w\sum_{k=-\infty}^{\infty}\delta\left(\frac{x}{a} - 1 - 2k\right) = f\left(\frac{x}{L}\right) \tag{14.64}$$

$$(-\infty < x < \infty)$$

Let us introduce the following dimensionless values

$$y = \frac{x}{L},\ \xi = \frac{y}{\varepsilon},\quad W = \frac{w}{a},\ \gamma = \frac{cL^4}{aEI}$$

where $\varepsilon = a/L << 1$. As a result the above Eq. (14.64) for the beam's center line takes the form

$$\frac{d^4W}{dy^4} - \frac{1}{2}\gamma\,\mathrm{sgn}[\tau(\xi)]e'(\xi)\,W = \frac{\gamma}{c}f(y) \tag{14.65}$$

This equation becomes equivalent to (14.60), after the following substitutions

$$z = \begin{bmatrix} W \\ W'' \end{bmatrix},\quad q = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},\ p = \frac{\gamma}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ g = \frac{\gamma}{c}f(y)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $r \equiv 0$. Note that q and g do not have "imaginary parts" in hyperbolic elements (14.51); therefore $G \equiv g$, $Q \equiv q$, $P \equiv 0$, and $F \equiv 0$. Inserting these values in (14.62) and (14.58), one finally obtains

$$z = B^0(y) + \frac{1}{2}\varepsilon^2\tau^2(\xi)\, pB^0(y) + O\left(\varepsilon^3\right)$$

or

$$\begin{bmatrix} W \\ W'' \end{bmatrix} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4}\gamma\varepsilon^2\tau^2\left(\frac{y}{\varepsilon}\right)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}\begin{bmatrix} B_1^0 \\ B_2^0 \end{bmatrix} + O\left(\varepsilon^3\right) \tag{14.66}$$

where the term $\varepsilon^2 B^2(y)$ is ignored compared to the leading-order term $B^0(y)$; however, the terms of order ε^2 describing the discreteness effects are maintained. The matrix-column $B^0 = [B_1^0,\ B_2^0]^T$ is determined from Eq. (14.63). In a component-wise form, this equation reads

$$\frac{d^2}{dy^2}\begin{bmatrix} B_1^0 \\ B_2^0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -2 \\ \gamma & 0 \end{bmatrix}\begin{bmatrix} B_1^0 \\ B_2^0 \end{bmatrix} = \frac{\gamma}{c}\begin{bmatrix} 0 \\ f(y) \end{bmatrix} \tag{14.67}$$

This system is equivalent to

$$\frac{d^4 B_1^0}{dy^4} + \frac{\gamma}{2}B_1^0 = \frac{\gamma}{c}f(y) \tag{14.68}$$

and $B_2^0 = d^2B_1^0/dy^2$. Note that Eq. (14.68) is the result of a homogenization of Eq. (14.65) with respect to the fast spatial scale ξ. In other words, Eq. (14.68) describes an elastic beam resting on the effectively continuous elastic foundation. In order to illustrate the asymptotic solution, let us consider the case of sine-wave loading, $f\ (x/L) \equiv q_0 \sin(\pi x/L)$, where q_0 =const. Taking into account only the leading order "slow" and "fast" components gives the bending moment in terms of the original variables

$$M(x) = EI\frac{d^2w}{dx^2} = -M_0\left[1 - \gamma\left(\frac{a}{2\pi L}\right)^2\tau^2\left(\frac{x}{a}\right)\right]\sin\frac{\pi x}{L} \tag{14.69}$$

where $M_0 = 2q_0\pi^2L^2/(2\pi^4 + \gamma)$. The bending moment diagram is given in Fig. 14.5.

Finally, note that the homogenization procedure described in [31] gives the resultant equation in a slow spatial scale and a so-called cell problem in the fast scale. In the above approach, the analog of cell problem associates with the fast oscillating spatial scale given by the triangle wave function $\tau(y/\varepsilon)$. As a result, solution for the cell problem automatically unfolds on the entire structure so that the fast and slow components of elastic states are eventually expressed though the same coordinate in a closed form. Every cell of the infinite array of cells is associated with same standard interval $-1 < \tau < 1$. Further clarifications are given in the next section.

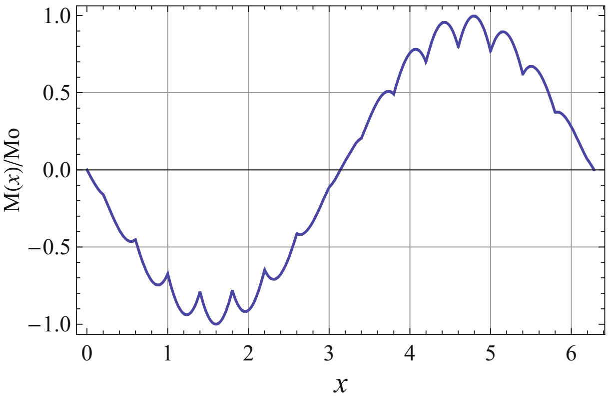

Fig. 14.5 Bending moment of the beam on the discrete elastic foundation; numerical values of the parameters are as follows: $L = \pi,\ a = 0.2$, and $\gamma = 1948.0$

14.4 Wave Propagation in 1D Periodic Layered Composites

Propagation of waves in periodic media has been of significant interest in various branches of optics, acoustics, and elastodynamics for several decades due to a widening area of practical applications for composite materials and extensive usage of periodic structures in civil engineering [12, 13, 31, 47, 118, 154, 159, 215, 216]. Periodicity in material properties can serve for passive control of wave propagation in different micro- and macro-systems. Basic physical formulations and analytical methodologies are systemized and documented, for instance, in [35, 36, 56]. Although linear problems may possess exact solutions, different approximate methods are quite popular by two major reasons. First, typical exact solutions are usually represented as a combination of local solutions describing separate layers (cells), while global characterization of propagating waves is of main interest. Second, the exact approaches are usually difficult to extend on even weakly nonlinear materials. Note that the wave dynamics of layered structures possess typical properties of waves in continuous materials and those in discrete lattices, as confirmed by the presence of so-called pass bands and stop bands in dispersion curves. Despite of the extra complexity, such specifics widen the set of analytical, numerical, and experimental tools. This section is based on reference [194].

14.4.1 Governing Equations and Zero-Order Homogenization

Let us consider longitudinal waves propagating along the infinitely periodic composite rod consisting of alternating layers of two elastic materials as shown at the top of Fig. 14.6. The governing one-dimensional wave equation is

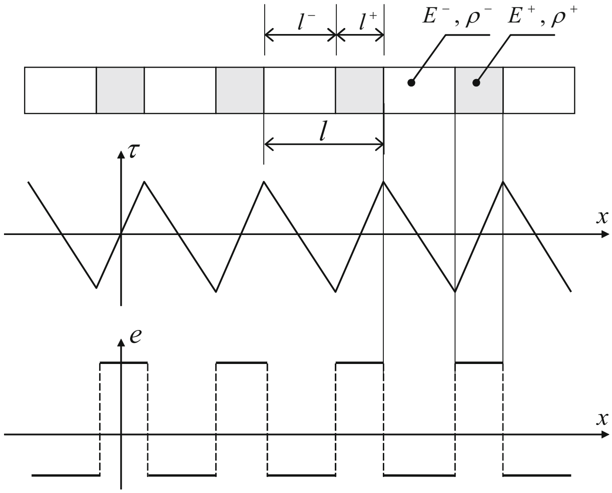

Fig. 14.6 1D periodic composite media with the corresponding basis functions, τ and e; see also Fig. 4.1 of Sect. 4.1.8 for details

$$\rho^{\pm}\frac{\partial^2 u^{\pm}(t,x)}{\partial t^2} - E^{\pm}\frac{\partial^2 u^{\pm}(t,x)}{\partial x^2} = 0 \tag{14.70}$$

where $E^{\pm}$ are Young's moduli; $\rho^{\pm}$ are mass densities; and $u^{\pm}(t,x)$ are displacements; the superscript "+" or "−" indicates different types of layers. Since both types of layers are assumed to be linearly elastic, the longitudinal stress is given by Hooke's law

$$\sigma^{\pm}(t,x) = E^{\pm}\frac{\partial u^{\pm}(t,x)}{\partial x} \tag{14.71}$$

In the case of the perfect bonding between the layers, Eq. (14.70) must be considered under the following continuity conditions at the layer interfaces $x = x_n$

$$u^-(t,x_n) = u^+(t,x_n) \tag{14.72}$$

$$\sigma^-(t,x_n) = \sigma^+(t,x_n) \tag{14.73}$$

$$(n = 0, \pm 1, \pm 2, \ldots)$$

Due to linearity of the boundary value problem, (14.70) through (14.73), the Floquet-Bloch approach [36] gives the exact dispersion relation [13]. However, when elastic waves are considerably longer than typical cells of the material, the idea of homogenization can be used for easing estimations of global elastodynamic properties of composite structures. In such case, a natural small parameter characterizing the rate of heterogeneity and the corresponding scales for spatial coordinates can be introduced as

$$\varepsilon = \frac{l}{L} << 1, \qquad \eta = x, \qquad \xi = \frac{x}{\varepsilon} \tag{14.74}$$

where ε is a unitless heterogeneity parameter, l is a cell (layer) thickness, L is the wave length, η is identical to the original coordinate x associated with the spatial scale of the propagating wave, and ξ is another coordinate, associated with the spatial scale of heterogeneity.

Asymptotic homogenization procedures are usually designed in such way that ξ is a *local* coordinate attached to a typical cell of the material, and the corresponding "cell problem" is assumed to depend slowly upon the *global* coordinate η. As a result, the effect of structural periodicity becomes somewhat shadowed in the solution despite the fact that considering a single *arbitrary* cell is justified by periodicity.

Note that a *quasi static* homogenization, corresponding to the limit $\varepsilon \longrightarrow 0$, can be conducted by calculating the effective Young's modulus and mass density of a single elementary cell composed of two different layers. For instance, neglecting the inertia term in Eq. (14.70) gives general solution in the form $u^{\pm} = A^{\pm}x + B^{\pm}$, where $A^{\pm}$ and $B^{\pm}$ are arbitrary constants and $x = 0$ corresponds to the boundary between two different layers. Two of the four constants are eliminated from this solution due to the continuity of displacement and stress at the boundary $x = 0$. Then, calculating the effective strain of the entire cell of two layers, $\epsilon = \left(u^{+}|_{x=l^{+}} - u^{-}|_{x=l^{-}}\right)/l$, gives the effective Young's modulus

$$E_0 = \frac{\sigma^{+}}{\epsilon} = \frac{\sigma^{-}}{\epsilon} = \frac{E^{-}E^{+}}{E^{-}(1-s) + E^{+}s} \tag{14.75}$$

where the parameter $s = l^{-}/l$ describes the "inner geometry" of cells, $l = l^{+} + l^{-}$.

The effective mass density of the cell is given by

$$\rho_0 = \frac{\rho^{+}l^{+} + \rho^{-}l^{-}}{l} = \rho^{-}s + \rho^{+}(1-s) \tag{14.76}$$

Due to the structural periodicity of the composite rod, all the cells must have the same effective Young's modus and mass density. This actually means that the rod is homogenized, and therefore its partial differential equation takes the form

$$\rho_0 \frac{\partial^2 u_0(t,x)}{\partial t^2} - E_0 \frac{\partial^2 u_0(t,x)}{\partial x^2} = 0 \tag{14.77}$$

However, this equation is justified only for very long waves, since it ignores the dispersion of waves caused by their scattering at the boundaries between different layers. Also the heterogeneity effect on wave shapes cannot be captured by the function $u_0(t,x)$.

14.4.2 Structure Attached Triangle Wave Coordinate

Following publication [194], we introduce the periodic nonsmooth coordinate

$$(-\infty, \infty) \ni \xi \longrightarrow \tau \in [-1, 1]: \qquad \tau = \tau\left(\frac{4\xi}{L}, \gamma\right) \tag{14.78}$$

where τ is a triangle wave whose geometry is linked to the structural periodicity of composite as shown in Fig. 14.6.

The period of function (14.78) with respect to the original coordinate x is equal to the length of one cell, l, because $4\xi/L = 4x/l$. Such a space folding coordinate transformation incorporates micro-structural specifics of the material into the differential equations of elastodynamics on the preliminary phase of analysis. The shape of function (14.78) is controlled by the parameter

$$\gamma = -1 + 2s, \quad s = \frac{l^-}{l} \tag{14.79}$$

where the ratio s is defined in (14.76) in such a way that, when the thickness of different layers is the same, $l^- = l^+$, then $\gamma = 0$, and hence function (14.78) describes the triangle wave.

Recall that function (14.78) is non-differentiable at points $\{\xi : \tau = \pm 1\}$ and non-invertible on the entire period. Therefore, using τ as a new independent variable requires a specific complexification of the unknown displacement function u as

$$u(t, x) = U^+(t, \eta, \tau)e_+ + U^-(t, \eta, \tau)e_- \tag{14.80}$$

where the algebraic basis is

$$e_\pm = \frac{1}{2}[1 \mp \gamma \pm (1 - \gamma^2)e] \tag{14.81}$$

and e is a generalized derivative of the triangle wave function (14.78)

$$e = e\left(\frac{4\xi}{L}, \gamma\right) = \partial\tau\left(\frac{4\xi}{L}, \gamma\right) / \partial\left(\frac{4\xi}{L}\right) \tag{14.82}$$

In Eq. (14.81), the parameter of asymmetry γ normalizes the range of change for the basis elements to the standard intervals, $0 \le e_+ \le 1$ and $0 \le e_- \le 1$, as illustrated in Fig. 14.7.

Differentiating (14.80) with respect to the original coordinate x gives yet again the element of the same algebraic structure

$$\frac{\partial u}{\partial x} = D_+U^+e_+ + D_-U^-e_- \tag{14.83}$$

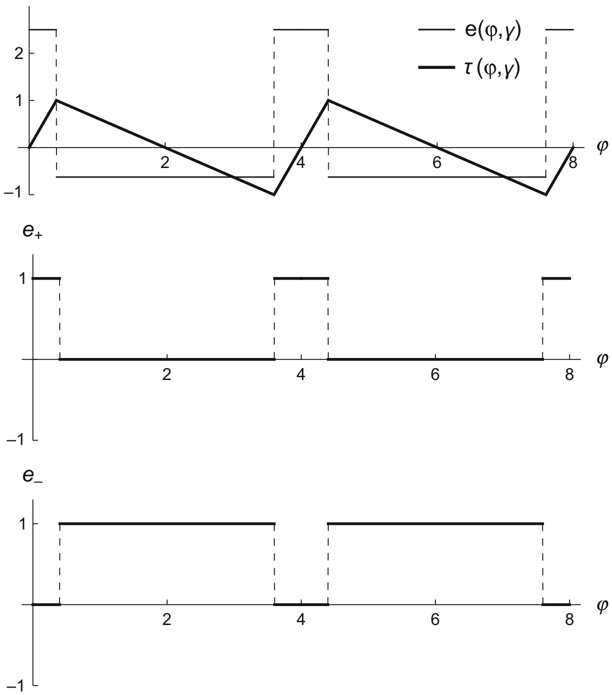

Fig. 14.7 Example of idempotent basis generated by the function $\tau(\varphi, \gamma)$ for $\gamma = 0.6$

where $D^{\pm}$ are linear differential operators,

$$D_{\pm} \equiv \frac{\partial}{\partial \eta} \pm \frac{4}{\varepsilon L(1 \mp \gamma)} \frac{\partial}{\partial \tau}$$

and the following displacement continuity condition is imposed

$$(U^{+} - U^{-})|_{\tau=\pm 1} = 0 \tag{14.84}$$

Condition (14.84) eliminates the δ-function singularities caused by differentiation of the stepwise discontinuous functions $e_{\pm}$ in (14.80). In terms of the original variables, such elimination of singularity is obviously equivalent to the displacement continuity condition.

In compliance with (14.80), the mass density ρ and Young's modulus E are represented in the same algebraic form

$$\begin{aligned} \rho &= \rho^{+} e_{+} + \rho^{-} e_{-} \\ E &= E^{+} e_{+} + E^{-} e_{-} \end{aligned} \tag{14.85}$$

Taking into account (14.83) and (14.85) gives the following expression for stress

$$\sigma = E\frac{\partial u}{\partial x} = E^+D_+U^+e_+ + E^-D_-U^-e_- \equiv \sigma^+e_+ + \sigma^-e_- \tag{14.86}$$

where orthogonality of the basis elements, $e_+e_- = 0$, was used; see Fig. 14.7.

Analogously, combining (14.80) and (14.85) brings the inertia force to the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho^+\frac{\partial^2 U^+}{\partial t^2}e_+ + \rho^-\frac{\partial^2 U^-}{\partial t^2}e_- \tag{14.87}$$

Expressions (14.86) and (14.87) enable one of describing the composite rod by a "single" partial differential equation in the standard form

$$\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial \sigma}{\partial x} = 0 \tag{14.88}$$

Now applying the differentiation rule (14.83)–(14.84) to the stress σ gives

$$\left(\rho^+\frac{\partial^2 U^+}{\partial t^2} - E^+D_+^2U^+\right)e_+ + \left(\rho^-\frac{\partial^2 U^-}{\partial t^2} - E^-D_-^2U^-\right)e_- = 0$$

or

$$\rho^\pm\frac{\partial^2 U^\pm}{\partial t^2} - E^\pm D_\pm^2U^\pm = 0 \tag{14.89}$$

under condition

$$(\sigma^+ - \sigma^-)|_{\tau=\pm1} = (E^+D_+U^+ - E^-D_-U^-)|_{\tau=\pm1} = 0 \tag{14.90}$$

The boundary condition (14.90) occurs in a similar way to (14.84) as a result of elimination of singularity caused by differentiation of the stress function (14.86). From the physical standpoint, this is equivalent to the continuity of stress at bonding interfaces. Note that, during the procedure of asymptotic integration described below, it is convenient to deal with the continuity conditions in the form

$$(U^+ - U^-)|_{\tau=1} \pm (U^+ - U^-)|_{\tau=-1} = 0 \tag{14.91}$$

and

$$(E^+D_+U^+ - E^-D_-U^-)|_{\tau=1} \pm (E^+D_+U^+ - E^-D_-U^-)|_{\tau=-1} = 0 \tag{14.92}$$

Equation (14.89) under the boundary conditions (14.91) and (14.92) represents the final result of transition to the structure-based coordinate τ. As compared to the boundary value problem (14.70) through (14.73), the resultant problem still has the same dimension. However, representation (14.80) eventually provides a closed-form description combining both global and local specifics of wave shapes. In addition, the boundary conditions of continuity for the displacement and stress occur automatically as a result of elimination singularities of their derivatives.

Remark Note that, due to the "functional linearity" property, the above formulation remains valid for nonlinear cases as well. For instance, let the stress-strain relationship for every "positive" layer be described by

$$\sigma^{+}(t,x) = F\left(\epsilon^{+}, \frac{\partial \epsilon^{+}}{\partial t}\right) \tag{14.93}$$

where $\epsilon^{+} = \partial u^{+}/\partial x$ and F is a nonlinear function.

Assuming that every "negative" layer is linearly elastic and conducting the derivations gives the boundary value problem

$$\begin{aligned} \rho^{+}\frac{\partial^2 U^{+}}{\partial t^2} - D_{+}\left[F\left(D_{+}U^{+}, D_{+}\frac{\partial U^{+}}{\partial t}\right)\right] &= 0 \\ \rho^{-}\frac{\partial^2 U^{-}}{\partial t^2} - E^{-}D_{-}^{2}U^{-} &= 0 \end{aligned} \tag{14.94}$$

and

$$\begin{aligned} \left[F\left(D_{+}U^{+}, D_{+}\frac{\partial U^{+}}{\partial t}\right) - E^{-}D_{-}U^{-}\right]|_{\tau=\pm 1} &= 0 \\ (U^{+} - U^{-})|_{\tau=\pm 1} &= 0 \end{aligned} \tag{14.95}$$

Note that nonlinearities may lead to inevitable technical complications unrelated to the key elements of the suggested formulation.

14.4.3 Algorithm of Asymptotic Integration

Let us seek solution of the boundary value problem (14.89), (14.91), and (14.92) in the form of asymptotic expansions

$$U^{\pm} = u_0(t,\eta) + \sum_{i=1}^{4} \varepsilon^{i} U_i^{\pm}(t,\eta,\tau) + O(\varepsilon^5) \tag{14.96}$$

where ε is the heterogeneity parameter defined in (14.74).

Note that the adopted asymptotic order provides sufficient details of the corresponding homogenized equation, whose asymptotics appear to be delayed by two steps of iterations. The time variable preserves its original scale, and zero-order (generating) term $u_0(t, \eta)$ is assumed to be the same for both components $U^{\pm}$ and hence independent on the coordinate τ. When substituted in (14.80), this term gives

$$u(t, x) = u_0(t, \eta)(e_+ + e_-) + O(\varepsilon) = u_0(t, \eta) + O(\varepsilon) \tag{14.97}$$

due to the property $e_+ + e_- = 1$, as explained by Fig. 14.7.

Further, substituting (14.96) in Eq. (14.89) and the boundary conditions, (14.91) and (14.92) and then matching terms of the same order of ε give a sequence of linear boundary value problems. At every step of iterations, the mathematical structure of differential equations remains the same and takes the form

$$\frac{\partial^2 U_i^{\pm}(t, \eta, \tau)}{\partial \tau^2} = f_i^{\pm}(t, \eta, \tau) \tag{14.98}$$

where the dependence $f_i^{\pm}(t, \eta, \tau)$ on τ is polynomial, which is known as soon as all the previous iterations have been processed, and the dependencies on t and η are combined of different derivatives of $u_0(t, \eta)$.

General solution of Eq. (14.98) can be represented in the integral form

$$U_i^{\pm}(t, \eta, \tau) = \int_0^{\tau} f_i^{\pm}(t, \eta, s)(\tau - s)ds + A_i^{\pm}(t, \eta)\tau + B_i^{\pm}(t, \eta) \tag{14.99}$$

where $A_i^{\pm}(t, \eta)$ and $B_i^{\pm}(t, \eta)$ are four arbitrary functions of the slow arguments.

On the i-iteration, the boundary conditions are not uniquely solvable for all the four unknowns, $A_i^{\pm}(t, \eta)$ and $B_i^{\pm}(t, \eta)$. Three of the four boundary conditions determine $A_i^{\pm}(t, \eta)$ with the following coupling

$$B_i^{+}(t, \eta) = B_i^{-}(t, \eta) \tag{14.100}$$

while the fourth boundary condition can be represented in the form

$$(E^+ D_+ U^+ - E^- D_- U^-)|_{\tau=-1}^{\tau=1} = 0 \tag{14.101}$$

where the typical symbol of double substitution is used.

Due to the property, $e_+ + e_- = 1$, the terms $B_i^{\pm}(t, \eta)$ in (14.99) contribute some correction into still arbitrary generating term $u_0(t, \eta)$ (14.96) and thus can be ignored. Boundary condition (14.101) plays a specific role. It is intentionally kept unsatisfied as long as the generating term $u_0(t, \eta)$ is maintained arbitrary. Once a sufficient number of iterations have been processed, the corresponding truncated series is substituted in (14.101) that leads to the homogenized equation for $u_0(t, \eta)$. Note that no operators of averaging are imposed on the original differential equation (14.88). Instead, the homogenized model is generated by the boundary condition

(14.101), which is one of the two continuity conditions for the stress function (14.90).

14.4.4 Homogenized Equation and Solution

Conducting the first four steps of the asymptotic procedure, as described in the previous section, gives the asymptotic solution in the form

$$
\begin{aligned}
u(t,x) &= u_0(t,x) + \varepsilon u_1(t,x)\tau + \varepsilon^2 \left[u_2^+(t,x)e_+ + u_2^-(t,x)e_-\right] \frac{\tau^2-1}{2} \\
&\quad + \varepsilon^3 \left[\left(u_3^+(t,x)e_+ + u_3^-(t,x)e_-\right)\frac{\tau^3}{6} + \left(A_3^+(t,x)e_+ + A_3^-(t,x)e_-\right)\tau\right] \\
&\quad + O(\varepsilon^4)
\end{aligned}
\tag{14.102}
$$

where $\eta \equiv x\ (-\infty < x < \infty)$, $\tau = \tau\,(4x/l, \gamma)$, and $e_\pm = e_\pm\,(4x/l, \gamma)$, and different functions of the arguments t and x are sequentially expressed through derivatives of the generating solution $u_0(t,x)$ as

$$
\begin{aligned}
u_1 &= \frac{L}{4}\frac{\left(1-\gamma^2\right)\left(E^- - E^+\right)}{(1-\gamma)E^- + (1+\gamma)E^+}\frac{\partial u_0}{\partial x} \\
u_2^\pm &= \mp\frac{L(1\mp\gamma)}{16E^\pm}\left[8E^\pm\frac{\partial u_1}{\partial x} \mp L(1\mp\gamma)\left(\rho^\pm\frac{\partial^2 u_0}{\partial t^2} - E^\pm\frac{\partial^2 u_0}{\partial x^2}\right)\right] \\
u_3^\pm &= \mp\frac{L(1\mp\gamma)}{16E^\pm}\left[8E^\pm\frac{\partial u_2^\pm}{\partial x} \mp L(1\mp\gamma)\left(\rho^\pm\frac{\partial^2 u_1}{\partial t^2} - E^\pm\frac{\partial^2 u_1}{\partial x^2}\right)\right] \\
A_3^\pm &= -\frac{3(1\pm\gamma)E^\pm u_3^\pm + (1\mp\gamma)E^-\left(u_3^\pm + 2u_3^\mp\right)}{6\left[(1-\gamma)E^- + (1+\gamma)E^+\right]}
\end{aligned}
$$

The asymptotic order of solution (14.102) is high enough for practical estimations of the heterogeneity effects on wave shapes. One more iteration has to be processed though in order to see such effects in terms of the homogenized equation, which is obtained by substituting the components $U^\pm$ (14.99) into the boundary condition (14.101)

$$
\rho_0\frac{\partial^2 u_0(t,x)}{\partial t^2} = E_0\frac{\partial^2 u_0(t,x)}{\partial x^2} + (\varepsilon L)^2 E_2\frac{\partial^4 u_0(t,x)}{\partial x^4} + O(\varepsilon^4) \tag{14.103}
$$

The effective parameters are determined by collecting terms with different derivatives and summarized as

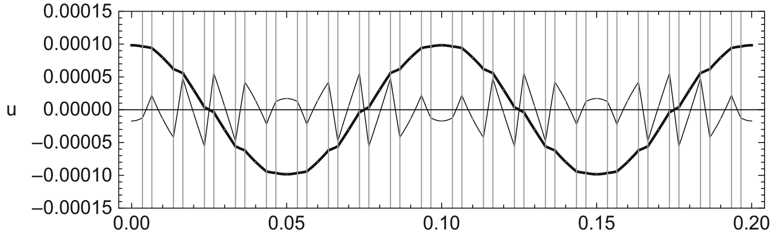

Fig. 14.8 Spatial shape of the propagating wave through the composite rod obtained from third-order asymptotic solution (14.102) (thick line) and correction to the homogenized solution with the amplitude zoomed by factor ε^{-1} (thin line); $E^+ = 7\cdot 10^{10}$ N/m^2, $E^- = 21\cdot 10^{10}$N/m^2, $\rho^- = 7800$ kg/m^3, $\rho^+ = 2700$ kg/m^3, $l = 0.01$ m, $L = 0.1$ m, $(\varepsilon = 0.1)$, $\gamma = -0.4$

$$\begin{aligned} \rho_0 &= \frac{1}{2}\left[(1+\gamma)\rho^- + (1-\gamma)\rho^+\right] \\ E_0 &= \frac{2E^-E^+}{(1-\gamma)E^- + (1+\gamma)E^+} \\ E_2 &= \frac{1}{6}\frac{\left(1-\gamma^2\right)^2 E^-E^+\left(E^-\rho^- - E^+\rho^+\right)^2}{\left[(1-\gamma)E^- + (1+\gamma)E^+\right]^3\left[(1+\gamma)\rho^- + (1-\gamma)\rho^+\right]^2} \end{aligned} \tag{14.104}$$

Note that substituting (14.79) in E_0 and ρ_0 gives (14.75) and (14.76), respectively. Therefore, Eq. (14.103) is reduced to Eq. (14.77) as $\varepsilon \longrightarrow 0$. Otherwise, the term of order ε^2 describes the effect of wave dispersion due to the structural heterogeneity of the rod. After substitution (14.79), the effective parameters (14.104) coincide those obtained in [64] and [12] by different methods. However, the main target of this section is "closed-form" solution (14.102) describing both local and global wave shapes within the same expression. As a result, it is possible to visualize cell to cell transitions over the long wave length as shown in Fig. 14.8. Some features, such as beating effects in the nonsmooth component of solution, would be difficult to observe by other means. Besides, Fig. 14.8 validates the asymptotic property of expansion (14.102) since the maxima of correction terms, $u - u_0$, even with magnifying factor $\varepsilon^{-1} = 10$ are about 50% below the wave amplitude.

14.5 Acoustic Waves from Nonsmooth Periodic Boundary Sources

This section deals with two-dimensional acoustic waves propagating from a discontinuous periodic source located at the boundary of half-infinite space. It is shown that introducing the triangular wave function as a specific spatial coordinate

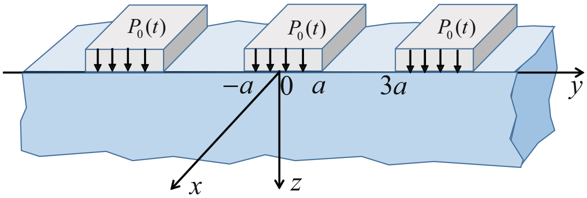

Fig. 14.9 A half-infinite acoustic media excited by the stepwise discontinuous periodic pressure source at the upper boundary

naturally eliminates discontinuities from the boundary condition associated with the active boundary.

For illustrating purposes, let us consider the case of two-dimensional stationary waves propagating in the half-infinite media from a piecewise-linear periodic boundary source as shown in Fig. 14.9.

Let us describe acoustic waves by the linear wave equation in the standard form

$$\frac{1}{c_f^2}\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \tag{14.105}$$

where P is a pressure deviation from the static equilibrium pressure; x, y, z, and t are spatial coordinates and time, respectively; and c_f is the speed of sound in the media.

Further, the plane problem is considered when $P = P(t, y, z)$, and therefore, $\partial^2 P/\partial x^2 = 0$. Such an assumption can be justified by sufficiently long piezoelectric rods whose characteristics are constant along the x-coordinate. Suppose that the pressure generated by the rods near the boundary is $P_0 = A \sin \Omega t$, where A and Ω are constant amplitude and frequency, respectively.

Let the boundary condition at $z = 0$ to have the form

$$P(t, y, 0) = \begin{cases} P_0(t) & \text{for } (4n-1)a \le y \le (4n+1)a \\ 0 & \text{for } (4n+1)a \le y \le (4n+3)a \end{cases} \tag{14.106}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Note that, based on what is actually known near the fluid-source interface, the boundary condition can also be formulated for pressure derivatives. From the mathematical standpoint, this do not affect much the solution procedure though.

Let us seek the steady-state solution, which is periodic with respect to t and y and remains bounded as $z \to \infty$.

Since the boundary condition is periodic along y-coordinate with period $T = 4a$, then, according to the idea of nonsmooth argument transformation, the triangular wave periodic coordinate is introduced as $y \to \tau(y/a)$. As a result, the boundary condition (14.106) and yet unknown solution are represented as, respectively,

$$P(t, y, z)|_{z=0} = \frac{1}{2}P_0(t) + \frac{1}{2}P_0(t)\,\tau'\left(\frac{y}{a}\right) \tag{14.107}$$

and

$$P(t, y, z) = P_1(t, \tau(y/a), z) + P_2(t, \tau(y/a), z)\tau'(y/a) \tag{14.108}$$

where the components P_1 and P_2 are considered as new unknown functions.

Taking into account the expression $[\tau'(y/a)]^2 = 1$ gives first generalized derivative of the original unknown function in the form

$$\frac{\partial P}{\partial y} = \frac{1}{a}\frac{\partial P_2}{\partial \tau} + \frac{1}{a}\frac{\partial P_1}{\partial \tau}\tau'\left(\frac{y}{a}\right) + \frac{1}{a}P_2\tau''\left(\frac{y}{a}\right) \tag{14.109}$$

Since the function $P(t, y, z)$ has to be continuous with respect to y in the unbounded open region $z > 0$, then the periodic singular term τ'' in (14.109) must be eliminated by imposing condition

$$P_2|_{\tau=\pm 1} = 0 \tag{14.110}$$

Analogously, second derivative takes the form

$$\frac{\partial^2 P}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 P_1}{\partial \tau^2} + \frac{1}{a^2}\frac{\partial^2 P_2}{\partial \tau^2}\tau'\left(\frac{y}{a}\right) \tag{14.111}$$

under the condition

$$\frac{\partial P_1}{\partial \tau}|_{\tau=\pm 1} = 0 \tag{14.112}$$

Note that both derivatives, (14.109) and (14.111), as well as the original function (14.108) appear to have the same algebraic structure of hyperbolic numbers. Obviously, differentiation with respect to t and z preserves such a structure as well. As a result, substituting the second derivatives into differential equation (14.105) and collecting separately terms related to each of the basis elements $\{1, \tau'\}$ give two partial differential equations for the components of representation (14.108)

$$\frac{1}{c_f^2}\frac{\partial^2 P_i}{\partial t^2} = \frac{1}{a^2}\frac{\partial^2 P_i}{\partial \tau^2} + \frac{\partial^2 P_i}{\partial z^2} \tag{14.113}$$

$$(i = 1, 2)$$

Substituting then (14.108) in (14.107) gives the corresponding set of boundary conditions

$$P_i\,(t, \tau, z)\,|_{z=0} = \frac{1}{2}P_0\,(t) = \frac{1}{2}A \sin \Omega t \tag{14.114}$$

Now Eqs. (14.113) and boundary conditions (14.110), (14.112), and (14.114) constitute two independent boundary value problems for the components P_1 and P_2. *However, the result achieved is that no discontinuous functions are present anymore in the boundary conditions.*

Solving the above boundary value problems by the standard method of separation of variables gives finally

$$\begin{aligned} P(t, y, z) &= \frac{1}{2} A \sin \Omega \left(t - \frac{z}{c_f} \right) \\ &+ A \left\{ \sum_{k=1}^{m} \frac{(-1)^{k-1}}{(k-1/2)\pi} \sin \Omega \left(t - K_k z \right) \cos \left[\left(k - \frac{1}{2} \right) \pi \tau \left(\frac{y}{a} \right) \right] \right. \\ &+ \left. \sum_{k=m+1}^{\infty} \frac{(-1)^{k-1}}{(k-1/2)\pi} \sin \Omega t \exp \left(-\chi_k z \right) \cos \left[\left(k - \frac{1}{2} \right) \pi \tau \left(\frac{y}{a} \right) \right] \right\} \tau' \left(\frac{y}{a} \right) \end{aligned} \tag{14.115}$$

where

$$K_k = \sqrt{\left(\frac{\Omega}{c_f} \right)^2 - \left(k - \frac{1}{2} \right)^2 \left(\frac{\pi}{a} \right)^2}, \qquad k = 1, ..., m$$

$$\chi_k = \sqrt{\left(k - \frac{1}{2} \right)^2 \left(\frac{\pi}{a} \right)^2 - \left(\frac{\Omega}{c_f} \right)^2}, \qquad k = m + 1, ...$$

and m is the maximum number at which the expression under the first square root is still positive.

A three-dimensional illustration of solution (14.115) is given by Figs. 14.10 and 14.11 for two different magnitudes of the frequency Ω. Besides, it is seen that shorter waves are carrying the information about the discreteness of the wave source for a longer distance from the source.

Note that solution (14.115) could be obtained in terms of the standard trigonometric expansions by applying the method of separation of variables directly to the original problem, (14.105) and (14.106). However, the derivation of solution (14.115) implies no integration of discontinuous functions, since all the discontinuities have been captured in advance by transformation (14.108).

It is also worth to note that the terms of series (14.115) are calculated on the standard interval, $-1 \le \tau \le 1$, which is covered by one half of the total period, whereas the standard Fourier expansions must be built over the entire period. This is due to the fact that representation (14.108) automatically unfolds the half-period domain on the infinite spatial interval.

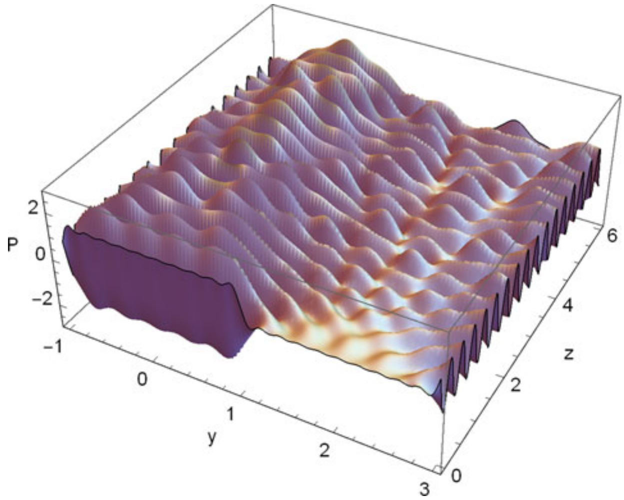

Fig. 14.10 Acoustic wave surface for the set of parameters : $c_f = 10.0$, $a = 1$, $\Omega = 172$, $t = 3$, and $A = 2$

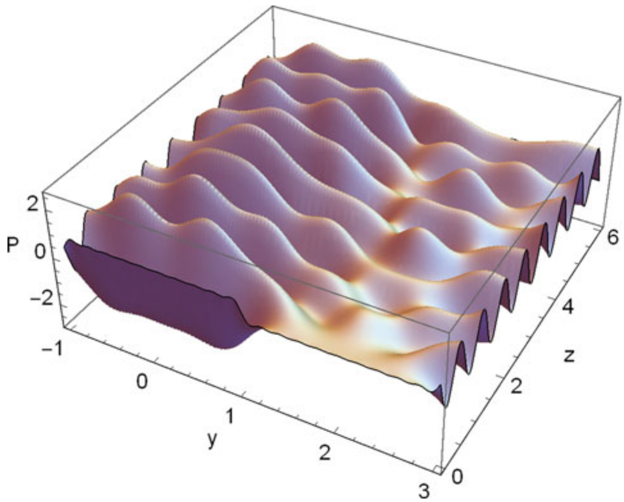

Fig. 14.11 Acoustic wave surface for the set of parameters: $c_f = 10.0$, $a = 1$, $\Omega = 86$, $t = 3$, and $A = 2$

14.6 Spatiotemporal Periodicity

As a possible generalization of the approach, let us consider, for instance, the boundary condition in the form

$$P(t, y, z)|_{z=0} = f(t, y) \tag{14.116}$$

where the function f is periodic with temporal period $T_t = 2\pi/\Omega$ and spatial period $T_y = 4a$.

Introducing the triangular wave spatial argument, $\tau_y = \tau(y/a)$, gives

$$f(t, y) = F_1(t, \tau(y/a)) + F_2(t, \tau(y/a))\tau'(y/a) \tag{14.117}$$

where

$$\begin{aligned} F_1(t, \tau_y) &= \frac{1}{2}[f(t, a\tau_y) + f(t, 2a - a\tau_y)] \\ F_2(t, \tau_y) &= \frac{1}{2}[f(t, a\tau_y) - f(t, 2a - a\tau_y)] \end{aligned} \tag{14.118}$$

In a similar way, introducing the triangular wave temporal argument, $\tau_t = \tau(2\Omega t/\pi)$, into both of the components, F_1 and F_2, gives eventually expression of the form

$$f(t, y) = f_0(\tau_t, \tau_y)e_0 + f_1(\tau_t, \tau_y)e_1 + f_2(\tau_t, \tau_y)e_2 + f_3(\tau_t, \tau_y)e_3 \tag{14.119}$$

where components $f_i(\tau_t, \tau_y)$ are uniquely determined by rule (14.118) applied to each of the two arguments, and the following basis is introduced

$$\begin{aligned} e_0 &= 1 \\ e_1 &= \tau'(2\Omega t/\pi) \\ e_2 &= \tau'(y/a) \\ e_3 &= e_1 e_2 \end{aligned} \tag{14.120}$$

Basis (14.120) obeys the table of products

$$\begin{array}{ccccc} \times & e_0 & e_1 & e_2 & e_3 \\ e_0 & 1 & e_1 & e_2 & e_3 \\ e_1 & e_1 & 1 & e_3 & e_2 \\ e_2 & e_2 & e_3 & 1 & e_1 \\ e_3 & e_3 & e_2 & e_1 & 1 \end{array} \tag{14.121}$$

Now, the acoustic pressure is represented in the similar to (14.119) form

$$P(t, y, z) = P_0(\tau_t, \tau_y, z)e_0 + P_1(\tau_t, \tau_y, z)e_1 \\ + P_2(\tau_t, \tau_y, z)e_2 + P_3(\tau_t, \tau_y, z)e_3 \tag{14.122}$$

Regarding the problem described in the previous section, the components of representation (14.122) can be obtained as an exercise by introducing the argument τ_t directly into solution (14.115). However, formulations based on representation (14.122) become technically reasonable whenever the boundary pressure is adequately described by the functions τ_t and τ_y or their different combinations, for instance, polynomials. In such cases, polynomial approximations with respect to the bounded arguments may appear to be more effective as compared to Fourier expansions. Let us, for instance, $P_0(t)$ describes the periodic sequence of rectangular spikes of the amplitude A,

$$P_0(t) = \frac{1}{2}A\left[1 + \tau'\left(\frac{2\Omega}{\pi}t\right)\right] \equiv \frac{1}{2}A(1 + e_1) \tag{14.123}$$

Then, boundary condition (14.107) takes the form

$$P(t, y, z)|_{z=0} = \frac{1}{4}A(1 + e_1)(1 + e_2) \\ \equiv \frac{1}{4}A(e_0 + e_1 + e_2 + e_3) \tag{14.124}$$

where the basis elements $\{e_0, e_1, e_2, e_3\}$ are given by (14.120) and the table of products (14.121) is taken into account.

Now, substituting representation (14.122) in (14.124) gives the boundary conditions for its components at $z = 0$ as follows

$$P_i(\tau_t, \tau_y, 0) = \frac{1}{4}A; \quad i = 0,...,3$$

Finally, the three-dimensional case can be considered by adding periodicity of the source along the x-direction at the boundary $z = 0$ and introducing the corresponding triangular wave argument, say τ_x. The corresponding rules for algebraic manipulations would be analogous to those generated by the arguments τ_t and τ_y. However, necessary details are illustrated below on another model.

14.7 Membrane on a Two-Dimensional Periodic Foundation

Consider an infinite membrane resting on a linearly elastic foundation of the stiffness $K(x, y)$ under the transverse load $q(x, y)$. Assuming that both the stiffness

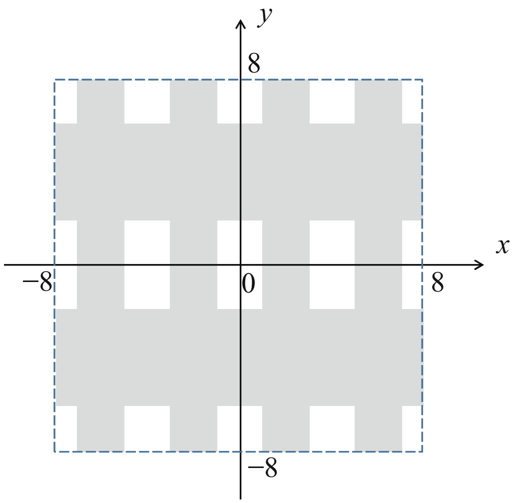

Fig. 14.12 Fragment of the map of periodic elastic foundation; $a = 1.0$ and $b = 2.0$

K and load q are measured per unit membrane tension T, the partial differential equation of equilibrium is represented in the form

$$\Delta u - K(x, y)u = q(x, y) \tag{14.125}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

where $u = u(x, y)$ is the membrane transverse deflection.

The foundation is assumed to be stepwise discontinuous and periodic along each of the coordinates as described by the function

$$K(x, y) = \frac{k}{4}\left[1 + \tau'\left(\frac{x}{a}\right)\right]\left[1 + \tau'\left(\frac{y}{b}\right)\right] \tag{14.126}$$

With reference to Fig. 14.12, function (14.126) is defined on the infinite plane, such that

$$K(x, y) = \begin{cases} 0 \ (x, y) \in \text{any "dark field"} \\ k \ (x, y) \in \text{any "light field"} \end{cases} \tag{14.127}$$

In the same way, Fig. 14.13 provides maps for the elements of basis

$$e_0 = 1$$
$$e_1 = \tau'(x/a)$$
$$e_2 = \tau'(y/b) \tag{14.128}$$

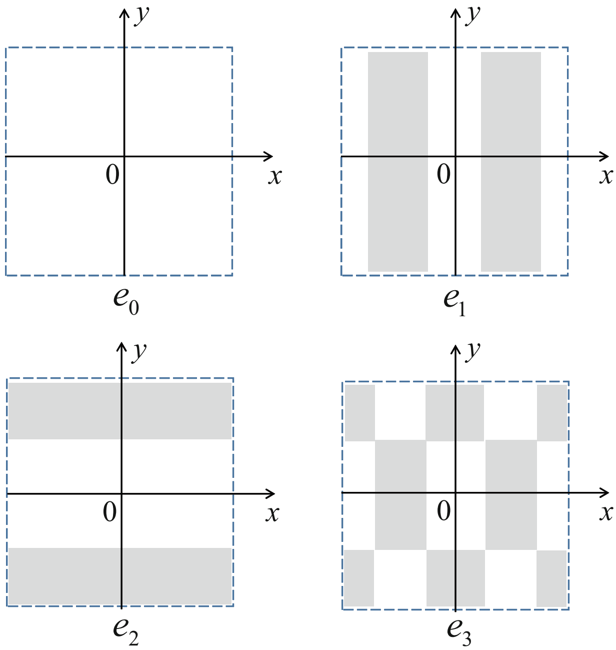

Fig. 14.13 The standard basis map: each of the elements is equal to unity within light domains and zero within dark domains; square areas $[-4 < x < 4, -4 < y < 4]$ are shown under the parameters $a = 1$ and $b = 2$

$$e_3 = e_1 e_2$$

The above table of products (14.121) is still valid for basis (14.128). As a result, function (14.126) takes eventually the form

$$K(x, y) = \frac{k}{4}(e_0 + e_1 + e_2 + e_3) \tag{14.129}$$

Now let us represent the membrane deflection in the form

$$\begin{aligned} u(x, y) = {} & X(\tau_x, \tau_y, x, y)e_0 + Y(\tau_x, \tau_y, x, y)e_1 \\ & + Z(\tau_x, \tau_y, x, y)e_2 + W(\tau_x, \tau_y, x, y)e_3 \end{aligned} \tag{14.130}$$

where $\tau_x = \tau(x/a)$ and $\tau_y = \tau(y/b)$ are triangular waves whose lengths are determined by the periods of foundation along $x-$ and $y-$ direction, respectively; scales of the explicitly present variables, x and y, are associated with the scales of loading $q(x, y)$, which is assumed to be slow as compared to the spatial rate of foundation.

Note that both linear and nonlinear algebraic manipulations with combinations of type (14.130) are dictated by the table of products (14.121). For example, taking into account (14.129) and (14.130) gives

$$Ku = \frac{k}{4}(X + Y + Z + W)(e_0 + e_1 + e_2 + e_3) \tag{14.131}$$

High-order derivatives of (14.130) are simplified by using the table of products (14.121) and introducing specific differential operators as follows.

First, using the chain rule gives

$$\frac{d\tau_x}{dx} = \frac{1}{a}\tau'(x/a) = \frac{1}{a}e_1 \tag{14.132}$$

$$\frac{d\tau_y}{dy} = \frac{1}{b}\tau'(y/a) = \frac{1}{b}e_2 \tag{14.133}$$

Then, taking into account (14.121), (14.128), (14.132), and (14.133) gives first derivatives of (14.130) in the form

$$\begin{aligned}\frac{\partial u}{\partial x} &= \left(\frac{1}{a}\frac{\partial Y}{\partial \tau_x} + \frac{\partial X}{\partial x}\right)e_0 + \left(\frac{1}{a}\frac{\partial X}{\partial \tau_x} + \frac{\partial Y}{\partial x}\right)e_1 \\ &+ \left(\frac{1}{a}\frac{\partial W}{\partial \tau_x} + \frac{\partial Z}{\partial x}\right)e_2 + \left(\frac{1}{a}\frac{\partial Z}{\partial \tau_x} + \frac{\partial W}{\partial x}\right)e_3 \\ &+ \frac{1}{a}(Y + We_2)\frac{de_1(x/a)}{d(x/a)}\end{aligned} \tag{14.134}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \left(\frac{1}{b}\frac{\partial Z}{\partial \tau_y} + \frac{\partial X}{\partial y}\right)e_0 + \left(\frac{1}{b}\frac{\partial W}{\partial \tau_y} + \frac{\partial Y}{\partial y}\right)e_1 \\ &+ \left(\frac{1}{b}\frac{\partial X}{\partial \tau_y} + \frac{\partial Z}{\partial y}\right)e_2 + \left(\frac{1}{b}\frac{\partial Y}{\partial \tau_y} + \frac{\partial W}{\partial y}\right)e_3 \\ &+ \frac{1}{b}(Z + We_1)\frac{de_2(y/b)}{d(y/b)}\end{aligned} \tag{14.135}$$

Last addends in (14.134) and (14.135) include derivatives of the stepwise discontinuous functions $e_1(x/a)$ and $e_2(y/b)$. Such derivatives are expressed through Dirac δ-functions and therefore must be excluded from the expressions (14.134) and (14.135) due to continuity of the original function $u(x, y)$. The δ-functions are eliminated under the boundary conditions

$$\begin{aligned}Y|_{\tau_x=\pm 1} &= 0 \\ W|_{\tau_x=\pm 1} &= 0\end{aligned} \tag{14.136}$$

and

$$Z|_{\tau_y=\pm 1} = 0$$
$$W|_{\tau_y=\pm 1} = 0 \qquad (14.137)$$

The rest of terms in (14.134) and (14.135) represent linear combinations of the basis $\{e_0, e_1, e_2, e_3\}$. In order to formalize the differentiation procedure, let us associate expansion (14.130) with the vector-column

$$\mathbf{u} = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \qquad (14.138)$$

In a similar way, let us introduce the vector-columns $\mathbf{u}_x$ and $\mathbf{u}_y$ associated with derivatives (14.134) and (14.135) under conditions (14.136) and (14.137), respectively,[1]

$$\mathbf{u}'_x = \mathbf{D}_x \mathbf{u} \qquad (14.139)$$
$$\mathbf{u}'_y = \mathbf{D}_y \mathbf{u}$$

where

$$\mathbf{D}_x = \begin{bmatrix} \partial/\partial x & a^{-1}\partial/\partial \tau_x & 0 & 0 \\ a^{-1}\partial/\partial \tau_x & \partial/\partial x & 0 & 0 \\ 0 & 0 & \partial/\partial x & a^{-1}\partial/\partial \tau_x \\ 0 & 0 & a^{-1}\partial/\partial \tau_x & \partial/\partial x \end{bmatrix} \qquad (14.140)$$

and

$$\mathbf{D}_y = \begin{bmatrix} \partial/\partial y & 0 & b^{-1}\partial/\partial \tau_y & 0 \\ 0 & \partial/\partial y & 0 & b^{-1}\partial/\partial \tau_y \\ b^{-1}\partial/\partial \tau_y & 0 & \partial/\partial y & 0 \\ 0 & b^{-1}\partial/\partial \tau_y & 0 & \partial/\partial y \end{bmatrix} \qquad (14.141)$$

These differential matrix operators automatically generate high-order derivatives of combination (14.130) provided that necessary smoothness (boundary) conditions hold. For instance, the components of expansion for Δu are given by the elements of vector-column $(\mathbf{D}_x^2 + \mathbf{D}_y^2)\mathbf{u}$ under conditions

[1] Note that $\mathbf{u}'_x$ is not $\partial \mathbf{u}/\partial x$.

$$\left(\frac{1}{a}\frac{\partial X}{\partial \tau_x}+\frac{\partial Y}{\partial x}\right)|_{\tau_x=\pm 1}=0$$
$$\left(\frac{1}{a}\frac{\partial Z}{\partial \tau_x}+\frac{\partial W}{\partial x}\right)|_{\tau_x=\pm 1}=0 \tag{14.142}$$

$$\left(\frac{1}{b}\frac{\partial X}{\partial \tau_y}+\frac{\partial Z}{\partial y}\right)|_{\tau_y=\pm 1}=0$$
$$\left(\frac{1}{b}\frac{\partial Y}{\partial \tau_y}+\frac{\partial W}{\partial y}\right)|_{\tau_y=\pm 1}=0 \tag{14.143}$$

Consider now the particular case $a=b=\varepsilon \ll 1$. Following the differentiation and algebraic manipulation rules as introduced above, and substituting (14.130) in (14.125), gives

$$\Delta_\tau X+2\varepsilon\left(\frac{\partial^2 Y}{\partial \tau_x \partial x}+\frac{\partial^2 Z}{\partial \tau_y \partial y}\right)+\varepsilon^2(\Delta X-F)=\varepsilon^2 q(x,y)$$
$$\Delta_\tau Y+2\varepsilon\left(\frac{\partial^2 X}{\partial \tau_x \partial x}+\frac{\partial^2 W}{\partial \tau_y \partial y}\right)+\varepsilon^2(\Delta Y-F)=0$$
$$\Delta_\tau Z+2\varepsilon\left(\frac{\partial^2 W}{\partial \tau_x \partial x}+\frac{\partial^2 X}{\partial \tau_y \partial y}\right)+\varepsilon^2(\Delta Z-F)=0 \tag{14.144}$$
$$\Delta_\tau W+2\varepsilon\left(\frac{\partial^2 Z}{\partial \tau_x \partial x}+\frac{\partial^2 Y}{\partial \tau_y \partial y}\right)+\varepsilon^2(\Delta W-F)=0$$

where $\Delta_\tau=\partial^2/\partial\tau_x^2+\partial^2/\partial\tau_y^2$, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, and symbol F denotes the following group of terms related to the elastic foundation

$$F\equiv\frac{1}{4}k(X+Y+Z+W) \tag{14.145}$$

Boundary conditions (14.142) and (14.143) can be simplified due to (14.136) and (14.137). As a result, the complete set of boundary conditions takes the form

$$\frac{\partial X}{\partial \tau_x}|_{\tau_x=\pm 1}=0,\quad \frac{\partial X}{\partial \tau_y}|_{\tau_y=\pm 1}=0$$
$$Y|_{\tau_x=\pm 1}=0,\quad \frac{\partial Y}{\partial \tau_y}|_{\tau_y=\pm 1}=0$$
$$\frac{\partial Z}{\partial \tau_x}|_{\tau_x=\pm 1}=0,\quad Z|_{\tau_y=\pm 1}=0 \tag{14.146}$$
$$W|_{\tau_x=\pm 1}=0,\quad W|_{\tau_y=\pm 1}=0$$

Note that Eq. (14.144) has constant coefficients, whereas the stepwise discontinuities of foundation have been absorbed by the triangular wave arguments τ_x and τ_y. The corresponding boundary conditions (14.146) generate a so-called "cell problem" [100] within the rectangular domain $\{-1 \leq \tau_x \leq 1, -1 \leq \tau_y \leq 1\}$. Therefore, the arguments τ_x and τ_y locally describe the fast varying component of membrane shape. In contrast, the explicitly present coordinates x and y describe the slow component in the infinite plane $\{-\infty < x < \infty, -\infty < y < \infty\}$. Finally, the increase in number of Eqs. (14.144) of-course complicates solution procedures from the technical standpoint. However, the obvious symmetry of equations helps to ease the corresponding derivations. For instance, Eqs. (14.144) can be decoupled by introducing new unknown functions $U_i = U_i(\tau_x, \tau_y, x, y)$ $(i = 1, .., 4)$ as follows

$$\begin{aligned} U_1 &= X + Y + Z + W \\ U_2 &= X + Y - Z - W \\ U_3 &= X - Y + Z - W \\ U4 &= X - Y - Z + W \end{aligned} \tag{14.147}$$

Linear transformation (14.147) however makes boundary conditions (14.146) coupled. New boundary conditions are given by the inverse substitution in (14.146)

$$\begin{aligned} X &= \frac{1}{4}(U_1 + U_2 + U_3 + U_4) \\ Y &= \frac{1}{4}(U_1 + U_2 - U_3 - U_4) \\ Z &= \frac{1}{4}(U_1 - U_2 + U_3 - U_4) \\ W &= \frac{1}{4}(U_1 - U_2 - U_3 + U_4) \end{aligned} \tag{14.148}$$

Note that transformation (14.147) can be effectively incorporated at the very beginning of transformations by using the idempotent basis as described in the next section.

14.8 The Idempotent Basis for Two-Dimensional Structures

The two-dimensional idempotent basis is introduced as follows

$$i_1 = e_1^+ e_2^+ = \frac{1}{4}(e_0 + e_1)(e_0 + e_2) = \frac{1}{4}(e_0 + e_1 + e_2 + e_3)$$

$$i_2 = e_1^+ e_2^- = \frac{1}{4}(e_0 + e_1)(e_0 - e_2) = \frac{1}{4}(e_0 + e_1 - e_2 - e_3)$$
$$i_3 = e_1^- e_2^+ = \frac{1}{4}(e_0 - e_1)(e_0 + e_2) = \frac{1}{4}(e_0 - e_1 + e_2 - e_3) \quad (14.149)$$
$$i_4 = e_1^- e_2^- = \frac{1}{4}(e_0 - e_1)(e_0 - e_2) = \frac{1}{4}(e_0 - e_1 - e_2 + e_3)$$

where the standard basis e_i is defined by (14.128) and the table of products (14.121), and the following notations for one-dimensional idempotent basis are used

$$e_k^{\pm} = \frac{1}{2}(e_0 \pm e_k) \quad (14.150)$$
$$e_k^+ e_k^- = 0; \quad (k = 1, 2)$$

The main reason for using basis (14.149) is that its table of products has the normalized diagonal form

$$i_k i_n = \delta_{kn} \quad (14.151)$$

where δ_{kn} is the Kronecker symbol.

The geometrical meaning of property (14.151) follows from the maps in Fig. 14.14.

In this basis, representation (14.130) takes the form

$$u(x, y) = \sum_{k=1}^{4} U_k(\tau_x, \tau_y, x, y) i_k \quad (14.152)$$

As a result,

$$f\left(\sum_{k=1}^{4} U_k i_k\right) = \sum_{k=1}^{4} f(U_k) i_k \quad (14.153)$$

where f is practically any function, linear or nonlinear.

First-order partial derivatives of representation (14.153) are obtained as follows

$$\frac{\partial u(x, y)}{\partial x} \quad (14.154)$$
$$= \sum_{k=1}^{4} \left[\frac{1}{a} \frac{\partial U_k(\tau_x, \tau_y, x, y)}{\partial \tau_x} e_1 i_k + \frac{\partial U_k(\tau_x, \tau_y, x, y)}{\partial x} i_k + U_k(\tau_x, \tau_y, x, y) \frac{\partial i_k}{\partial x} \right]$$

Further, taking into account (14.149) and (14.150), gives

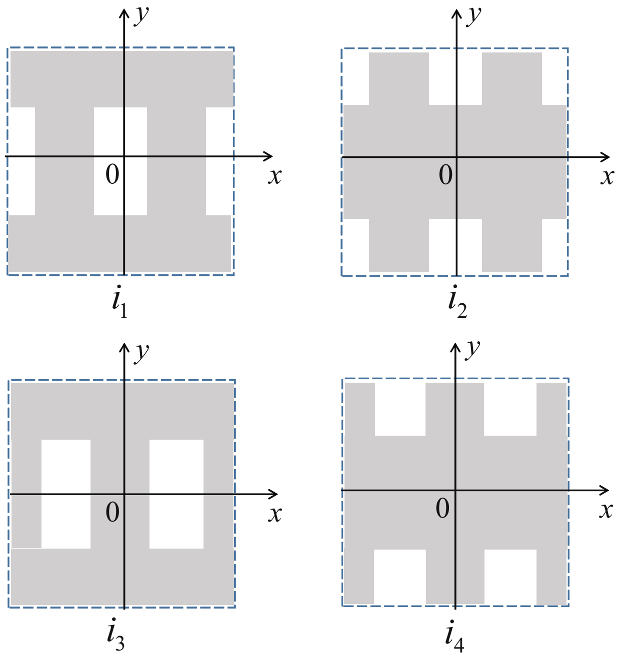

Fig. 14.14 The map of idempotent basis: each of the elements is equal to unity within light domains and zero within dark domains; square areas $[-4 < x < 4, -4 < y < 4]$ are shown under the parameters $a = 1$ and $b = 2$

$$
\begin{aligned}
e_1 i_1 &= (e_1^+ - e_1^-)e_1^+ e_2^+ = i_1 \\
e_1 i_2 &= (e_1^+ - e_1^-)e_1^+ e_2^- = i_2 \\
e_1 i_3 &= (e_1^+ - e_1^-)e_1^- e_2^+ = -i_3 \\
e_1 i_4 &= (e_1^+ - e_1^-)e_1^- e_2^- = -i_4
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial i_1}{\partial x} &= \frac{\partial e_1^+}{\partial x} e_2^+ = \frac{1}{2}\frac{\partial e_1}{\partial x} e_2^+ \\
\frac{\partial i_2}{\partial x} &= \frac{\partial e_1^+}{\partial x} e_2^- = \frac{1}{2}\frac{\partial e_1}{\partial x} e_2^- \\
\frac{\partial i_3}{\partial x} &= \frac{\partial e_1^-}{\partial x} e_2^+ = -\frac{1}{2}\frac{\partial e_1}{\partial x} e_2^+ \\
\frac{\partial i_4}{\partial x} &= \frac{\partial e_1^-}{\partial x} e_2^- = -\frac{1}{2}\frac{\partial e_1}{\partial x} e_2^-
\end{aligned}
$$

As a result, derivative (14.154) takes the form

$$\begin{aligned}\frac{\partial u}{\partial x} &= \left(\frac{1}{a}\frac{\partial U_1}{\partial \tau_x} + \frac{\partial U_1}{\partial x}\right) i_1 + \left(\frac{1}{a}\frac{\partial U_2}{\partial \tau_x} + \frac{\partial U_2}{\partial x}\right) i_2 \\ &+ \left(-\frac{1}{a}\frac{\partial U_3}{\partial \tau_x} + \frac{\partial U_3}{\partial x}\right) i_3 + \left(-\frac{1}{a}\frac{\partial U_4}{\partial \tau_x} + \frac{\partial U_4}{\partial x}\right) i_4 \\ &+ \frac{1}{2}(U_1 - U_3)\frac{\partial e_1}{\partial x} e_2^+ + \frac{1}{2}(U_2 - U_4)\frac{\partial e_1}{\partial x} e_2^-\end{aligned} \tag{14.155}$$

Analogously, one obtains

$$\begin{aligned}\frac{\partial u}{\partial y} &= \left(\frac{1}{b}\frac{\partial U_1}{\partial \tau_y} + \frac{\partial U_1}{\partial y}\right) i_1 + \left(-\frac{1}{b}\frac{\partial U_2}{\partial \tau_y} + \frac{\partial U_2}{\partial y}\right) i_2 \\ &+ \left(\frac{1}{b}\frac{\partial U_3}{\partial \tau_y} + \frac{\partial U_3}{\partial y}\right) i_3 + \left(-\frac{1}{b}\frac{\partial U_4}{\partial \tau_y} + \frac{\partial U_4}{\partial y}\right) i_4 \\ &+ \frac{1}{2}(U_1 - U_2)\frac{\partial e_2}{\partial y} e_1^+ + \frac{1}{2}(U_3 - U_4)\frac{\partial e_2}{\partial y} e_1^-\end{aligned} \tag{14.156}$$

Let us introduce vector, associated with expansion (14.152), and the corresponding differential matrix operators as, respectively,

$$\mathbf{u} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \tag{14.157}$$

and

$$\mathbf{D}_x = \begin{bmatrix} \frac{1}{a}\frac{\partial}{\partial \tau_x} + \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{1}{a}\frac{\partial}{\partial \tau_x} + \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{1}{a}\frac{\partial}{\partial \tau_x} + \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & -\frac{1}{a}\frac{\partial}{\partial \tau_x} + \frac{\partial}{\partial x} \end{bmatrix} \tag{14.158}$$

$$\mathbf{D}_y = \begin{bmatrix} \frac{1}{b}\frac{\partial}{\partial \tau_y} + \frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & -\frac{1}{b}\frac{\partial}{\partial \tau_y} + \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{1}{b}\frac{\partial}{\partial \tau_y} + \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -\frac{1}{b}\frac{\partial}{\partial \tau_y} + \frac{\partial}{\partial y} \end{bmatrix} \tag{14.159}$$

Substituting (14.152) into the original Eq. (14.125), using the differentiation rules for idempotent basis, and assuming that $a = b = \varepsilon$, gives

$$\begin{aligned}
&\Delta_\tau U_1 + 2\varepsilon\left(\frac{\partial^2 U_1}{\partial\tau_x\partial x} + \frac{\partial^2 U_1}{\partial\tau_y\partial y}\right) + \varepsilon^2\Delta U_1 = \varepsilon^2[q(x,y) + kU_1] \\
&\Delta_\tau U_2 + 2\varepsilon\left(\frac{\partial^2 U_2}{\partial\tau_x\partial x} - \frac{\partial^2 U_2}{\partial\tau_y\partial y}\right) + \varepsilon^2\Delta U_2 = \varepsilon^2 q(x,y) \\
&\Delta_\tau U_3 - 2\varepsilon\left(\frac{\partial^2 U_3}{\partial\tau_x\partial x} - \frac{\partial^2 U_3}{\partial\tau_y\partial y}\right) + \varepsilon^2\Delta U_3 = \varepsilon^2 q(x,y) \\
&\Delta_\tau U_4 - 2\varepsilon\left(\frac{\partial^2 U_4}{\partial\tau_x\partial x} + \frac{\partial^2 U_4}{\partial\tau_y\partial y}\right) + \varepsilon^2\Delta U_4 = \varepsilon^2 q(x,y)
\end{aligned} \tag{14.160}$$

where the notations Δ_τ and Δ have the same meaning as those in Eqs. (14.144).

Equations (14.160) are decoupled, at cost of coupling the boundary conditions though

$$\begin{aligned}
&\frac{\partial(U_1 - U_3)}{\partial\tau_x}|_{\tau_x=\pm1} = 0, \quad \frac{\partial(U_1 - U_2)}{\partial\tau_y}|_{\tau_y=\pm1} = 0 \\
&(U_1 - U_3)|_{\tau_x=\pm1} = 0, \quad (U_1 - U_2)|_{\tau_y=\pm1} = 0 \\
&\frac{\partial(U_2 - U_4)}{\partial\tau_x}|_{\tau_x=\pm1} = 0, \quad \frac{\partial(U_3 - U_4)}{\partial\tau_y}|_{\tau_y=\pm1} = 0 \\
&(U_2 - U_4)|_{\tau_x=\pm1} = 0, \quad (U_3 - U_4)|_{\tau_y=\pm1} = 0
\end{aligned} \tag{14.161}$$

Note that both boundary value problems (14.144) through (14.146) and (14.160) through (14.161) implement the transition from two to four spatial arguments: $\{x, y\} \to \{\tau_x, \tau_y, x, y\}$. The arguments τ_x and τ_y naturally relate to cell problems and incorporate the corresponding elastic components within the class of closed-form solutions.

Finally, let us introduce two-dimensional idempotent basis generated by the triangular asymmetric wave; see Fig. 14.15. First, following definitions of Chap. 4, let us introduce one-dimensional idempotents associated with x and y coordinates

$$\begin{aligned}
e_i^+ &= \frac{1}{2}\left[1 - \gamma_i + \left(1 - \gamma_i^2\right)e_i\right] \\
e_i^- &= \frac{1}{2}\left[1 + \gamma_i - \left(1 - \gamma_i^2\right)e_i\right] \\
&(i = 1, 2)
\end{aligned} \tag{14.162}$$

where $e_1 = \partial\tau(x/a, \gamma_1)/\partial(x/a)$ and $e_2 = \partial\tau(y/b, \gamma_2)/\partial(y/b)$.

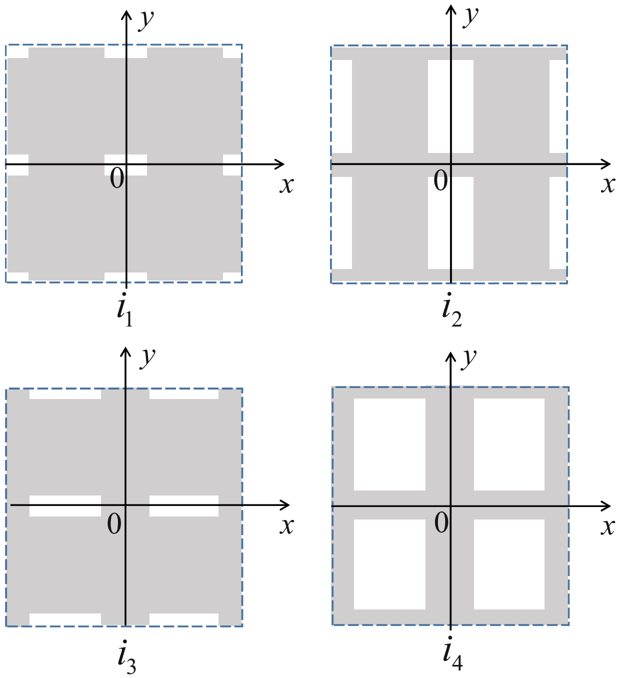

Fig. 14.15 The map of idempotent basis generated by the asymmetric triangular waves with parameters: $a = b = 1.0$, $\gamma_1 = 0.2$, and $\gamma_2 = 0.6$; each of the elements is equal to unity within light domains and zero within dark domains; square areas $[-4 < x < 4, -4 < y < 4]$ are shown

Now the two-dimensional idempotent basis is given by $i_1 = e_1^+ e_2^+$, $i_2 = e_1^+ e_2^-$, $i_3 = e_1^- e_2^+$, and $i_4 = e_1^- e_2^-$, although further expansions shown in (14.149) are not valid any more.