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Results from the POEM5 Conference, 2022

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Editorial

The POEM conference – A Mathematics Education **P**erspective **o**n **E**arly **M**athematics Learning Between the Poles of Instruction and Construction – has become a strong tradition since the first one was held in 2012. Every other year it brings together researchers within the field of early childhood mathematics education to exchange current research findings and to collaborate and discuss new ideas. After a break due to the Covid-19 pandemic, the fifth edition of the conference was held in Gothenburg, Sweden, in May 2022. The conference gathered more than 60 researchers and included 24 presentations. The theme of the conference was "Teaching mathematics as to be meaningful – foregrounding children's play and perspectives". To give inspiration and depth to this theme, professors Bert van Oers and Ingrid Pramling Samuelsson were invited as keynote speakers.

Bert van Oers (Vrije Universiteit Amsterdam) gave a lecture on the development of mathematical thinking in young children's play, focusing particularly on the role of communicative tools. He has been researching this topic over the past two decades and developing theory for the early childhood mathematics education community. In his paper, he summarizes some of the main findings of this research programme on emergent mathematical thinking in young children. The research programme is based on a cultural-historical activity theory perspective (CHAT) and focuses on identifying the productive conditions that may promote mathematical thinking in young children (4–8 years old). He lists several conditions (play format, productive dialogue, schematizing, narrative competence and intertextuality) that have been empirically shown to promote meaningful mathematical thinking in young children, helping them communicate about mathematical objects.

Ingrid Pramling Samuelsson (University of Gothenburg) shared her view on the notions of "Child Perspective" and "Children's Perspectives" in mathematics learning in the early years. In her lecture, she asked if the two views of child perspective and children's perspectives can be of help in teaching young children mathematics. With more than 40 years of research in the field of early childhood education, Pramling Samuelsson promotes children's perspectives as key in all education activities. In early childhood education, there are several other recurring notions that are central, such as "listening to children" and, not least, the notion of play, but

Pramling Samuelsson raises our awareness as to whether these become merely rhetoric or actual parts of praxis. In teaching according to Developmental Pedagogy, this comes through as a joint venture between children and teachers, in which the teachers know the direction for learning to point out to children, why the children's experiences and ways of talking about a content are key objectives for the teacher to challenge for the children's further development of understanding.

The presentations of both keynote speakers address issues that have been put at the centre of their research agenda for a long time, providing the early childhood mathematics education research community valuable insights for the development of theory and practice. The many scholars they have collaborated with and the research initiatives originating from their work are amply reflected in the research presented at the POEM conference and in these proceedings.

The overarching theme of this year's POEM conference is also well reflected in the papers presented at the conference and in these proceedings. We recognize that all parts of the theme "Teaching mathematics as to be meaningful – foregrounding children's play and perspectives" can be found in the relation between teachers, children and the mathematical content, and this is further framed within the context of play. Four sub-themes appear in these proceedings: *Play and learning, Children's perspectives on mathematics, Teachers' competencies* and *Theorizing aspects of early mathematics education*.

Play and Learning

Globally, most guidelines and curricula for early childhood education mention play as one of the key features for young children's learning. Still, there are quite different views on the definitions of play (and on whether it is even possible to define play) and in what ways play should become part of children's learning. We chose to emphasize play in the theme for this year's POEM conference, and three of the papers presented specifically focus on play. The broad spectrum of how to understand play and learning is reflected in these papers.

Amrar, Clerc-Georgy and Dorier study pretend play (playing hospital in the classroom) and point to the close link found between pretend play and mathematics, as both are cultural and semiotic activities. They particularly discuss the need to dialectically scaffold play and mathematical thinking where the teacher guides children through the reflective process. The other two papers within this sub-theme focus on digital play and learning of mathematics. During the last two years of the pandemic and lockdown in many countries, digitalization has become a familiar part of education, including in the early years, as a means to uphold education standards. But even before covid restrictions, the notion of digitalization appeared in curricula and guidelines for early childhood education and mathematics teaching. Christiansen raises the question of whether digital tools' entrance into kindergarten might entail a risk of changing children's pedagogical environment to one with less focus on children's play. She addresses this question in a study of 1-year-olds' interaction with three applications that provide the children with a playful environment. The kindergarten teacher in the study is shown to seize the opportunity that arises when the child faces digital challenges and uses it as an opportunity to discuss specific mathematical objects. A similar interest in research is found in the paper by Birklein and Steinweg. They take play as a framework for learning a step further in a study comparing differences in free versus supervised use of an application developed to support children's mathematical competencies; more specifically, they examine how the supervision in the implementation influences the children's behaviour and their progress through the game. The common assumption that a lack of limitations and instructions will inevitably lead to excessive media consumption is shown not to be true in their study. This highlights the need for more research taking different approaches, including both explorative and designed studies, in order to deepen our understanding of play and learning as intertwined phenomena.

In these three papers where play is foregrounded, whether in a digital or physical environment, the authors conclude that the teacher has a key role as a mediator of mathematical objects because the play environments do not give sufficient support in themselves for children exploring new mathematical meaning.

Children's Perspectives on Mathematics

One way to foreground children's perspectives is to focus on how children understand and deal with different mathematical ideas. This is in focus in four of the papers.

Eriksson, Hedefalk and Sumpter investigate the tension between division and fair share when 5-year-olds are faced with different sharing tasks, varying the sizes of dividend and divisor and sharing equal parts (groups) and unequal parts. Results of their observations show that children use different strategies when performing division. The study also shows how children use ethical reasoning claims to convince their peers in the sharing tasks rather than mathematical reasoning. The other three papers within this sub-theme are focused more specifically on problem solving as a mathematical activity. In the first, Vogler, Henschen and Teschner focus on the different heuristics emerging in peer interactions in block-play situations and on the extent to which the interactions and collective problem solving create conditions for mathematical learning. They find that children use various heuristic procedures in their interactions during block play such as decomposing into sub-problems, systematic guessing and testing and working backwards when working on problems in block-play situations. The paper by Meaney, Severina, Gustavsen, Hoven and Larsen focuses on children's mathematical and computational thinking. Problems to be solved are identified through children's actions, or rather observed signs of uncertainty, which guide the authors in their study of what mathematical understanding children use when engaging in problem solving with robots in naturalistic settings. By focusing on the children's uncertainty, the authors identify problems from the children's perspective rather than against predetermined outcomes. The authors show how children's different understandings about counting contribute to their abilities to solve these problems. Finally, Palmér and van Bommel focus on how young children explain their choice of representation when working on a problem-solving task on combinatorics and how the representations used are compatible with children's understanding of abstraction. The results indicate that difficulties in representing the context of the problem-solving task may force some of the children to work with a representation on a level of abstraction not suitable for them. This in turn influences how they manage to solve the problem-solving task.

Altogether, these four papers contribute to the field of early mathematics education research from the perspective of the learner, which in this research domain includes young children to whom mathematical concepts and skills are novel. The children's perspective should thereby be significant to further the understanding of children learning mathematics.

Teachers' Competencies

Five of the papers focused on the competencies of early childhood education teachers. Three of the papers direct attention to teaching skills and what informs educational choices. In their paper, Carlsen, Erfjord and Hundeland present a modification to a framework "Knowledge Quartet" developed in a school context, and they suggest that this framework could be used as an analytical tool to investigate the mathematical competence of kindergarten teachers. The framework is trialled on observations from a case study where a mathematical activity includes children's guided inquiries into features of two-dimensional geometrical shapes. A broader approach to teacher competence and, in particular, factors that may influence educational choices, is found in the large-scale interview study by Torbeyns, Verbruggen and Depaepe, who investigate early childhood education teachers' use of educational technology in mathematics education and its association with school and teacher characteristics. They find a broad variety of factors that influence the use of technology and that teachers need to have well-developed ICT competencies, both digital skills and pedagogical-technological competencies, to enable constructive mathematical learning processes. Beck and Vogler empirically investigate whether teachers being trained in "design patterns of mathematical situations" have a favorable influence on the responsiveness in the interactions between teachers and children. The results indicate that the implementation is a promising form of preparation for mathematically rich and responsive interactions.

The other two papers within this sub-theme concern teachers' assessment competence regarding children's knowledge and skills. In their paper, Benz, Reuter, Maier and Zöllner study in-service kindergarten teachers and their selection of and reflection on suitable diagnostic situations and diagnostic tools. Diagnosing is considered one main facet of teachers' professional competence, generating information about children's understanding and eliciting students' cognitive skills. The participating teachers express positive effects of implementing adaptive learning support, that is, showing advanced professional competence. In *Rinvold and Skorpen's* paper, the focus is on kindergarten teachers' play-responsive assessment of children's mathematical proficiencies. They found that situations reflecting real-life activities where children are given a large amount of freedom are important assets in the assessment planning process. This is discussed both in play-responsive participation in children's self-initiated play and when the kindergarten teachers organize activities for the sake of observation.

Similar to the papers in the first sub-theme, "Play and learning", these four papers emphasize the key role of the teacher as the mediator for children's opportunity to learn mathematics.

Theorizing Aspects of Early Mathematics Education

To move the field of early childhood mathematics education forward, and to legitimate it as a part of the broader mathematics education research area, there are also attempts to theorize aspects of mathematics learning and teaching as well as to develop the methodologies within this research area. The five studies within this sub-theme are based both in different theoretical frameworks and in empirical observations. For example, Reikerås investigates the relation between language and mathematics skills observed in 2-year-olds' play and everyday activities, a relation that is significant to gain a better understanding of in the thematic and play-oriented early childhood education. The study is large scale and thus may provide statistical correlations, something that qualitative studies (which make up most of the contributions) cannot do.

Some of the papers take a specific interest in unpacking theoretical concepts such as multimodality and mathematizing in relation to empirical observations. In the paper by Billion and Huth, the starting point is empirical findings from a case study of a 4-year-old boy's mathematical play with wooden figures while he interacts with three peers and a teacher. The situation was an open offer for the learners, who were free to address several different mathematical ideas. The study gives rise to potentially rethinking the notion of multimodality as not always an interwoven but sometimes a phased-wise parallel proceeding construction in mathematical interaction.

Two papers take up the notion of mathematizing, an expression introduced by the Dutch mathematician Hans Freudenthal. Palmér and Björklund focus on mathematizing and, in particular, on how to understand the "real world" of very young children, which significantly contributes to mathematizing taking place. The results show that "real world" can involve both imagination and play, but mathematization is only noticed when there is a problem that, from the perspective of the child, needs to be solved. Björklund and Elofsson bring in the notion of playfulness as part of mathematics teaching as a motor in the mathematizing process. Their paper discusses the significance of playfulness as a feature of mathematics teaching in early education, and particularly brings to the fore nuances in different ways of making playfulness an asset in teaching.

Finally, Lembrér focuses on methodological choices in research on early mathematics education, discussing two different data collection methods used when investigating parents' views on mathematics education for young children. She found that the different narratives produced were affected by the ways the data were collected and gave different insights into parents' views on early mathematics education. Such inquiries of methodological outcomes are of course valuable to all future studies involving people of all ages.

The five papers in this sub-theme are of importance to moving the field of early childhood mathematics education forward, not only empirically but also theoretically and methodologically.

Closing the Fifth POEM Conference

As mentioned in the introduction and as is apparent in the presentation of papers described above, the theme of the conference is recognized in all studies because they all concern, in one way or another, the relation between teachers, children and the mathematical content. A special focus of this year's conference was the framing of these relations within the context of play. We found this theme important in relation to the debate on whether teaching should be integrated with or separated from children's play, a debate we find to be contradictory as both teaching and play are highlighted as central in the curricula of preschool education in most countries. Thus, within the context of play, we find some papers putting the teaching of mathematics to young children as the essential issue of discussion. This means research in early childhood mathematics education has a strong ambition to facilitate mathematical learning in ways that adhere to the children's experiences and lived world. Other papers focus on meaningfulness in the learning process, particularly from the child's perspective. Meaningfulness indeed appears in the presented research as an essential feature for learning, but by emphasizing play, as in some of the papers, this becomes even more evident. One thing we noticed in the papers of this year's conference is that more papers than before include toddlers. As early mathematics in some countries are limited to children over the age of 3, we wish to stress the importance of POEM also making visible the significance of mathematics education with toddlers.

Even though the papers described above are presented under the four sub-themes of *Play and learning, Children's perspectives on mathematics, Teachers' competencies* and *Theorizing aspects of early mathematics education*, these themes are visible in all of the papers. The division into sub-themes is based on what is being foregrounded in the papers, but a focus on, for example, teachers is not possible without also including content and children and vice versa. The papers, together with the keynote lectures, move the field of early childhood mathematics education forward – empirically, theoretically and methodologically. We are eagerly looking forward to forthcoming POEM conferences to see where this young but strongly up-and-coming area in mathematics education research is heading.

Contents

The Development of Mathematical Thinking in Young Children's Play: The Role of Communicative Tools

Bert van Oers

On the History of Developmental Education: The Case of Mathematical Thinking

In the Netherlands we have been working since the 1980s at the implementation of a curriculum for primary education, called *Developmental Education*. This curriculum aims at a broad identity development of pupils and teachers and requires that all learning should be culturally meaningful and make personal sense for the learners. In this way, we (i.e., the teachers, teacher trainers, researchers) aim at enabling pupils to become agentive critical participants in all kinds of cultural practices. *Developmental Education* is not a mandatory curriculum in the Netherlands: every school is free to opt for this approach or not, and can get funding from the government for its implementation and professionalization of its teachers.

Developmental Education is based on Vygotskij's cultural-historical theory of human development, elaborated with Leont'ev's Activity Theory and El'konin's theory of child development (El'konin, 1978; Leont'ev, 1978; van Oers, 2012a; Vygotsky, 1978). On the basis of this Cultural-Historical Activity Theory (CHAT), it is assumed that higher psychological functions emerge from meaningful interactions with adults or more knowledgeable peers. Rather than being naturally emerging and waiting for maturation stimulated by experiences with the outside world, psychological functions develop in a process of a person's enculturation that starts out from participation and communication with more knowledgeable others in cultural practices that make sense for them, such as household, hospital, supermarket, museum, post-office, library, artist studio, gardening, etc. Due to this meaningfulness, children are mostly interested to be engaged in these well-known activities,

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and are willing to acquire new actions with new cultural means (like instrumental tools or concepts, etcetera) to make it look 'like the real world' for them. Participation in real life cultural practices always requires the adoption of a specific role within that practice, and eventually it calls for the appropriation of the tools that are constitutive for this role, like for instance a stethoscope for a doctor, measuring devices and numbers for a gardener or carpenter, symbolic representations for mathematicians. Tools are unavoidable for enacting a role in a cultural practice. As Vygotskij has pointed out, language is an essential tool for executing context-based actions and cognitive functions and for communicating with others and oneself about tool use in specific situations (e.g., Vygotskij, 1982/1934).

In this article, I want to focus on the emergence of mathematical thinking as a meaningful activity, that is to say as a cultural activity that has cultural relevance and also makes sense for the learner. Studying the genesis of mathematical thinking (as in this article) first of all implies that we have to explain how 'mathematics' enters into the child's mind in the first place. In one of his books on child development, Vygotskij (1984, pp. 226–227) explains this in general terms as steps in the process of development of children's consciousness. Freely summarized here in terms of an emerging mathematical consciousness, this process can also be sketched as a result of the interaction of a child and an adult. As a first step, the child is **acting spontaneously** with objects in his environment, often on the basis of exploratory needs but also on the basis of imitations of significant others. In this stage the children may be reciting number words, but this action is not yet 'mathematical' for them, although it may look like that from an adult perspective (Munn, 1997, p. 16). These 'counting'- acts and included objects at first do not have specific intellectual meanings for the child beyond the fact that she/he wants to manipulate the objects and by so doing look like the surrounding adults. The observing adult then may **interpret this** in terms of her/his own cultural cognitive system and (for example) mention the name of the object or of the act carried out on this object. This is the second step: the act and object by themselves are then turned into a shared topic that can be enriched by the adult by adding new meanings ('predicates'). This is the moment that mathematical qualifications enter the consciousness of the child. The child hears and may repeat 'ten', 'count', 'number', or 'more' etcetera. Finally, in due time, the child may begin **using these new meanings for her/himself** and for communication with others. This third step is the real starting point of genuine mathematical thinking *of* the child ('mathematising').

From the perspective of the cultural-historical activity theory, we conceive of mathematising as a motivated cultural activity with specific tools and goal-oriented actions to solve problems that participants in this activity encounter. However, in our own observations in primary schools, we discovered that the goals for the use of mathematical tools (like measuring, adding, subtracting, etc.) could be of different kinds: first children used mathematical tools for solving problems in the context of everyday settings and cultural practices (like: finding out how much to pay in the shoe shop, or determining number of blocks one needs for building a wall around a castle). We can call this *extra-mathematical* use of tools. On the other hand, when growing older the children sometimes also become interested in the structure of the mathematical tools and the operations themselves, and start using the tools they have acquired for solving problems of the activity of mathematising itself, like 10 year old children who explored the structure of the number line and discovering (with the help of the teacher) even and uneven, the features of '10', or even what happens when we go endlessly to the left on the number line and arrive 'under zero'? We call these *intramathematical* reasoning. For young children (until eight), however, we never witnessed spontaneous intramathematical reasoning in the classroom. Their involvement with mathematics was only extramathematical: the use of mathematical tools for exploring concrete situations and for the achievement of practical goals in the cultural activities they play. This is the type of activity I will focus on in this article.

From these general theoretical assumptions about the enculturation of children in school, I have studied with my colleagues from the *Vrije Universiteit Amsterdam* (VU University Amsterdam) the development of cultural abilities like literacy and mathematising from preschool on. In this article I will review some outcomes of our research program on the development of mathematical thinking in 4–8 years old children over the past decades. This research program promotes research that is at the same time essentially theory-driven, evidence-based, and practically implemented. Methodologically, this means that we are not satisfied with positive empirical evidence alone, but also examine how this empirical result can be conceptually explained in terms of our CHAT theoretical framework, and how this outcome can be implemented by teachers in their everyday classroom. Hence, an important part of this research is accomplished in collaboration with teachers in their classrooms as long term quasi-experimental research or case-studies. The implementation process is most of the time also guided by teacher trainers who are expert in the (theoretical background of) *Developmental Education* (van Oers, 2012a, 2013a; van Oers & Pompert, 2021).

Productive Conditions

Over the past decades, we have discovered and researched a number of conditions that have turned out to be productive in promoting some form of mathematical thinking in young children. Such productive conditions (both interactional and situational) are assumed to be effective for the promotion of mathematical thinking, as they call for specific actions or imply affordances for such actions. From the CHAT perspective on human practical, perceptual, verbal and mental actions, we assume that all these actions are object-oriented, goal-directed and tool-based. This latter implies that all mathematical actions require the use of symbols (tools) that can be used to discover (intramathematical or extramathematical) new knowledge of a situation or – more often than not – the use of the operations that can be carried out on these symbols. The organized system of mathematical actions is called a mathematical activity, or just 'mathematising' (using a term from Freudenthal, 1973, p. 134). In an earlier overview of our research on young children's mathematical activity (van Oers, 2014[1]), I circumscribed 'mathematising' as 'the activity of producing structured objects that allow further elaborations in mathematical terms through problem solving and (collective) reasoning/argumentation' (van Oers, 2014, p. 112). The relevant objects children find in their direct environment can be collections of things, situations, patterns of events, and the like. In the above-mentioned 2014-article, I also argued that mathematising is a complex cultural practice which can only be accessed by newcomers (like young children) when they are allowed to play it. In play, children know and want to follow some of the *rules* of this practice, children are *involved* participants (i.e., they participate on their own willful accord), and enjoy some *degrees of freedom* in the choice of object, rules and tools. An essential condition to safeguard the mathematical nature of the activity, is the co-participation of an adult or more knowledgeable peer in this activity, provided she/he does not disturb the qualities of play (rules, degrees of freedom, involvement) (van Oers, 2004, 2012b).

A seminal insight of Vygotskij is his conception of social origin of psychological functions (including mathematical thinking). Psychological functions show their social origin in the fact that they maintain their communicative function when they are verbalized, and explain to the Self or someone else what the thinker has in mind. But even when they are internalized, we may theoretically assume that this inner speech is preserved in moments of orientation and control or evaluation (see for example Vygotskij, 1982/1934). It was Gal'perin in particular who elaborated this idea in a cohesive argument and empirical research (see Gal'perin, 1969, 1976).

In our research program, the above described tenets turned out to be powerful ideas to lead our research. Van Houten et al. (2013) demonstrated a positive correlation between children's narrative competence (as measured by a standardized test) and their abilities in early arithmetic. This correlation may be caused by intermediating factors like sociocultural background and cannot by itself be interpreted as proof for a causal relationship between arithmetic and narrative competence. Nevertheless, it strongly suggests that the ability to communicate coherently about aspects of a situation (including mathematics) is a core factor in both types of competences. In another study of mathematical thinking in young children in primary school, we could further support this assumption about the importance of communicating in the process of promoting mathematical thinking (van Oers, 2013b). Characteristically, this thinking is based on 'connected discourse' (Luria, 1969, p. 138*ff*). That is to say, according to Luria, it shows that coherence is linked to the actual speech situation, and is syntagmatic, i.e., combines different terms into a meaningful proposition (Luria, 1969). In different previous studies we found that children like to use drawings in their play and integrate these with symbols representing number or quantity(−change), often to make sure to others that their drawing communicates extra information about the drawing and the context of its use. The use of drawings (and schemes) combined with symbols contributes to a

[1] Based on a presentation at the POEM 2012 conference in Frankfurt: Mathematics Education **P**erspective **o**n **e**arly **M**athematics – Learning between the Poles of Instruction and Construction

further mathematisation of young children's language (van Oers, 2002). As a matter of fact, these studies convergently support the productive role of connected discourse for the formation of emergent mathematical thinking.

Pondering further on the idea of communicative activity as a way to promote coherent mathematical thinking in children, we also wondered whether the use of more structured communicative tools (like schemes) could further support young children's thinking on mathematical objects, like number, quantity, addition or subtraction, spatial relations etc. For the study of this question, we guided teachers in early years classroom (5–6 year olds) to participate in children's play activities and introduce useful schematic representations (schemes like ground plan, construction plan, diagram) that were recognized by the children as interesting and helpful for solving problems in their activities (see for instance, van Oers, 1994, 1996). In their shoe-shop in the classroom, children for example piled up different sizes of shoes and copied this contraption by drawing a kind of histogram (see Fig. 1a); the teacher also offered them a so-called "feet measuring thing" (see Fig. 1b), which was appreciated by the children as a handy thing to figure out the size of their and the teacher's shoes. In our studies we always took care that the presented schemes made sense for the children, were linked to a current and meaningful activity of the children, and were integrated into their own speech.

In one of our further studies on this topic we conducted a longitudinal, quasi-experimental study with a pretest-post-test control group design (N = 133). In a 5-year olds classroom the teachers provide different types of schemas that fitted in the current activities of the children's play. During the whole year the children had a variety of schematizing experiences (reading schemes, construction schemes, action plans). In the next year these children started their first mathematics lessons and their scores on mathematics tests (Post-test and delayed post-test) were compared to a randomly chosen control group of the same age that took the same tests. Our experimental groups scored significantly better on post-tests (both for arithmetic and schematizing) than the control group which had not got any experience with schematizing (Poland, 2007; Poland & van Oers, 2007; Poland et al., 2009; van Oers & Poland, 2007). However, in a delayed test after one-and-a-half year the

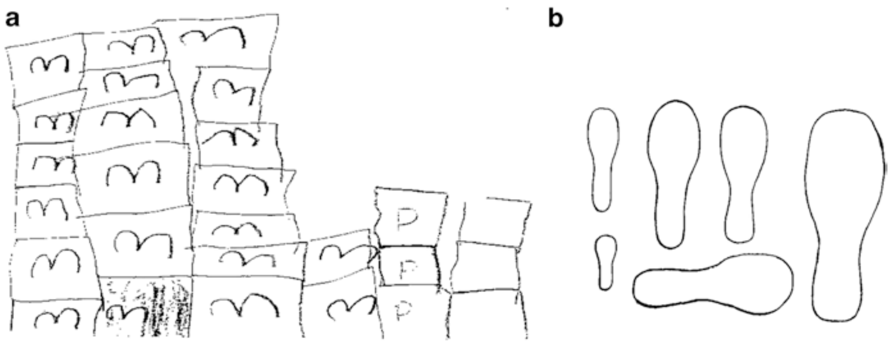

Fig. 1 (**a**) histogram of shoe boxes, (**b**) "feet measuring thing"

children's scores on an arithmetic test were not significantly different anymore. Obviously, good arithmetic scores can also be achieved by drill and practice.

In sum, in a series of theoretically connected studies on children's manifestations of acts that we as adults could identify as productive conditions that could be linked to the meaningful promotion of mathematical thinking in the context of children's play: playful enactment of the use of numbers and quantities, the use of symbolic tools like schematic representations, narrative competence, presence of an helpful adult or more knowledgeable peers for learning how to communicate about number, quantity and change. However, most of the studies mentioned above were carried out in classrooms with young children in the age of 5–8. We now have reasons to assume that mathematical (co-)thinking starts earlier than at the age of 5 years (see for example Carruthers & Worthington, 2006; Worthington, 2021; Worthington & van Oers, 2017).

When and How Do We Start?

In the educational literature, it is an empirically well-established fact, that children spontaneously engage during everyday situations with symbols, patterns and objects that are generally recognized in our culture as mathematical, long before they enter formal schooling in arithmetic (McMullen et al., 2019; Ramani et al., 2015; Rathé et al., 2016a, b; Wijns et al., 2020). The children (4–5 years old) encounter these symbols, etc. in their cultural environments: on the streets (e.g., numbers on cars, busses), supermarket (price list), in the newspapers, picture books etc. Children notice these numbers without directions from adults (see Rathé et al., 2022).

Despite the empirical evidence for young children's awareness of numerocity and numbers, a more detailed description of the course of this evolution from spontaneous use of notions of numerocity and number to number concepts and operations, remained unspecified. My basic hypothesis was that the previously described Vygotskian three step approach to the development of mathematical thinking may be applicable in the younger ages too. The first interest in exploring the tenability of this hypothesis was to figure out how adults could help younger children improving their communicative ability regarding numerosity, number and changes in this intellectual domain, without direct instructions and/or impairing the quality of children's spontaneous play.

In our own research group, it was Maulfry Worthington (UK), who addressed this problem with a well-balanced series of studies into the emergence and development of young children's personal mathematical inscriptions in the context of their spontaneous play. This work was rooted in the work that she did before in collaboration with Carruthers in the UK (see Carruthers & Worthington, 2005, 2006). In their collaborative previous work, these researchers aimed at discovering and describing the range of these mathematical marks from early play exploration to later written calculations. Worthington's research[2] consisted of longitudinal (1 year), ethnographic research in case-studies taken from the contexts of children's homes and nursery classrooms, particularly formatted as pretend play (see Worthington, 2021). In these play contexts, she observed how these young children built up their ability to *communicate* about quantity and number and to *construct* the types of graphical means they used. Her data collection was based on interviews with children, teachers and parents, and on participatory observations. The analyses were mainly qualitative, but in some cases data could be categorized into different classes of graphical means which permitted quantitative analyses too (e.g., regarding the use of different types of symbolic representations – derived from the work of Pierce's semiotic view, Buchler, 1995 – in different groups of children). All participating children were selected from a nursery school and were 3–4 years old. This school was located in a large multicultural city in the southwest of England.

Worthington's research project yielded a number of very interesting findings. Without trying to be exhaustive, a few of them must be mentioned here as they fit perfectly well into the body of my present argument regarding the development of young children's mathematical thinking. First of all, Worthington could identify many cases of children who attempted to communicate about numbers and counting in their play setting, like the boy who had constructed a parking place completely with tickets and prices on each ticket (40 p., 60 p. etc.). It also turned out that the numberings were not suggested or imposed by the teacher, but were clearly drawn from children's 'funds of knowledge', built up spontaneously in their home contexts, the adult world of his parents, events in the outside world, observations of the teacher's behavior in her daily classroom activities (see Worthington & van Oers, 2016, 2017). Furthermore, Worthington could identify different types of graphical means constructed by the children, at first using their own words, scribble marks, drawings, or even alphanumerical number-like signs copied from their environments. Later on, the children started using iconic (like wavy lines for written texts), indexical (like arrows), and symbolic means, like tallies or numbers (categories borrowed from Pierce), and a new category referring to culturally developed symbols (like numbers, diagrams, plans, etc.).

Finally, the symbols evolved into the culturally accepted mathematical symbols. The observations of the nursery children's spontaneous play and communications about number revealed that their ability to read the intentions of abstract mathematical language and recognize the patterns of their combinations, was growing step by step with the help of the teacher (Worthington et al., 2019). By communicating about numbers and quantities, it also turned out that the communicative signs for the children actually represented 'texts' about number, i.e., things that they could tell about or explain in a coherent narrative about numbers.[3] More detailed analyses of the communications between the children and the teacher revealed another

[2] The research was for a doctoral Thesis at the VU Amsterdam, under my supervision and dr Marjolein Dobber's.

[3] This narrative nature of symbolic representations was previously also demonstrated in other situations, see van Oers, (1997).

mechanism of utmost importance in this respect. When playing with the children, teachers often provided new symbols or proper forms of the signs and spontaneous symbolizations of the children, but they did so only when the children showed need for such signs and symbols. No direct instructions. In due time, the children adopted the suggestions of the teacher and used them to expand their own texts. Hence we can say that the children's shared numerical notions were actually *intertextual*. Constructing such intertextuality with the children should be regarded as an essential dynamic element for the emergence of mathematical meanings in young children (Worthington et al., 2023). Adult supervision of these intertextual mathematical constructions (e.g., when using diagrams, graphs, or reflecting about number: 'what can we legitimately say about number?') was essential for the correct ways of linking symbols in propositions about mathematical objects (a quality correctly named 'grammaticity' by Worthington, 2021, p. 106ff). Adults' supervision that doesn't reduce the value of the parameters of children's play (rule-based, free to some extent, and involvement), is an essential productive condition for the communication about numbers, numerosity, quantity, etc. for the development of mathematical thinking and the construction of mathematically acceptable topic-predicate structures.

Discussion and Prospect

How can we conceptually understand the above presented productive conditions? What are the psychological dynamics of these processes? The potentially seminal relationships between the emergence of mathematical thinking and children's ways of communication and narration about number, has been discussed by several academics (Pimm, 1987; Krummheuer, 1997; Lorenz, 2012; Maier & Schweiger, 1999; Sfard, 2008). As Pimm (1987, p. 76) pointed out: 'Part of learning mathematics is learning to speak like a mathematician'. A lot of theoretical argumentation and empirical evidence has been accumulated since then. However, it is still important nowadays to clarify how this understanding is to be implemented into everyday classrooms with young children.

In our own research program we have spent serious efforts in adding to this understanding of the development of mathematical thinking with the help of appropriate communicative tools. It is clear from our observations that each of the productive conditions mentioned (play format, schematizing, narrative competence, intertextuality, interaction with adults) engages children in **communication** about quantity, numbers, relationships, patterns of signs, representational processes. Such communications not only invite children to think again and reflect, encouraged by questions like 'Are you sure?' (van Oers, 1996), but also to build topics of joint attention (Tomasello, 1999), which can be enriched in the ongoing discourse by new predicates produced by the child's own thinking or offered by peers and adults. So, like Vygotskij already pointed out in the last chapter of his *Thinking and Speech* (Vygotsky, 1982/1934), the construction of *topic – predicate structures* belongs to the core of conceptual thinking (see also van Oers, 2006 for further argumentation and examples). Communication ('speech') is to be considered an essential means for conveying and analyzing meanings, according to Vygotskij. With respect to mathematical thinking, this is consistent with Sfard's (2008) conceptualization of mathematizing. In such collaborative (communicative) constructions of increasingly elaborated and more sophisticated mathematical topic – predicate – structures, (young) learners get involved in *intertextual interactions* in which they weave together their own texts and those of others, provided they make sense for the learning child.

This ought to be a basically playful endeavor that highlights the rules of the game, but also allows the player enough 'degrees of freedom' in his actions and choices for explorations of the meanings and of the (intramathematical or extra-mathematical) limitations of the (mathematical) rules. Like in the explorations of the number line when children examine what it means to go 'under zero'; does this make sense?

The role of the teacher is essential here in order to guarantee the cultural relevance of the children's inventions for the enrichment of the (mathematical) topic-predicate structures. It is important to realize that this does not prohibit the teacher to teach the children. However, this should always be *embeddedteaching,* embedded in the context of the children's playful engagements with mathematics, trying to answer their own questions. All assistance of the teacher should be connected to the children's questions and interests in the context of their playful mathematics.

How to implement these ideas into the classrooms? In our own work in schools in the Netherlands for the implementation of the educational concept of *Developmental Education* (van Oers, 2012a, c; Slob et al., 2022; van Oers & Duijkers, 2013) we have elaborated a play-based curriculum approach in collaboration with teachers, teacher trainers and researchers, which helps teachers to organize their classroom work in the context of playful sociocultural practices. In the context of such practices, children encounter different kinds of problems (including mathematical ones) that they can try to solve with the help of teachers and peers. In this implementation strategy the main focus is on the innovation of teachers' thinking about curriculum and meaningful learning. In our research thus far we may legitimately draw the conclusion that teachers can indeed initiate and promote dialogical discourses with young children about events and phenomena in their daily life (van der Veen, 2017). From this perspective it is also plausible to assume that such dialogical discourses concerning mathematical topics in everyday practices can indeed promote mathematical thinking in the context of playful participation in varying sociocultural contexts. We have strong reasons to continue along this path in the future.

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A Child Perspective and Children's Perspectives on Mathematics Learning in Early Childhood Education

Ingrid Pramling Samuelsson

Children's Education Worldwide

The United Nations Agenda 2030 points out in Goal 4.2: "By 2030, ensure that all girls and boys have access to quality early childhood development, care and pre-primary education so that they are ready for primary education". Globally, the gross pre-primary (1 year before school) enrolment rate increased by 27% points in the last 19 years, from 34% in 2000 to 61% in 2019. Despite this progress, as of 2019, there were at least 175 million children aged 3–6 years old who were not enrolled in education, according to a UNICEF (2022) global report on early childhood education (ECE). It is great that ECE is on the agenda and that more and more children get a chance to participate in education, but the whole field of education and care before schooling is very complicated, and it is difficult to compare findings among countries. For example, the names for these kinds of activities vary: day-care, crèche, preschool, play school, infant school, kindergarten, toddler group, early child development, early childhood care and education, child-minding services, etc. There is no uniformity whether on what settings before school are called or what they contain. Further, the same notion can have different meanings in various countries (Pramling Samuelsson et al., 2018). In Sweden, preschool stands for all education from toddlers to school entrance, whereas preschool in, for example, the Netherlands represents part-time education for children between two-and-a-half and four as a preparation to start school, while a full day programme is called day-care (Preschool and day-care in the Netherlands, 2022). The reason for the large variation may be that children's learning before school is most often linked to family policy and not to educational policy. Nevertheless, the common rhetoric remains:

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"Education starts at birth! Parents are the child's first teachers; ECE is only complementary or compensatory to family experiences, depending on how the family functions".

However, all children should mean ALL—all ages and wherever they live. Agenda 2030 is for all children, and even though Goal 4.2 states readiness for school, it does not indicate whether early years education should be like school education. Professionals have to decide what is meant by preparing children for school and life. Goal 4.7 points out the knowledge and skills needed to promote sustainable development. One of these notions is *global citizenship* as a process in which evidently language and mathematics become necessary for first becoming a local citizen and then becoming a global citizen. Thus, although the goal may not point out mathematics or literacy in some form, the target aspect is citizenship.

How Can a Child Perspective and Children's Perspectives Be of Use in Young Children's Learning?

Agenda 2030 is not the only central aspect of early education; the UN Convention of the Right of the Child (UNICEF, 1989) is equally crucial. Specifically, two articles of the Convention stand out:

- *Art 3: In all actions concerning children, whether taken by public or private social welfare institutions, courts, administrative authorities or legislative bodies, the best interests of the child shall be a primary consideration.*

This article posits that adults' thoughts are the best for a child, based on both general knowledge of child development and knowledge of the child as a person. This can be viewed in ECE mathematics as the adults' plan according to the age of the children and the personal knowledge of the individual children in the group. This is what Sommer et al. (2010) called *a child perspective* – interpreting what is best for each child based on general knowledge about the specific child and earlier experiences. This may be what is often labelled "developmentally appropriate" education in the United States (Sanders & Farago, 2018).

- *Art. 12: States Parties shall ensure that children who are able to form their own opinions have the right to express them freely in all matters affecting the child, in which case the views of the child shall be given importance in relation to the age and maturity of the child.*

Sommer et al. (2010) perceived this as the *children's perspectives*—listening to and interpreting the child's expressions. Here, the child is involved in an active way, giving his or her thoughts, ideas, etc. Children are given agency for their own experiences, which makes it necessary to create opportunities to listen to children.

In Article 3, the child is viewed as an object, even though adults intend to meet children's needs, and in Article 12, each child is viewed as a subject who has agency. In the first case, teaching a child mathematics refers to giving children tasks that are appropriate for their age, according to what child development tells us that children are capable of doing. In the second case, mathematics teaching in ECE is a question of giving tasks and communicating with children to capture their ideas and learn from that to take a new step towards challenging children (Shier, 2001). From this perspective, solving a problem is a joint negotiation between the teacher and children, in which there is an opportunity to solve the problem in various ways and not always according to what the teacher thinks is correct. An example of the last case could be when children get the task to divide 10 buns between three people, and some children take an ethical perspective instead of a mathematic one, claiming that it is fairer for the father to get five buns because he needs more food, or for the mother to get only one bun since she always is on a diet (Doverborg & Pramling Samuelsson, 1999). Here, considering the children's perspectives is a key component of teaching!

Listening to children is important since it also has become both related to democracy and participation, but studies have shown that participation from the teachers' perspectives ranges widely from children participating in their own play, since they decide on what to do by themselves, to voting about what to do in various learning situations, etc. (see Williams et al., 2016). Listening to children has been used as a pedagogy in terms of "The Pedagogy of Listening" (Åberg, 2018), which originally emerged from the preschools in Reggio Emilia but has also been used as a rhetoric for all good work in preschool.

Data from an empirical study based on open questions about sustainability work in preschool in Sweden by Engdahl et al. (2021) were re-analysed with a focus on how the preschool teachers use the often-intertwined notions of *listening to children* and to *follow children's interest,* which in everyday work is related to taking children's perspectives. The new analysis revealed three categories of conceptions, representing various ways of thinking and talking about how teachers say they take advantage of listening to children or meet their interests (Björklund, 2020) which will be described and illustrated by quotations from the teachers in the following segments.

Listening Is Central to Discussions About Democracy and Participation

In this category, teachers relate the perspectives of listening to children and following children's interest to the social developmental aspects as a representation of making children participating in various activities, and by that giving them possibilities to be involved in a democratic process, becoming able to give their views.

> During the past year, we have chosen to focus on collaboration between children in our activities. We have focused on abilities such as feeling responsible, listening, following, leading, expressing thoughts, and explaining. This approach increased awareness of democracy.
>
> We educators have a responsibility to listen to the children, give them influence, and create participation.
>
> Everyone's opinions are equally valuable!
>
> We practice listening to each other and respecting each other.

From the teachers' view, taking the above perspective, the reason for listening to children is to develop the social aspects—that is, children's rights to become a participant in matters about themselves, which today is clearly spelled out as a goal in the Swedish curriculum (National Agency for Education, 2018).

To Follow Children's Interests and to Listen to Children for Their Learning

In the following category, we can see the perspective of a theory of learning in which children learn when they are active in thinking and reflecting, which has always been a perspective in Swedish preschool, where play and learning both have been central as well as care and education as two sides of the same coin. In this analysis, the teachers further expressed the fact that the child's interest or idea should always be the beginning of learning about something. The teachers' task is to listen to what children are interested in to know what to work with.

> In addition, there are expectations and faith in the future that will characterise the meeting between the children's questions and challenges about climate and sustainable development. Therefore, the preschool should listen to and meet the children where there is interest.
>
> The children's interest and curiosity control the content of the project, and the children become involved.
>
> To slow down, see/feel the joy in what we have and are, dare to stop and start from children's questions, thoughts, dilemmas, and problems.
>
> The project started with us being out and discovering and exploring to capture children's interest in sustainable development.
>
> Children's experience is that projects become the best if we take care of the children's interests. Then you can really see a genuine interest and a desire to learn. That's why I start with the children's ideas.
>
> The role of play in young children's learning and wellness is central to teachers becoming aware of their interests.

In this category, children's interests are central to learning. Teachers indicate that they have to listen to children or observe play to understand what topic/theme to work on, since children's world is the starting point for starting with a specific content area.

Negotiation as the Beginning of Both Learning in and About Democracy and Various Content Areas

In the following category children's perspectives are always related to communication and negotiation with the teacher. If the ideas or interest comes from the children or not is not the main question, but how the teacher managed to get children interested in something or to share their ideas in the education setting.

> We should of course use children's interest, but you cannot request that exactly all children who come to preschool are interested in just this; therefore, it may sometimes be necessary that we who work in preschool arouse an interest by challenging …
>
> When we try to listen to children's interests, it usually becomes a Matthew effect—the more you have, the more you get—and the children who already have a lot of influence get more influence. In addition, we educators have a tendency to pick up what the children are interested in that we recognise. This means that we rarely start from the interests of the children who come from other cultures and have different experiences than we educators.

One could claim that listening to children and using their interest in pedagogy is more problematised than the two approaches above. The teacher and the children are both necessary for educating children. The teachers know about the curriculum and have to be active in pointing out content areas. If children should always decide, nothing new may come up; instead, all children need to be challenged. This third category more clearly shows that the teacher's role is a pedagogical approach to learning and development, which is more in line with *developmental pedagogy,* the approach to preschool developed in the research group at the University of Gothenburg (see, e.g., Pramling & Pramling Samuelsson, 2011). Teaching in ECE is always a joint venture between children and teachers, in which the teachers know the direction to point out to children for learning (Doverborg et al., 2013), and the children's experiences and ways of talking about the content are key questions for the teacher to challenge further development of understanding. In developmental pedagogy, the content can be described in two steps: an area, such as the content of mathematics, and the learning objects—that is, the specific aspect of mathematics addressed in every new teaching situation that could, for example, be number conceptions, sizes, and patterns.

All these aspects that teachers bring up are, of course, relevant. Sometimes, children become occupied with something to which the teacher could link his or her planning, but most of the time, the teacher is the one who knows the experiences or the curriculum goal the children need to work towards. However, democratic and negotiating lenses are both necessary for influencing children's learning. Most importantly, taking children's experiences and ideas into consideration can be viewed as a touchdown in time regarding the status of the children in their learning process of the aspect of mathematics work right now on the spot (Pramling Samuelsson & Pramling, 2009).

Empirical Research Paves the Way to Education Based on Science?

ECE has a tradition based on child development, a question of children's more general personal development, which is important in the early years. However, as Sommer (2006) pointed out in his description of childhood psychology, the large general theories about child development have been exchanged with many minitheories in various areas, with children's knowledge of mathematics learning being one area (Björklund & Palmér, 2018; Björklund, 2019). This means that content has become important in ECE, which was not previously the case. In the past, the approach of teaching was the key question, which is still central but related to various content areas and to children's sense-making of the content (Björklund & Pramling Samuelsson, 2020; Pramling, 1983). Hence, empirical work in various content areas is so important for developing didactics for early years that it takes children's agency into consideration, rather than relying on child development in relation to various content areas.

In what follows are two empirical examples of aspects of mathematics in which the children's worlds of understanding a specific task vary and become visible in their acting and communication. The scenarios provide the teacher with an understanding of how children make sense and offer opportunities to understand how he or she has to challenge children for further development of understanding a concept. The first scenario evaluated young children's understanding of the *first and the last*. The teacher and each child in the study played with animals, and the teacher suddenly asked, "Should we let the animals go for a joint walk?" She then took one animal at the time and said, "First comes the cow, then the panda, followed by the lamb and last comes the pig". After that, she asked the child, "Which animal is the first one, and then which is the last one?" In the study, 240 children between ages 1 and 3 years participated, and the analyses of children's understanding were categorised into four categories: (1) know both first and last, (2) know first, and say that the last is the second one (which may be because we had 4 items, 3 may have been easier for the youngest), (3) know only first, and (4) did not bother at all, continue to play (Sheridan et al., 2009).

The second empirical case tried out an example of representation in mathematics. The teacher and a boy (3.4 years) counted animals—two tigers of various sizes—and the teacher posed the question, "Can you write on this paper, how many tigers you have, so we can remember?" The boy had already counted the tigers. He looked at the tiger and said, "I cannot draw the ears of the tiger!" Teacher: "It is not important!" The boy continued, "I cannot draw the mouth of the tiger, either". The teacher, realising that the boy thought he had to draw the tigers, responded by saying, "But do you think there is any other way to write on your paper how many tigers you have?" Then the boy drew one long line and one short, and then he took each tiger and put it on the respective line, according to the length of the tigers.

These two examples can be viewed both as making children's way of experiencing these mathematical questions visible from their perspectives and as touching down in time regarding the status of the child's knowledge development at the present moment. However, we do not know whether these tasks created new knowledge in children's minds. Thus, there is a need for research on how the process of children's learning can become visible to the teacher.

Let me exemplify with one more example that has very little to do with mathematics (that is, indirect distance): *how to find out something*. The participants are 300 children between 2 and 8 years of age. The researcher's question is, "If you want to find out how far it is to the moon, how would you go about that?" The four categories of conceptions analysed from the data are: (1) I would build a space shuttle and go there, (2) I would ask someone who has been there. (3) I would ask my parents or my teacher, and (4) I will find it out via media, books, the internet, etc. We can see that in children's minds, conquering knowledge ranges from finding out by independent enquiry to acquiring free knowledge from others (Pramling, 1983).

In a group of children, one can always discern numbers of qualitatively different categories of conceptions. These categories become visible by making children's perspectives visible. Sometimes these categories represent a step towards a more advanced understanding of something in relation to culturally accepted conceptions; other time conceptions may be horizontal, just representing various ways of expressing the same understanding. The example of how to find out something represents more or less advanced understanding by the children, which helps the teacher see what the next step of understanding for children might be.

How Is Play Related to Learning?

All ECE curricula mention play as a key factor in young children's learning. However, it is seldom problematised but is just taken for granted. Article 35 in UNCRC (1989) *also recognizes the child's right to rest and leisure, to **play** and recreation adapted to the child's age and the right to participate freely in cultural and artistic life.* If there is anything children relate to, it is play. All children play if they are not hindered from doing so! The play, as such, is what children are mostly interested in, as well as coming to ECE to meet their friends. Some children are ultimate players, whereas others do not depend on their early experiences. Fleer (2015) claimed that early interaction in families sometimes lack the interactive ground for developing playful children. Thus, even though there may be aspects of play that children are born with, acting in imaginative play is learnt from the environment. Further, by being included in imaginary play, children also have to have cultural knowledge to take part (Mauritzson & Säljö, 2003).

Some researchers claim that we, as adults, destroy children's play if we intervene, and that we should not relate play to learning (Steinsholt, 1999; Øksnes & Sundsdal, 2018; Hangaard Rasmussen, 2016), whereas others consider it advantageous to relate play and learning in ECE (Lillemyr, 1995; Pramling Samuelsson & Asplund Carlsson, 2008). Therefore, how are preschool staff dealing with play in practice? Studies about what teachers say as well as what they do can be described from three perspectives. The first and most common is that play is the child's own world, and teachers do not get involved in children's play. Second, teachers are external observers of the play who then try to help children expand their play by giving them new props, reading books on the topic of their play—they try to inspire but not take part. Third, teachers become involved in children's play as playmates. The debate of whether or not to participate in children's play is often based on ideology—what teachers think is best for children (Pyle & Danniels, 2017). With this debate as rhetoric, we began to work in our research group to find empirical evidence of what happens if teachers go into children's play in a praxis-oriented project called *Play-based didactics: Developing early childhood didactic theory in collaboration between researchers and preschool teachers.* The following is a summary of the application:

> Discussions about the preschool's activities often end up in an advocacy of either teaching or play (in preschool often referred to as 'free play'). A premise for this project, however, is that such a dichotomy is not fruitful for understanding and promoting children's development. Instead, a central challenge for contemporary Swedish preschool is to develop a form of teaching that is in line with the preschool's tradition and history, i.e. to design play-based preschool didactics. In this project, we further develop the preschool's own theory formation: Developmental pedagogy.

The journey from developmental pedagogy (Pramling, 1994) to the theoretical foundation of this last project has investigated play in various ways. In our earlier studies, we took play for granted; we described it but did not problematise it—play was self-evident therein. However, in 2003, when we published a meta-study of our research in the book *The Playing Learning Child* (Pramling Samuelsson & Asplund Carlsson, 2003/2014), it became obvious that we looked at children as playing learning children; we did not separate play from learning. As teachers, we have to learn that this is the case, and we have to integrate play and learning in praxis. Then came the question: Is it possible to integrate play and learning into a goal-directed practice? However, this seems to involve two different views: play as children's world and learning as the teachers' and the curriculum's intention. The simple answer to this question became: It all depends on the teacher's communication—whether they could allow both themselves and children to use both fantasy and reality in the communication (Johansson & Pramling Samuelsson, 2006; Pramling Samuelsson & Johansson, 2006). The latest way to look at play and learning developed from the project mentioned above is through *play-responsive teaching* (Pramling et al., 2019).

These changes in how play and learning are conceptualised have been viewed more carefully and extended by Pramling Samuelsson and Björklund (2022). The latest publication in process is one in which teachers who participate in a network based on this last research project have developed agency to become authors of their own work, as they have interpreted play-responsive teaching in their everyday work with children. The book is called *The Teaching Playing Preschool Teacher.* The teacher becomes both the one who leads children towards understanding various mathematical notions and one who learns from children's ideas all the time. Examples of mathematics in play-responsive teaching can be found in Pramling et al. (2019), specifically Chaps. 8 and 10.

A short summary of what play-responsive teaching means

- Teaching is a shared activity in which both children and the teacher are engaged jointly. This means that it is not a question of giving children a task and then waiting for an answer. There is a negotiating dialogue going on about meaning.
- Participants shift between and relate **as if** (imagination) and **as is** (accepted, culturally established knowledge) without the play being interrupted. Fantasy and reality are not separated in the communication, although children learn to under-

stand what is the one and the other perspective – but communication becomes enjoyable.
- Teachers become participants and co-creators in play where they can introduce cultural resources (as is knowledge) or suggest additional fantasies (as if) that can help children expand the play or develop it in new directions.
- In this kind of play-responsive teaching, children's agency is promoted so that they can become genuine actors in their life and learning (rather than **passive** recipients of instructions).
- Teachers also become genuine active participants, since they find it interesting to see what comes up in children's minds. They have an interest in what triggers children's active engagement.

What Did We Learn From the Project?

First of all, the participating teachers became quite skilled in participating in children's play, even though it was hard for some of them in the beginning, as they video-recorded children play without being involved, or tried to focus on the social aspect to get all children involved. In the end, they said that the children often asked them to come and participate in their play. They could, in other ways, participate without destroying the play! Further, the children enjoyed and asked them to participate when they got used to having teachers involved.

The notions of *intersubjectivity, narrative, and meta-communication* became important to talk about, and the teachers used them to interpret their video recordings. Intersubjectivity clarifies what communication with a joint focus or content can look like, and allows for understanding the intersubjectivity in short situations of shared focus. Narratives seem to constitute a frame for play content. Especially for young children, it was often related to stories or songs they were familiar with as the focus of play. Meta-communication is known as a source for developing awareness in children by putting words and reflections into what they are doing. Children do not always create an understanding of just doing something; thus, engaging the mind of children also means reflecting and communicating about it.

As *didactic consequences*, we observed that teachers had to plan to become able to get involved in children's play. This does not happen on the spot, and the other staff must be aware that the teacher now intends to be involved in children's play and cannot do everything else, such as answering the phone, welcoming late children, or taking care of the dishes. Further, teachers must be convinced that play and learning can influence each other and that there are possibilities to focus on questions in play that are related to the curriculum goals. This means seeing themselves as active in communication with some children all the time, and not only in the thematic work in which they intend to distribute knowledge. Teachers must be responsive to children's actions (verbally and in acting) and give children preconditions for being responsive to the teacher's activities. Some of these key notions are the exchange between "as if" (fantasy) and "as is" (culturally established) knowledge. Teachers must become able to talk in play or be different characters, as well as meta-communicate about the play or the content in the play activities in which both the teacher and children are involved. Just as teachers have to be skilled in communicating with children about various content areas, they have to learn the characteristics of play to be play-responsive.

Conclusions

In a recent analysis of documents during the history of Swedish preschool's development, it becomes obvious that preschool has developed through a hard political battle, first as a struggle between men and women and later between different political parties. It also became visible that changes in both access to preschool and pedagogy depend on political decisions (Klingvall & Pramling Samuelsson, 2022). Hence, lobbying is an important aspect of being a researcher if one intends to influence change. Bringing play into children's learning has support in the text of the Swedish curriculum (National Agency for Education, 2019, p. 9):

> Children should be given the conditions both for play, which they themselves take initiative and play introduced by someone in the work team. All children should be given opportunities to participate in shared games based on their conditions and abilities. When someone in the work team follows or leads play appropriately, either outside the games or by participating themselves, factors that limit play can be noticed and work methods and environments conducive to play develop. An active presence makes it possible to support communication between the children and to prevent and manage conflict.

As play has been more extended and related to learning in the curriculum, Swedish preschools have become more knowledge-oriented. The reason for this is, of course, that we have so much more research today showing the capacity of the child under appropriate conditions. Education has, by tradition, been related to school, where in many places in the world, knowledge is distributed to children. Changing this conception is difficult, even though research reveals otherwise regarding young children's play and learning.

One aspect concerns the contemporary emphasis in ECE on paradigmatic (categorical, scientific) knowing at the expense of narrative knowing (Bruner, 1990; Singer & Singer, 2005). Due to fear that some children may be falling behind in their knowledge (language development, subject-matter expertise, etc.), increased focus has been put on paradigmatic knowing. However, narrative knowing (typically nurtured through play) is arguably more important and certainly more foundational regarding mathematizing. Through narrating, people communicate, make sense (of themselves, each other, and the world), remember, form identities, and imagine other possibilities (as if and what if?), and it is therefore critical also to a sustainable future—imagining other ways of living and solving problems. Communication is central in all early learning, and a recent research overview, *A Systematic Review of Educators' Interactional Strategies that Promote Rich Conversations with Children aged 2–5 years* (Houen et al., 2022), shows that it is not only open questions that contribute to learning; play-responsiveness does as well.

The formal education of ECE is non-formal, in which the teacher needs to have professional knowledge of all content areas, such as mathematics, to know what to focus children's attention on, thereby giving children opportunities to learn. Putting children's attention towards various mathematical notions can be achieved by capturing the situation in which a child shows interest, or planning tasks of various kinds. These have been done with great success by designing learning situations based on variation theory (Björklund, 2016). Challenging children in a play-responsive way is important, since play-responsiveness does not have to only take part in imaginative play—it can be used in any situation or with any content area. Play and learning should always be integrated into each other in ECE.

With mathematics, similar to other content areas, children need to both live and experience mathematics in meaningful situations in daily life and communicate mathematics. It is in this communication that teachers can direct children's attention to aspects that can help them develop an understanding. Even though the intention during preschool years is to develop children's understanding of various notions in mathematics, the teacher must always be aware of the fact that children need concrete objects that can be used as support to explain or argue for something (Nergård, 2022).

Regarding both a child perspective and children's perspectives in ECE, both are necessary in practice. For the teacher to have appropriate professional knowledge in mathematics and child development is the basis for early education (child perspective). However, within this frame are the children's perspectives of, for example, numbers, patterns, and forms, aspects that need to be met and challenged in teaching and influencing children's learning. This call for specific knowledge by the teachers in didactics in various areas.

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Preschool Teacher Practices During a Collective Play-Related Thinking: Dialectic Between Pretend Play and Mathematics

Linda Amrar, Anne Clerc-Georgy, and Jean-Luc Dorier

Introduction

From an historico-cultural perspective, pretend play holds an important place and is thought to create a zone of proximal development (Vygotsky, 1966/2016). In fact, when children play, they "jump above the level of [their] normal behavior" (Vygotsky, 1966/2016, p. 18) by taking advantage of the degrees of freedom to manipulate and explore their environment (van Oers, 2014). Vygotsky (1966/2016) suggests that during pretend play, children create an imaginary situation in which they substitute objects meaning, take roles, and define rules associated with these roles. Furthermore, pretend play constitutes the leading activity for children aged between 3 and 7 (Vygotsky, 1966/2016). Meaning that even if children do not engage most of their time in pretend play activities, it is thought to drive their development.

Pretend play is associated with the development of different cognitive and socio-affective competencies, some of which are associated in particular with the development of mathematical thinking (Amrar & Clerc-Georgy, 2020). During free play, more than half of the play time is devoted to the exploration of mathematical concepts (Seo & Ginsburg, 2004). During pretend play, a wide range of mathematical

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activities are explored (van Oers, 1996). According to van Oers (1996), when mathematical activities initiated by children are seized by the teacher, they become mathematical teaching opportunities. Together, these studies show the importance of scaffolding the pretend play components dialectically with the mathematical activities initiated by children during play.

However, many preschool teachers restrain from intervening during child-initiated activities, by fear of interrupting the children's play. As a result, many opportunities to sustain the development of children are overlooked, which could explain the dichotomy between play and learning amongst teachers and children. Pyle and Danniels (2017) identify two preschool teacher's profiles depending on whether the play is investigated or seen as separate from learning. Clerc-Georgy et al. (2020) propose an alternative view to overcome this dichotomy. In their paper, the authors support the idea of a dialectical relation between play and learning, highlighting that scaffolding should be directed towards the development of play and curriculum at the same time. Scaffolding interventions promote learning and can take place either inside or outside the play (Fleer, 2015, 2017b; Wassermann, 1988). To date, few studies have investigated what preschool teachers do when they seize dialectically mathematical teaching opportunities and pretend play. This will be the focus of this case report describing the practices used by a teacher during a CPRT which takes place in the aftermath of a children-initiated pretend play activity.

Pretend Play and Mathematics From an Historico-Cultural Perspective

Pretend play and mathematics are both cultural and semiotic activities. According to Ernest (2006), a semiotic activity involves the use of signs, the use of rules associated with these signs and the attribution of meaning to the signs. Mathematics share all the component of a semiotic activity as signs are used continually during a mathematical activity (Dijk et al., 2004). Regarding play, Vygotsky points out: "Play is the main path to cultural development of the child, especially for the development of semiotic activity" (Vygotsky, 1930/1984, p. 69, as cited in van Oers, 1994). However, Vygotsky (1966/2016) highlights the need to be cautious with the comparison between pretend play and mathematics through the lens of signs. Indeed, the signs used in play are not specific and identical across the play of children, however, in mathematics, a consensus around the use and meaning of signs has been reached across a community of mathematicians. Hence the importance of not intellectualizing children's play when considering their relation to semiotic activities.

The change in the meaning of objects and actions is specific to the substitution component of pretend play and is used as an indicator of the maturity of the play (Bodrova & Leong, 2012). The role of the teacher is to sustain the development of this component bearing in mind to decrease the mount of support, allowing the children to take responsibility for this change in meaning (Kravtsov & Kravtsova, 2010). Furthermore, the substitution component of pretend play is the best predictor of performances in mathematics and reading (Hanline et al., 2008). More specifically, results of this study have shown that the more the children are able to substitute objects and actions while pretending, the higher their mathematical performances will be when they reach the age of 8. The hypothesis of the authors is that both activities engage the use of signs. This result shows the importance of developing the substitution component of pretend play, considering the long-term effect associated with the maturity of this component.

The close link between pretend play and mathematics makes it a particularly interesting activity to sustain the development of the mathematical thinking of children. In a study by van Oers (1996), teachers were supposed to ask semiotic questions to children (e.g., “Are you sure?”, “How could you be sure?”) in order to stimulate the mathematical actions of children during a pretend play activity based on a shoe shop scenario. The results show that children initiated different types of mathematical actions such as classification, 1–1 correspondence, measuring, and schematizing. Another result of this study is that teachers can use semiotic questions to seize the mathematical activities of children, which then become mathematical teaching opportunities. This study is of particular interest as it shows that teachers can sustain the mathematical actions during pretend play.

These studies highlight the semiotic nature of the link between pretend play and mathematics. They also reveal the fundamental role of the teacher in seizing opportunities to scaffold mathematical actions and pretend play activities initiated by children. However, they do not take into account how teachers can scaffold dialectically the development of pretend play and the mathematical actions initiated by children simultaneously.

Scaffolding During Pretend Play

According to Wood et al. (1976), scaffolding is the process at stake when a teacher supports a student during the completion of a task too difficult to be solved by the student alone. The three main components of scaffolding are: transfer of responsibility, contingency and fading (van de Pol et al., 2010). Scaffolding refers to the progressive transfer of the amount of responsibility to perform a task that is gradually transferred from the teacher to the student. Meaning that scaffolding is aimed at progressively giving ownership of the task completion to the student. Contingency refers to the adjustment of the support to the current level of the student. According to the clues provided by the students’ answers, the teacher will elaborate an idea on the current understanding of the student and adapt the scaffolding. Fading refers to the decreasing amount of support provided by a teacher to a student. When scaffolding fades, the assistance provided is progressively lessened to allow the student to succeed without support. In order to be characterized as scaffolding, an interaction needs to bring into play these three components (van de Pol et al., 2010).

Scaffolding techniques can be used during an interaction aiming at communicating about the play. It has been shown in the work of Wassermann (1988) and Truffer-Moreau (2020). In the Play-Debrief-Replay (PDR) model, Wassermann (1988) holds the view that play is an area of explorations which can be shared and expanded during a debriefing session. During play, different groups of children explore a scientific material proposed by the teacher. The explorations carried out by the children are used as the basis for the debriefing part. After the debriefing session, a replay session takes place where children can play again, bearing in mind the information discussed during the debriefing. The debriefing session share characteristics with the CPRT described by Truffer-Moreau (2020). A CPRT is a reflexive interaction between a teacher and a group of children, elaborating on the scientific concepts investigated during a child-initiated activity. The author highlights that the CPRT "acts as a pivot between children-initiated activities and adult-initiated activities" (Truffer-Moreau, 2020). The CPRT focuses on scientific concepts explored by children during a free play activity. Compared to the PDR model, the material and the group of children are not restricted during the play activity and the replay session does not necessarily take place directly at the end of the debriefing session. A teacher can decide to set up a CPRT at any time during the play. Thus, the teacher is required to be sensitive to children activities as well as to the dimensions of the curriculum associated with the scientific concept explored. During a CPRT, the teacher scaffolds children's thinking on the basis of what happened during the children-initiated activity. The teacher guides an interaction involving the group of children based on the scientific concepts explored during the play activity. The CPRT is part of a broader systemic structure called pedagogical structure including training activities and two leading activities, which are play activity and learning activity (Elkonin, 1999; Vygotsky, 1935/1995). In the pedagogical structure, the knowledge holds a central place as it constitutes the core and the binder of all the structure's components. In the PDR model, a replay session takes place directly after the debrief session. The aim is to allow children to replicate their findings or to test other hypothesis discussed during the debriefing. In the CPRT model, play is included in a broader structure. The teacher can thus decide to put in place a structured activity on the scientific concept seized during the CPRT shortly after the CPRT.

Research Question

The studies presented thus far provide evidence that pretend play constitutes a central activity for children's development during which mathematical contents appear. These contents can be seized by the teacher to foster their appropriation through scaffolding techniques such as CPRT. However, little is known about the teacher's practices at stake during a CPRT dealing dialectically with a pretend play activity and a mathematical content seized by the teacher. The aim of this research is to explore in terms of scaffolding the practices used by a teacher who seize mathematical teaching opportunities from pretend play activities during a CPRT. The research question is: What are the practices associated with scaffolding used by a teacher during a CPRT in order to seize mathematical teaching opportunities?

Data

The data are part of a larger study exploring the mathematical teaching opportunities arising from children-initiated activities. This extract has been selected as it is considered to be a CPRT. The classroom is situated in the French-speaking part of Switzerland. The teacher has 7 years of experience in teaching. Eleven children aged between 5 and 6 took part in the activity. The teacher and all of the children's parents signed the ethic form and the data were recorded according to ethical considerations. For ethical reasons, fictitious names are used in this case report. The researchers acted as observers and did not interact with the children.

Before the free play activity begins, the teacher provided materials (e.g., X-ray pictures, fabrics, big jigsaw pieces made of foam, paper, pens...). During a free play activity, the children initiated a hospital pretend play. They undertook different roles (e.g., receptionist, doctors, patients) and recreated different areas in the classroom (e.g., reception, patient's house, waiting room). The teacher acted as the doctor and moved between being inside and outside of the play. After approximately 1 h, the teacher gathered the children to do a CPRT. She placed an A3 paper-sheet on the floor in front of the children. In Fig. 1, the drawing of the plan of the hospital partly recreated during the CPRT based on the hospital play activity initiated by children is shown.

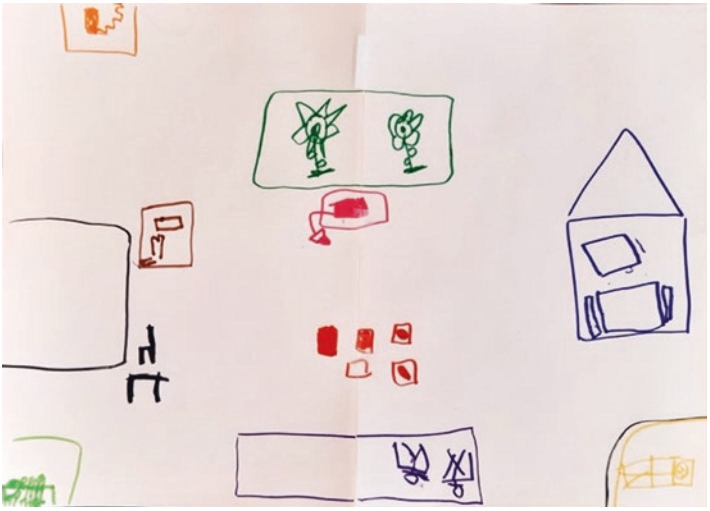

Fig. 1 Plan of the hospital

Methodology and Analysis

In this case report, a qualitative approach will be used based on the interaction analysis of Jordan and Henderson (1995) and the scaffolding framework analysis by van de Pol et al. (2010). The interaction analysis investigates the orderliness and projectability through the decomposition into units of coherent interactions. This method is particularly useful in studying interactions in a school context as it allows the identification of the different practices at stake during teaching/learning interactions. Derry et al. (2010) suggest focusing on "a particular pedagogical or subject content". In this case report, a CPRT focusing on the elaboration of the plan of the hospital created by children during a pretend play activity is analyzed. Jordan and Henderson (1995) highlight the importance of identifying the beginning and the end of the interaction to analyze. The definition of a CPRT is used to identify the boundaries of the interaction analyzed. The beginning of the activity takes place when all the children sit down together. The end of the activity takes place when the modality of the activity changes from a collective to an individual activity. The interaction has been transcribed including the gestures of the teacher and the children prior to the analysis.

Results

The analysis reveals four practices used by the teacher to seize mathematical teaching opportunities during a CPRT, which will be detailed below: (1) guiding a two-level intersubjectivity, (2) fostering the development of an imaginary situation, (3) raising awareness on challenges and (4) creating meaning between symbolization in the play and symbolization in a cultural tool.

Guiding a Two-Level Intersubjectivity

At the beginning of the CPRT, the teacher guides a two-level intersubjectivity in order to create a common understanding: an intersubjectivity between the teacher and the children (Björklund et al., 2018), and an intersubjectivity between the children themselves. At the teacher-children level, the teacher is responsive to the children's answers and ensures that a sufficient intersubjectivity emerges. At the children-children level, the teacher guides the interactions to allow children to share their perspectives, mutualize their knowledge, and reach a common understanding. This two-level intersubjectivity (teacher-children; children-children) is directed towards the adjustment of all of their perspectives on the discussed play activity. The teacher makes room for perspectives to be expressed and orchestrates the discussion to establish a sufficient intersubjectivity at these two-level.

Excerpt: “location of the waiting room”

13	Teacher:	And then we said that there is the waiting room. The waiting room, where it is?
14	John:	[points in the direction of the waiting room in the classroom].
15	Teacher:	Yes, you are right, it is here. So, I will mark it here [delineates with a dark pen a rectangle representing the waiting room on the paper sheet]. So, here, it is the waiting room.
16	Children:	Yes.
17	Teacher:	Ok. Here, [points the dark rectangle] it will be the waiting room.

In this excerpt, the teacher asks questions about the location of the hospital’s waiting room created by the children and documents the plan by drawing a rectangle on the paper sheet. This is done systematically for each area every time an agreement is reached. The teacher asks about the location of the waiting room. John points in the direction of the area dedicated to the waiting room during the play activity. As all the children agree, the teacher adds the delineation of the waiting room on the plan. In this excerpt, the plan supports the two level intersubjectivity as it contains information on what is agreed on between the teacher and the children but also between the children themselves.

Fostering the Development of an Imaginary Situation

The second practice requires fostering the imagination of the children while reflecting on the play. To expand the learning at stake, the teacher builds the CPRT on the basis of what happened during the play activity and fosters the imagination of children to encourage them to go further. While doing this, the teacher creates a bridge between play activity and learning (Fleer, 2017a).

Excerpt: “adding a school to the hospital”

40	Veronica:	And after, me, I know. I know what we are going to do [raises her hand]. Over there, over there [points in the direction of the back of the classroom], it is the classroom, over there over there in the classroom.
41	Teacher:	There is a school. But is it part of the hospital or is it another area?
42	Veronica:	It’s part of the hospital.
43	Teacher:	There is a school in the hospital.
44	Veronica:	Yes.
45	Oscar:	No.
46	Tom:	Yes. There is a school to make the children for example I fell at ski and they are going to teach you how to ski better so that you won’t hurt again.
47	Teacher:	Oh ok. [takes a pen to delineate the school area on the paper sheet]

In this excerpt, Veronica adds a school in the hospital. During the play activity, the children did not create a school which could explain Oscar’s objection. Instead of asking if the school was set up during the activity, the teacher encourages Veronica to give more information about the school. Tom makes a link between the function of a hospital and a school to make a case for Veronica's proposition. Once the agreement is reached, the teacher adds the new area of the hospital on the plan. The plan thus becomes more complex and supports children as they create meaning. The teacher fosters the imagination of the children by being responsive to their propositions and by encouraging them to capitalize on their knowledge to explore further through imagination.

Raising Awareness on (Potential) Challenges That Can be Resolved Using Mathematical Tools

The third practice used by the teacher while talking about the imaginary situation, is to make children aware of (potential) challenges that can be resolved using mathematical tools. The teacher can also mention an event experienced by the children during the imaginary situation, and discuss the challenges through a mathematical perspective.

Excerpt: "shed light on a challenging situation"

122	Teacher:	You see now me I can remember well because we just did it now. But the next time, now we will have to tidy the classroom. Next time, if I want to do the same hospital again, how will I do to remember that this was the twine area, this was the reception?
123	Charlotte:	So we write.
124	Teacher:	Oh. You would like to write. Do you have the place to write re-ce-ption. (insisting on the syllables, showing the place it would take to write the word on the paper sheet)
125	Child:	No!
126	Charlotte:	So, above it.
127	Leo:	Up! Up!
128	Teacher:	Ok. Could we draw instead of writing something to remember that this was the reception?
129	Children:	Yes!

In this excerpt, the teacher anticipated the difficulty that children would face when trying to match an area on the plan and its corresponding area in the classroom, when rebuilding the complex hospital they created, if they do not add a coding system to the plan. The teacher raises the awareness on the limit of memory span and the need to find a strategy. Charlotte suggests writing it down and the teacher raises another challenge related to the space required to write down long words. Charlotte insists by saying that it is possible to write above the delimited areas on the plan. Finally, the teacher suggests making drawings associated to each delimited area on the plan. All the children agree with this solution. In this excerpt, the teacher raises awareness on a potential challenge and guides the discussion towards the use of symbolization. The teacher takes the responsibility to shed light on a challenging situation and suggests a solution involving the use of a mathematical tool: symbolization.

Creating Meaning Between Symbolization in the Play and Symbolization in a Cultural Tool

This practice requires the creation of meaning between symbolization in the play and symbolization in a cultural tool. During pretend play, children substitute the meaning of objects and actions to assign them a new meaning. During the CPRT, the teacher guides the interaction toward the creation of meaning between the substitution during play and the use of symbols in a cultural tool. Children are encouraged to reflect on the symbols they used in the play and expand their abstract thinking to symbols used in cultural tools. Here, the CPRT acts as a bridge between symbolization in the play and symbolization in a cultural tool.

Excerpt: "finding symbols to represent areas on the plan"

130	Teacher:	What could we draw then? [pointing to the square associated with the reception on the plan]
131	Maya:	A computer.
132	Teacher:	Oh. Draw a computer.
133	Diego:	And a chair. [pointing to the reception area: a chair, and a keyboard on a table]
134	Teacher:	And to remember… And a chair. And to remember this? We've said that it was the waiting room.
135	Emily:	A bed or a chair.
136	Teacher:	For example, chairs.

In this excerpt, the teacher encourages children to find symbols that could be drawn on the plan to remember the function associated with each area delineated on the plan. The teacher asks questions to make the children aware of the challenges that need to be resolved collectively and transfers the responsibility of creating the symbols to the children. The teacher guides the children to move from symbolization in the play to symbolization in the plan. For different areas, children suggested the props used in these specific play areas. To represent the reception area, Maya suggests drawing a computer. Diego adds a chair as there is a chair in front of the table with the keyboard. On the plan, a keyboard and a computer mouse have been drawn to represent the reception (pink). When the teacher asks about the waiting room, Emily suggests a bed or a chair as they created beds and used chairs from the classrooms to represent the waiting room area during the play activity. As beds are already used to represent the patients' room (yellow), the teacher repeats "chairs". In the Fig. 1, we can see some iconic signs sharing characteristics with props used in the play. For example, the waiting room is represented by a chair and a table (black). When the teacher creates meaning between symbolization in the play and symbolization in a cultural tool, it scaffolds symbolization.

Discussion

The aim of the study was to identify the practices used to seize mathematical teaching opportunities during a CPRT. The results suggest that the teacher used four different practices sustaining dialectically mathematical and play-related thinking of children during a CPRT taking place after a pretend play activity. The teacher ensures that a sufficient intersubjectivity emerges at two level creating a common understanding of the subject at stake: between the teacher and the children and between the children themselves. The teacher fosters the development of the imaginary situation and raises awareness on challenges. The fourth practice is used to create meaning between symbolization in the play and symbolization in a cultural tool.

During the CPRT, children were encouraged to reflect on the imaginary situation created at a metacommunicative level and using a mathematical perspective. During a CPRT, the teacher undertakes the role of the expert and guides the children through the appropriation of cultural tools while leading a reflexive discussion, which maintains the link between imaginary situation and learning. In this CPRT, the teacher starts a plan of the hospital created by children during a pretend play activity by delineating the areas imaginated by children and then let them individually draw a symbol representing an area. Dijk et al. (2004) found that the first signs used by children often represent a real object. This result highlights the importance of introducing symbolization activities in the early years as they play an important role in the development of the abstract thinking of children. This is in line with the idea stated by Vygotsky (1966/2016) that learnings occur at a social level before being internalized at the individual level. During a CPRT, the teacher guides a discussion to create this dynamic using different practices.

In our study, the findings suggest a role for the teacher in sustaining and scaffolding dialectically pretend play and mathematical activities initiated by children in order to build a cognitive structure (van de Pol et al., 2010). Children's cognitive structure will build upon the pretend play and mathematical explorations they initiated, as well as the reflexive interactions guided by the teacher. This article adds to the growing body of research highlighting the need to scaffold dialectically pretend play and the scientific concepts explored during an imaginary situation (Fleer, 2017b). The specificity of this study is that a pretend play activity initiated by children could provide the basis for a mathematical reflection guided by the teacher beyond the play activity.

The major limitation of this study lies in the fact that the findings come from a case study. Thus, other practices could exist, which have not been found in our analysis. There is, therefore, a definite need for further studies to explore the practices used by teachers to sustain dialectically the imaginary situation and the mathematical content initiated by children in a larger corpus of CPRT. Although the current study is a case study, the findings suggest that different practices can be used by teachers to seize mathematical teaching opportunities during a CPRT taking place outside of a pretend play activity. The practices found in this article could be used for the initial training of teachers, guiding them through the appropriation of CPRT as a useful tool to foster children's learnings.

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Supporting One-Year-Olds' Digital Locating: The Mediating Role of the Apps and the Teacher

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Introduction

Digital tools are becoming increasingly more common all over the globe, with young children encountering them at home as well as in early childhood education (Otterborn et al., 2019). Although there is some research on older children using digital tools, little is known about the youngest children, especially when it comes to mathematics. Yet digital tools receive a lot of political attention, with an OECD report stating that "linking the way children interact with ICT inside of school to the way they already use it outside of school can be a key to unlocking technology's potential for learning" (Schleicher, 2019, p. 10). Similarly, a link between play and learning is commonly made in Early Childhood Education and Care (ECEC) practice and research (Nilsson et al., 2018). For example, Lundtofte's (2020) literature review of children's tablet play found that children's use of tablets is often connected to digital literacy and learning.

In this article, I investigate how two one-and-a-half-year-old children and a kindergarten teacher[1] engaged with playful digital apps together in a Norwegian kindergarten. Across the Scandinavian countries, digital practices are part of the national curricula for ECEC (Børne-og Socialministeriet, 2018; Ministry of Education, 2017; Skoleverket, 2018) and, consequently, teachers are expected to facilitate their use. In Norway, although kindergarten is voluntary for children aged

[1] Kindergarten teachers in Norway have a bachelor's degree in kindergarten teacher education, and make up 42.3% of the staff in kindergarten (UDIR, 2022).

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1–5 years old, in 2021, 87% of children between 1 and 2 years old attended kindergarten (SSB, 2022). Given that toddlers encounter digital practices in kindergarten and elsewhere, more research is needed to explore how this age group engages in digital activities, particularly those which facilitate very young children's mathematical engagement. This article aims to construct knowledge about how the design of digital apps supports kindergarten teachers in doing this. Therefore, the research question is: in what ways can digital apps mediate the way kindergarten teachers support the youngest children's engagement in the mathematical activity of locating?

Teaching in Digital Environments

Although digital tools such as mobile technologies have been around for some time, their entry into early childhood education remains contested, with opinions often divided into two groups: those who see digital tools as a threat, and those who think they can make the world better (Danby et al., 2018; Palaiologou, 2016). Given this contested space, it can be challenging for teachers to implement digital tools in their teaching. Vangsnes et al. (2012) suggested that teachers rarely bring the Nordic pedagogy of valuing play into digital game-playing, indicating that the digital tools' entry into kindergarten might entail a risk of changing the children's pedagogical environment to one with less focus on children's play (Christiansen & Meaney, 2020). Fleer (2014) argued that that there is a need for further research to recognize new opportunities for play in digital settings.

Tablets with multitouch capabilities are often considered an intuitive tool suitable for children's developing coordination skills. For example, Geist (2012) found that when toddlers use digital touch screens, they showed a high level of skills in using the screens independently, and explored options in similar ways to how they used other play materials such as building blocks. Nevertheless, digital apps for children are often given a strong learning focus, being marketed as "fun learning" (Kvåle, 2021). In apps with a linear build-up of tasks, children can often guess their way to the correct answer and seem to be motivated by the digital stardust or confetti they get for finding the right answer, rather than by learning ideas or concepts (Nilsen, 2018). Children can also change the purpose of digital activities from what was planned by teachers. Lafton (2019) found that children playing a digital memory game ignored the intended rules of the game, and instead created their own game with different rules. Digital apps with strong framing, those with clear control over the communication and the pace of the app, can sometimes limit children's participation (Palmér, 2015). Lembrér and Meaney (2016) suggested that children's agency could be strengthened by giving them control over the game, which could lead to them needing or wanting to explain what they were doing. This need could support children's reflective mathematical thinking. Playful apps with a weak framing, where the app provides the children with control, could engage children in mathematical language and thinking when they played by themselves (Christiansen, 2022).

Theoretical Framework

This article is part of a wider study where I have used Bishop's (1988) six fundamental mathematical activities to describe children's mathematics. In this article, I focus on the mathematical activity of locating as this is mostly what these very young children engaged with when using digital apps. To investigate how digital apps mediate kindergarten teachers' support, I use Ladel and Kortenkamp's (2011, 2014) artefact-centric activity theory (ACAT) which has been developed to capture the complexity of children's use of digital tools.

The Mathematical Activity of Locating

More than 30 years ago Bishop (1988) identified locating as one of six mathematical activities that are present in all cultures. In Norway, the curriculum known as the Framework Plan for Kindergartens (Ministry of Education, 2017) includes the learning area of quantities, spaces and shapes, which is based on Bishop's (1988) activities (Reikerås, 2008). Researchers have also found that situations involving Bishop's (1988) activities are present in children's culture (for example, Helenius et al., 2016; Johansson et al., 2014).

Bishop (1988) described locating as more or less sophisticated activities connected to navigating and communicating about the environment which are closely linked to language as language develops through taking part in these situations. Lowrie (2015) found that video games for 5-year-old children required visuospatial skills similar to those needed in real life-situations, suggesting that digital games can provide opportunities to engage children in locating activities. Although verbal language can both make locating ideas easier to spot and focus the children's attention on the mathematical aspects of a situation, children are also able to engage in the different mathematical activities without using verbal language (Flottorp, 2010; Meaney, 2016). Children's engagement with locating ideas does not have to be done through verbal language. What remains unknown is how apps might mediate the kind of support, including the development of verbal mathematical language, that teachers could provide to increase young children's opportunities to engage in locating tasks.

Artefact-Centric Activity Theory

Children's engagement with multitouch screens such as tablets is complex. To deal with this complexity, Ladel and Kortenkamp (2011, 2014) developed artefact-centric activity theory (ACAT), based on Engeström's (2015) activity theory. ACAT theorises the relationship between the child, the app, the mathematics and others who participate in the activity. ACAT describes how the app mediates the processes the children engage in, rather than assessing their learning outcomes (Ladel & Kortenkamp, 2014). In this theory, the artefact (in this case the digital app) is at the centre and is affected by and affects four components: subject, group, rules and object. The model describes how the interaction between the subject (child) and the object (mathematics) is mediated through the artefact (the digital app). By using ACAT, it is possible to show how the app's design affects children's mathematical involvement and to discuss how digital apps mediate the way kindergarten teachers support the youngest children's mathematical engagement (Fig. 1).

The ACAT model consists of two triangles. In the first triangle, the artefact (the digital app) is connected to the nodes for rules (how the app is designed) and the object (the mathematical content). The triangle describes how the mathematical object is internalized in the app and then externalized through the tablet's design. The rules (the design principles of the app) describe how the mathematical object is presented in the app (what kind of activities and feedback are available).

In the second triangle, the focus is on how the artefact (the digital app) is used by the subject (the child who is playing) and the relationship to the group (the social group around the child). The focus is on what the subject internalizes from engaging with the app, and how the subject externalizes that engagement through their words

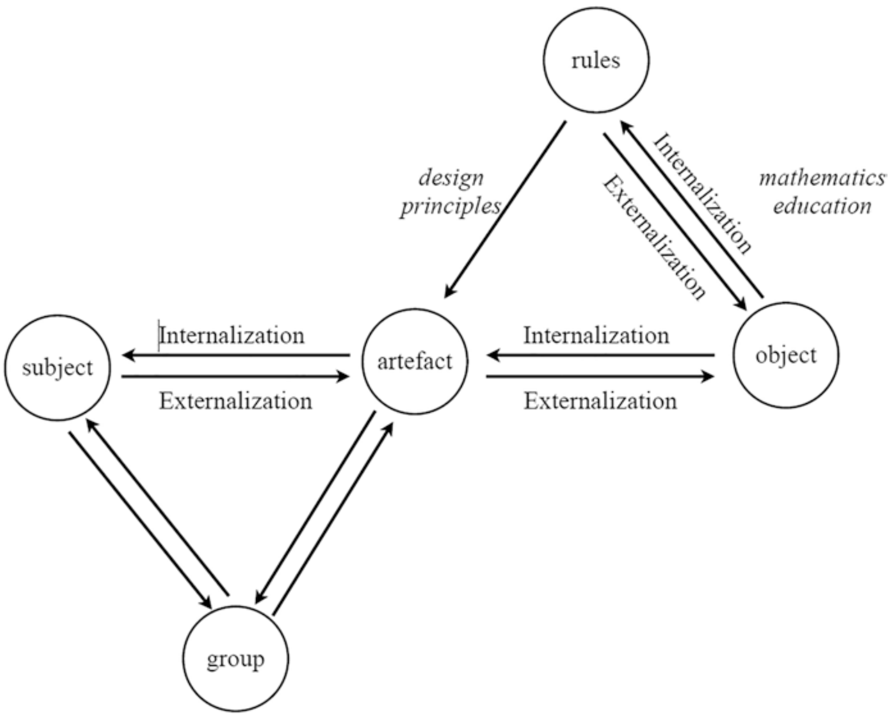


Fig. 1 ACAT from Ladel and Kortenkamp (2011, p. 66)

and actions. The group influences the child's engagement because it can facilitate or hinder, or be facilitated or hindered, by the way that the app presents the mathematical content (as a basis for what happens in the first triangle).

Methodology

As part of a wider project, video recordings were made of children engaging with digital tools at a Norwegian kindergarten over a 2-month period. For this article, I analyse 20 min of video, where two one-and-a-half-year-old children, Ole and Trine, played with three digital apps with a teacher.

Informants and the App

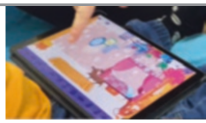
The teacher was informed about the wider project in a series of staff meetings before filming began. She gave her informed consent to be part of the project. The parents also gave written consent for their children's participation after a parents' evening was held to inform about the project. The children were given time to get to know the researcher before the video recordings began. The video recordings were carried out in the rooms the children usually used in the kindergarten, and they were always with one of their permanent teachers. There was an agreement to turn off the camera if the children seemed uncomfortable with being filmed. The two children were learning Norwegian as their first language. In the videos, the children engage with three different apps which they chose from what the kindergarten staff made available on the tablets. These apps were: My PlayHome, an app designed to excite and captivate children (My PlayHome, 2022); Toca Kitchen, an app designed for play (TOCA BOCA, 2022); and Crocro's Friends Village, an app designed to "both entertain and encourage them [children] to learn, develop, and flourish" (Samsung, 2022). Both the children had previously tried all three apps.

Analysis

Transforming the video data into written text was the first step in the analysis. I started by transcribing verbal utterances and adding in screenshots of the actions connected to what was happening on the screen, inspired by Cowan (2014) (see Table 1). To identify how the app mediated the kindergarten teacher's support of the children's locating, I examined the transcripts from the videos carefully many times, and returned to the video if I was uncertain about something.

I then organized the analyses into two parts, each part focused on one of the triangles in ACAT. Everything related to how the children used the app to engage with

Table 1 Transcript of datamaterial

Verbal engagement	Bodily engagement	Screenshot from the video
Teacher: Oh [pauses] You need to press the dog. Then he will move. Can you see him walking along? Do you see that?	The teacher traces her finger along the tablet, from the dog and towards the end of the screen.	

the mathematical activity of locating (actions and vocabulary linked to locating and navigating in the virtual space of the app) was coded as part of the rules–object–artefact triangle.

Following earlier research on children engaging with apps (Christiansen & Meaney, 2020; Geist, 2012; Kvåle, 2021; Nilsen, 2018). I wanted to examine apps which did not have (mathematics) learning as a stated aim of the app. Therefore, the chosen video recordings did not include the children using apps that were described by their developers as educational.

After watching the children's engagement, I identified locating as a mathematical object that was made available to the children through playing with the app. It may be that the teacher did not think that she was focusing on mathematics or locating when engaging with the apps with the children. However, this way of engaging in mathematics through playful participation is in line with the Norwegian framework plan which states that children should learn through play (Ministry of Education, 2017). In the video, the teacher appeared to focus on communicating about the children's navigating, and communicating about the environment, which is in line with Bishop's (1988) understanding of locating.

Everything related to how the child and the group around the child interacted with the app was coded as part of the subject–group–artefact triangle. In ACAT, the role of the teacher is not explicit, but the teacher is part of the "group" in the subject-group-artefact triangle. In the recordings, the children each had a tablet, but they mostly focussed on Ole's screen. Therefore, I have made Ole the subject of the analysis. Trine and the teacher are part of the social group around Ole. The choices for this coding are discussed in more detail in the findings and discussion section.

Findings and Discussion

To answer the research question, I describe how the three digital apps supported the kindergarten teacher in facilitating the youngest children's locating. I found that the pace of the apps appeared to have an impact on what kind of conversation the teacher and the children engaged in. If an app operated at a slow pace, there was an increase in opportunities for the teacher to facilitate children's engagement with aspects of locating. Lembrér and Meaney (2016) had noted a similar result with preschool children using an interactive table and a balancing app.

Crocro's Friends Village

The first triangle of ACAT focus on rules-object-artefact. In the video, I identified aspects of locating externalised through the app as a result of its design, especially the feedback provided to the child when they interacted with the app. Ole touched the screen on the sloth's floor (see Fig. 2, left). The app provided feedback in that it zoomed into this floor of the building. Ole was then in the sloth's virtual home. It was unclear if this was a deliberate move by Ole. The reaction of the app provided Ole with the opportunity to see how his actions allowed him to move between the outside and the inside of the building. This indicates two kinds of locating; a virtual one where Ole moved inside a building from being outside, and a physical one in which moving his finger around the screen changed what appeared on the screens. At this point, neither the teacher nor Trine made any comments verbally or with gestures about either aspect of locating.

On the next screen, there was a fishbowl in front of the sloth and fish were flying from left to right, over the sloth's head, along with some boots (see Fig. 2, centre). Ole dragged his finger across the screen several times where the fish were flying, but the app did not respond. Ole seemed aware of the movement of the objects, suggesting that he had an opportunity to engage with aspects of locating such as moving from left to right. He also seemed to expect that touching the moving fish would result in something happening. The object flying over the app appeared to fly at a pace which required visuospatial skills beyond what Ole had at this time.

After a few seconds, Ole stopped touching the screen. The teacher asked, "What will happen if we press the apple?" and pointed to the apple at the bottom of the screen. In this way, the teacher supported Ole in locating other items which may have been difficult to see because of the many small items on the screen. Ole pressed the apple and the fishbowl was replaced with a salad bowl (Fig. 2, right), where

Fig. 2 Screenshots from Crocro's Friends Village

different fruit and footballs appeared and disappeared. Ole tapped the fruit but again the app did not respond. Then the teacher suggested, “Maybe he wants to have it inside his stomach”? Ole dragged the fruit towards the sloth’s mouth: the sloth ate it, smiled and gave the ‘ok sign’. The design of the app provided feedback to Ole which supported him to see that feeding fruit to the sloth would make it respond, unlike tapping on the fruit. He then engaged with locating aspects of moving items around the virtual space. The actions of the sloth seemed to motivate Ole to continue exploring by pulling something else from the salad bowl. The screen changed back to the fishbowl. Ole went back to tapping the screen, which did not result in the sloth responding.

The second triangle focused on how the app (the artefact) was used by Ole (the subject) and his interactions with Trine and the teacher when playing with the app (the group). When the teacher suggested that “maybe he wants to have it inside his stomach” Ole was guided towards trying something new, and the teacher was able to focus on locating by asking, “Should he (the sloth) put it in his mouth?” Trine then put her finger inside her mouth and the teacher held her hand in front of her mouth. Trine did the same and the teacher said “Now we will be full up. In our stomachs” and rubbed her stomach. This exchange would have helped the children to understand about different parts of their own bodies, such as their mouth where food goes in, and their stomach where food ends up. Locating different parts of their bodies and knowing their names, such as stomach, are new and relevant aspects for many young children to learn.

In Crocro’s Friends Village, there was a lot of quickly-moving objects. Ole did not have the fine motor skills needed to navigate the app, and consequently he did not always get feedback from the app that supported his engagement with aspects of locating, apart from watching the items move from left to right and tapping on the fish on the screen. The teacher’s communication, even if it was not explicitly about aspects of locating, appeared to support Ole to keep exploring. However, it seemed as if the somewhat fast-moving items, and a large number of small items on the screen, made it challenging for the teacher to engage in a discussion about this with Ole.

Toca Kitchen

In Toca Kitchen there were also internalized possibilities to engage with locating. These possibilities were externalized through a virtual environment in which the children had a range of options. The feedback was different depending upon the choices made. When opening the app, the children could choose a boy, a girl or a monster (Fig. 3, left). Trine pressed the monster, but as had been the case with Ole’s choice of the sloth’s apartment when playing Crocro’s Friends Village, it is unclear whether Trine’s choice was deliberate. Nonetheless, once the monster was chosen, the app then showed a virtual kitchen. Pressing the monster showed the children how moving their fingers and tapping on the screen would lead to changes on the

Fig. 3 Screenshots from Toca Kitchen

screen. Both aspects of locating to do with locating themselves in the virtual environment produced changes in the app.

Trine tapped the screen and a door opened showing a range of cooking implements, such as a saucepan and a blender (see Fig. 3, centre). Again, it seemed that Trine did this randomly. She then touched the middle of the screen and the door to the kitchen equipment closed. Ole stretched his finger towards the monster, slightly touching the fridge door which opened (Fig. 3, centre). The teacher said "He was hungry. He's got a melon, a mushroom and some bread." She pointed to each item. "What do you want to eat, Trine"? Trine said something inaudible and pressed the onion. After one wrong try, Trine pressed the onion which made it fall from the fridge onto the table. Similarly, Ole struggled to feed the monster, suggesting that the app was difficult to navigate. The aspects of locating were to do with dragging the food from the fridge to the monster's mouth. At times, the feedback from the app did lead to the monster eating the food, but when unsuccessful led to the fridge door closing. Ole and Trine externalized locating through words and actions only to a small extent.

However, the teacher focused Ole and Trine's attention on locating by adding words to the onion going *into* the monster's mouth. Using language to discuss environments is, according to Bishop (1988), part of locating. The teachers' support is also connected to supporting Ole and Trine in exploring the options available within the app. When the onion fell out of the fridge, the teacher opened the cooking menu and suggested that the children use it by asking "what do we have to do now"? However, when neither of the children responded to her invitation to explore the cooking options, she closed the menu.

In Toca Kitchen, opportunities for engaging in locating were externalized through the opportunities to move the objects, and because moving the object towards the monster's mouth would result in the monster eating the food. Although this app moved at a slower pace than Crocro's Friends Village, it appeared to be difficult for the children to navigate; for instance when one wrong tap would close the fridge. The children were interested in, and managed to give the monster food to eat, and the teacher and the children discussed how the food disappeared and went into the stomach, also by using gestures such as rubbing their stomach.

My PlayHome

In My PlayHome, locating is externalized through movable objects in the room and through the opportunity to navigate between the rooms by touching the arrows or dragging items toward the arrow (see Fig. 3). Ole tapped a dog, making it move around. The teacher said, “There it was a dog”, and Ole continued to tap the dog and make it move. Suddenly he also started moving a mirror hanging on the wall. The teacher then stretches in and said “You are moving the mirror. Here, can you move it”? The teacher thus focused the children’s attention on the movable parts in the app. Ole continued to tap the dog and say “tap woof-woof”. The teacher said: “Oh, you have to tap woof-woof, then he will move. Can you see him walking along”? The teacher traced her finger in the direction the dog had moved, focussing on how Ole changed the dog’s positioning in the app. Ole then repeated the teacher’s movement and instead of tapping the dog he dragged it along the screen making it move further than before. He did it again, and this time he moved it onto the wall. The teacher said “Oh, now he jumped”, and again she pointed to where the dog jumped from and where he jumped to. Ole said: “Oh, no. Go down” and tried to move the dog down from the wall by dragging his finger across the dog. Although Ole moved the dog further than planned, he experienced that he could drag items around in the room. The teacher focussed Ole’s attention on locating, by using words and gestures that describe how the dog moved in the virtual room.

Ole closed the app and opened it again, this time in a room with a chair and a grandfather. Ole said “Grandad sit there” and pointed to a chair. The teacher replied, guiding Ole to explore how he could get the granddad to sit: “Should he sit there? Then we have to move him. Can you tap on him”? Ole started tapping him, but switched to dragging. He dragged him up the ceiling at the top of the screen making him move one floor up into a bathroom (from bottom left to top left in Fig. 3). The teacher said “Oh, now he is in the bathroom” and Ole said “Into the shower” and tried to drag him into the shower but ended up dragging him into the bedroom next door (from top left to top right in Fig. 3).

Ole ended up in a different virtual room with a bed. Ole dragged a baby onto the bed and pulled the blanket up (see Fig. 4). The teacher said “Oh, you pulled the blanket” and Ole replied “Oh no, baby!” sounding distressed. The teacher said “He’s gone. He’s gone. Where is he”? Ole said “There” while pointing at the bed. He pulled the blanket back, making the baby visible again, and this made Trine say something inaudible while sounding excited. Ole moved the baby off the bed, put it back, and continued to pull the blanket back and forth. Trine and the teacher were commenting on the baby appearing and disappearing.

When using My PlayHome, the children appeared to be interested in making things move, and both Ole and the teacher externalized language related to locating. The teacher’s language focused the children’s attention on locating, through using verbal language such as the baby being *in* the bed; the dog *moving*, *jumping*, *walking along;* the baby *was gone* and *there*. My PlayHome has the option of undoing actions. Unlike Crocro’s Friends Village and Toca Kitchen, where Ole and Trine

Fig. 4 Four screenshots from My PlayHome, illustrating the topology of the house

only had the option of doing the same thing over again, in My PlayHome they could move items back and forth. Ole explored this option when he covered and uncovered the baby with the blanket, encouraged by his own reaction when the baby 'disappeared', the teacher's question "Where is he?" and Trine's excitement when the baby was still in the bed. Even if Ole navigated between rooms without doing so intentionally, this is a relatively slow-moving app where the children actively have to make something move by dragging it, although they sometimes make them move further than planned. The slow pace gave the children the opportunity to explore the virtual space at a slower pace than in the two previous apps, and this also provided space for the teacher to mediate the locating through active engagement around the app. The teacher not only voices what the children are doing, but also uses gestures to underline the locating, such as showing the children that they are moving the mirror and tracing the dog's movement on the screen. The tracing of the dog's movement appears to encourage Ole to try to drag items instead of just tapping them.

Conclusion

The findings show that the apps provide opportunities for engaging in locating. This is because they provide opportunities for users, such as Ole and Trine using their fingers to make things move in the app, and such as opening cupboard doors in Toca

Kitchen. When Ole and Trine interact with the apps, they do so mostly through body movements, and sometimes using verbal language to express what they are doing or want to do. The locating in the virtual word can sometimes lead to discussing locating in the actual world, for instance when Trine put her finger in her mouth, showing where the food went. With the very young children, it seems that the teacher has a key role in mediating locating.

Although Ole did explored the apps to some degree, as Geist (2012) found toddlers can do, moving small objects in the virtual space required good fine motor skills and this has an impact on the aspects of locating that the children could engage with. In Crocro's Friends Village, the need for fine motor skills was combined with fast-moving objects. In Toca Kitchen, many things were happening on the screen at the same time. These design features of the apps appeared to make it harder for the children to engage with the opportunities for exploring aspects of locating by playing with the app. When the children use My PlayHome, the pace is much slower. Although there is no built-in *need* for explanation such as Lembrér and Meaney (2016) suggest there should be when (older) children take part in the game, My PlayHome provides opportunities to discuss what is happening, especially when Ole pulls the blanket over the baby, and he can pull it back and forth many times to both see and discuss what is happening.

Although playful apps have been shown to engage older children in mathematical language and thinking when they play by themselves (Christiansen, 2022), these younger children did not use a lot of verbal language when they engaged with any of the three apps. As the digital apps did not provide any verbal input, the teacher was the only one providing models of mathematical language. It was the teacher who voiced what the children did in relationship to aspects of locating in the virtual environments. In fast-moving apps, it was harder for the teacher to focus the children's attention on these opportunities, as something new is happening all the time. The teacher had a key role as the mediator of the mathematics; the mathematical object (locating) is externalized to the children through what is seen on the screen and how they use their fingers to interact with it, but also through the teacher's verbalization. The differences in how the three apps provided opportunities for children's interactions on the screen.

The teacher focused on the aspects of locating that the children were interested in and allowed the children to decide what to do and how to do it. For example, with My PlayHome, the teacher followed what the children did, rather than limiting their exploration so that they were guided towards solving tasks. The teacher appeared to value the play, both in that she did not insist that the children engage with apps with what Kvåle (2021) call 'fun learning', and because she let the children explore freely while she highlighted aspects of locating. The kindergarten teacher seizes the space for action that arises when the child faces digital challenges and use it as an opportunity to discuss the mathematical object of locating.

The teacher has an important role when young children engage with apps designed for play as the teacher becomes a mediator of the mathematical content in the app, just like the app itself. The teacher both highlights the mathematics in the app (in this case locating) and underlines the way the app externalizes the feedback, such as when she says to Ole that the dog is moving when he taps it. As the teacher has this key role as a mediator of the mathematical object, I suggest that the model of ACAT should be expanded by adding a third triangle where the connection between the group (in this case Trine and the teacher) is highlighted.

Figure 5 illustrates how ACAT can be expanded to include the teacher's mediating role. The red arrows illustrate the relationship between the teacher and the mathematical content; the object designed in the app enables the teacher to focus on the mathematics, and the teacher focuses the children's attention on the mathematical opportunities designed in the app.

In this study, I have shown that although digital apps have the potential to engage even very young children in locating and the teacher might have a key role as a mediator of the mathematical object. The design of apps which provide an open virtual environment for children to explore at their own pace can provide spaces where the teacher can focus the children's attention on locating by adding words to what the children are engaged in. Of the apps that have been investigated in this study, the non-linear and slow-moving apps without verbal language appeared to provide the best opportunities for such spaces for verbal language contributions from the teacher, who provided a supportive environment for the children's own explorations. Although ACAT was useful in helping me make sense of the data, it was clear that the teacher's role was not sufficiently highlighted in supporting the children in engaging with the mathematical objects of learning. Theoretical models

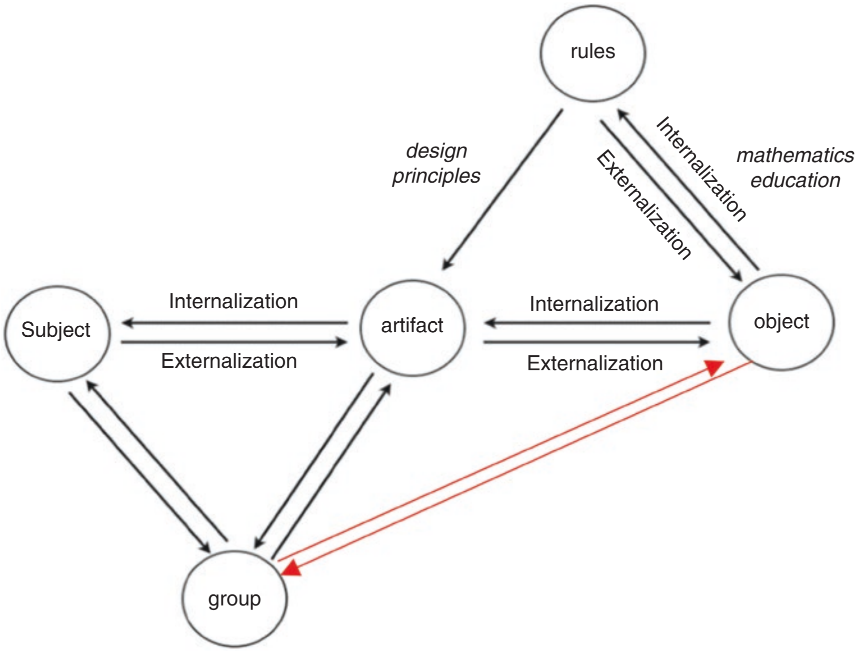


Fig. 5 Elaboration of the ACAT from Ladel and Kortenkamp (2011, p. 66)

used to analyse children's digital engagements need to recognize the importance of the teacher. For future research, I would suggest looking into how digital apps can support other mathematical activities and whether the design elements in other apps can support engagement with all of Bishop's (1988) mathematical activities.

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I Can Do It on My Own?! Evaluation of Types of Implementation of Digital Game-Based Learning in Early Mathematics Education

Laura Birklein and Anna Susanne Steinweg

Introduction

Nowadays, typical everyday learning and play situations – particularly at home – have changed. Not only but especially because of lockdowns and remote-schooling phases due to the pandemic, digital media and digital solutions have become a natural part of everyday life, even for young children. The EfEKt study evaluates different types of implementation of the app MaiKe (see below), which has been developed to support mathematical competencies of children aged 4–6 (Steinweg, 2016). The study is carried out in Germany, where special conditions in terms of media use have to be considered. The debate whether the use of digital media in early childhood education is appropriate or not is still vividly ongoing and divides educators as well as parents into two seemingly irreconcilable camps (for a detailed and reference-based description see Birklein & Steinweg, 2018). On the one hand, fears prevail that free media use will take over and displace real world play situations without stimulating any development in (mathematical) competencies. On the other hand, digital media and especially digital learning games are considered helpful for learning progress. As the project is conducted in Germany, this debate needs to be embraced in its research questions. The change in life in terms of digital transition processes may offer opportunities, but should be considered reflectively and be one focus of research in early mathematics.

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Theoretical Framework

The EfEKt study is closely embedded in the theoretical discourse on digital learning, fundamental ideas in early mathematics, and last but not least learning supervision. Moreover, the theoretical framework is oriented towards relevant research. In the following three sections the main theoretical references are outlined.

Digital Learning

One way to support and encourage children's mathematical thinking and learning might be to enrich existing play and learning environments with digital games. Despite persisting scepticism of parents and educators regarding the use of digital media in early education, there is a growing consensus that the potentials of multimedia resources cannot be simply ignored (Palme, 2007). At best, the broad availability of mobile devices (miniKIM, 2020) and technical innovations, such as the touch screen technology of smartphones and tablets, which is especially suitable for younger children, open up new learning opportunities and thereby also learning opportunities for mathematics.

More and more studies and meta-analyses (e.g., Knogler et al., 2017; Vogel et al., 2006) show effects of the use of digital media – usually compared to traditional learning activities. However, a closer look at the studies reveals that effects on learning processes depend on various context parameters. The potentials of digital learning environments can only be fully exploited under certain preconditions regarding content selection, adequacy of activities, and suitability of types of implementation of digital games into existing learning environments in kindergarten or families. Various research studies in the past few years addressed these questions (e.g., Lembrér & Meaney, 2016; Moyer-Packenham et al., 2018; Papadakis et al., 2018).

Fundamental Ideas in Early Mathematics Education

Everyday and play situations are considered promising opportunities to facilitate mathematical activities, mathematical discussions, and fruitful interactions, which, therefore, support the development of mathematical competencies (e.g., Gasteiger, 2010). Research emphasises fundamental or 'big' ideas of mathematics (Brownell et al., 2014; NAEYC & NCTM, 2010; Sarama & Clements, 2009). Furthermore, research in psychology identifies some of these ideas as key predictors of mathematical learning outcomes in school (e.g., Dornheim, 2008; Jordan et al., 2009; Lembke & Foegen, 2009). Various approaches on integrating these mathematical key ideas in early education have been proven to be effective in evidence-based research studies (e.g., Gasteiger, 2015; Gerlach et al., 2013; Krajewski et al., 2008).

These predictive competencies are taken up in the presented project, as the MaiKe app contains tasks designed on the mathematical key ideas numbers and operations, space and shape, patterns and structure, quantities and measurement. A detailed and reference-based description of all other tasks offered by the app MaiKe is provided in Steinweg (2016). Also, the learning effects are tested with a standardized test based on the key ideas (LauBe, 2015).

For example – concerning the big idea of number – activities in the app invite children not only to count or to order numbers, but to compare structured and non-structured sets in order to stimulate the ability to perceive structures in sets and to use them to determine cardinality (Schöner & Benz, 2018). Sprenger and Benz (2020) identify two different but interrelated processes: perception of sets and determination of cardinality. The set can be perceived 'in structures', 'as a whole' or 'as individual elements' and the cardinality can be determined by 'known facts', 'derived facts', or 'counting strategies'. Counting strategies and perception of sets as individual elements mark the least advanced responses. The use of the structure indicates more sophisticated competencies and is a precondition for subitizing strategies.

Learning Supervision and Support in Early Mathematics Education

The importance of educators' professional learning support or supervision are beyond discussion (e.g., Gasteiger & Benz, 2018). This holds true for playing learning games together with educators and parents as well. For example, Schuler et al. (2019) could show significant differences between the influence of direct and indirect support on the percentages of shown verbal and non-verbal mathematical activities. Their findings give evidence that play situations with direct support stimulate children to verbalise their mathematical activities to a particular extent, and that these play situations are cognitively more challenging. The distinction made here between direct and indirect refers to differences in the adults' behaviour. On the one hand, adults act as direct guides, and on the other hand, they supervise the play situation indirectly. The type of indirect supervision (in the sense of the presence of adults) often corresponds to the behaviour of parents in play situations at home. This holds especially true for digital (learning) games. The project, therefore, takes up the interesting question of whether the mere presence of an adult has an impact on play behaviour and especially learning outcomes.

Today's everyday play situations include digital games. If children – as is customary in German kindergartens most of the time – are allowed to choose for themselves which game they want to engage with, tablet apps and digital options come into play. Again, these play opportunities can be supervised or guided by adults. There is evidence that children tend to benefit from different settings depending on their learning development:

> There was some indication, however, that instruction by a teacher was more effective for children just beginning to recognize numerals, but the opposite was true for more able children. Children might best work with such programs once they have understood the concepts; then, practice may be of real benefit. (Clements, 2002, p. 162)

Further studies comparing the two approaches to learning support through digital learning games might be helpful.

Methodolgy

The EfEKt study exemplarily evaluates the implementation of one particular math app. The app *MaiKe* (*Ma*thematik *i*m *K*indergarten *e*ntdecken [Discovering mathematics in kindergarten]) is designed as an enriched digital learning environment along the key ideas of mathematics (see above) to encourage children's mathematical thinking and learning processes (e.g., Birklein & Steinweg, 2018; Steinweg, 2016). MaiKe offers six so called different worlds with 10 games each. Overall approximately 480 tasks are presented. Game access is given progressively depending on the progress the child makes. Throughout the worlds, complexity and difficulty of the games increase. MaiKe uses a digital form of feedback called AUC-feedback (answer-until-correct) (e.g., Narciss, 2012). Consequently, for all tasks the number of (unsuccessful) solution attempts is unlimited – which allows trial & error approaches. Each and every mathematical content in the realms of number, geometry, patterns, and measurement is revisited across worlds (spiral curriculum).

The EfEKt evaluation study uses a pre- and post-test design with different intervention groups (setting A and B) and a control group. Children in the control group receive no special support apart from the usual kindergarten's daily activities.

In *setting A*, several tablets are provided on which only the app MaiKe can be played. The tablets are available to the children at any time at their own request for free play of the app (other apps are not enabled). A short introduction on how to use the tablets and how to open the app was given to everyone beforehand. In the further course of the entire project, the kindergarten educators are explicitly asked to assist only if children ask for help.

In contrast to setting A, regular play sessions organised by the researcher take place in *setting B*. The play time and duration is therefore fixed by the project design. Researcher and child meet in one-on-one situations. In principle, the interaction is structured by the individual progress of the child playing the app. The researcher acts as a supervisor and does not give any help in solving the tasks. The adult's conversational impulses are limited to some interview questions placed in each case in the analogous situations and in selected game progressions to each participating child (guided interview). In this sense, the researcher interacts with the children whilst playing in order to get a deeper insight into the children's thinking and learning processes.

Six kindergartens participate in the study and are randomly assigned to one of the settings described above in groups of equal size. The intervention takes place over a period of 1.5 years in total (Fig. 1). The whole sample of participating children (n = 66) is divided into two different age groups. The pre-school children, who constitute half of the sample (age group 1) are on average 6 years old at the beginning of the study. They participate in the intervention for 6 months before the post-test takes place shortly before they start school. The younger children (age group 2) are on average 1 year younger at the beginning of the intervention and take part over the whole period of the study (1.5 years). The age group 2 children completed an intermediate test half a year before their school entry.

Methods of Measurement and Analysis

As shown in Fig. 1, different data sets are collected through different measures that can be used for further analysis and to answer quantitative and qualitative research questions (Birklein, 2020).

In both settings A and B, a specially designed research version of the app MaiKe provides automatically written log files of the individual use. The files make additional usage data within the intervention phase available for analysis. All participants have their own account for using the app. In this way, it is possible to back up the game score and to create individual log files. The log files document the start and end time of each game, the percentages of correct swiping actions as well as trial & error attempts, and the duration of the time played. The log file data is only available for the researcher and not for children, parents, or kindergarten educators.

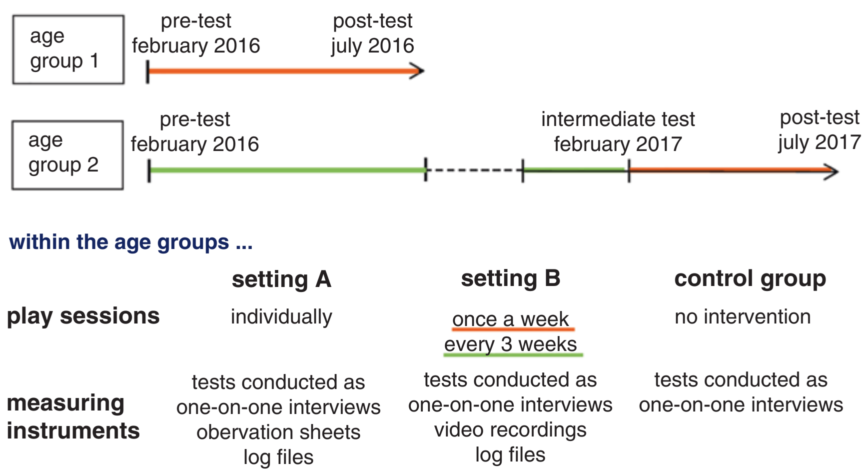

Fig. 1 Design of the study

In setting A, kindergarten teachers fill in observation sheets for each participating child, to document the playing behaviour and any unusual situations that may occur. These data provide additional information for the analysis of the log files.

The development of the children's mathematical competencies are assessed by means of a standardized school entry test, which focusses on fundamental mathematical ideas (in particular numbers & operations, shape & space, measurement & seriation) with special consideration of predictive competencies (LauBe, 2015). The test is designed to be conducted one-on-one (see Fig. 1). The children answered mainly orally or with the help of material. The results are quantitatively analysed via construction of confidence intervals (CIs) with a confidence level of 95% (e.g., Hazra, 2017). The CIs include both, sample size and standard error, and, therefore, provide more information than a comparison of mean values. The CIs allow to compare the groups with each other (CIs for independent samples) and also enable a comparison within the groups over the different measurement times (CIs for paired samples).

The researcher-child-interaction in setting B is recorded by video and documented in transcripts to capture interactions and reactions during the play sessions. These data form the basis for qualitative analysis of individual learning processes, e.g., simultaneous number perception (e.g., Schöner & Benz, 2018) and are documented in detail in Birklein (2020). The qualitative analyses provide an insight into typical reactions when dealing with the MaiKe app and into individual learning and development processes. The case studies provide additional findings that are not represented by the quantitative data and that open up further interesting research perspectives.

In this research focus, special attention is paid to tasks on determining the cardinality of (structured) sets. Eight games throughout the MaiKe app contain different forms of representation of such quantities (finger pictures, dot-fields, etc.). In the available video data, verbal expressions and interactions of the children with the app are analysed in order to get some indication of whether the children perceive a structure in the representations and if they use the structure to determine the cardinality of the given sets. The chosen method combines quantitative with qualitative elements (e.g., Schmidt, 2015). First, a theory-based category system is developed to capture the children's strategies for solving the tasks. The category system is based in particular on Benz's theoretical model (Sprenger & Benz, 2020; Schöner & Benz, 2018). In contrast to Benz, the study presented here includes no eye-tracking. The assigned strategies, therefore, result from the qualitative analysis of children's reaction and interaction (verbal utterances, gestures, and actions). The developed coding guideline includes descriptions and typical examples for each category, which are in particular 'strategy use', 'gestalt matching' (if applicable), 'counting all', and 'trial & error'.

The categorization of children's reactions offers the possibility to identify patterns in the data and to map development processes in order to find out whether there are changes in children's strategies during the intervention period that indicate learning processes regarding the perception of sets and determination of cardinality.

Research Questions

The EfEKt project addresses various issues on digital learning game use and its effects that build on and complement each other. A complete overview of all questions and research results can be found in Birklein (2020). This paper addresses two of the research questions in more detail, and provides exemplary insights into a third question.

The first question takes up the current debate in Germany and addresses the concerns about extensive media use when it is not limited by adults (e.g., in terms of time). The question quantitatively analyses the use of an app by comparing two different types of implementation (with and without supervision by an adult):

1. *Does the type of supervision in the implementation influence the children's behaviour and their progressing through the game?*

The findings of the first question may also provide a possible basis for interpreting results of the second research question, which investigates the quantitative effects on the development of mathematical competencies of children. Moreover, the question addresses the second (German) concern that digital games may have no or negative effects on learning progress:

2. *Does the intervention affect the development of mathematical competencies of the participating children compared to a control group?*

The third question focuses on individual learning processes while playing the MaiKe app that can be identified by qualitative analysis:

3. *What qualitative indications of advancing (more sophisticated) thinking can be identified?*

Results

The findings on each of the three research questions are presented in one of the following sections.

App Use in Comparison of the Different Settings

The log file data is analysed to answer the question, whether there are differences between the settings in terms of time of use and game play. Table 1 exemplarily shows the relevant data for the (younger) children of age group 2:

The children's *progressing through the game* shows that in the free play setting A, the children reach on average the seventh game of the so called fifth world (column #1), i.e. they successfully complete 47 of 60 available games in total. The

Table 1 Comparison of average usage data between setting A and setting B of age group 2

	#1	#2	#3	#4	#5
Ø	Score	Games (with repetitions)	Total time (over 1.5 years)	Number of play sessions	Duration of a play session
Setting A	world 5 game 7	117.8	04:33:20	16.5	00:16:04
Setting B	world 6 game 8	92.9	03:41:11	22.7	00:09:41

children in the more guided setting B reach on average the eighth game in the sixth world, which means they complete 58 different games. Nevertheless, more games are played overall in the free play setting A (column #2). This indicates that children in setting A repeat games far more often than the ones in setting B.

Looking at the *behaviour* in terms of, for example, the total time children spend playing the games of the MaiKe app over the entire project period of 1.5 years, the values hardly differ (column #3). Although the children in setting A use the offer less frequently on their own initiative (16.5 times, column #4), they played a little longer than the children in setting B (on average 5 min more playing time, column #5). The findings are overall comparable to those for group 1. Due to the shorter duration of participation in the project (half a year), these children have not progressed as far in the app of course.

The Development of Mathematical Competencies in Comparison of the Different Setting

The *mathematical competencies* of all participating children from the two settings and the control group are assessed by a standardized test (conducted in one-on-one interviews in all groups), which is usually used at school entry to monitor basic mathematical knowledge acquired in kindergarten. The diagram (Fig. 2) exemplarily shows the results for the children of age group 1, who participated 6 months in the project.

Although the average pre-test results differ between the groups (independent samples), the CIs rate these differences as random at this time. In contrast, the average results of the post-test indicate a statistically relevant difference in two cases of group comparison, i.e., between setting B and the control group as well as between setting A and the control group.

The changes within the groups (paired samples) over the different measurement times from pre- to post-test are statistically relevant for both intervention settings A and B. The development of mathematical competencies in the control group can be considered random.

Even though the mathematical competencies of children in setting B increase slightly more during the project period, this may – following the interpretation of

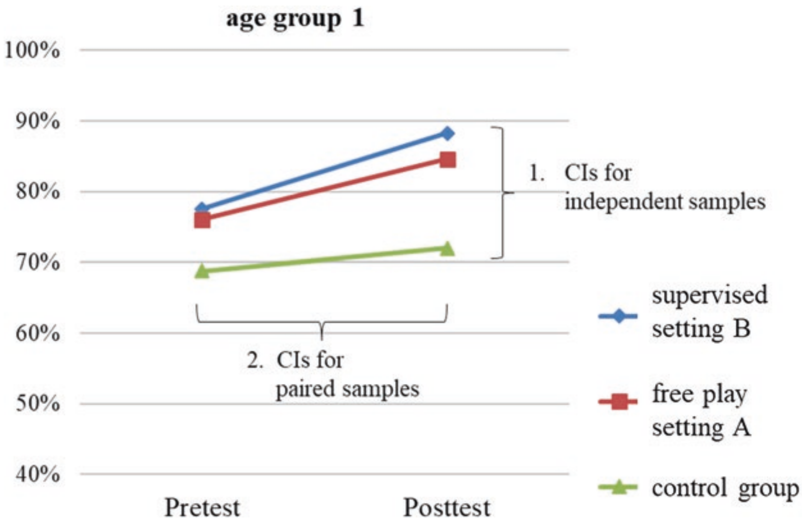

Fig. 2 Development of mathematical competencies of age group 1

the CIs – also be a random effect. In other words, there is no statistically relevant difference between the two implementation settings A and B.

On the Development of Thinking Processes

As an illustrative example of the qualitative analysis, a section of a progress chart for two games on determination of cardinality is presented in this paper. In the selected games analysed here, the required solution number needs to be assigned by swiping movements to the given dot-field on the right (Fig. 3). The remaining number figures in the middle have to be matched to the bin. The seventh game in the fourth world (game 4.7, Fig. 3, left side) is nearly identical to the third game in the sixth world (game 6.3, Fig. 3, right side). In the latter one an animation of a hand, which covers up the given dot-field after a short time (< 3 s) before the number figures appear on the screen, expands the challenge for the children.

The frequencies for the strategies 'structure use', 'counting all' and for a 'trial & error' approach are recorded across all participating children as percentages. In some cases the analysis of the children's interaction allows 'no assignment' to a specific solution strategy.

All children determine the number 8 in game 4.7; none of them uses a trial & error approach. The group of children identified as structure users (45%) is larger than the counting group (32%). In game 6.3 the major part of all children (86%) uses a structure to determine the amount of dots.

The arrows between the tables visualize the thinking development paths. The thickness indicates the occurring frequency. In these two games all children who

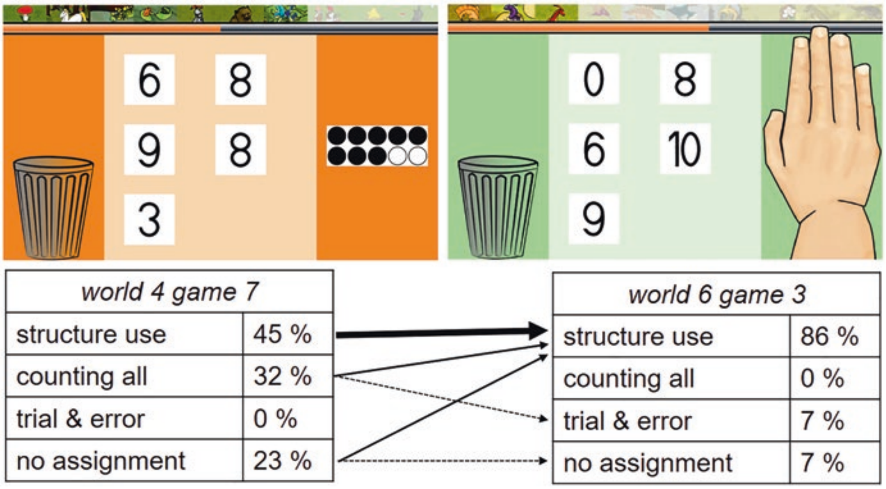

Fig. 3 Section of a progress chart

perceive structures and use them to determine the number in game 4.7 stick to this strategy in game 6.3 as well. Almost all children who counted the elements one by one in game 4.7 perceive structures in game 6.3.

While the arrow diagram of frequencies provides a general overview of solving strategies and development across the whole group of participating children, individual case studies reveal aspects that are not captured in the category system. Thus, it becomes apparent that the children whose solution strategy is assigned to the category ‘structure use’ do not always see the identical structure in given representations and that they use different strategies – sometimes quite flexibly. For instance, Heiko explains his rapid determination of the number 8 in relation to the total number of the dots in the field: “"Because 2 are missing, it’s 8”. Emilia refers to a previous task with nine given dots: “that there were 9 and then I just counted 1 off again”. Lea first derives the 8 from the predecessor: “Because after 7 comes 8”. In a later stage of the game, she also uses the power of 5: “Because 3 points and then 5, then it equals 8”. Across all the case studies, a variety of variants of structures in the category ‘structure use’ and for some children an impressive flexibility in the use of their strategies can be documented. Overall, the decomposition of 8 into 5 (or ‘a row’) and 3 is used most frequently.

Discussion

Comparing the MaiKe app use in the implementation settings, the results are quite alike. In particular, the average time spent playing the app in each session is very moderate. The findings falsify the common assumption that no limitations and no instructions inevitably lead to excessive media consumption. Fears are unfounded that free play – at least in the case of this particular MaiKe app – will lead to extensive use of digital media. These findings are an important argument in discussions with parents and educators in Germany.

The only noticeable difference between the free and supervised play settings is recognisable in the two interdependent values of game progress (score) and repetition rate. The exercise, and thus the learning progress, takes place through repetition and thus replaces the guidance by an adult. These findings are consistent with those of Clements (2002).

The results strongly indicate that the MaiKe app use shows statistically relevant positive effects on the development of mathematical competencies, which are supportive for starting school. This becomes evident in both settings (compared to the control group). Again, the supervision of an adult, as implemented in this project, has no verifiable effects. The children taking their own time and choosing to play the app whenever they want (free play) are in no way inferior in their development to those who have played in presence of the researcher at set times. Both types of organisation, therefore, can be suggested as possible options to integrate an app in the daily routine of a kindergarten (or at home). They both offer opportunities to enhance mathematical learning and are thus more effective than the usual offers in German kindergartens (control groups).

Nevertheless, several studies and projects focus on the importance of learning support for children's learning processes (e.g., Gasteiger, 2010; Klibanoff et al., 2006). The slightly higher development of mathematical competencies in supervised setting B in this study may be an indication in this regard. The difference is not statistically relevant in this study. This may be due to the fact of the very limited guidance in setting B; only in the context of prompting interview questions. Although, the interview impulses probably encouraged the children to deal with the mathematical contents in a deeper way, no further support was given. Therefore, there can be no talk of deliberate learning support – which research highlights as particularly effective – in setting B either. Further studies could explore types of attentive guidance versus free play situations.

The MaiKe app itself, which serves as means and material in this study, influences the test results in the basic competencies (matching the 'big ideas', see above). Due to the design of the app, children need to follow a certain way through the different games. If the solution rate is too low, the app first expects replays of the game before the next games are unlocked. In this sense the app itself acts as a learning guide.

Moreover, the game design influences the strategies used. The deeper insight in learning processes based on qualitative analyses indicates that the children use more efficient and sophisticated ways to determine quantities of sets over the course of the intervention period. Almost all children who were counting one by one in game 4.7 are actually able to use a structure for determinating the quantity in game 6.3 (Fig. 3). One possible reason for these developments may be that the games also become more challenging as they progress. For example, due to the hand animation in game 6.3 and the rapid coverage of the dot-field, less elaborated strategies (‘counting all’) are hardly possible any more. Children for whom such a strategy is not (yet) available therefore fall back on a trial & error approach.

Also, from this it can be drawn that free play (setting A) is not so free after all. At the same time, this also may explain why the development does not differ significantly in the different settings. The efficacy of playing with (learning) apps without further instructions necessarily depends on the app design. The tasks provided in the app must be mathematically and educationally sound – however, this condition might not be sufficient.

A promising research perspective may be the comparison or the combination of digital tasks with their physical material-based counterparts in terms of learning progress on mathematical key ideas and, where appropriate, understanding of mathematical concepts. Each game in MaiKe is inspired by common kindergarten material like building blocks, number cards, dice etc. The possibility of easily translating the digital into a material-based game was deliberately taken into account in the MaiKe app-design in order to generate ideas for real learning environments in kindergartens through the app. This kind of research is particularly worthwhile because there is a broad consensus that an app (like MaiKe) cannot replace a rich and stimulating play and learning environment, but it can complement it.

In further research, children’s prior knowledge may be taken into account. The experience gained in the context of this study as well as existing evidence suggest that children’s prior knowledge can have an impact on the way they deal with the mathematical content, on the reception of the digital feedback, on the willingness to repeat tasks and further rehearse their own knowledge, or on motivation in general.

In this study there is hardly any evidence of mathematical competence development in the children of the control group. This is at least remarkable and alarming from a mathematics educational perspective. In daily routine of kindergarten, too little attention is obviously paid to recognising mathematical learning opportunities and making them fruitful. German kindergarten educators are usually not trained in mathematics. This spin-off of the analysis of the results shows how important it would be to reorganise the training of kindergarten teachers.

When interpreting these and other results, it is important to note that they refer to this exemplary selected MaiKe app and a small sample. The specific concept as well as the design allow only a limited transfer to other digital learning environments. In order to allow generalizable conclusions, a study with a larger sample of apps and children has to be carried out to verify the indicated tendencies.

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The Tension Between Division and Fair Share

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Introduction

One of the key concepts in mathematics is division (e.g., Kiselman & Mouwitz, 2008), and although children most often tend to divide into equal parts, when it comes to sharing resources, it is not always straight forward (Wong & Nunes, 2003). Previous studies conclude that children's understanding often is a result of their experiences of sharing (e.g., Davis & Pitkethly, 1990; Desforges & Desforges, 1980; Squire & Bryant, 2002a, b), and looking at preschool children, they often learn about sharing in preschool, as well from home and from friends (Borg, 2017). At the same time, there are reports that these every day experiences can act as an obstacle for understanding of division as equal parts (Smith et al., 2013; Wong & Nunes, 2014). Even though one might think that division is a higher form of sharing, a fair share is not always the same thing as division (Hamamouche et al., 2020; Hestner & Sumpter, 2018). It is about how resources should or could be allocated (Chernyak & Sobel, 2016; Hestner & Sumpter, 2018; Smith et al., 2013). Context matters when deciding what is a fair share (Huntsman, 1984; Sigelman & Waitzman, 1991; Wong & Nunes, 2014); for instance, a study on 5 years old show that they take

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different aspects into account when deciding how to share resources, so that someone identified being in need could get a larger amount (Enright et al., 1984).

At the same time, there is a growing body of research indicating that children as young as one can understand sharing into equal parts (e.g., Geraci & Surian, 2011; Sommerville et al., 2013). The norm of fair share appears to be strong already from a young age. This norm has been discussed in studies on ethical reasoning about sharing resources: children express that they know that they should divide resources into equal parts, even when they decide not to do so (Smith et al., 2013). From a mathematical point of view, this is not division (Correa et al., 1998), but it could function as a starting point for ethical reasoning around sharing resources and thereby address issues with respect to sustainability (Hedefalk, 2015). Given that 'fair' is not an unequivocal concept, values are therefore an important topic for teaching sharing, independent if the aim is to discuss values or to talk about division. Such discussions are relevant already at preschool level: in the Swedish curriculum for preschool, it states that children should be provided the conditions to develop:

> The ability to discover, reflect on and work out their position on different ethical dilemmas and fundamental questions of life in daily reality (Skolverket, 2019, p. 13)

It is therefore relevant not to neglect or disregard children's reasoning where sharing is done in unequal parts. Instead, it is of interest to understand the arguments backing up the child's reasoning (Hedefalk et al., 2022). The aim here is to study preschool children's' collective mathematical reasoning about sharing. The research questions are: (1) What mathematical properties do children use in their reasoning?; and, (2) When is mathematical reasoning replaced with ethical reasoning?.

Background

Mathematical reasoning can be defined in many different ways (Lithner, 2008; Sumpter, 2016), and here, the choice is to see collective mathematical reasoning as a collective line of arguments that is produced when solving a task. This is seen as a collective effort that aims to create meaning (Eriksson & Sumpter, 2021; Sumpter & Hedefalk, 2018). Reasoning is therefore a social process with the assumption that mathematical reasoning is crucial for the understanding of mathematics (e.g., Herbert & Williams, 2021). Lithner (2008) suggested the following reasoning sequence with four steps as follows: (1) a task situation (TS) is met; (2) a strategy choice (SC) is made where the 'choice' should be interpreted in a wide sense; (3) the strategy is implemented (SI); and, (4) a conclusion (C) is drawn. We then apply Toulmin's (2003) model for each of these steps, which means that the task situation can be supported by identifying arguments (Eriksson & Sumpter, 2021), the strategy choice and implementation can be supported by predictive and verifying arguments (Lithner, 2008), finally, conclusion can be supported by evaluative arguments (Sumpter & Hedefalk, 2018). Each of these steps join in a chain of arguments, an argumentation, that has components described as data, warrant, backing, and conclusion, with the latter step differing from how the conclusion is presented in the reasoning sequence. In this way, based on Sumpter's (2016) integration of mathematical reasoning and argumentation, reasoning is seen as the vertical line of the reasoning sequence (TS – C) whereas argumentation is the horizontal line (i.e., the four different types of arguments). In order to analyse the content of the arguments, Lithner (2008) proposes the notion of 'anchoring' mathematical properties in the components of the arguments. The different mathematical properties are objects (e.g., natural numbers, rational numbers), transformations (e.g., division), and concepts (e.g., the integer concept) that consist of sets of objects and transformations. Thereby, collective mathematical reasoning is similar to how mathematical discussion is defined by Pirie and Schwarzenberger (1988): as a purposeful talk (in our case, solving a task), on a mathematical subject (here with the emphasis on relevant mathematical properties of the different arguments), in which there are genuine pupil contributions, and interactions.

Division can be defined as a/b where a is the dividend (numerator) and b is the divisor (denominator), and the result is described as a fraction, quotient, or ratio. Division can be seen as an inverse transformation to multiplication, that $a/b = k$ if and only if $a = bk$ where $b \neq 0$ (Kiselman & Mouwitz, 2008). In school mathematics, division is often viewed either as quotition or partition. The common core for either of these is that the shares (i.e., fraction, quotient, or ratio) are of equal size. This is the main difference between division and sharing, where the latter can accept unequal shares (Correa et al., 1998). Studies has shown that when solving mathematical tasks that involve sharing resources, children/teenagers can use both mathematical properties and ethical properties such as values (Chernyak & Sobel, 2016; Hedefalk et al., 2022; Hestner & Sumpter, 2018). One example of such study is Enright et al. (1984) where children age five were asked to share resources, and recipients that were identified as having greater need got larger shares. Studies has also shown that children as young as two, expect sharing to be in proportion to effort (e.g., Sommerville et al., 2013). Using the same starting point as for mathematical reasoning, we define ethical reasoning as a collective line of arguments that is produced when solving a task, but where the arguments are anchored in values (Sumpter & Hedefalk, forthcoming). This is similar to moral reasoning (Samuelsson & Lindström, 2020). We follow Samuelsson's (2020) criterions for deciding whether an ethical reasoning is sustainable or not by using his SIL methods: (1) coherence (S); (2) information (I); and, (3) vividness (L). This implies that sharing based on ethical reasoning can include division sharing in equal parts as well as sharing in unequal parts. The ethical argument is coherent when it does not contain logical flaws, is based on correct and relevant information and motivations that a listener is willing to accept (Samuelsson & Lindström, 2020). However, facts are not enough to make an ethical decision about sharing: the child needs to mentally make the sharing task vivid to try to understand another person's (or soft toy's) point of view in the sharing experience. One example of a statement lacking vividness is "It is fair", whereas the statement "It is fair since X and Y" provides a backing to the claim 'fair' (e.g., Toulmin, 2003) and thereby provides an element of vividness (e.g., Samuelsson, 2020) to the reasoning. If the argumentation consists of all three parts (S, I, and L), it is considered that the child has made an ethical argument about sharing (Sumpter & Hedefalk, forthcoming).

Methods

In order to analyse different types of collective reasoning, we used two tasks describing different scenarios of resources that needed to be shared among recipients. The tasks were the first two in a set of six that had been developed and tested earlier, where each task described different mathematical properties and different ethical issues (Sumpter & Hedefalk, forthcoming). The first task was an open task where the children were asked to divide 12 biscuits (in coloured paper) between three soft toys (a teddy bear, a dog, and a tiger). If the children decided on a solution that was not division, they were asked again as a follow up if they could make the sharing into equal parts. The reason for this was to see if division was an option at all, but the instruction was not to credit any solution as the correct solution. The second task was to divide four biscuits between the three soft toys. Again, the children were free to come up with any solution, but the instruction was that all biscuits needed to be shared (i.e., it was not ok to give back or toss away the surplus biscuit).

Six children worked in pairs together with one of their pre-school teachers. The instruction for the teacher was to ask questions to stimulate arguments such as "What are you thinking?", but not to give any evaluation of the solution (i.e., "This is in/correct"). The children were in the following pairs: (1) Noel (age 5y 8m) and Maya (age 4y and 9m); (2) Nova (5y and 2m) and Ida (5y 1m); and, (3) Adam (5y 6m) and Anna (5y 2m). All children are born in Sweden and have another language as a first language, apart from Noel, who arrived in Sweden 3 months prior to the recordings. Noel speaks almost fluent Swedish.

Their work was videotaped and these videotapes were transcribed verbatim, including actions according to principles presented by Mergenthaler and Stinson (1992). From these principles follows that an argument could also be a gesture or nonverbal action from the children. The second stage of the analyses was to organise the transcripts according to the mathematical reasoning structure, TS, SC, SI, and C (e.g., Lithner, 2008), and arguments for each step were identified. The arguments were then analysed using the notion of anchoring of mathematical properties, for instance the transformation division as a repeated subtraction, thus giving biscuits to each of the three soft toys, one at the time. The last stage of the analyses was to look at the arguments using Samuelsson's (2020) method-based model, originally developed for teaching ethics but here used as an analytical tool (e.g., Sumpter & Hedefalk, forthcoming). Here, we are interested in how the arguments change when children decide to make choices connected to ethical values about sharing. The arguments not based in mathematical properties were analysed using the three SIL criterions: (1) coherence (S); (2) information (I); and, (3) vividness (L). The study follows the ethical principles of the Swedish Research Council. That means, for example, that the parents have signed a letter of consent and that the names of the children are anonymised. The children were informed that they could end their participation at any point of the recordings.

Results

Table 1 presents an overview of the different types of collective reasoning:

Starting with the first task, the three pairs used different types of reasoning. The first pair, Noel and Maya, started with an ethical reasoning and there was a tension between them where Noel initially wanted to give more biscuits to two of the soft toys and less to the tiger:

Noel: Me not like tiger, it can eat me!
Teacher: Ok, that is your [way of] thinking. But you like the dog? And that is why it got more of the biscuits?
Noel: Yes, I gave it a lot, a lot, a lot [stressing the importance].

The reasoning is considered an ethical reasoning according to the SIL- method, since the argument for the sharing in unequal parts was justified with the argument from Noel that he does not like the tiger since it is dangerous (it can eat him up), and using opposite argumentation regarding the dog. The motivation was lively (as the child conveys the tigers hunger feelings and can see the consequence if it acts on that), informed (tigers eat humans) and coherent (logical reasoning in his way of thinking). As a second step, when the children were informed to share equally, a conflict arises when Noel wants to give more biscuits to the rabbit who is, according to Noel, “hungry”. His argument was lively (as the child conveys the rabbits hunger

Table 1 Mathematical and ethical reasoning

	Pair 1 Noel, Maya	Pair 2 Nova, Ida	Pair 3 Adam, Anna
Task 1	SCI: unequal parts. SIL SC2a: (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1) 9/3, r = 3 SC2b: 3/3 = 1 C: 4 = (1 + 1 + 1) + 1	SC: unequal parts C = {4, 3, 5}	SC1a: 9/3 3 + 3 + 3, r = 3 SC1b: 3/3 = 1 C: 4 = 3 + 1
Task 2	SCa: 4/3 = 1 reminder 1 SCb:{1, 2, 1} – SIL SCc: 1/n (n not determined), shared in 3 SCd: (4/(1/2))/3 Cd: {1.5, 1, 1.5}	SCa: 4/3 = 1 reminder 1 The surplus biscuit should be eaten up to preserve equal parts. SCb: 1/n (n not determined), shared in 3	SCa: 4/3 = 1 reminder 1 Tension: Anna wants unequal sharing, Adam argues for equal parts. SCb: ½ + ¼ + ¼, cardinal 3 SCc: ¼ + ¼ + ¼ + ¼ SCd: (1/4)/3 = 1/12 C: 1 + ¼ + 1/12

feelings), informed (he is aware of the amounts of biscuits and receivers) and coherent (logical reasoning that can be accepted). The other child, Maya, opposes and justifies that equal also means fair (i.e., the same number of biscuits for all stuffed animals). Maya's argument at that point was informed (i.e., she is aware of the amount of biscuits and receivers), coherent (logical reasoning that can be accepted) but it is not interpreted as lively, according to the SIL-method, since she does not express why and how it will affect the soft toys. This part of the reasoning was considered a mathematical reasoning where Maya's argument of equal parts is accepted by Noel with the transformation 12/3 = 9/3 + 3/3 where both divisions were made as repeated subtraction. The collective reasoning was thereby a result of a negotiation. The second pair, Nova and Ida, decided to keep the initial decision to share the biscuits in unequal parts although the teacher tried to encourage them to try division. Neither of them gave any arguments to why, and their reasoning was considered neither ethical reasoning nor mathematical founded reasoning. The third pair, Adam and Anna, had no problem to share nine biscuits in groups of three. It took some encouragement from the teacher for them to realise that it was ok to share the remaining three biscuits as well.

Looking at the second task, the first pair again struggled to agree between sharing in unequal parts and division. Maya, again, stated "it has to be fair" whereas Noel argued for a solution where the tiger and the rabbit got one biscuit each whereas the dog got two since "he is really hungry". The reasoning here is considered ethical reasoning according to the SIL-method. The result is sharing in unequal parts. The second pair, Nova and Ida, suggested that the remaining biscuit should be eaten up, without any further arguments. When encouraged to divide the remaining biscuits into smaller parts, they continued to cut the biscuit in smaller and smaller parts and then sharing these to the three recipients without any signal that it should be equal. The third pair experienced the same tension when Adam argued for division and Anna suggested sharing in unequal parts, where it was Anna who took the scissors first. Although they agreed on the strategy choice, to divide the surplus biscuit into pieces, they disagreed on how it should be done, see Table 2:

Adam expressed verifying arguments to support the implementation of the strategy, that everyone should have one [piece] of the remaining biscuit, where it appeared not so important that the parts are of equal size. Although the final solution was 1 1/3 biscuits (1 + ¼ + 1/12), the conclusion was not supported by any arguments. Also, given that Adam earlier argued for a solution with unequal sizes of the parts, it is more plausible to assume that the conclusion is a result of random actions more than a result of an informed strategy with an argumentation backed up with claims or warrants.

Table 2 Pair 3 reasoning about 1 divided by 3

Time	Person	Data	Reasoning Structure	Arguments
07:18	Teacher	What can you do then? [putting the scissors in front of the children]	TS initiated	
07:21	Anna	[Bend forward and picks up the scissors and the biscuit] Cut it!	SC:1 divided into parts	Predictive argument: dividing is necessary
07:23	Teacher	Uhm. Do it.	SC confirmed	
07:31	Anna	[Cuts the biscuit into halves]	SI: 1/2	Verifying action
07:36	Adam and Anna	Also this one you should cut. [Picks up one of the halves and cuts it into two bits] Everyone should have one bit [Gives one bit to the tiger, here named as lion. At the same time, Anna picks up the other half and the scissors]… and then [gives one bit to the dog]… and then [tries to take the bit Anna has in her hand]	SI: 1 divided into three parts, where the focus is on cardinal value of 3, not equal size.	Verifying arguments: everyone should have one bit. Predictive argument diving is necessary (equal size of the parts) Identifying argument: n should be 3
07:48	Anna	No!		
07:48	Adam	It should have it!		Stressing without further arguments.
		[Meanwhile, Anna cuts the half into quarters]	SI: 1/4	
07:50	Adam	Four! [sounds disappointed, open his arms and hands as to stress the conclusion]	C: 1 is divided in 4 parts	Evaluative argument identifying the new problem, once again there are one extra piece: 1 divided by 3.
07:57	Anna and Adam	Anna shares out the quarters, Adam takes the extra quarters and cuts into three bits which are shared under further discussion between the two.	TS: ¼ should be divided SC: (1/4)/3 = 1/12 SI: Straight forward C: 1/12 is added to 1¼	No arguments.

Discussion

Starting with the mathematical properties in the collective reasoning, the results showed a variation of mathematical components. Looking at the different transformations in how sharing was made, the most common strategy choice was division as repeated subtraction, one item to each recipient at a time. One pair immediately created the subset '9' of 12, and grouped the nine items into three groups of equal size. Here, we do not have further information on why sharing out the remaining three items was considered a difficulty, which is an interesting topic for further research. Regarding the transformation 1/3 was a challenge for all three groups, including a tension between the idea of equality and other counter arguments. Just as previous studies, this was by no way straight forward (e.g., Wong & Nunes, 2003). One child, Maya, tried several times to convey to her partner that when sharing resources, it has to be fair, and here, the norm of equal parts appeared to be strong (e.g., Geraci & Suriam, 2011; Smith et al., 2013; Somerville et al., 2013). However, since it is not clear what fair means in this particular situation and therefore we cannot draw any further conclusions about this situation. One situation where there is more information, is the situation where sharing is done in equal amount yet unequal sizes. Then, the main argument was based on the mathematical property of the object '3' which was cardinality. There was no argumentation, at least not explicit in words or actions, about the size of the parts. Similar reasoning was noted in Sumpter and Hedefalk (forthcoming). The implication is that if wanting to challenge young children and their reasoning about division, it might not be so much a question about a items shared by b recipients as much it is about the sizes of the parts, especially when $a < b$. This means that although it is of importance to understand division both as quotition or partition (e.g., Schmidt & Weiser, 1995), and the central mathematical properties of the quotient (ratio) is vital given the difference between division and sharing (Correa et al., 1998). Then, one vital step might be to explore the relationship between division, fraction, and measurement (e.g., Eriksson & Sumpter, 2021), instead of increasing the size of the dividend.

Looking at when mathematical reasoning was replaced with ethical reasoning, there were some instances where the context matter (e.g., Hester & Sumpter, 2018; Huntsman, 1984; Sigelman & Waitzman, 1991; Sumpter & Hedefalk, forthcoming; Wong & Nunes, 2014): the tiger was scary, the rabbit was hungry, and the dog was more worthy since a child liked it. The context here was mainly emotional, which is one part of ethical reasoning (Samuelsson & Lindström, 2020). However, although vividness did function as an analytical unit for our analysis, we anticipate that given the age of the children, it can be difficult to formulate arguments that a listener is willing to accept (e.g., Samuelsson & Lindström, 2020), especially if mathematics and values are interlaced (Hedefalk et al., 2022). Here, the context of the tasks was relatively neutral, and as a theoretical concept it might need some more methodological work in order to function with young children and situations where values play a bigger part in the reasoning.

As stated in the beginning, the Swedish curriculum for preschool stress that children should get the opportunity "to reflect on and work out their position on different ethical dilemmas" (Skolverket, 2019, p. 13). Although the two cases tested here did not explicitly invite to ethical arguments by providing information of one of the recipients having a greater need (e.g., Enright et al., 1984; Sumpter & Hedefalk, forthcoming), the cases did offer the children to express different arguments and provided several opportunities for compromises through negotiation of different strategy choices. The study of such reasoning is something that could be further developed, especially if wanting to use it as a starting point for exploring sustainability issues (e.g., Hedefalk, 2015; Samuelsson & Lindström, 2020).

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Collective Problem Solving in Peer Interactions in Block Play Situations in Kindergarten

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Introduction

Research in mathematics education addresses the fact that mathematical learning already takes place before children enter school. Children grasp first mathematical content in their everyday life, both in the home environment and at kindergarten, and approach the process of mathematisation. This happens mostly in play or in play situations in which the children can, for example, become aware of the "quantitative and spatial dimensions of reality" (van Oers, 2014, p. 115) in problem-solving situations. In this context, it is not only mathematical content, such as the arithmetic-based quantity comprehension and counting skills described by van Oers (2014) and the (spatial) geometric ideas that are acquired by the children in play, but also process-related skills, such as reasoning (Vogler, 2021; Krummheuer, 2013), modelling and problem solving (Sumpter & Hedefalk, 2015; Di Martino, 2019; Carpenter et al., 1993). These process-related competencies are considered by studies, especially in situations with elementary pedagogical professionals, in which adults usually play a significant role in the process of negotiation of meaning (Vogler, 2021). Although researchers such as Carpenter et al. (1993) and Lopes et al. (2017) describe that adults in their role as more competent persons of reference can act as a special role model in interactions, it is often also peer interactions that have a significant influence on early learning processes. Especially in regard to process-related competencies, such as problem solving and in those involved in processes of negotiating meaning, peer interactions benefit from the unique

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interactional proximity between the participants (Vogler et al., 2022; following Vygotsky, 1978). Moreover, peer interactions make up a large part of the interaction time in kindergarten. During this shared time, a variety of new experiences are made by the children. This also includes the exchange, deepening and networking of mathematics-related knowledge in different mathematical domains (Henschen et al., 2022; Henschen, 2020). It is therefore particularly surprising that research has increasingly been focusing less on peer interactions and more on interactions with adults. Exceptions to this trend include, for example, work by Henschen (2020), Helenius et al. (2016) and Flottorp (2011), which examine peer interactions and their contribution to mathematical learning. In this context, Helenius et al. (2016) explain that block play situations with Legos in the kindergarten setting is not only mathematical when explicit mathematical content is negotiated, but also when problem-solving processes emerge in the negotiation process between the children (Helenius et al., 2016). In line with said research by Helenius and colleagues, this paper examines two different block play situations in kindergarten regarding reconstructible (collective) problem-solving activities. It will describe which characteristics of problem-solving processes emerge among children in these situations and how these are interwoven in the negotiation process with various mathematical content and other competencies that are important for mathematical learning processes, such as argumentation. The aim here is to generate initial insights into the extent to which problem solving in particular has a remarkable influence on the conditions for the opportunity for early mathematical learning.

Theoretical Remarks

Early Mathematical Learning (Through Problem Solving) in Co-construction

There has been a lot of controversy for a long time about whether collective or individual problem solving is more conducive to learning in kindergarten: some researchers, among them Piaget, held the view that individual work was more productive because of the egocentrism of young children; however, both Vygotsky (1978) and Mead (1934) argued that collaboration was more beneficial. Azmitia (1988) and others have shown that collective problem solving in interactions leads to more sustainable learning success, especially for younger children. These studies support the co-constructivist perspective on early learning (in peer interactions) of Vygotsky (1978) and Youniss (1980), which also serves as the theoretical starting point for the analyses of mathematical learning and problem solving presented in this article. Key variables of this perspective are the situational negotiation processes of mathematical meaning in interactions. From this (co-)constructivist perspective, learning is to be seen as a process of becoming increasingly autonomous in these interactions of mathematical discourse. In line with this, conditions for the opportunity of learning of 'the new' (Krummheuer, 2013; Miller, 1987) are created by enabling children to actively and productively participate in the negotiation processes of discourse, thus opening up various scopes of participation called "leeways of participation" by Krummheuer (2013, p. 251; following Brandt, 2004).

Collective Problem Solving

Based on the findings of various studies, problem-solving situations appear to be particularly appropriate for enabling such actively productive participation in negotiation processes of mathematical meaning and thus co-constructive mathematical learning. Polya (1962) defines problem solving as the attempt to find an appropriate activity to reach a desired point without being able to achieve the actual expected end goal. Following Polya's definition, Avcu and Avcu (2010) explain that if mathematics is problem solving, then problem solving can be defined as the elimination of the problem situation through the use of critical thinking processes and the required knowledge. In this context, Baroody (1993) states that problem solving requires mathematical thinking. In the context of the mathematics classroom, Lopes et al. (2017) explain that in problem solving – starting from a problem formulated by the learners – the classroom becomes a place of questioning and contextualisation in which the children discover mathematical relationships based on their own everyday experiences. In this context, problem solving is distinguished from (routine) task solving because a knowledge structure (or epistemic structure) alone is not sufficient for problem solving; heuristic ways of thinking become necessary here. In their recommendations for teachers, Hatfield, Edwards, Bitter and Morrow (2007) emphasise that these strategies help students to make progress in solving challenging and difficult problems.

Heuristic Strategies of (Collective) Problem Solving

Some of these heuristics that are described by Polya (1945) in his work "How to solve it" are the "analogy" (principle), "decomposition" (and recombination), "symmetry", "generalisation", "invariance", "working backwards" and "working forward" (and combinations), systematic resp. "intelligent guessing and testing" and "representation change" ("draw a figure") (Schoenfeld, 1987, p. 284). As the focus of this article is mainly on the development of individual strategies in collective problem-solving processes in peer situations in kindergarten, only some of these strategies are described in more detail here. Thereby, the intelligent guessing and testing, which can also be reconstructed in the following analyses, "is guessing and trying processes to check the probable conditions" (Avcu & Avcu, 2010, p. 1284). Working backwards is described as a "useful and efficient strategy" (Amit & Portnov-Neeman, 2017, p. 3793), which can also be traced in the interactions of the children in our article: Here, the problem solver starts working backwards when the goal is clear but there are many possible starting points. Finally, the heuristic strategy of decomposition (and recombination) should be explained here. This involves decomposing a problem into subproblems, each of which can then be solved more easily. Elia et al. (2009, p. 607) describe following Charles and colleagues that various heuristics can be "introduced [and used] in primary or middle school mathematics teaching".

Problem Solving in the Early Years

Concerning early mathematical problem-solving in kindergarten, the question arises: How can very young children, who are mainly illiterate, solve mathematical problems? Lopes et al. (2017, p. 254) note that this type of question reveals the "misconception of early mathematical problem-solving" activities that solving mathematical problems means calculating or applying a set of rules (or an algorithm); this misconception has yet to be overcome. However, building knowledge through trial and error is also part of problem solving. Through exploration and experimentation, hypotheses can be analysed and solutions can be explored, making learning individual and meaningful for children. Children construct meaning through their efforts to discover or invent, so the novelty (knowing) described by Miller (1987) can be co-constructed. Vygotsky (1967) and Helenius et al. (2016) describe the central role of play in early mathematical learning and problem solving. More generally, Vogt et al. (2018, p. 592) outline that "innovative approaches to early mathematics should not only be developmentally appropriate and effective but should also be compatible with kindergarten pedagogy. Since kindergarten children are highly motivated to learn, but not in a formal way, play can be seen as a powerful tool for learning". For researchers such as Lopes et al. (2017) and Carpenter et al. (1993), adults in particular are in the role of the more competent others who create a playful environment in which children are confronted with problems.

Peer Interactions in Early Mathematical Learning

Chaiklin (2003, p. 42) also describes the unique role of the "more competent other person" in mathematical learning processes. They are considered to be the persons who, through their advantage and scope of (established) mathematical knowledge, introduce children to the culture of mathematics and through negotiation support children in participating in mathematical activities (such as solving problems) in play situations. In addition to adults, however, peers of the same age can also play an essential role as interaction partners in problem-oriented mathematical negotiation processes, as the studies mentioned above by Azmitia (1988), Di Martino (2019) and also Helenius et al. (2016) show.

Research Desiderata and Questions

While Flottorp (2011) deals with the extent to which various mathematical contents are negotiated in peer interactions, Henschen (2020) also highlights general process-related competencies and the mathematical content areas. Helenius et al. (2016) focus specifically on problem solving in peer interactions in (block-)play and discuss that this play can be categorised as mathematical, specifically through the process of problem solving. However, Helenius et al. (2016) focus less on how different heuristics emerge in such peer interactions and to what extent they create conditions for mathematical learning. In our paper, we try to close this research gap and to further elaborate on the approaches of Helenius et al. (2016). The following questions focus on the qualitative reconstructive analysis: (1) Which procedures in problem solving (heuristics) emerge in play situations among children in kindergarten? (2) How can these heuristic practices in the negotiation process of meaning create conditions for the opportunity for mathematical learning?

Data and Methodology

Data Corpus – Block Play with Peers in Everyday Kindergarten Life

The data basis for the research questions are videos of 30–90-min free play situations with building materials from everyday life in kindergartens. The ethnographic data was collected as part of Henschen's (2020) PhD project on block play. The videos were recorded by a student who was familiar to the children from previous internships during her studies; consent was obtained from the kindergarten and the children's parents. From these internships, both the student and the children were already familiar with the video recording of play situations. The children observed were mainly from a group of four children (Max 4;10, Ron 4;8, Emma 6;0 and Anna 5;3 years old) who worked with one of the materials available in the block play area all together, in different group compositions, and sometimes with other children. Therefore, no specific or new material was selected by the researcher. The play situations were filmed during some of these free play phases over a period of 4 weeks. Two paradigmatic examples of such play situations were selected for the comparative analyses.

Basis of the Methods of Analysis

Using (qualitative) thematic analysis (Kuckartz, 2014), Henschen (2020) developed two different coding frames. One coding frame addresses the connection between mathematical content and informal mathematics in children's block play (categories

Fig. 1 Meaning of the categories. (According to Henschen, 2020, p. 403)

of the content level). The other differentiates the ways children work in their block play (Henschen et al., 2022). Henschen developed the categories of the second coding frame based on literature (Siraj-Blatchford & MacLeod-Brudenell, 1999, p. 68; Bruce et al., 1992, p. 124): "making/monitoring", "constructing/building", "evaluating/titling" and "designing/adapting". They reflect that for block play and technical learning opportunities, certain steps in problem-solving processes need to be described. The following Fig. 1 shows how these categories can be understood and illustrated following Henschen's (2020) work.

The categories at the content level are "wrong way-right way", "small-large", "slanted-straight", "open-closed", "fixed-unfixed" and "equal-unequal" (Henschen 2020, p. 278). These categories can be understood as kinds of "natural categories" (Kuckartz 2014, p. 44) because the children's spontaneous use of language is also considered in the category designations (Henschen, 2020; Henschen et al., 2022). Although these categories are conceived as "natural categories", they allow the description of mathematical content in block play situations.

Methodological (Pre)considerations and Methods of Analysis

While Henschen (2020) develops categories that can be used to identify and describe mathematics in play situations, the research questions raised in this paper require a micro-analytical expansion of the methodological approach. This analytical focus is realised with the use of interaction analysis from the field of interpretative research in mathematics education, which among other sources can be traced back to work by Krummheuer (2013). The analysis can be seen as particularly suitable because it focuses on processes of negotiation of meaning in 'moments of crisis', in which, as can be concluded here, the above-mentioned problem-solving processes are initiated.

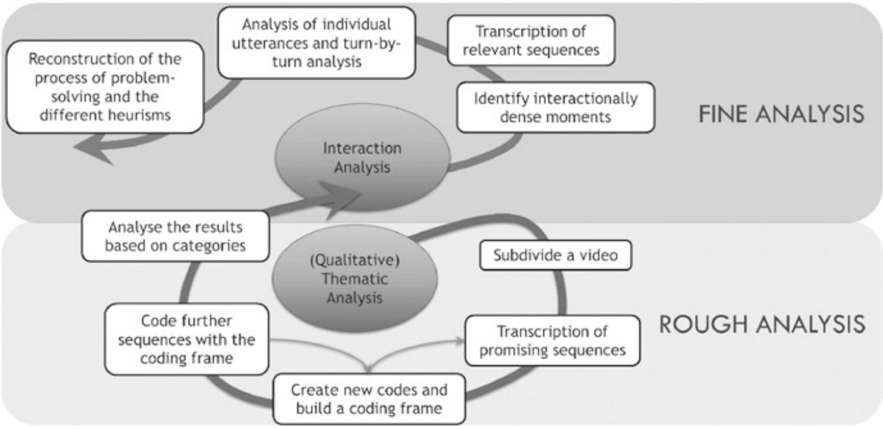

Fig. 2 Combination and integration of methods. (Following Henschen et al. 2022)

This expansion allows for a focus on problem solving, which is illustrated in Fig. 2.

In this context, Henschen's approach provides a fundamental rough analysis. This process of rough analysis was used to identify sequences that are mainly characterised by the density of mathematics-related negotiation in which problem-solving processes may emerge. These interactionally dense moments can be used as a basis for the fine analysis, which follows the qualitative interpretative paradigm and corresponds to an analysis of interaction. Within the framework of the analysis of interaction (Krummheuer, 2013), the thematic negotiation processes can first be worked out, on the basis of which subsequent processes and heuristics of problem solving (Schoenfeld, 1987 according to Polya) were reconstructed. Finally, conditions for the opportunity to learn can be deduced from these reconstructions. In the following, an analysis is realised using two paradigmatic examples. Especially the first part (analysis of the ladder example, see below) shows detailed results of the rough and fine analyses and how these analytical steps are interwoven and mutually enrich each other. The second part (analysis of the roof example, see below) provides the comparative element of the analysis (Krummheuer, 2007; Krummheuer & Brandt, 2001).

Analyses of Empirical Examples of Block Play Situations in Kindergarten

Analysis of the Ladder Example

Before the analysed situation begins, Ron and Max have already been working for some time with a pre-existing construction made of SONOS material. In minute 26, after Max has attached a construction with wheels (the children also refer to it as a "forklift") to the top of the structure. Ron remarks, "Why all the way to the top, then they can't get it down at all". Max then says that "they" have a "ladder", and Ron suggests that they build a ladder. After the two of them have made a short piece of the ladder (4 rungs), they turn to the structure with this piece.

004	Ron:	And then how do they get up there?

In connection with Ron's previous statement that the forklift cannot be brought down from the very top, the question can be understood as a problem definition. However, what is remarkable is the apparent everyday world reference to Ron's statement. Perhaps Ron disagrees with Max's suggestion to fix the (too) short piece of the ladder and therefore asks for clarification, which does not directly follow.

Looking at the rough analysis for this scene, the following categories can be assigned with regard to mathematical aspects. Here, Max and Ron refer to the localisation or direction of objects, which is an example of spatial orientation. In this context, the category "wrong way-right way" can be reconstructed. Linked to this finding, but going even further, the category "open-closed" can be assigned here. These topological category is addressed when the connection between two places/points or the accessibility is mentioned. However, in various subsequent scenes, it becomes evident that the children are still looking for a solution to the problem.

026	Max:	No, we'll stack it on the floor
027		*lays the ladder through the building* (Fig. 3)
028		We stack it like this.
029		*makes a stacking gesture with his hand*
030	Ron:	*takes out the ladder*
031		No, we stack it this high.
032		*places the ladder diagonally on the floor.*
033		Because otherwise, they wouldn't be able to get up there.

Max's utterance can be understood as a counter-proposal to the fixing. He suggests integrating the ladder into the building, with high flexibility and possible utility for the shared play context. In this context, Max uses the expression "stacking", as does Ron in the situation. In the scene, Max identifies the previously built ladder as a representative for several ladders to be stacked. Stacking identical objects by placing them appropriately on top of each other can be understood as an engagement with the geometric relation parallel to. Therefore, the category "slanted-straight" can be assigned to the action.

Ron builds on Max's idea by combining it with his idea of using the ladder diagonally on each floor. Ron adopts Max's expression "stacking" in the sense of positioning one on top of the other within the structure and varies the ladder's position in that it is placed diagonally on the floor. By limiting themselves to the task of "stacking", the children are able to explore and eliminate different possible solutions for positioning the ladder. This can be seen as a propaedeutic to the heuristic intelligent guessing and testing.

Fig. 3 Ladder within the building

While the rough analysis shows the connection to mathematical content and activities, the fine analysis can reveal the collective problem-solving process. It becomes obvious how both are interwoven with each other and can mutually stimulate each other. The problem or construction task (ladder) could also be seen here as a topological or spatial problem and is at least closely linked to the two categories "wrong way-right way" and "open-closed": How do you get from one place to another? How does one get up?

052	Ron:	but otherwise, they can't get up there at all
053	Max:	but they put the ladder together, you goof!#
054		*looks at Ron*
055	Ron:	all ladders?
056	Max:	Yes.

In line 52, Ron picks up on the outline of a problem, which mathematically shows a link to spatial orientation and the "topographical idea of connection" (Henschen, 2020, p. 318) from line 033: "and how do they get up there?" Max is referring to this initial question in line 053. It can be surmised that while Ron sees the solution as a construction task, Max shifts the solution to the narrative play action. A non-hierarchical variety of solutions emerges here, which also becomes apparent in the further sequence. Between line 057 and line 077, there is a shift towards the construction task; there is a connection between the two ladders that have been created in the meantime. Several components then extend this composite ladder. Subsequently, the problem shifts to how and whether the ladder can be fixed (anchoring problem). In the process, further ideas for solutions are integrated, such as the need to use or omit specific components (lines 089 and 094) or pay attention to the design of the structure (line 097). The negotiation subsequently intensifies, and a longer argumentatively structured process of negotiation of meaning can be traced (cf. Vogler et al., 2022). As a result of this argumentation process, Ron then also succeeds in convincing Max to accept a solution to the anchoring problem.

140	Ron:	But you can fix it like that and that's good, then it will hold better.
141		*points to the upper end of the ladder and the upper edge of the structure*
142	Max:	But we need another long one like that first.
(...)		
157	Ron:	wait, we have to#
158		*together with Max he pulls the ladder upwards out of the structure*
159	Max:	remove
160		*tries to fix the rod to the corner again*
161	Ron:	*together with Max, he grabs the ladder*
162		we have to really get the ladder in there#
163	Max:	installing correctly
164		*together they connect the ladder to the top corner of the building, the ladder now hangs parallel in front of the building* (Fig. 4)

From line 140, a turn in the interaction can be reconstructed. After the children had previously struggled over the relevance of their different solutions, Ron's argumentation now seems to resonate with Max and both children try to find a standard solution to the problem. This may be due to the fact that in line 141, Ron gesturally illustrates the distance or gap between the end of the ladder and the building, while the ladder is stuck diagonally through the upper floors of the building. At this moment, Max develops the idea of not only connecting the ladder and the building using a white connecting piece and using another long rod as an intermediate piece (line 142). In this context, with regard to the mathematical content measurement (category "small-large"), it is particularly noteworthy that the distance between the end of the ladder and the building corresponds to the length of the selected longer rod. It can be assumed that Ron's gestural illustration of the distance initiates an

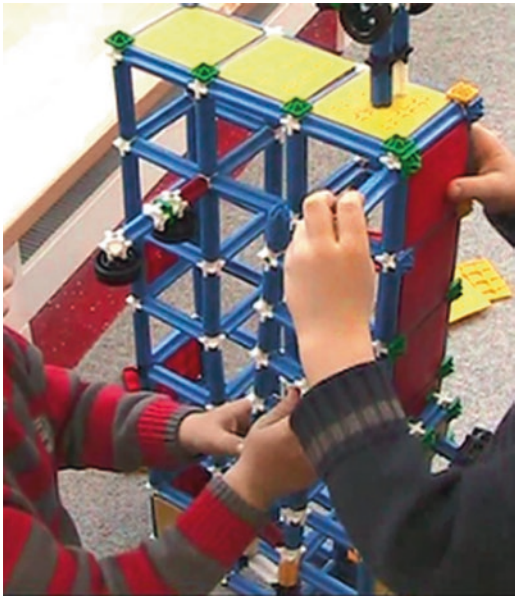

Fig. 4 Ladder hangs in front of the building

estimation process that leads to Max choosing a suitable rod that also enables the attachment. This situation enables the children to mathematically experience that the diagonal placement of the ladder in the building creates an angle between the pole attached to the ladder and the building that does not allow it to be attached to the building (lines 156–157). As the children have now interactively agreed on a common mathematical interpretation of the problem and a working consensus, the collective solution to the anchoring problem takes place from lines 158 to 164. After the upper point of the ladder has been clarified, the ladder is extended to the ground. While the question "And then how do they get up there?" focuses on the goal of arriving at the top, the ladder piece attached to the top now results in the opposite task: the ladder must lead downwards. This can be interpreted as an experience with the problem-solving strategy working backwards.

Analysis of the Roof Example

In addition to the analysis of the ladder example, a short example of decomposing into subproblems) as a heuristic way of thinking can be developed in the following scene. The starting point of the analysed scene is a situation in which three girls also interact with the SONOS material and a picture in which various buildings are depicted. The girls approach the problem of constructing a three-dimensional building with a gabled roof from a two-dimensional image (not to scale). In the scene presented below, Emma focuses on the depiction of the house with the gabled roof.

004	Emma:	*places the pole with three links (blue-white-blue) she is holding in her hand on a gable end in the illustration*
005		I'll do it like this first
006		*then holds the pole over the other gable end and takes it away again* (Fig. 5)

In line (4), Emma holds a rod from three components (blue-white-blue) over the image of the structure so that the rod first covers one side of the gable and then turns it to the other end of the gable. She accompanies this action verbally: "I'll do it like this first". Emma obviously indicates that she will first make the gable end of the roof. It can be assumed that Emma illustrates both the two gable ends as well as the angle between the two ends through her action in line 004, although she only uses one rod for illustration. In this context, the term "first time" may indicate that the girl is following a multi-stage plan to build the structure. She seems to be giving precise information about her work steps here. Emma could thus be passing on a kind of technical knowledge or way of working about building with the material. Here, the strategic solution of breaking down the set or chosen task into individual work steps in order to be able to cope with it can be reconstructed. In this, the heuristic of "decomposing" can be recognised as it is also presented in problem solving (Schoenfeld, 1987). However, the other two girls do not take up the procedure in the following. The construction of the roof fails.

Fig. 5 Emma holds the pole over the gable

Empirical Findings

From a perspective of mathematics education, it can be particularly emphasised that, in the overall view of the negotiation process, different mathematical content emerges in the situation with Ron and Max. As shown in the analysis, the children use and gain experience in measuring when estimating lengths, with spatial and positional concepts, (right) angles, connections between two points and planes, as well as with the geometric relations parallel, diagonal and perpendicular. They can be assigned on the basis of the above-mentioned categories "small-large", "open-closed", "wrong way-right way" and "slanted-straight".

However, the material forces the children to build in parallels and with right angles and to use rods and connecting elements alternately, to which they finally submit. Nevertheless, it is not possible to say here what role the Sonos material plays for the observed problem-solving processes and mathematical activities of the children. Henschen (2020, p. 422) found no clear differences in terms of "category density" in her work, which examined situations with Sonos material as well as those with wooden building blocks. She merely found that in some of the analysed situations, when using different materials (building blocks or Sonos material), different principles are addressed by the children. When building with the Sonos, the children tend to talk about constructing techniques, e.g., plugging components together to form a corner or extending a rod by plugging two rods together; when building with building blocks, they focus, for example, on wall building patterns or trying out the domino effect.

Finally in the analysed example, the two children come to a common solution to their initial problem through their experiences with the material, the argumentatively rich and ongoing process of negotiation, as well as by interpreting the structures. In the analysis of the process of meaning negotiation it also becomes obvious that, in addition to mathematical content, processes that are particularly relevant for mathematics also emerge, one of these being the process of problem-solving as described by Helenius et al. (2016). In the play actions of the children that we analysed, specific problem-solving strategies (heuristics) could be reconstructed in this context. In this way, the first steps of *systematic or intelligent guessing and testing* and *working backwards* could be reconstructed in the boys' situation. In the example of the girls, it was also possible to establish that children also use the heuristic of *decomposing*. While in the scene with Ron and Max a certain interweaving of the collective problem-solving process with the collective argumentation process can be traced in which the outline of a problem only emerges as a shared interpretation through the argumentative negotiation (Vogler et al., 2022), no such collective problem solving occurs between the girls. In comparison, it can be assumed that only the multi-layered, collective problem-solving process in connection with the argumentative negotiation leads to successful problem solving for the children.

Conclusion and Outlook

Following Helenius et al. (2016), this article has shown that peer interactions in block play can enable learning processes that relate to mathematical content and processes. Beyond the findings of Helenius et al. (2016), it was possible in this article to reconstruct various heuristic procedures in the peer interactions, such as decomposing into sub-problems, systematic or intelligent guessing and testing, and working backwards. In the example of the successful problem-solving process of Ron and Max, it is particularly remarkable in this context that it is a *collective* problem-solving process that is characterised by its close interconnection with a collective argumentation process: Through the presentation of a problem, a common focus is initially found, which then leads into an argumentatively structured, multi-faceted process of negotiation. Finally, through interactive co-construction, the problem is solved this way. Concerning the (content-related) learning process, it is particularly interesting in connection with the successful problem-solving process of the two boys that different mathematical content is experienced and negotiated in the collective problem-solving and argumentation process. The diversity of these processes of negotiation about their mathematical content and methods can be seen as particularly sustainable for a networked acquisition of mathematical knowledge (cf. Krummheuer & Schütte, 2014).

Although the analyses presented here can only provide first insights into the problem-solving processes in peer interactions, the paper on the one hand gives an idea of how productive collective problem solving can be for mathematical learning across content. On the other hand, initial insight has been given into how great the importance of peer interaction can be as a condition for the opportunity of mathematical learning. This is due to the fact that argumentative processes of negotiation can emerge in the collective problem-solving process in free play, which can lead to taken-as-shared meanings.

Consequently, these peer interactions in free play need to be safeguarded. The challenge is finding a way to support the meanings and strategies developed by the children in the peer interactions and also to transfer them to joint activities with adults (for example, in situations with professionals in the kindergarten). However, future research must clarify how to determine when an appropriate moment for adults to enter this situation of peer interaction could be in order to support problem-solving processes. In this context, from the perspective of mathematics didactics research it seems indispensable to systematically observe peer interactions and to explore attempts at interactional support. The research presented here can provide a first approach to this.

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Mathematical and Computational Thinking in Children's Problem Solving with Robots

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Introduction

In this paper, we investigate two young (3–4 years old) children's interactions with a programmable floor robot. With the integration of computational thinking (CT) into mathematics in the Norwegian school curriculum (Kunnskapsdepartementet, 2019), there is a need to investigate whether there are overlaps between CT and mathematical ideas in how young children engage in programming tasks, in barnehage, Norwegian early childhood centres. Although young children's programming of robots has been researched for at least a decade (see, for example, Highfield, 2010), the focus has mostly been on intervention studies to do with CT (see, for example, Bakala et al., 2021). In their systematic review of previous research, Jung and Won (2018) found only one article that focused on how preschool children engaged with mathematical ideas and this was Highfield's (2010) intervention study. Although Palmér (2017) stated in her study which linked programming to mathematics, "there is a lack of studies on programming conducted in everyday preschool practices" (p. 76), hers was also an intervention study. There is, therefore, a need for research that investigates "what already is" as well as "what ought to be", as Palmér (2017) described the distinction in research types between naturally occurring situations and intervention studies. "What already is" research is important for understanding the children's point of view, which can then inform intervention studies.

Therefore, the research question is "what CT and mathematical understandings do children use when engaging in problem solving with robots at barnehage?" To answer this research question, we analyse the problems two children identified when working with robots, to determine potential relationships between

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mathematics and CT. To determine the potential intersections between, we first describe previous research on CT and mathematics in early childhood education.

The Intersection of Mathematics and Computational Thinking From Using Robots

We begin by briefly describing Bishop's (1988) six mathematical activities used in early childhood education and care (ECEC) research and how they related to research to do with robots, before describing aspects of computational thinking used by young children. Then we consider how computational thinking and mathematics have been related.

Bishop's (1988) six mathematical activities form the basis for mathematics in the Norwegian barnehage curriculum (Reikerås, 2008) and have been used extensively in research on ECEC in Scandinavia (see, for example, Fosse et al., 2020; Helenius et al., 2015). The six mathematical activities are: Playing Explaining; Designing; Locating; Measuring; and Counting. In early childhood mathematics, Playing has been connected to playing games through rule following and rule negotiation and also to problem solving (Helenius et al., 2016). Explaining is to do with how children show and describe their understanding (Fosse et al., 2020). Designing is about using mental images of shapes to design an artefact (Helenius et al., 2015). Locating is about how young children explore and describe themselves and other objects in space, through words, actions and drawings, including maps (Helenius et al., 2015). Measuring for young children is often associated with comparing attributes, either directly or indirectly, such as by using pencils to determine the length of something (Helenius et al., 2015). Counting involves understandings about discrete amounts and the relationship between those amounts, through, for example, one-to-one correspondence, dividing and combining objects into different groups, and using basic arithmetic (Helenius et al., 2015).

In studies related to the use of robots in ECEC, Locating was the most common of Bishop's (1988) six activities. For example, Highfield (2010) identified spatial concepts, including positional language and angle rotation, both aspects of Locating as they were to do with locating objects in space. In research from the first years of school in Panama, Muñoz et al. (2020) showed a similar use on location concepts when working with robots. In another intervention study, Angeli and Valanides (2020) investigated the computational thinking of five-to-six-year olds in ECEC. They hypothesised that children would not have difficulties with the commands to move forward and backward, but may have had difficulties with turning right and left. However, their pre-test results showed that only the command to move backwards was unfamiliar to children. The intervention provided experiences with the commands which seemed to lead to higher post-test results. Similarly, Di Lieto et al. (2017) found improvements in preschool children retaining visual-spatial knowledge in their working memory. Palmér's (2017) study also focused on improving children's spatial thinking and showed that engagement in programming activities with a robot likely resulted in changes in the post-test results of the eight preschool children in the study. Nevertheless, Clarke-Midura et al. (2021) posited that young children's developing coordination system, connected to difficulties matching their own movements to that of a robot, could be similar to the impreciseness shown in young children's early number sense understandings, suggesting there was a developmental progression that children moved through.

Other mathematical activities were present in some studies. In her description of spatial concepts, Highfield (2010) included transformational geometry, such as rotation, which is part of Bishop's (1988) mathematical activity of Designing. Highfield (2010) also identified concepts and processes that were similar to the mathematical activities of Measuring and Counting (Helenius et al., 2015). Palmér (2017) noted that to programme young children needed to have number understandings, in particular one-to-one correspondence, to relate the number of presses on the robot to the number of squares it was expected to move.

Young children's problem solving was also mentioned in most of the earlier studies about using robots in early childhood centres. Problem solving has been linked to Bishop's (1988) mathematical activity Playing, because problem solving often requires imagining "what if" scenarios (Helenius et al., 2016). Problem solving in programming robots has been highlighted as important (Fessakis et al., 2013). For example, Di Lieto et al. (2017) stated "educators claim that robotic 'hands-on' experimentation facilitates the transformation of abstract concepts into concrete and verifiable operations, promoting new perspectives for thinking and developing problem-solving skills" (p. 17). Given the emphasis on children engaging in problem solving in the barnehage curriculum (Fosse et al., 2020), it is valuable to consider the connections to CT.

Although definitions of CT are still debated, Bakala et al. (2021) stated that in research on robots in early childhood education, the most frequently included components of CT were, "algorithmic thinking, abstraction, decomposition, sequencing, generalization, and debugging" (p. 2).

Algorithmic thinking is often described as the ordering of actions for completing the whole task and so are linked to sequencing. Palmér (2017) considered that there was a relationship between the sequencing of actions and mathematics, "the children showed an ability to sequence, which includes, planning and putting objects (commands) in the correct order, which is important in both literacy and mathematics" (p. 83). Muñoz et al. (2020) found that at least half of the 4-to-5-year-old children could provide an appropriate sequence of actions for moving a robot, before their intervention began.

Decomposition is the ability to identify the parts of a program. Angeli and Valanides (2020) found that most of 5-to-6-year-old children "decomposed the task in a number of subtasks equal to the number of commands in the task and chose to execute one subtask at a time" (p. 10). They considered that this showed that children had the capability to break tasks down into small, more manageable steps. Palmér (2017) also noted that the children in her study decomposed the tasks into different sets of sub-tasks.

Debugging involves identifying issues in the running of the program and fixing them (Bakala et al., 2021). In earlier research, Palmér (2017) noted that it was often conflated with problem solving because debugging is usually described in relationship to fixing problems in the programs. Over half of the children in Muñoz et al.'s (2020) study were able to debug problems in programming a Bee-Bot at the start of the intervention. When preschool children could identify the problem, Lavigne et al. (2020) found that they were more able to fix it. In both Lavigne et al.'s (2020) study and Bakala et al.'s (2021) literature review, children were noted as successfully debugging or using more sophisticated debugging strategies, with the help of the teacher. These studies worked with older ECEC children, "children as young as 5 years old are able to debug through trial-and-error practices but could achieve more sophisticated debugging strategies if provided with the necessary scaffolding and learning opportunities" (Bakala et al., 2021, p. 9). Younger children, or children without the help of adults, may struggle with debugging, perhaps because they could not identify the problem or because they did not have the strategies to fix these problems.

Although most earlier research about young children programming robots illustrated links between mathematics and CT, these connections were rarely discussed. By starting with children's own problems with programming robots, our aim is to describe where the connections between mathematical understandings and CT understandings were important in their problem solving.

Methodology

As a part of a wider study about the use of digital apps in a barnehage, four, short video recording were captured serendipitously of a Blue-Bot robot being programme. A Blue-Bot can be programmed to move around a mat (see Fig. 1), by pressing buttons that represent the actions of going forward (Forward), going

Fig. 1 Task's layout and children

backwards (Backward), turning left 90° (Left Turn), and turning right 90° (Right Turn). A Start button when pushed starts the Blue-Bot moving through the programmed sequence of actions and a button which clears the program from the Blue-Bot's memory (Clear).

In the videos, two children (C1 in red dress, 4 years old, and C2 in grey, 3 years old) were attempting to programme the robot, with a teacher (T). The barnehage had a focus on using digital tools, but field notes indicated that the robot was a recent addition to the barnehage and the participants had limited previous experience with them. As the videos, showed children in a nutralistic setting, it provided an opportunity to explore a "What already is" situation.

To focus on how the children made sense of the programming of the robot in the naturalistic setting, we decided to identify when the children were unable to solve problems immediately. To do this, we looked for signs of uncertainty in the children's spoken utterances and in body language. It was decided to focus on young children's body language as it was likely to provide more information than their spoken utterances alone (Johansson et al., 2014). As a group, we watched the videos several times together to gain agreement on when the children showed uncertainty. We identified particular body actions that appeared in three of the four videos, which we agreed showed the children's uncertainty. These included gestures, like an open mouth or a finger in the mouth (see C1 in Fig. 2), averting the child's gaze from the adult and moving themselves away from the mat. Once uncertainty was identified, we considered what occurred before and after to determine what the problem was which had caused the uncertainty and if and how the problem was resolved in the interaction. If the tracing back indicated that the problem was not related to mathematics or CT, it was not analysed any further. Five problems were identified as concerning CT and mathematics.

Although some examples of other mathematical activities were apparent in the data, the problems that caused the children's undertainty were mostly about Bishop's (1988) mathematical activities of Locating and Counting, with all the problem solving situations being considered to be about Playing. If the problem was about identifying the route, positioning Blue-Bot on the map, or orientating it in the situation, it was classified as being about Locating (Bishop, 1988). When the child's problem was about the number of squares the Blue-Bot had to move, it was considered to be about the mathematical activity of Counting (Bishop, 1988).

In regard to CT, we deemed the problems to be about sequencing and decomposition, and to a lesser extent debugging. Sequencing was identified when the child struggled programming the Blue-Bot's actions in order. When the child focused on the individual actions of the programme, we classified this as decomposition. Debugging occurred when the children identified a problem with the programme, when the robot did not move to where they expected or wanted it to go, and tried to resolve it.

Fig. 2 Uncertainty shown by holding finger near the mouth

Results and Discussion

In this paper, we present three of the five problems, identified in the video recordings, which illustrated most clearly a potential relationship between understandings about sequencing and Locating, and between decomposition and debugging with Counting.

Problem 1

This problem occurred after C1 and C2 had already worked with the teacher to programe the Blue-Bot to move along a complex path to get to the castle square on the mat. After attempts to programme the whole sequence in one go, the teacher had supported the children to programme individual actions. In this episode, the teacher tried again to have C1 sequence a series of actions together, which would make the robot move four steps forward, turn right and then go another four steps forward. This program involved C1 engaging in algorithmic thinking through sequencing the set of actions and in decomposition, breaking the robot's path down into the different actions. For C1, integrating the turn into the program caused her uncertainty around how the two actions of going forward four squares were related to what she considered to be the robot's eight-square path.

The teacher began by asking where the robot would go next, starting from the castle square. C1 chose the green flower and counted to eight while pointing once to each square, to show the path (Figs. 3 and 4).

T asked, "How many do we have to count before it will turn?". C1 counted and pointed at the squares, "one, two, three, four {the square in the corner}" (See Fig. 5). T stated "Four!", while C1 continued, "five". T interrupted her, "Four! {T moved closer and pointed at the square} There are four {points again, looking into C1's eyes}, and then turn". C1 nodded twice slowly, then sat with her gaze on the mat, suggesting that she was confused about why she had to stop at four squares, when the whole path was eight squares.

Struggling with integrating the turn could be a problem about Locating (Bishop, 1988), although it was clear that C1 understood the proposed path for the robot. Therefore, it seems more likely that the confusion was over splitting the eight square path into two parts.

T then suggested programming the robot, C1 opened her mouth (see Fig. 6), but then slowly nodded, suggesting she remained uncertain. C1 began programming by pressing the Clear button. She then followed T's instructions to press the Forward button four times. T then told her to press the Right Turn button and asked her which direction the Blue-Bot had to turn. C1 looked at the corner and touched it with her right hand, before moving her finger over the Blue-Bot. Holding her hand over the corner square, T asked again about the direction. C1 touched the fifth square, saying, "This one", then she turned to the Blue-Bot and pressed the Right Turn button.

Fig. 3 Marking the starting point

Fig. 4 Marking the end point

Fig. 5 Coming to the turn

C1's actions and words reinforced that she was not confused about the direction of the robot's path (Figs. 7 and 8).

While pointing towards the remaining four squares, the teacher said, "Should we count how many times it has to go forward to the flower?" C1 nodded, then C1 and T pointed at two different positions on the mat (Fig. 9). This suggests that C1 did not understand that the path had to be split into two parts. T seemed to recognise that C1 was confused and so reinforced that the robot's path had to be split into two actions (decomposition). She said, "It stands here. {C1 moved closer to the corner}. It stands here and turns {T pointed to the next square (the same movement is shown in Fig. 10)}. Then you have to count from here {T pointed at the next square again.

Fig. 6 Showing uncertainty with open mouth

Fig. 7 T shows the turn direction

C1 nodded twice, with a slightly opened mouth} and onward" {T moved her index finger to indicate three moves towards the right}.

Tapping the next square, T continued, "This is one, {C1 held her hand on the corner square (see Fig. 10)} one. {T moved her finger to the right. C1 kept her hand on the corner.} Because it stands here {T pointed at the corner}." C1 moved her finger to the next square, saying "one" and proceeded to point and count "two, three". With C1, T pointed to the last square (Fig. 11). C1 said "four" and T agreed, "Four to the flower". As shown earlier, C1 did not show difficulties matching the number words to each of the squares as she moved along the path. However, it is unclear if she considered the final number to represent the total amount, or a position on the path.

C1 returned to the robot and T gave a direction, framed as a question "Will you press four times the Forward button?". C1 began to place her finger on the button, but then removed it, "I have already done it!". T replied, "Then we have turned.

Fig. 8 C1 shows the turn direction

Fig. 9 Two starting points for counting

First, we went four forward {T moved her hand along the route}, then turned right {T made a rotation gesture over the corner square}, then four more forward to come to the flower {T showed the rest of the route}. Now we are going to try how this will go. If you now press the Forward button four times". C1 pressed the button four times (see Fig. 12), but with her mouth slightly open, suggesting she remained uncertain. Although the uncertainty could be because the number of squares was the same for both parts of the path, it seemed more likely that what was unclear was each lot of four steps was related to the eight steps. This suggests it was decomposition, not algorithmic thinking, that C1 struggled with in the programming.

After pressing the start button (see Fig. 13), the robot began to move. C1 moved her hand to the end of the mat (Fig. 14), as she seemed to be uncertain that the robot would turn. When it did, C1 looked confused.

Fig. 10 T showing next square

Fig. 11 Joint counting

C1 could show the robot's proposed path, but she struggled with decomposing it into individual actions (four steps forward, turn right, four steps forward) and this impeded her programing the robot appropriately. As the Bee-Bot remained at the starting point, the relationship between the different parts of the path and the pressing of the buttons were hidden. Bakala et al. (2021) noted the high cognitive demands of programming on children as they had to remember the sequence of commands being put into the robot. This could explain some of the difficulties that

Fig. 12 Programming the robot

Fig. 13 C1 after pressing Start

C1 experienced with understanding how both forward actions were related to the eight squares she had counted.

In regard to her mathematical understandings, the child's uncertainty seemed only to some degree to be about Locating – how the turn affected where the robot went. Rather understandings about Counting seemed to more likely to be contributing to her uncertainty. Although she showed one-to-one correspondence between the counting words, the squares and the pressing of the buttons, C1 seemed not to recognise that the total amount of squares, eight, was the same as two groups of four. This requires understanding about addition to do with total amounts being composed of smaller amounts and how this relates to reciting counting words

Fig. 14 Blocking with her hand

(Baroody, 1987). It may be that the child used the counting words to mark the order of squares and as a result has an ordinal, rather than a cardinal understanding of number, which has been noted as typical for children of this age (Bruce & Threlfall, 2004). Nevertheless, by holding her hand at the end of the mat, C1 seemed to be predicting that the robot would not turn (Fig. 14), suggesting that she saw that her number of presses of the go-forward button would result in the robot moving further than the original four. This suggests that C1 did have some understandings of cardinality (Bruce & Threlfall, 2004). These results suggest that for this child the CT aspects of decomposition and algorithic thinking are connected to Counting, highlighting the need for children to have understandings about cardinality and early addition.

Problem 2

In this episode, C2's problem seemed to be about the Blue-Bot not stopping on the boat square, her chosen end point, which was three squares up from her starting point in the bottom left-hand corner. C2 had no difficulty locating the straight path of the robot. However, the relationship between the number of squares, reciting the counting words and the number of pushes of the Forward button caused some difficulties.

C2 with the Blue-Bot nearby, counted, "One {touches the yellow square}, two {touches the blue square}, three, four, five {holds fist on the boat square for three counts}." T checked, "Will it go to the boat? {T touched the boat three times}." C2

Fig. 15 Blue-Bot going off the mat

Fig. 16 T clearing the previous program

replied "Yes, like this" while putting the Blue-Bot on a yellow square. T asked about where the Blue-Bot should be, as C2 pressed the Start button. The robot ran through the previous program and consequently moved off the mat. C2 tried to stop it with her hand (Fig. 15) and T had to assist C2 to stop the Blue-Bot running through the rest of its program (Fig. 16). C2 seemed surprised when it did not stop on the boat square, using her hand to impede its progress. This suggested she was uncertain about why this had occurred.

T finally stopped the robot and cleared its memory, "C2, where should we start?" C2 said "One, two, three, four {Touched the squares individually as she said the number word (see Fig. 17)}, five {touches boat square for second time}". T placed the Blue-Bot on the corner square, where C2 started to count. This suggested that C2 could identify a path for the robot by pointing at the squares.

Fig. 17 C2 touching yellow square on "Two"

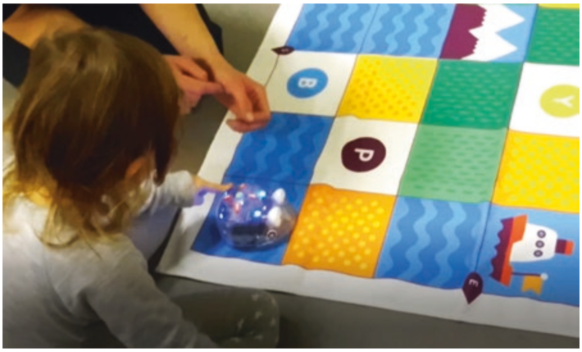

Fig. 18 C2 pressing the Clear button

T then told C2 how to program the robot to move three squares, "We have to press the Clear button first {T pointed and C2 pressed the Clear button (Fig. 18)}. Then we have to count how many times it is to there {T pointed to the squares}". C2 counted "One, two, three." As C2 could match the counting words to her pointing to individual squares, it seemed that she had understood the path the robot was to take and had some number understandings connected to one-to-one correspondence.

Although T asked C2 to press the Forward button three times, C2 kept pushing the button. T stated "You have to count! Wait. {C2 stopped}. Can you press the Clear button again? Then you can press the Forward button three times." C2 counted to three again, but looked like she might keep going. T said "Stop! Only three right?{C2 kept pushing the button while counting to seven} Oi! That was many. {T shrugged her shoulders}. Shall we see what happens?" C2 pressed the Start button and the Blue-Bot went past the boat square. T said, "He passes by! (Fig. 19)" C2 laughed and stopped the robot with her hands (Fig. 20).

After a lot of support from T, the robot was eventually programmed to go forward three steps. However, C2 caught the Blue-Bot with her hand as it approached the boat square as though she was unsure it would stop.

Fig. 19 Surprise at the robot going passed the boat

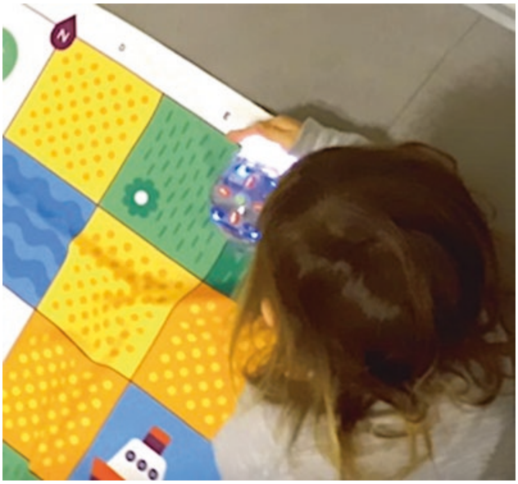

Fig. 20 Blocking the robot with her hand

C2 did not show uncertainty in the same way as C1, but indicated there was a problem when the robot did not go where she had expected, by physically picking it up while the program was running (Fig. 15), turning to the teacher when the robot did not stop where she wanted it to, putting her hand in front of the Blue-Bot to stop it from leaving the mat (Fig. 19), or holding it from behind (Fig. 20). C2 was aware that her programming of the robot did not result in it stopping on the boat square.

Although Lavigne et al. (2020) found that children were more able to debug a program if they could identify errors, it seems that this is only the case when they have the necessary skills or interest in learning how to fix the bugs. In C2's case, she seemed uninterested in matching the counting words to the squares the robot had to pass. The child's wish for the robot to stop on the boat square seemed to be secondary to her delight in reciting the number words. So, although C2 could identify the problem, fixing it did not seem to be incentive enough for her to focus on the one-to-one correspondence, even with the support of the teacher. Children of this age can be taught to recite the counting words, without an appreciation of how the counting words relate to amounts (Bruce & Threlfall, 2004). Programming the robot so it would stop where she wanted it to stop did not seem to be sufficient incentive to learn more about how counting could support her problem solving.

Problem 3

In the first problem, C1 showed uncertainty about incorporating the Right Turn command in between the two actions of going forward four steps. In this episode, C1 solved the issue by not including a turn into the robot's path. In so doing, she adopted a typical problem solving strategy of simplifying the problem so that it became more manageable, a common strategy promoted for older children to use at school (Barham, 2020).

This interaction began as the others had, by the teacher asking the child to chose a starting and finishing square. T asked, "Where will it go now?" C1 looked at the mat, stretched her index finger and slowly moved her hand towards the square with a tree (Fig. 21), then replied, "Tree. {C1 touched the middle of the tree, lifted her hand up then turned and smiled at T}." T checked with C1 that this was to be the end point, "To the tree? {C1 nodded and smiled}. Where should we start?" After a pause, C1 stated, "We start there! {She touched the yellow square above the tree square (Fig. 22)}."

T then suggested that C1 put the Blue-Bot on the start square. As C1 did this, T asked "And what [button] do we have to push first?". C1 looked at the mat, "This one {C1 pointed to the square with the tree}." Her mouth was open, suggesting she was a little uncertain. T then gave a direction in the form of a question, "But C1, first, we have to press the Clear button. Right?" C2 seemed to remain uncertain by

Fig. 21 Pointing to the end point

Fig. 22 Pointing to the starting point

Fig. 23 Showing uncertainty about pushing Clear

holding her mouth open as she pressed the Clear button (see Fig. 23). T reinforced her movement with, "Yes."

C1 moved her hand towards the Forward button, then took it away before holding it over the Turn Right button. She then moved her hand away from the robot (Fig. 24) and turned to T. C1 said, "No turn! {C1 smiled}." T replied, "No turn {T shook her head}. Okay. But what then?" C1 replied with, "It is one. {C1 pointed with index finger at the tree square while looking at T, suggesting that she was referring to the path being one square long}." T responded by asking, "Straight forward?" T and C1 nodded to each other. C1 followed with, "I have to push once. {C1 pressed the Forward button once}." After some reassurance from the teacher, C1 pressed the Start button and the Blue-Bot moved to the tree square and stopped.

According to Muñoz et al. (2020) 4-to-5-year-old children can provide an appropriate sequence of actions for moving a robot without help. However, C1 who was

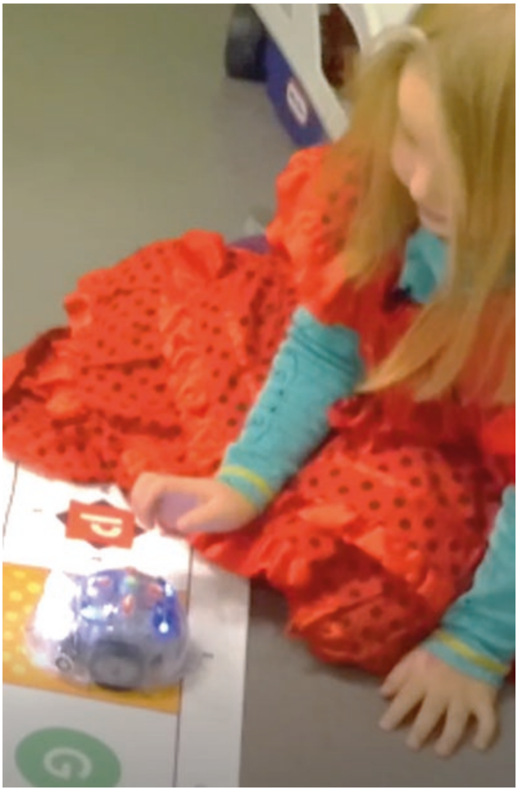

Fig. 24 Uncertainty about which button to press

4 years old solved the issue from Problem 1 by identifying a one-step path, which eliminated the need to incorporate a turn and split a path into two (or more) shorter ones. This can be seen in her exclamation "No turn!" She also simplified the number of steps the robot had to travel to the smallest amount possible, suggesting that she might have been aware that her understandings of how numbers worked was insufficient to solve more complex problems, such as Problem 1.

Conclusion

Earlier research on children's engagement with programming floor robots has mostly been through intervention studies (see for example, Muñoz et al., 2020). In our small study, we found similar overlaps between mathematics and computational thinking to those noted earlier, such as location with sequencing and decomposition (see Angeli & Valanides, 2020). However, by focusing on the children's uncertainty, we identified problems from their perspective. As a result, we have been able to show how different understandings about Counting contributed to their possibilities and willingness to solve those problems. Although Palmér (2017) noted the importance of number understandings, she highlighted one-to-one correspondence. However, C1 and C2 both showed some understanding of the need to match each number word to each push of the Forward button. However, C2 seemed uninterested in matching the number words to the squares in her path, often counting the final square more than once even if she pointed and counted simultaneously. C2 seemed to get more enjoyment from just reciting the counting words than programming the robot, so it would stop at the chosen square. C1 on the other hand showed that she was interested in having a program that resulted in the robot arriving at the end stop appropriately. Her problem seemed to be in inserting the turn because the 8 step path that she saw now consisted of two four-step paths (with the turn in the middle). This seemed to be connected to a lack of understanding about how eight steps could be made up of smaller amounts. C2 overcame this issue by identifying a path for the robot which did not require a turn.

Floor robots only have limited possibilities to move (forward or backward and turn left or turn right), so it was surprising to find that young children's understandings about Counting (Helenius et al., 2016) have not been documented as contributing to their understandings about sequencing, decomposition and debugging previously. Yet, as can be seen in our two examples, if the children do not have the appropriate Counting understandings, it becomes very difficult to determine by themselves or even with the teacher's help how to resolve the problem. Although the teacher in both episodes ensured that the problems were solved, it is unclear if either child understood how this had been achieved.

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Young Students' Choice of Representation When Solving a Problem-Solving Task on Combinatorics

Hanna Palmér and Jorryt van Bommel

Introduction

This paper is about the representations 6-year-old students use when working on a problem-solving task on combinatorics. Most studies on young students and representations have focused on numbers and quantitative thinking (Sarama & Clements, 2009). In a longitudinal study, a connection was found between the representations the students used and the extent to which they managed to solve a combinatorics problem-solving task (Palmér & van Bommel, 2017, 2018). However, when working on this problem-solving task, iconic representations did not generate more complete solutions than pictographic representations. Quite the opposite; pictographic representations seemed to imply more systematization and less duplication (van Bommel & Palmér, 2021). This is somewhat surprising as iconic representations are considered to be on a higher level of abstraction than pictographic representations (Heddens, 1986). For example, low achieving students have shown to more often use pictorial and iconic representations that are also poorly organised whereas high achieving student more often use well-structured abstract representations (Mulligan, 2002). Thus, the connections found between representations and how students solved the combinatorial problem did not apply to results from previous studies of young students' use of representations (see e.g., Hughes, 1986; Mulligan & Mitchelmore, 2009; Piaget & Inhelder, 1969).

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In the longitudinal study, some hypotheses for these results were posed (Palmér & van Bommel, 2018; van Bommel & Palmér, 2021). For example, it was suggested that the time issue and the connection between pictographic representation and the context of the task may explain why students who use pictographic representations make few duplications. Also, it takes longer to draw a pictographic representation than to draw an iconic representation, giving students who drew a pictographic representation more time to reflect on the task. Iconic representations are easier to draw, which makes the process faster and which may be why the solutions with iconic representations quite often contain several duplications (Palmér & van Bommel, 2017). To elaborate further on the previous somewhat contradictory results and on the hypotheses described above, after students solved the combinatorial problem-solving task they were interviewed about their choice of representation. The question focused on in this paper is: What rationales do young students express for their choice of representation?

The paper begins with a background on combinatorics and representations. After that, it focuses on the methodological aspects of the study, followed by results and analysis. The paper ends with a discussion in which the limitations of the study are also addressed.

The Combinatorics Task and Representations

The problem-solving task on combinatorics is one of several tasks used in a Swedish longitudinal intervention study investigating the potential of using problem solving as the start for the mathematics education of 6-year-olds. In Sweden, 6-year-olds have not yet begun formal schooling but attend what is called preschool class, a year of schooling intended to provide a smooth transition between preschool and school (Swedish National Agency for Education, 2014). Problem solving is part of the Swedish preschool class curriculum, but combinatorics is not. However, research has proven that within a proper and meaningful context young students can indeed work with combinatorial tasks finding permutations (English, 2005). Combinatorics can also be connected to pattern (a predictable regularity) and structure (how elements are organized and related) where Mulligan and Mitchelmore (2009) emphasise that awareness of pattern and structure is critical to mathematical learning.

The combinatorial task the students worked on entailed determining how many different ways three toy bears could be arranged in a row on a sofa. Thus, it was an enumerative combinatorial task involving counting permutations, in this case for n = 3. To make the task meaningful for the students, it was presented as a conflict between the toy bears, where they could not agree on who should sit at which place on the sofa. One toy bear then suggested that they could change places every day. The students' task was to find out how many days they could sit in different ways on the sofa. As an introduction to the task, the students were shown three plastic bears in three different colours.

In studies on representations, one focus is on linkages and development between informal and formal representations (e.g., Hughes, 1986; Heddens, 1986; Carruthers & Worthington, 2006). Heddens (1986) focused on the connection between objects (concrete representation) and signs (abstract representation). He introduced representations of two levels of abstraction between these two representations, pictures and tally marks. He referred to pictures of objects as semi-concrete, and to tally marks (where the symbols or pictures do not look like the objects they represent) as semi-abstract. When documenting permutations in this study, all students used pictographic or iconic representations. The pictographic representations were drawings of the plastic bears in the three colours (example Fig. 1, uppermost), and the iconic representations were lines or dots in the three colours (example Fig. 1, bottommost).

Connecting these two representations to abstraction, the use of pictographic representations, as in drawing bears, implies a semi-concrete level, while using iconic representations such as lines or dots implies a semi-abstract level, which is then considered more abstract than the semi-concrete level (Heddens, 1986).

Fig. 1 Example of pictographic (uppermost) and iconic (bottommost) representations

Also older students often start by copying the picture of the items to be combined when working with combinatorics. These representations gradualy become more systematic and refined throughout the solving process (Rønning, 2022). Listing items systematically is one difficulty for young students when working on combinatorial tasks (English, 2005). English (1991) found three approaches used by young students when working with combinatorics: the random stage, the transitional stage and the odometer stage. The random stage entails trial and error, where checking becomes important to avoid duplicates. At the transitional stage, students start to adopt patterns in their documentations, but the pattern is not kept all through the task. Instead, the students revert to the trial-and-error approach. At the odometer stage, the students use an organized structure for selecting combinations throughout the whole solving process. One item is held constant while the others are varied systematically.

In the study focused on here, these stages were combined in the analysis with the degree to which students produced duplicates. In earlier interventions with this task, the students showed a preference for iconic over pictographic representations. Only a few students (4 out of 114) managed to find all permutations (Palmér & van Bommel, 2018). The pictographic representations seem to imply more systematization and fewer duplications than the iconic representations (van Bommel & Palmér, 2021). A new consideration was formulated regarding students' rationales for choosing specific representations.

To summarize, based on above the theoretical framing in this study consists of the two dimensions of representations and systematisations. When analysing representations, Hughes' (1986) notions pictographic and iconic representation are used (Fig. 1). As some children use both pictographic and iconic representations this gives thee possible outcomes. When analyzing systematization, English's (1991) notions random stage, transitional stage and odometer stage are used. These stages are in turn divided into two outcomes based on whether or not the students produce duplications (random and transitional stage) or to what degree all solutions were found (for odometer). Finally, the three outcomes of representations are connected with six outcomes of systematization (see Table 1).

Table 1 Categorization of documentations based on representation and systematization

		Pictographic	Pictographic/Iconic	Iconic	Total
Random	With duplications	1	2	2	5
	No duplications	16	1	3	20
Transition	With duplications	1	2	2	5
	No duplications	1	1	4	6
Odometer	Not all solutions	7	1	0	8
	All solutions	1		3	4
Total		27	7	14	48

Method

As mentioned, the problem-solving task on combinatorics is from a longitudinal intervention investigating the potential in using problem solving as the start for the mathematics education of 6-year-olds. This intervention is conducted through educational design research (Anderson & Shattuck, 2012). The intervention has been ongoing for several years, involving more than 40 Swedish preschool classes in different design cycles with different foci. The empirical material in this paper is from one small pilot design cycle within this intervention focusing on the students' rationales for the representations they use when working on the combinatorial tasks.

Selection of Preschool Classes

Five classes were selected for this design cycle based on their teachers' interest in participating. The teachers working in these preschool classes are educated as preschool teachers, which implies that they have completed a 3-year university course in preschool teacher education. The teachers have participated in several of the previous design cycles and hence were familiar with the aim of the study and the problem-solving task on combinatorics. As preschool class is only 1 year of education, there are new students in the classes each year, so the same lessons can be re-used each year. Thus, the teachers had implemented the problem-solving task on combinatorics several times before with other students. The students were familiar with problem solving but not with combinatorics. In line with the ethical rules described by the Swedish Research Council (2017), the students' guardians were given written information about the study and approved their children's participation. Altogether, 48 students from these five preschool classes got approval and thus participated in this design cycle.

The Problem-Solving Lesson

When introducing the problem-solving task, the students were shown three small plastic bears in three different colours. After the introduction, the students worked individually. They were given white paper and pencils in different colours but no instructions regarding what or how to document on the paper. After working individually, the students were divided into pairs to compare and discuss their documentations. However, they were not allowed to change anything on their documentations. Finally, all students were gathered for a joint discussion based on their documentations.

Interviews and Analysis

To explore the students' choice of representation, a short interview was conducted after the problem-solving lesson. The interview guide was developed by the researchers and communicated to the teachers. The teachers have been taking an active role in the larger education design research study and, as in previous design cycles, it was they who conducted the interviews (Palmér & van Bommel, 2021). The researchers instructed the teachers, in writing and verbally, on how to carry out the interviews and how to take notes. In the interviews, the teachers showed the students their documentation from working on the task and asked them, *Why did you choose to draw bears/dots/lines/.. when solving the task?* The teachers then documented the students' answers.

The researchers analysed the students' answers to the interview questions together with their documentation from the lesson. Based on the limited selection of students, this analysis was qualitative, explorative and mainly at a group level (see Cohen et al., 2018). First, the students' documentation was categorized deductively, identifying whether a pictographic and/or an iconic representation was used, and also identifying the stage of systematization used and the number of solutions: random with or without duplicates, transition with or without duplicates, and odometer with all solutions or not (see English, 1991, 1996). After that, the students' answers to the question on their choice of representation were categorized inductively, based on the representation used. Thus, answers from students using pictographic and iconic representations respectively were analysed collectively, and during the analysis we looked for patterns.

Results

When working on the task, all students used pictographic and/or iconic representations (Table 1).

When working on the combinatorial task, the students need to pay attention to each object as well as to the relation between the objects; therefore both pictographic and iconic representations are well suited. The students showed a preference for pictographic representations. As shown in Table 1, 27 students used a pictographic representation (Fig. 2 uppermost), 14 students used an iconic representation (Fig. 2 bottommost), and seven students started with a pictographic representation but switched to an iconic representation while solving the task (Fig. 2 middle).

As in previous cycles (Palmér & van Bommel, 2017, 2018; van Bommel & Palmér, 2021), few of the 27 students using a pictographic representation made duplications. Of the 27 students using a pictographic representation, 17 used trial and error when working on the task, but only one produced duplicates. The students using an iconic representation were fewer (14) and more distributed in terms of systematization stage when documenting their solutions. The use of iconic

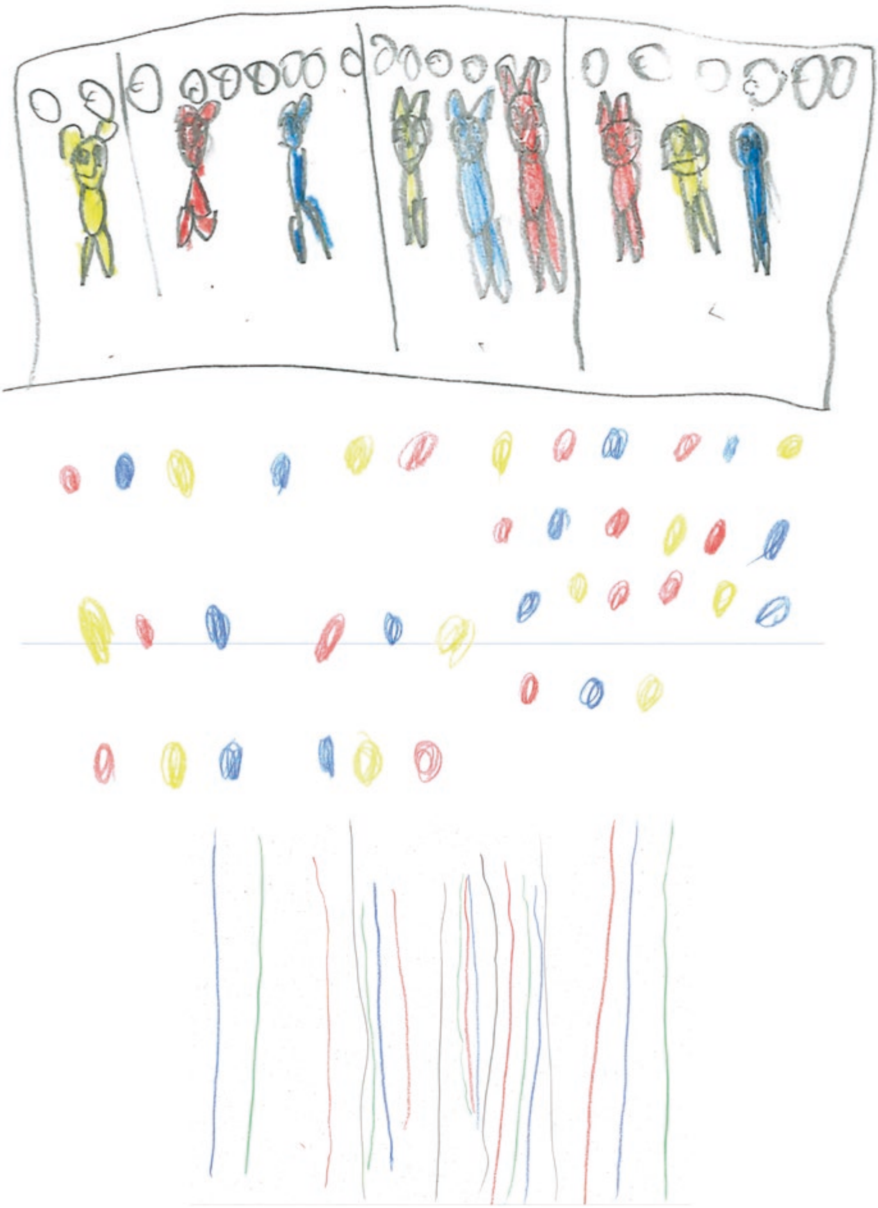

Fig. 2 Examples of pictographic and iconic representations in students' documentations

representations implies a semi-abstract level, more abstract than the use of pictographic representations (Heddens, 1986). Also as in previous studies (Palmér & van Bommel, 2017, 2018; van Bommel & Palmér, 2021), a larger proportion of the students who used iconic representations made duplications. Further, three students using iconic representations found all permutations. We do want to point out that this study includes few participants, however, these results are well in line with previous studies (Palmér & van Bommel, 2017, 2018; van Bommel & Palmér, 2021).

In the interview, the students who used a pictographic representation gave very homogeneous answers to the question of choice of representation. Their answers indicated that the rationale for drawing bears was that the task was about bears. These students answered, for example, *Because it is bears* and *You showed us bears.* Thus, for these students the choice of representation seems to have been quite obvious: you document what the task is about, capturing the context of the task.

For students who used iconic representations, the rationales for their choice of representation can be divided into two themes. Some of these students answered that they drew lines or dots because they were not able to draw bears; it was too difficult. Thus they put forward a technical reason to choose an iconic representation: for example, *I chose to do dots as it was so hard to draw bears.*

Other students answered that they drew lines or dots simply because it was easier or faster than drawing bears. Here we see a rationale indicating an understanding that icons (dots, lines) can represent bears. The students expressed that to solve the task, the bears per se are not important; for example, *You can draw circles instead of bears* or *It works just as well with squares.*

The seven students that used both pictographic and iconic representation showed various degrees of systematization, and there were no clear patterns between their solutions and their answers to the interview question. However, when we focused on why they changed representation, a pattern occurred. Most often, these students explained their choice of representation as changing to an easier or faster representation, for example, *I started with the bears but it was faster to draw lines.* Compared to the themes above, the technical issue of not being able to draw bears obviously does not apply as they started by drawing bears. Here they express the insight that icons can replace the pictographic representation, although they did not articulate it as specifically as the students who used only iconic representations did. The rationale was more a practical one: *It was faster to draw lines*.

Discussion

The starting point for this small-scale pilot study was previous results of a problem-solving task on combinatorics where iconic representations did not generate a higher level of correct solutions than pictographic representations, which may be considered somewhat surprising (Palmér & van Bommel, 2017, 2018; van Bommel & Palmér, 2021). The sample in this study is too small to allow for generalizations, but based on the students' answers in the interview, we will make some refined hypotheses to be further investigated.

The division of systematization stages within the different forms of representation in this study is similar to that in previous cycles (Palmér & van Bommel, 2017, 2018; van Bommel & Palmér, 2021). In the interview, the students who used a pictographic representation (drawing bears) expressed that the context of the task led their choice and answered that they drew bears because the task was about bears. These answers indicate that the students chose a representation on a level of abstraction that is suitable for them, in this case a representation on the semi-concrete level, bridging the gap between the concrete and the abstract level (Heddens, 1986). When documenting their solutions with a representation at this level of abstraction, these students make very few duplications regardless of stage of systematization (i.e., random, transition, odometer; see English, 1991, 1996).

The 14 students who used iconic representation were more distributed in terms of systematization when documenting their solutions. In the interviews, some of these students expressed an understanding that the bears per se were not important to solving the task. Others gave answers indicating a technical reason for choosing this type of representation – they wanted to draw bears but resorted to drawing lines or dots because drawing bears was not possible for them. This latter group of students may have been induced to work with a representation at a level of abstraction that was not suitable for them. Thus, the context of the task, which was intended to make the task meaningful for the students, may have instead hindered some of them from using a representation at an appropriate level of abstraction. This may explain why students who used iconic representation are more distributed in terms of systematization when documenting their solutions.

As English (2005) suggested, combinatorics can be used with very young students. English points out that a proper context will make it possible to work with combinatorics and finding permutations. Our research points out that it is the context that influences the choice of representation, which in turn can both hinder and support students in completing the problem-solving task. Adjusting the context with regard to complexity to draw this context will give us a better insight into whether or not students' choice of representation reflects their level of abstraction.

Some of the students expressed that they chose a pictographic representation, drawing bears, because that is what the task was about. This might indicate a socio-mathematical norm that the students are consciously or unconsciously aware of. In our study, it is the students' own teachers who conducted the interview, and these socio-mathematical norms might be difficult to challenge or contest. Our collaboration with the teachers has been ongoing (Palmér & van Bommel, 2021), and the teachers are well aware of what we want to capture and what they are expected to do. It is important to realize that we cannot be sure that students' responses to the interview questions would be the same if the interviews had been conducted by the researchers instead of the teachers.

We have mentioned that the small number of participants in this study does not allow us to make generalizations. However, it is important to consider that this small-scale pilot study was not about reaching final conclusions but about seeing in what direction we could further develop our interventions and give directions for further design cycles (see Anderson & Shattuck, 2012). If we focus on the different levels of abstraction, we could work on the concrete level and let students work with actual toy bears, or we could omit any representation and ask students to merely colour (e.g., In how many ways can a predesigned flag be painted using three different colours?). Both options would give us an opportunity to focus on whether or not the students are able to find all permutations and whether or not duplicates are created. However, our results show that the students' level of abstraction and their representations might not be compatible, and thus it would be interesting to investigate further. With a focus on representations, we could develop the intervention by letting students work on a similar problem-solving task on combinatorics but where the object is easier to draw, such as pens or buttons (e.g., In how many ways can three pens/buttons of different colours be arranged in a row?). At an earlier stage of the intervention, we already designed an application where the students could work with pictographic representations (van Bommel & Palmér, 2021), prolonging the semi-concrete phase and giving the students an opportunity to internalize the problem before starting to document it. A decline in duplicates in students' documentation was observable, but the pictographic and iconic representations still showed differences in stages of systematization. Thus, a focus on objects that are simpler to draw could be an option and would give us a better insight into whether or not the form of representation chosen by the students is in line with and supports their own level of abstraction.

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The Knowledge Quartet as a Theoretical Lens to Explore Kindergarten Teachers' Teaching of Mathematics

Martin Carlsen, Ingvald Erfjord, and Per Sigurd Hundeland

Introduction

The study[1] reported here addressed kindergarten teachers' discursive practices as we were focusing on the mathematical competencies that a kindergarten teacher (KT) drew on in her teaching of a mathematical activity designed by researchers. Analyses were conducted based on the Knowledge Quartet (KQ) developed by Rowland et al. (2005). According to Maher et al. (2022), the KQ encompasses both the static and the dynamic aspects of (mathematics) teachers' knowledge. We are particularly interested in the dynamic features of kindergarten teachers' mathematical competencies. Norwegian kindergartens are situated within a social pedagogical tradition (Norwegian Directorate for Education and Training, 2017). Thus, a kindergarten teacher is supposed to nurture and empower mathematical explorations amongst the children, let them get familiar with and achieve experience with respect to mathematical concepts within number, geometry, and measurement. The study reported here had a forerunner in a study conducted by Hundeland et al. (2017).[2] In that study, we found that the four dimensions of the KQ were intertwined in the dialogues, as one move might simultaneously exemplify more than one dimension.

[1] The research reported was conducted within the Agder project. The Agder project was funded by the Research Council of Norway (NFR no. 237973), The Sørlandet Knowledge Foundation, The Development and Competence Fund of Aust-Agder, Vest-Agder County, Aust-Agder County, University of Agder and University of Stavanger.

[2] Dialogue 1 and Dialogue 3 below were included in Hundeland et al. (2017) as well. However, in this study these dialogues have been reanalysed.

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In this study we are broadening our scope in scrutinizing the teaching of a KT even further, as we aimed to elaborate the ways that a KT revealed her subject knowledge when teaching a mathematical activity for 5-year-olds. As will be seen, the re-analysis also resulted in new insights into the subtleties of KT's revealed mathematics knowledge.

Even though the research literature is sparse when it comes to preschool teachers' knowledge for teaching mathematics, there are a number of studies addressing preschool teachers' pedagogical content knowledge and competencies more broadly. For the sake of space, we will here only mention two relevant studies (Other studies may be found in Levenson et al. (2011) and in the two studies' literature reviews).

Oppermann et al. (2016) studied how preschool teachers' mathematical content knowledge, among others, are related to their sensitivity to mathematical issues arising in play-based situations. These researchers argue that "offering high quality mathematical education is a challenging task for preschool teachers and requires a number of competencies" (p. 174). In particular, the preschool teachers' mathematical content knowledge and pedagogical content knowledge are important.

Bruns et al. (2017) examined the effects of a professional development course involving early childhood teachers by conducting a pre-test/post-test study. These authors found "that the course affected teachers' mathematical pedagogical content knowledge" (p. 76). Furthermore, they argue that early childhood teachers' mathematics-related competence is highly relevant to children's mathematical learning. Bruns et al. argue, based on the research literature, that mathematical pedagogical content knowledge needed for teachers of early childhood comprises "knowledge about ways to analyse mathematical development, to create mathematical learning environments for young children and give adaptive support in natural learning settings." (p. 78).

We argue, based on the study of Oppermann et al. (2016) and Bruns et al. (2017) that insights into KTs' knowledge for teaching mathematics in kindergarten are highly relevant and important. Moreover, we argue that the KQ adds to these insights being a suitable framework for analysing KTs' knowledge for teaching mathematics.

This study aimed to scrutinise some of the subtleties involved when a KT carried out and involved a group of children in a mathematics activity. We deliberately use the term teaching of mathematics to address the KT's practice, even though the tasks of teaching in kindergarten are different from tasks of teaching in school (cf. Erfjord et al., 2012). For example, in the Norwegian kindergarten mathematics lessons of individual work is rare. The children do neither have explicit learning goals to achieve as pupils have in school. The Framework plan for the enterprise of kindergartens in Norway (Norwegian Directorate for Education and Training, 2017) rather emphasises that "Kindergartens shall highlight relationships and enable the children to explore and discover mathematics in everyday life, technology, nature, art and culture and by being creative and imaginative. The learning area shall stimulate the children's sense of wonder, curiosity and motivation for problem-solving." (p. 53). This quote is in line with what Wells (1999) calls inquiry, as a "willingness to wonder, to ask questions" (p. 121), and so forth. Nevertheless, the role of a KT in teaching a mathematical activity carries several similarities with the role of a mathematics teacher in the classroom: The KT plans the teaching of the activity, leads the mathematics work, acts in the moment in order to adapt questions and tasks for the children. Furthermore, she organises the activity and the interplay of the different children's contributions as a plenary session comparable to a classroom situation in school. The Framework plan for the enterprise of kindergartens in Norway (Norwegian Directorate for Education and Training, 2017) does not use the term 'teaching', but as argued above, a KT's practice may still be labelled teaching (cf. Sæbbe and Mosvold (2016) for a further discussion of the concept of teaching in Norwegian kindergartens).

The Knowledge Quartet as an Analytical Framework

The Knowledge Quartet was launched by Rowland et al. (2005) in order to address and characterise mathematics teachers' knowledge in mathematics teaching. Originally, Rowland et al. used videotapes from pre-service teachers' mathematics classroom lessons as well as post-teaching stimulated-recall interviews in their analyses of mathematics teachers' utilization of pedagogical and mathematical knowledge. We adopted the KQ as our analytical lens in the characterisation of kindergarten teachers' revealed mathematical and pedagogical knowledge when teaching 5-year-olds. Our particular use of the KQ addressed one KT's teaching activity involving two-dimensional geometrical shapes.

As the term indicates, the KQ encompasses four dimensions which address mathematics teachers' knowledge revealed in the classroom (Rowland et al., 2005). Rowland et al. took a grounded approach to their data and identified these four dimensions along which the mathematics pre-service teachers' "mathematics-related knowledge" (p. 255) was analysed. The KQ focuses on teachers' observable mathematics-related knowledge emerging in situations in the mathematics classroom. These four dimensions are Foundation, Transformation, Connection, and Contingency.

Foundation is a dimension of the Quartet used to address the knowledge background of the (kindergarten) teacher, knowledge of both mathematics and mathematics education. Furthermore, Foundation is informing the three other dimensions. Foundation addresses the propositional knowledge, i.e. the KT's knowledge of relevant mathematical concepts and their inherent relationships, the KT's knowledge of pedagogical and mathematics education research informing the practice of mathematics teaching in kindergarten, and the KT's view upon the purpose and relevance of mathematics education for kindergarten children and these children's mathematics learning. We analytically use this dimension to evaluate the mathematics and the didactical insights revealed by the KT's teaching.

Transformation is one of the two dimensions used to address knowledge-in-action, i.e. the (kindergarten) teacher's knowledge of mathematics and mathematics education as revealed in orchestrations of mathematical activities. Transformation comprises the KT's choices regarding demonstrations given, representations used, and examples provided in her teaching. Furthermore, transformation addresses the KT's ability to transform the mathematics "in ways designed to enable students to learn it" (Rowland et al., 2005, p. 265). We analytically use this dimension to evaluate the KT's choices of representations and examples when characterizing various two-dimensional geometrical shapes.

Connection is the other dimension of knowledge-in-action as it addresses how the KT draws connections between involved mathematical concepts, overtly making connections between involved mathematical procedures, discussing various meanings for the involved concepts and diverse ways of carrying out the involved procedures. We analytically use this dimension to evaluate how the KT characterises the different shapes, also by names, as well as how she makes mathematical connections between the geometrical shapes and their features.

Contingency is the dimension used to address knowledge-in-interaction, i.e. the (kindergarten) teacher's unfolding of knowledge of mathematics and mathematics education in interaction with the children. Contingency thus addresses the teacher's ability to respond mathematically appropriate to situations that have not been planned for or anticipated (Rowland & Zazkis, 2013), i.e. to act in the moment. Furthermore, Contingency encompasses whether the KT takes advantage of emerging mathematical learning opportunities and whether the KT deviates from her goals of the activity. We analytically use this dimension to evaluate the KT's 'on her feet' responses regarding the children's suggested ideas and to what extent she makes the children aware of particular mathematical ideas.

In the mathematics education literature, there are a number of studies who have utilised the KQ as an analytical lens (see for example Petrou and Goulding (2011), Liston (2015), Rowland et al. (2015), and Maher et al. (2022)). Maher et al. studied mathematics teaching comprising differential calculus and discrete probability. They found that "there is a complex interplay among aspects of the Knowledge Quartet, including the impact of foundational knowledge on contingent moments" (p. 233). However, none of these have investigated the mathematics teaching of a kindergarten teacher.

Methods and Context

The research design of our study bared characteristics of a case study (Bassey, 1999), as we delved into particularities and details of the KT's mathematics teaching providing a thick account. We observed and videotaped the KT's teaching and transcribed the videos for analytical purposes. The analytical process was driven by our use of the analytical codes associated with the four dimensions of the KQ. Phase 1 consisted of collective reflection on our data material, which encompassed videotapes of two sessions of four KTs, having the KQ's dimensions and associated codes in mind while making a first attempt to use the codes in analyzing our data. In phase 2 we collectively watched video excerpts from the four KTs, a phase resulting in choosing one of the KTs, called Wilma, as our analytical case. This choice was due to Wilma's teaching as being suitable for employing KQ as an analytical tool, and thus in answering our research question. In particular, we chose to analyse Wilma's teaching because it more than the others revealed all the four dimensions of the KQ. Furthermore, Wilma's teaching, more than the others, revealed more mathematical concepts and ideas, and her teaching, more than the others, explicitly revealed mathematical dialogues between the KT and the involved children. In watching the video of Wilma, we were able to observe her revealing of foundational knowledge, knowledge-in-action and knowledge-in-interaction. In phase 3 we transcribed in detail the video of Wilma's teaching. Phase 4 consisted of our collective conducting of in-depth analyses of the teaching.

Context

Wilma is a well experienced kindergarten teacher at the age of 45. She is educated as kindergarten teacher from university training (180 ECTS). The mathematical activity explored in this study concerned children's inquiries into features of two-dimensional geometrical shapes, the various shapes' names, and their conceptual relationships with each other. The session lasted for 26 minutes, and it involved six five-years-old children. In our design of the activity, materialised as a written activity description, we emphasised the aims of the activity, gave suggestions for how to teach the activity, provided explicit examples of mathematics questions to ask the children, and were explicit about the manipulatives to use, triangles, squares, rectangles, circles, trapezium, and rhombus. As regards the aims, we wrote: *The children are supposed to get experience in recognizing properties to different two-dimensional shapes. Furthermore, the children are supposed to practice mathematical argumentation with respect to features of the various shapes.* Concerning the teaching, we wrote: *Let the children investigate the shapes and their characteristics. Let the children discover the shapes' differences.* Moreover, we also encouraged the KT to make the activity her own, benefitting utilisation of her own experience and competence. Drawing on the KQ, earlier research and our methodical approach, we want to find answers to the following research question:

In what ways do a kindergarten teacher's subject knowledge come into play when teaching a mathematical activity for 5-year-olds?

Analysis and Results

We argue in accordance with Rowland and colleagues that "the quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom" (Rowland et al., 2003, p. 97). In the following, we present an analysis of Wilma's teaching of the geometry activity informed by the four dimensions of the KQ.

Foundation

The analytical contributory codes of Foundation are: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy, use of terminology; use of textbook; reliance on procedures (Rowland et al., 2005, p. 265). We found examples of Wilma's foundation revealed in the initial phase of her teaching:

Dialogue 1

Wilma: Today I have brought this box (Shakes a cubic box so that it makes sounds)
Sam: Oh, yes. The one you showed us before. But I don't remember what's inside it
Jack: It's shapes
Wilma: Yes, that's correct. And with mathematical terminology we call them geometrical shapes. Are you able to pronounce that?
Sam: I think it is cookies (Smiles as he says it)
Wilma: John, are you able to pronounce that? Geometrical shapes?
John: Geometrical shapes (Several children repeat and say simultaneously "Geometrical shapes")
Wilma: Yes, that is what they are called with mathematical terminology. Inside this box there are several of such shapes (She opens the box and shows it to all the children so that they may look inside the box)
John: It looks like a puzzle
Wilma: Yes, it looks like a puzzle. That's true
Sam: Yes. Are we going to puzzle with them?
Wilma: At least we are going to work with them, yes we are
Ken: Can you pour them out?
Wilma: I was thinking pouring them out. Then I want you to take a look at them. Currently, there are quite a few shapes and some of them are almost identical. Now you may take a look at them. (She pours the shapes out on the table; the children take some shapes each and say "that is small" and "a triangle")

In this dialogue we argue that particularly the codes awareness of purpose, theoretical underpinning of pedagogy, use of terminology, and overt subject knowledge may be used to analyse Wilma's teaching. Wilma's shaking of the box with shapes inside, showing the shapes and eventually pouring them out on the table, establish curiosity and engagement among the children. Moreover, the similarities made between shapes and puzzles nurture the children's interest and curiosity. Wilma's playful way of teaching this initial phase, certainly demonstrates her awareness of the overall purpose of the activity. Wilma furthermore attempts to make the children inquire into the different shapes and the shapes' features. Wilma wants the children to "take a look at them", and by doing that she signals that she wants the children to study the various shapes, distinguish between different shapes, recognise the shapes' characteristics, similarities between the shapes and so on. Wilma is thus empowering the children in using inquiry as a tool to make sense of the mathematics, simultaneously as she nurtures the children's curiosity and interest. These actions testify as exemplifying her theoretical underpinning of pedagogy.

Moreover, Wilma introduces congruent and similar shapes, and a large variety of shapes (various triangles, various quadrilaterals, circles of numerous sizes, ellipses, hexagons and octagons). This variety exceeds the variety suggested by us. Thus, Wilma makes her subject knowledge regarding geometrical shapes overt when elaborating and deviating from the written description.

Wilma emphasises mathematical terminology by her twice expressing of the term "geometrical shapes". She wants the children to appropriate the term. This emphasis on terminology also occurs a few minutes later, where Wilma emphasises that what Susie calls an oval shape mathematically is called an ellipsis. Wilma is making a link between shape (oval) and name (ellipsis) overt:

Dialogue 2

Susie: (Picks up a small oval shape and shows it to the other children) This is oval
Wilma: Yes, that's true. That one is oval. Do you know what it is called with mathematical terminology?
Susie: (Susie shakes her head)
Wilma: Sam and Jack, look at the one Susie now has in her hands. Susie said that it was oval. With mathematical terminology that shape is called an ellipsis.
Susie: Ellipsis?
Wilma: Yes, ellipsis.

The codes associated with Foundation are applicable and useful when analyzing this dialogue. By drawing on the codes of KQ, we get glimpses into parts of Wilma's foundational knowledge with respect to two-dimensional geometrical shapes and how she utilises this knowledge in her teaching of five-year-olds in kindergarten. Both mathematical and didactical insights are revealed.

Transformation

The analytical contributory codes of Transformation are: choice of representation; teacher demonstration; choice of examples (Rowland et al., 2005, p. 265). The dialogue below exemplifies how Wilma's knowledge-in-action was revealed. The children picked up various shapes that they found interesting, pentagons, ovals, and quadrilaterals; shapes they were not familiar with from before:

Dialogue 3

Wilma: Do you know what? These two quadrilaterals actually have other names with mathematical terminology. They have four edges (She counts "one, two, three, four" aloud while simultaneously pointing at the edges).
Susie: But what are they called then?
Wilma: That one is called a rhombus (she points at the rhombus while speaking).
Susie: Rhombus.
Wilma: Rhombus. And that one, do you notice that two and two edges are equal (she points at the parallelogram she shows). That edge and that edge (slides her finger along the two opposite parallel edges), are equal, and that edge and that edge are equal (slides her finger along the two other opposite parallel edges). Its name is actually a parallelogram.
Sam: A paragram?
Wilma: Yes, a parallel o gram.

This dialogue shows how Wilma transforms the mathematical content involved in the activity. She focuses at discussing two particular shapes which the children are unfamiliar with, a rhombus and a parallelogram. We interpret her utterance "These two quadrilaterals actually have other names" as an attempt to elaborate the children's conceptual reasoning concerning quadrilaterals. She makes a conceptual juxtaposition by counting the edges of these shapes aloud, making it overt that the shapes are indeed quadrilaterals but at the same time particular kinds. Wilma implicitly distinguishes these quadrilaterals from the more familiar quadrilateral shapes rectangle and square.

Wilma's teaching is interpreted as to illustrate how the dimension of Transformation is unveiled in a kindergarten setting. The associated codes were useful in characterizing how she transforms the mathematics to create appropriation opportunities for the children. Wilma chooses to use concrete, manipulative materials (choice of representation) and clearly shows the two shapes to all the children while she focuses on their mathematical names as well as their features (teacher demonstration). Wilma's action of showing one example of each of the new quadrilaterals while simultaneously describing the shapes as quadrilaterals by counting their edges (choice of examples), further illustrates her transformation of the mathematics involved.

Connection

The analytical contributory codes of Connection are: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness (Rowland et al., 2005, p. 265). We interpret Dialogue 3 to also exemplify how the dimension of Connection characterises Wilma's teaching. Wilma is making connections between concepts when she elaborates on the children's conceptualization of quadrilaterals. She establishes a shared focus of attention by pinpointing particular features of the parallelogram. Wilma uses her sliding index finger to emphasise that two and two edges are pairwise parallel. She says, "that edge and that edge are equal", and we interpret this as an attempt to make a connection between the characteristics of the edges and their fundamental role for classifying the shape as a particular quadrilateral. Wilma uses the term "equal" and not "equal length" and "parallel". The parallelism and equal length of the pairwise edges are in that sense only implicitly communicated. However, using the term "equal" together with the gesture of sliding her finger along the edges, we interpret as communicating that the edges are of equal length. From a mathematical point of view, if a quadrilateral has two and two opposite edges of equal length, these edges necessarily are parallel, and the quadrilateral is indeed a parallelogram.

We interpret Wilma's choice regarding variety of shapes as well as congruent and similar shapes as exemplifying how she anticipates the internal mathematical complexity and how she recognises the conceptual appropriateness of these two-dimensional geometrical shapes. Wilma's decisions regarding variety and number of shapes also testifies to how the dimension of Connection characterises Wilma's teaching.

At the end of Dialogue 3 we once again find an example of Wilma's focus on mathematical terminology, a focus demonstrating how she uses her foundational knowledge. Wilma offers opportunities for the children to appropriate the name "parallelogram" twice, showing her eagerness in naming mathematical objects correctly. The dialogue below further demonstrates how Connection characterises Wilma's teaching:

Dialogue 4

Wilma: Yes, Ken. Do you want to show that shape? Then we first have to discuss the shape of it

Ken: One, two, three, four edges (Ken shows, while rotating, a trapezium, close to a square)

Wilma: Four edges, Yes, it does have that. But what is different with this? If we compare it with that (She picks up a square). What is different? We may put them down at the table for all to see (She puts down the square while Ken puts down the trapezium next to it in the middle of the table). What is different?

Susie: That one is more askewed (points at the trapezium)

Wilma: Yes, that one is more askewed. It looks like there are two lines that are tilted… And then there are two lines that are equally straight (Slides her index finger along the parallel edges). This shape is called a trapezium. That is a difficult word.

Sam: You are quite precise

Wilma: Do you think I'm precise? Well, that's good. It is important to be quite precise.

In Dialogue 4 we once again see how Wilma tries to make the children appropriate connections between concepts, this time by comparing and distinguishing between a square and a trapezium. Ken has chosen a shape that he wants to show the other children, a shape that he finds out has four edges by counting. Counting has been used by Wilma (Dialogue 3) as a strategy to characterise shapes. Ken here adopts that strategy and implicitly argues the shape to be a quadrilateral, an implicit claim confirmed by Wilma. Then Wilma continues and starts to compare the chosen trapezium with a familiar and close to congruent square. She obviously wants the children to look at the two shapes that are almost identical, in order for the children to come up with the features that distinguish the trapezium from the square. In doing that, Wilma obviously also has anticipated the complexity and involved concepts to be appropriate for these children.

Three times Wilma asks the question "What is different?". Susie recognises that the trapezium "is more askewed". We interpret this utterance as Susie's way of telling the others that two of the edges are not parallel and that this fact makes the trapezium differ from the square. This interpretation is supported by the response Wilma gives, that "there are two lines that are tilted" and that "two lines that are equally straight. Comparing the two shapes edge by edge, the trapezium has two edges that are not parallel, a feature that makes that shape having a separate name: "This shape is called a trapezium". Interestingly, Sam comments on Wilma's argument and claims her to be quite precise. We interpret Sam's utterance as his way of showing that he has recognised that Wilma is mathematically accurate in her way of reasoning and orchestrating the activity. Sam seems to be involved in an initial process of appropriating the stringency and accuracy of mathematics.

Contingency

The analytical contributory codes of Contingency are: responding to children's ideas; use of opportunities; deviation from agenda (Rowland et al., 2005, p. 266). From the four dialogues analysed, we saw several examples of how Wilma responded to the children's contributions. Now we want to illustrate how Contingency characterised her teaching. In the following we do this by inclusion of single moves Wilma made, where she responded to the children through repetition, questioning and affirmation to make the children aware of the mathematical ideas involved.

On several occasions, Wilma was responding to children's ideas by acting in the moment. The children came up with two mathematical ideas in their dialogic contributions, the concept of sorting and the concept of geometrical shapes. Concerning sorting, Sam asked the question: "Can we sort them?". Some moves and seconds later, Wilma responded to Sam's question by asking a question in return: "Sam, what does it mean to sort?". She nurtured Sam's reasoning and wanted him to be explicit about his thinking regarding the mathematical concept of sorting. Later in their dialogue, Wilma followed up the emphasis on the concept of sorting by affirming one child's actions: "But he has sorted. That's excellent". As regards the concept of geometrical shapes, Wilma emphasised the mathematical features of the various shapes. This was exemplified in Sam's utterance: "Yes, but these are small (points at the short edges of the rectangle). These two are equally long". Wilma's response to this contribution was to address a question to all six children: "Does anybody know what the shape is called when two edges are quite long and two edges are shorter?". Wilma's response, in the form of a question, we argue, nurtured curiosity and interest among the children. Furthermore, her question also testified to how she contingently thought "'on her feet' and respond appropriately" (Rowland et al., 2005, p. 266) to the children's contributions. Wilma took the opportunity to involve all the children in collective reasoning regarding this shape.

Similarities and differences between the shapes were also focused on in Wilma's teaching, as we saw in Dialogue 4. Additionally, on another occasion, Susie argued that two congruent triangles may be joined in order to make a rectangle: "Jack's shapes have such…, but Ken's do not have such when he puts them together". Wilma responded to Susie's contribution by giving a question in return: "What happens when you put them together?"

In Wilma's teaching, we also found instances where she did make use of opportunities. Wilma made the children pay attention to the mathematical concept of counting as a strategy to classify shapes: "John, perhaps you can count how many edges they (two regular hexagons and one regular pentagon) have?". Another example where Wilma made use of opportunities occurring in the midst of teaching, was when Wilma nurtured the children's mathematical reasoning through questioning: "How did you figure out that one (points at one of the hexagons)?"; and "Do you want to tell the other children?". Additionally, Wilma's emphasis on naming unfamiliar shapes always occurred after the children had sorted the various shapes and a discussion had been going on regarding the shapes' mathematical features.

Discussion

We set out in this study to explore the following question: In what ways do a kindergarten teacher's subject knowledge come into play when teaching a mathematical activity for 5-year-olds? We used the Knowledge Quartet (Rowland et al., 2003, 2005) as an analytical lens through which we analysed one kindergarten teacher's teaching of a mathematical activity on two-dimensional geometrical shapes. From the analyses it was evident that the KQ was a powerful and useful framework when characterizing a KT's orchestration. From the analyses we saw that the KT employed her foundational knowledge (Foundation), knowledge-in-action (Transformation and Connection) and knowledge-in-interaction (Contingency) in several ways. In this sense, our study empirically confirms the claimed importance of early years teachers' mathematical content knowledge and pedagogical content knowledge (Bruns et al., 2017; Oppermann et al., 2016).

Almost all analytical contributory codes of KQ were found useful and illuminating in our analyses. The KT's foundational knowledge was, as far as what was revealed in the analyses, characterised by her emphasis on the use of mathematical terminology and her substantial subject knowledge regarding two-dimensional geometrical shapes. She also revealed her foundation knowledge through the overt acknowledgment of the activity's purpose and her theoretical underpinning of pedagogy through establishment of curiosity, interest, and engagement on behalf of the children. The children were given opportunities to inquire into the mathematics (Wells, 1999). Thus, Wilma's foundational knowledge fundamentally informed her teaching.

The codes (cf. Rowland et al., 2005) have furthermore proven to be analytically valuable tools, showing that the KT revealed her knowledge-in-action through use of manipulatives and purposeful choice of examples in demonstrating the features of various geometrical shapes. Moreover, the knowledge-in-action revealed was characterised by Wilma making connections between the concepts of quadrilaterals such as square and trapezium as well as drawing attention to features such as number of edges, length of edges and parallelism between edges associated with the various shapes. Furthermore, these utterances and actions showed how she "capitalises on these contingent situations" (Rowland & Zazkis, 2013, p. 137), in order to create opportunities for the children to appropriate the involved mathematical concepts.

Reflecting on the analyses made by Rowland et al. (2005) of mathematics teaching in a British school classroom, we observed in the analyses above that the KQ cannot be directly used as a template for analyzing a KT's teaching of a mathematical activity in a Norwegian kindergarten setting. Some of the codes were not found applicable. KTs working in accordance with the Norwegian curriculum for kindergarten (Norwegian Directorate for Education and Training, 2017), rarely teach mathematical activities characterised by long theoretical introductions and demonstrations. That would be regarded as inappropriate. In our analyses we rather observed that the KT's moves were quite short concerning time, often about 5–10 seconds. Additionally, it is rare in the Norwegian kindergarten setting to give children extensive time for inquiring into mathematical ideas without the KT interfering occasionally. The KQ analytical codes use of textbook, reliance on procedures, teacher demonstration, and making connections between procedures, are thus partially inapplicable since these are argued to be rarely present in mathematics teaching in kindergarten. In our analyses, we have thus not found these codes to be useful.

Our analyses have also shown that it was challenging to apply only one code for some of the moves and parts of the dialogues. In the dialogues analysed above, we observed that the dimensions were to some extent intertwined. This result is in accordance with the finding of Maher et al. (2022). This is also an argument in accordance with the developers of KQ (Rowland et al., 2005). One move may be argued to exemplify several codes, and hence, dimensions, simultaneously. The four dialogues analysed were thus not mutually exclusive adopting the KQ as an analytical lens. Nevertheless, they are argued to be characteristics of the kindergarten teacher's revealed foundational knowledge, knowledge-in-action and knowledge-in-interaction.

Studies of KTs' mathematical competencies are important since it is those competencies they have to draw on in order to nurture the children's processes of appropriating the mathematical concepts involved in activities (cf. Moschkovich, 2004; Rogoff, 1990). Wilma's use of knowledge-in-action and knowledge-in-interaction established opportunities for the children to make the mathematical concepts of various two-dimensional shapes, their names and features, their own. Future research opportunities may be found in scrutinising participating children's appropriation processes with respect to the mathematics offered. Further studies of KTs' mathematical competencies and how these are revealed in practice is also a promising road ahead.

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ECE Teachers' Use of Educational Technology in Early Mathematics Education and Its Association with Teacher and School Characteristics

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Introduction

Children's early mathematical skills are among the strongest predictors of their later academic achievement (Duncan et al., 2007). Supporting early mathematical development is therefore considered a key objective of early childhood education (ECE). A potential way to support the development of early mathematical skills is the use of educational technology (ET). Educational technology (ET) is generally defined as "electronic tools and applications that help deliver learning content and support the learning process" (Cheung & Slavin, 2013a, p. 279). In elementary and secondary education, cumulative evidence points to the potential of ET for improving learning outcomes in a variety of content domains (e.g., Cheung & Slavin, 2013a, b). An increasing number of studies recently pointed to the beneficial effects of ET for ECE as well (Griffith et al., 2020; Verbruggen et al., 2021). However, studies on ECE teachers' use of ET in early mathematics education and the factors that are associated with this ET use are limited. We aimed to complement current insights into this topic by systematically analyzing ECE teachers' use of ET in early mathematics education, in association with potentially influencing teacher and school characteristics.

Studies on elementary and secondary school teachers' use of ET in their (mathematics) instruction point to the complex interplay between, on the one hand, teachers' actual use of ET and, on the other hand, general and ICT-related teacher and

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school characteristics that contribute to this use. In line with these findings, Vanderlinde and van Braak (2010) developed the e-capacity model, distinguishing among different teacher level and school level conditions that are assumed to contribute to the effective integration of ET in regular classroom instruction. The e-capacity model consists of four concentric circles (see Fig. 1), with teachers' actual use of ET in the inner circle. The surrounding circles define the ICT-related teacher conditions, the ICT-related school conditions, and the school improvement conditions that are assumed to contribute to the implementation of ET in regular classroom instruction. As shown in the inner circle of Fig. 1, teachers can implement ET in their instruction in view of different goals, i.e., (a) to stimulate their students' basic ICT skills (acquiring knowledge and skills in ICT), (b) as an information tool (offering information to students via ET), and (c) as a learning tool (practicing domain-specific knowledge and skills in other domains than ICT). According to the e-capacity model, the implementation of ET is influenced by teachers' ICT competencies and their professional development in the domain

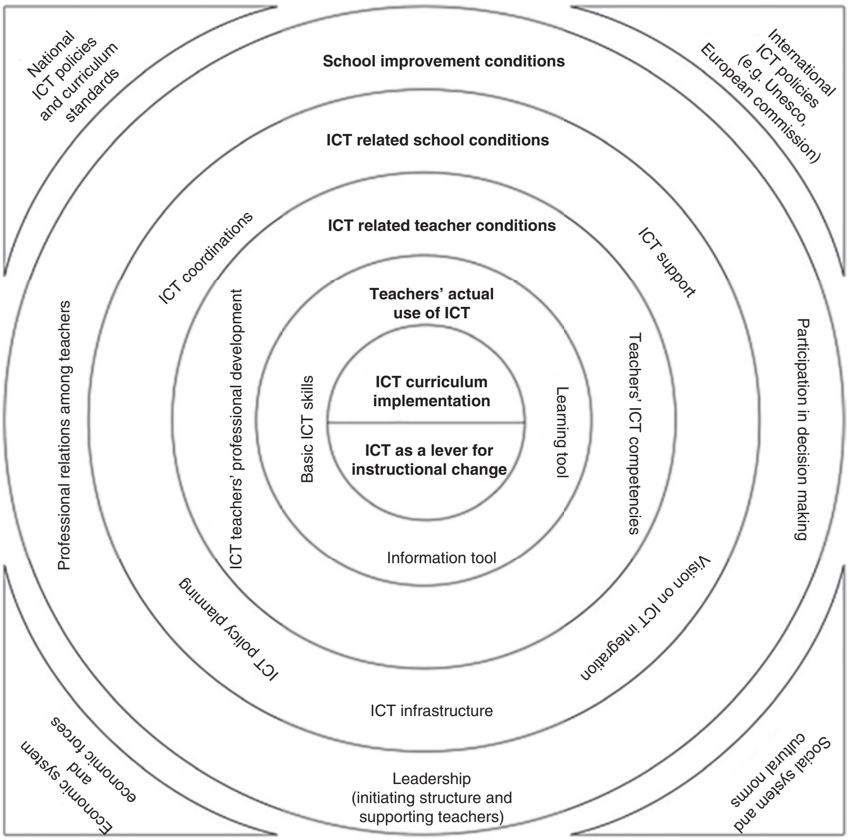

Fig. 1 The e-capacity model (Vanderlinde & van Braak, 2010, p. 544)

ICT. Teachers' ICT competencies are typically described in terms of their basic ICT skills and technology-related teaching skills (Fraillon et al., 2014; Sailer et al., 2021). Teachers' basic ICT skills refer to their "ability to use [digital technologies] to investigate, create, and communicate in order to participate effectively at home, at school, in the workplace, and in society" (Fraillon et al., 2014, p. 17). Teachers' technology-related teaching skills encompass their technological pedagogical knowledge that is required to effectively use digital technologies in their instruction in general and in specific content domains (Sailer et al., 2021). Professional development related to both basic ICT skills and technology-related teaching skills helps teachers to continuously align their mastery of these important skills with the rapidly-changing digital technologies sector. The next circle in the model refers to the ICT-related school conditions that are assumed to contribute to the effective use of ET in regular classroom instruction. Its major components involve the quantity and the quality of the school's ICT infrastructure, the school's vision and policy related to the integration of ET in regular classroom instruction, and the organization of coordination and support to facilitate this integration. General school improvement conditions, including the organization of the leadership at school, teachers' participation in decision making and their professional relations, form the outer circle in the model.

Previous empirical research provided evidence for the assumed relations within the e-capacity model in the context of ECE in general. These studies addressed ECE teachers' use of ET across all content domains, without specific focus on the domain of early mathematics. In other words, they pointed to ET use in ECE including the domain of mathematics, but without specific focus on or analyses related to only the domain of mathematics. These studies revealed that ECE teachers generally implement ET (across the different content domains of the ECE curriculum), but that the frequency of its implementation is rather limited. When implementing ET in their classrooms, ECE teachers were shown to use a variety of programs to reach a rich diversity of educational goals (e.g., Blackwell et al., 2013, 2014; Kerckaert et al., 2015; Masoumi, 2015; Nikolopoulou, 2014; Nikolopoulou & Gialamas, 2015; Romero-Tena et al., 2020). The programs used ranged from domain-specific programs focusing on the acquisition of young children's competencies in a specific content domain and/or on the remediation of difficulties in the acquisition of these competencies to domain-general software (e.g., Powerpoint) that can be used in diverse domains. Turning to the goals of ET use, it was shown that ECE teachers aim to address diverse aims, ranging from entertainment and communication to stimulating young children's development in specific content domains. Departing from the e-capacity model described above, Kerckaert et al. (2015) studied Flemish ECE teachers' use of ET in terms of goals and contributing variables. Their study revealed that ECE teachers integrate ET in their classrooms in view of two major goals, namely (a) to support preschoolers' acquisition of basic ICT skills and attitudes, and (b) to stimulate the development of preschoolers' competencies related to domain-specific contents and to support individual learning needs. These researchers further found that ECE teachers used ET most frequently in view of supporting preschoolers' acquisition of ICT skills and attitudes, and less frequently in view of stimulating preschoolers' development in specific content domains as mathematics.

Additionally, and in line with the assumed associations in the e-capacity model, the previously conducted studies on ECE teachers' use of ET in their classrooms pointed to the contribution of both teacher and school characteristics to this use (Blackwell et al., 2013, 2014; Kerckaert et al., 2015; Nikolopoulou & Gialamas, 2015; Pynoo et al., 2013; Romero-Tena et al., 2020). However, findings revealed a complex and mixed picture of a rich diversity of both general and ICT-related variables that might be associated with ECE teachers' ET use. Turning to teacher characteristics, studies pointed to the association between ET use and teachers' attitudes and beliefs about the affordances of ET, ICT competences, ICT professional development, computer experience, experience in education, work situation, degree/qualification, and the grade in which they teach. With respect to school characteristics, the limited number of available studies identified the type and location of the school, the school's ICT policy and infrastructure as important variables. But current findings about the contributions of these teacher and school characteristics to ECE teachers' general (across all content domains) ET use are mixed.

Given the small number of studies on ECE teachers' use of ET in general and the absence of studies analyzing this use in specifically the domain of mathematics, their mixed findings and the period in which they were conducted (i.e., studies conducted 5–10 years ago, in a rapidly changing ET society, requiring continuous and up-to-date studies on ET use), we aimed to complement current insights into ECE teachers' use of ET in early mathematics education by systematically analyzing (a) ECE teachers' use of ET in mathematics education (RQ1), (b) the different types of programs they use (RQ2), (c) the aims they try to address with this use (RQ3), and (d) the teacher and school variables associated with ET adoption (RQ4). We relied on the conceptual framework of Vanderlinde and van Braak (2010) on teachers' use of ET and the variables that are assumed to contribute to the adoption of ET in educational practice. Taking into account the findings of Kerckaert et al. (2015) on the aims with which Flemish ECE teachers implement ET in their classrooms, we defined the major goals of ET use as (a) supporting preschoolers' acquisition of basic ICT skills and attitudes, and (b) stimulating the development of preschoolers' competencies related to domain-specific contents and to support individual learning needs. As teachers' attitudes and beliefs are shown to contribute to their classroom instruction as well (Blömeke et al., 2015; Gasteiger & Benz, 2018; see also Kerckaert et al., 2015), we added ECE teachers' attitudes towards the use of ET in preschool instruction and their mathematical self-efficacy as additional contributing variables to the study. As such, we aimed to complement and extend current insights into ECE teachers' ET use by (a) analyzing ET use in the specific content domain of early mathematics education (and not in general, across all content domains), (b) relying on the theoretical framework of Vanderlinde and Van Braak (2010), including the most relevant potentially contributing variables in view of ET use.

We conducted the study in Belgium, Flanders, where ECE is organized for children aged 2.5–6 years. ECE teachers are trained as generalists during a 3-year professional Bachelor program including both theoretical and practical learning opportunities (180 ECTS credits, with 60 ECTS credits per year). ECE teachers are expected to stimulate children's development in a broad range of curricular domains, including mathematics, in informal learning situations (Eurydice, 2020). Typically, these informal learning situations consist of age-appropriate play-based learning activities integrating core competencies from different curricular domains.

Method

Participants were 342 ECE teachers from 219 different schools. Almost all participants were female (98%) and owned a computer at home (99%). Their age ranged from 22 to 62 years (M = 41 years), and they had at least 1 but not more than 41 years (M = 18 years) teaching experience. All participants gave their active informed consent. The study procedures were approved by the Social and Societal Ethics Committee of KU Leuven (G-2019 11 1814).

All participants were individually offered a structured interview or questionnaire focusing on ET use in mathematics education and its associated teacher and school characteristics. Although we aimed to conduct individual face-to-face interviews with all teachers, we had to change to digital interviews and digital questionnaires due to the national COVID-19 regulations for 200 of the 342 teachers (respectively 74 and 126 teachers). The method applied in the context of this study thus varied across participants, ranging from individual face-to-face interviews with the first 142 teachers via individual digital interviews with 74 teachers to digital questionnaires with the last 126 teachers (due to COVID-19 regulations). As we originally designed the study as an interview study with individual face-to-face interviews and as most teachers were individually interviews, we will further refer to interviews (and not questionnaire) as method applied in the study. Moreover, as there were no differences between these three groups of participants in ET use (Kruskal-Wallis test, $H(2) = 2.30$, $p = 0.32$), we grouped all participants in our analyses.

During the interviews, teachers were first asked about their adoption of ET in mathematics education (yes/no). In case they did not adopt ET in mathematics education, teachers were invited to report on the reasons for not using ET in their mathematics instruction. In case teachers reported to adopt ET, they were asked which concrete programs they did use. Additionally, for the (maximum) three programs they most frequently used, we offered them a series of questions addressing the goals and frequency of implementation, supported with a Likert-scale to facilitate the comparability and analysis of their answers. Concretely, they were asked to report whether and how often they used it (a) to foster preschoolers' basic ICT skills and attitudes (scale consisting of 4 items, e.g., "I use this program to teach my preschoolers basic ICT skills", to be rated on a 5-point Likert-scale, ranging from "never" to "daily"; max. score 20) and (b) to offer mathematical contents and to support individual learning needs (scale consisting of 6 items, e.g., "I use this program for illustrating certain topics in the domain of mathematics", to be rated on a 5-point Likert-scale, ranging from "never" to "daily"; max. score 30). Finally, we asked them a series of questions - again supported by a Likert-scale for ease of responding and scoring - on teacher and school variables that might contribute to the adoption of ET in their mathematics education. These questions were organized along eight different scales as described in the e-capacity model (Vanderlinde & Van Braak, 2010): (a) ICT policy, e.g., "In my school, there is a clear ICT policy plan" (11 items, 6-point Likert-scale, "completely disagree" to "completely agree", max. score 66), (b) ICT infrastructure, e.g., "In my classroom, there are sufficient computers available for the preschoolers" (7 items, 6-point Likert-scale, "completely disagree" to "completely agree", max. score 42), (c) ICT professional development, e.g., "I try to keep informed about everything that has to do with ICT in education" (4 items, 6-point Likert-scale, "completely disagree" to "completely agree", max. score 24), (d) ICT competences, e.g., "How well can you use ICT for the following purposes? Using ICT for lesson preparation" (18 items, 5-point Likert-scale, "not" to "excellent", max. score 90), (e) attitudes towards (i.e., perception of) ET use in education, e.g., "To what extent do you agree with the following statement: ICT improves the quality of education" (6 items, 6-point Likert-scale, "completely disagree" to "completely agree", max. score 36), (f) years of computer experience at home, i.e., "How many years of experience do you have with computers in your private life/ spare time?", (g) years of computer experience in classroom, i.e., "How many years of experience do you have with computers in the classroom?", and (h) self-efficacy in the domain of mathematics, e.g., "How confident do you feel about your ability to solve the following math problem? Calculating the price of a TV with 30% discount" (5 items, 4-point Likert-scale, "very unconfident" to "very confident", max. score 20) (Oppermann et al., 2016). Cronbach's alpha per scale ranged from .74 to .87, indicating sufficient to good internal consistency. Table 1 presents the descriptives (internal consistency, means, SD and range) per scale.

Table 1 Cronbach's alpha, range, mean and SD per scale

Scales	Cronbach's α	Range	*M*	*SD*
ICT policy (max. score = 66)	.81	18–65	46.08	8.90
ICT infrastructure (max. score = 42)	.76	7–41	27.14	6.56
ICT professional development (max. score = 24)	.76	4–24	12.70	4.01
ICT competences (max. score = 90)	.87	32–90	61.88	10.53
Attitudes towards ET use in education (max. score = 36)	.78	9–36	26.09	4.41
Years of computer experience at home	–	0–40	19.47	6.30
Years of computer experience in classroom	–	0–30	8.86	5.66
Years of experience in education	–	1–41	18.38	11.08
Self-efficacy in the domain of mathematics (max. score = 20)	.74	5–20	15.51	3.29
ET use in the domain of mathematics - basic ICT skills and attitudes (max. score = 20)	.87	4–20	13.15	3.86
ET use in the domain of mathematics - mathematical contents and individual learning needs (max. score = 30)	.79	6–26	14.75	5.12

We analyzed our data using SPSS Version 27.0. To answer RQ1 and RQ2, we descriptively analyzed teachers' use versus non-use of ET in mathematics education and, in case of using ET, computed how frequently they reported the use of (a) programmable ET (i.e., robots and programming languages, such as Beebot and Scratch Junior), (b) specific practice programs (i.e., programs to practice knowledge or skills in one specific mathematical subdomain, such as Tangrams), (c) comprehensive practice programs (i.e., programs to practice knowledge or skills in multiple mathematical subdomains, either within the same environment or in different related environments, such as Math Garden or Lego apps), (d) digital stories (e.g., digital storybooks on YouTube), and (e) other ET programs that cannot be classified into one of the four other groups (i.e., other content domain, domain-general programs, or no information about the name and/or type of the program). We computed the frequency of using ET to enhance basic ICT skills and attitudes versus to support other learning contents and individual learning needs per type of ET program (see RQ2, distinction among 5 different types of programs) to answer RQ3. Finally, we conducted binary logistic analysis (backward method) to answer RQ4, predicting ECE teachers' adoption of ET in mathematics education on the basis of all teacher and school variables included in the interview questionnaire.

Results

Our analyses indicated that almost 2/3 of the teachers (217 teachers) used ET in mathematics education, and thus also that about 1/3 did not (RQ1). Teachers mainly referred to constraints in the school's ICT infrastructure, namely (a) insufficient ICT infrastructure at school (44 teachers), (b) too expensive (38 teachers), (c) not included in teaching methods and materials used in my classroom or at school (38 teachers), and to insufficient knowledge and skills to effectively implement ET in their instruction, namely (a) never thought of using ET in mathematics it before (53 teachers), (b) no or insufficient information about selection and implementation of programs (48 teachers).

The 217 teachers adopting ET reported a total of 326 different ET programs (RQ2). Comprehensive practice programs were most frequently reported (100 teachers), followed by general (e.g., Powerpoint) or unspecified (unclear) programs (87 teachers). Digital stories (56 teachers), programmable ET (35 teachers) and specific practice programs (38 teachers) were less often mentioned. Most teachers reported using multiple programs, either within the same type of programs (64 teachers) or across multiple types of programs (56 teachers).

Next, with respect to the aim of using ET in mathematics education (RQ3), teachers reported the acquisition of basic ICT skills and attitudes at least as frequently as the support of mathematical competencies and individual learning needs for each of the different types of programs. Specific and comprehensive practice programs were used in view of both aims on an on average monthly basis, and thus as frequently offered to the children to support their ICT skills and attitudes as to

Table 2 Results of the binary logistic regression predicting ET adoption (n = 342)

	B	*S.E.*	Wald	Sig.	*Exp(B)*	Nagelkerke R^2
Constant	−4.12	1.24	11.14	<.01	0.02	
ICT infrastructure	0.06	0.03	4.85	.03	1.06	
ICT competences	0.05	0.02	9.26	.00	1.05	
Computer experience home	0.05	0.03	3.91	.05	1.06	
Grade (youngest)	−1.86	0.34	30.32	.00	0.16	

Note. Only the results for the final model are presented (only significant associations)

enhance their mathematical development or address their individual learning needs. But for the three other types of programs, teachers mainly aimed at fostering the acquisition of ICT skills and attitudes, with less attention for their development in the domain of mathematics. Digital stories and general or unspecified programs were used to support basic ICT skills and attitudes every week (on average), but only every trimester respectively every month in view of mathematical support and individual learning needs. Programmable ET was used least frequently, with an average monthly use in view of fostering ICT skills and attitudes and an average trimester use in view of enabling mathematical contents and individual learning needs.

Finally, we analyzed the association between teachers' adoption of ET (yes/no) and the teacher and school characteristics included in the interview questionnaire using binary logistic regression analysis (RQ4). As shown in Table 2, this analysis revealed that the school's ICT infrastructure and teachers' ICT competences and computer experience at home were positively related to teachers' use of ET. Teachers who reported being equipped with more ICT infrastructure in their school and possessing more ICT competences and computer experience at home were more likely to adopt ET.

Discussion

We aimed to complement and extend current insights into ECE teachers' ET use in early mathematics education by (a) analyzing ET use in the specific content domain of early mathematics education (and not in general, across all content domains), (b) relying on the theoretical framework of Vanderlinde and Van Braak (2010), including the most relevant potentially contributing variables in view of ET use. We therefore interviewed 342 teachers about their adoption of ET, the programs they use and their aims of using these programs, as well as the school and teacher characteristics that might contribute to ET adoption. A first major finding of the present study is that the majority of teachers reports to adopt ET in early mathematics education. However, a substantial amount of them refrains from doing so: about 1/3 of the teachers did not include ET in their early mathematics education. Teachers who did not adopt ET in their early mathematics instruction mainly pointed to difficulties related to the available ICT infrastructure at their schools and to their own ICT knowledge and skills as major obstacles to effectively use ET. This was confirmed in our regression analyses on the teacher and school variables that might contribute to the (non-)use of ET in early mathematics education, identifying the school's ICT infrastructure, teachers' ICT competences and their computer experience at home as key variables for including ET in ECE mathematics education. Teachers who did include ET in their mathematics education were better equipped at school in terms of ICT infrastructure, and had better developed ICT competences and more computer experience at home than teachers who did not include ET in their early mathematics education. These findings are in line with previous studies on the topic, and have important implications for educational policy and teacher training and professional development. First, these findings point to the need for sufficient financial support for improving the schools' ICT infrastructure. As discussed in Sailer et al. (2021), the availability of sufficient ICT, both in terms of quantity and in terms of quality of infrastructure, functions as a threshold to effectively implement ET in educational practice. However, as these researchers' findings indicate, sufficient and qualitatively-strong ICT infrastructure is a necessary but insufficient condition for effective ET use. Having available the necessary equipment, teachers also need to have well-developed ICT competences, both in terms of digital skills and in terms of pedagogical-technological competencies. Whereas teachers' digital skills enable them to design instructional activities that allow both passive and active learning processes in their students, teachers need sufficient pedagogical-technological competencies to fully employ the potential of ET in their classrooms and enable constructive learning processes as well. Teacher training and professional development initiatives need to provide ECE teachers ample theoretical and practical learning opportunities to acquire these important ICT competencies. As such, the effective use of the available ET can be promoted, resulting in rich learning opportunities for young children including ET.

As a second major finding, we point to the rich diversity of ET programs that ECE teachers use in mathematics education, ranging from domain-general to domain-specific programs. Teachers generally preferred the use of comprehensive practice programs that allow to stimulate the development of their preschoolers along a range of mathematical competencies in their mathematics education. These programs were used to enhance children's mathematical development as well as their basic ICT skills and attitudes. The latter finding also applies to specific practice programs that focus on a specific skill within the domain of mathematics: when using these programs ECE teachers mentioned focus on mathematical aims as frequently as focus on ICT-related goals. However, general programs as programmable ET, digital storybooks or domain-general and unspecified programs were used more frequently to foster the acquisition of basic ICT skills and attitudes than to support children's mathematical development and individual learning needs. Teachers' rather limited focus on mathematical aims when using ET in their mathematics education might be due to the instrument we used to assess this theme. Concretely, the items included in the scale for supporting mathematical contents and individual learning needs did not only question teachers' use of ET in view of general mathematical goals for all children, but also addressed contents related to specific individual remediation and support for children with learning difficulties. As the latter items apply to only a limited number of children and reflect a very specific use of ET in the domain of early mathematics, they might have resulted in an underestimation of teachers' mathematical aims when including ET in their mathematics education. Future studies that include a broader range of items, addressing both general (for all children) and specific (in view of differentiation) aims of using ET, are needed to evaluate this hypothetical explanation. Also, ECE teachers' views on the most powerful learning environments for children in the domain of mathematics might help to explain this surprising finding. It is possible that the teachers who participated to our study evaluated informal play situations not including ET as more powerful learning environments for enhancing core mathematical competencies in young children than ET-based programs. Consequently, they might primarily aim for mathematical development in playful situations not involving ET, and offer ET-related learning opportunities as an add-on that allows to also, and even primarily, foster the acquisition of ICT skills and attitudes, next to the mathematical competencies included in the program. As we only included teachers' attitudes towards using ET in education in general, and did not address their beliefs about the role of ET in supporting preschoolers' early mathematical development and in early mathematics instruction, future studies are needed to address this hypothetical explanation. As discussed in Lowrie and Larkin (2020), it is important to involve preschool teachers in discussions about ET integration in their regular classroom practices (in the domain of STEM). These discussions do not only add to the meaningful embedding of ET in play-based learning environments for preschoolers (with learning-supportive offline activities preceding and following the ET use), but also help teachers to build constructive beliefs about the potential of ET in preschool education. Future studies that actively involve preschool teachers in the design of technology-enhanced learning environments for young children in the domain of early mathematics and that include observation methods and in-depth interviews with ECE teachers will help to both foster and better understand ECE teachers' meaningful integration of ET in the domain of early mathematics and in early childhood education more generally. Although our interview data provide a comprehensive overview of the reported ET use in early mathematics education in a large and representative sample of ECE teachers, these self-report data do not give us a deep and detailed understanding of their actual practices. Observation studies with a smaller sample of teachers are needed to get a deepened understanding of the actual classroom practices. Together, the findings of these observation and design studies will increase our scientific insights into ECE teachers use of ET to foster young children's mathematical development, and offer building blocks for improving teacher training and professional development initiatives and, as such, current practices in early childhood education.

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Fostering Responsiveness in Early Mathematics Learning

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Challenges in Early Mathematics Learning

The early years are significant for learning mathematics (Sarama & Clements, 2008). In addition to early learning at home, learning mathematics in kindergarten is an important factor because it is the first experience of institutional learning and builds a crucial basis for future schooling (Claesens & Engel, 2013). In this context, preschool teachers can be seen as more competent other persons who support children's early mathematical learning. This early learning takes place co-constructively in interactions. Therefore, the OECD (2017) sees the interactions between preschool teachers and children as a key variable of an early and co-constructive learning process. It seems particularly desirable and effective that in such interactions the mathematically rich and creative ideas of the children are integrated; but especially German preschool teachers are mostly uncertain of how to realize their function as early learning partners and supporters in mathematically rich and adaptive contexts. To meet this challenge, the "design patterns of mathematical situations" (Vogel, 2014, p. 232; in the following: mathematical design patterns) provide detailed descriptions for adult persons of how to realize a "mathematical situation of play and exploration" (Vogel, 2014, p. 223). The mathematical design patterns are developed in the context of the erStMaL project (early Steps in Mathematics Learning; e.g. Vogel, 2014; Acar-Bayraktar et al., 2011). They provide information about the materials used, a description of multimodal opportunities to initiate mathematical negotiations of meaning and a guide to formulate reactions that are adaptive to the

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children's different mathematical ideas. In addition, they give detailed information about the mathematical content. From the researcher's perspective on these mathematical design patterns, the question arises as to how and to what extent the use of mathematical design patterns can foster a supportive and adaptive simultaneous (inter-)acting of the preschool teachers. A fruitful theoretical and empirical point of access to this question can be achieved through the concept of responsiveness, according to Koole and Elbers (2014), and adaption in mathematics education by Robertson et al. (2015), Bishop et al. (2016, 2022) or Beck and Vogler (2021, 2022).

Responsiveness as a Construct for Analyzing Adult-Child Interactions

In order to be able to integrate children's contributions into a co-constructive mathematical negotiation process, it is a necessary condition - if one follows the debate on "teacher noticing" (Sherin et al., 2011), for example - to "understand" or interpret children's contributions in terms of whose mathematical content. This "understanding" can be considered a responsive act: "One must engage with another's idea and respond to it in order to understand it—though the manner and quality of engagement can vary widely" (Bishop et al., 2016, p. 1173). From (our) interactionist perspective, responsiveness can be characterized as a coordination process between two persons, for example a child and their person of reference, in which the person of reference orientates themself toward the child's needs in order to respond to them in a prompt and coherent way (Papoušek, 2008). For early educational contexts, interactions can be described as responsive, which is based on the children's ideas, and in which the learners can co-determine the orientation of the process of negotiation of mathematical meaning. This means in concrete terms that the person of reference can flexibly assess when they need to expand or narrow spaces in the interaction context, when they need to instruct and when they need to allow free activities. Also in terms of mathematical (classroom) discourse, Bishop et al. (2022, p. 10) note that "responsiveness to students' mathematical thinking is a characteristic of classroom discourse that reflects the extent to which students' mathematical ideas are present, attended to, and taken up as the basis for instruction". In a responsive process, for example, a teacher takes up an unanticipated or unusual mathematical idea of a child's contribution. He or she supports the child in an adaptive way, e.g. by encouraging him or her to clarify and justify its utterance. Therefore, a high degree of responsiveness is an example of a successful scaffolding process (Vogler & Beck, 2020; Beck & Vogler, 2022). The responsiveness of parents, (educational) caregivers or (preschool) teachers is considered a key prognostic variable with regard to the social, cognitive and linguistic development of infants and children (Bornstein et al., 2008; Eshel et al., 2006). Against this background, the general question arises as to what the reality of interaction in early childhood practice looks like concerning responsive action, or how and with what kind of assistance adults in the context of kindergarten can be able to act responsively to the (mathematical) needs of the children in their everyday lives. This general question guides the research presented in this paper. This gives rise to concrete research aims and questions, which will be described, theoretically anchored and empirically examined in the following.

To pursue the questions, in our paper responsiveness is established as a construct for analyzing adult-child-interactions. On the level of interaction, responsiveness is in our paper distinguished by the fact that, in the interactive interplay of "initiation, reply and evaluation" (I-R-E, Mehan, 1979, p. 37), the teacher adapts different interactive elements to the learners' attainment level. In this context, as Bishop et al. (2016, 2022) also hint, an adaptation can take place in subject-specific discourses – such as mathematics – in terms of (1) mathematical content and (2) interactive discursive skills. In this conversational analytical approach we adopt in our paper (following Mehan), an aim is to empirically describe the extent to which an adult's actions, in these asymmetrical learning situations, are aligned with the assumed attainment level of a child to offer him or her the most productive participation opportunities possible through a responsive structuring of the interaction process (Beck & Vogler, 2021; Robertson et al., 2016). A high level of responsiveness generates interactional potential in a learning situation (Vogler, 2019), which opens up possibilities for interpretation and action for the children (Beck & Vogler, 2021). Both the adult's and the children's subsequent actions are analyzed in our observations, as the combination is necessary to reconstruct responsiveness. The other aim is to examine, what impact the use of design patterns have of the responsiveness in mathematical adult-children-interactions.

Design Patterns of Mathematical Situations

As stated above, preschool teachers are challenged by adaptively and responsively offering mathematically rich situations to children (Benz, 2012). Therefore, the starting point for our considerations is the question of whether mathematical design patterns may support preschool teachers by providing relevant information about mathematical topics and potential interactive moves for the initiated mathematical negotiations of meaning in a mathematical interaction. A "design pattern" is a standardized description system (Vogel, 2014, p. 225 f.), which makes it possible to include both the idea of mathematical content with which the children can deal, as well as the material-space arrangement and multimodal impulses in a teaching-learning environment, for supporting early co-constructive mathematical learning in various content areas. In contrast to many other recommendations for action and material, the mathematical design patterns focus primarily on the learning child, and their learning in the interactive situation. Two cornerstones of the pattern are the mathematical content and the multimodal impulses that anticipate the interactive negotiation process. The mathematical content formulated in the situation pattern is intended to support an accompanying person (B) in interacting in a mathematically rich and technically appropriate way by enabling him or her to interpret and classify the children's utterances and actions from a professional perspective.

On this basis of the mathematical design pattern, an adult should consequently be able to react adequately to the children's contributions. The multimodal impulses in the description pattern are intended to support the accompanying person or the professional staff in being able to act adaptively, professionally and co-constructively in the concrete interaction situation. Since the impulses consider detailed interactive and multimodal action scenarios, which in particular take up the children's mathematics-related interpretations and attributions of meaning and 'develop' them as co-constructive structures, the patterns are intended to support responsive action in the interaction in terms of both content and discursive action. To gain initial insights into whether such responsiveness manifests itself in the respective mathematical design pattern, we explore the questions in our contribution, how and to what extent the use of mathematical design pattern fosters a responsive (inter-)acting in child-adult interactions conducted in kindergarten.

Data and Methodology

To address the research questions, we analyze two contrasting cases of "mathematical situations of play and exploration" from the erStMaL project (Vogel, 2014). These two situations can be seen as paradigmatic examples of different manifestations or levels of responsiveness. They can be understood as representatives of specific patterns of responsiveness.

In this project, we have investigated adult-child interactions in 12 kindergartens. 144 children aged 3–5 were observed. Therefore, we have developed mathematical situations of play and exploration, which are conducted by a guiding adult (B) with small groups of children in preschool. The accompanying persons introduced themselves intensively to the children before the play situations were carried out to create a relationship of trust.

Every play situation has its origin in one of the five mathematical domains: numbers and operations, geometry and spatial thinking, measurement, pattern and algebraic thinking or data and probability (Acar-Bayraktar et al., 2011). They are described in the design pattern mentioned above so that the guiding adult can prepare him or herself before performing the situation. The performance of the mathematical situation in preschool was videotaped and transcribed.

Interaction analysis is used to qualitatively analyse the transcribed sequences (Krummheuer, 2012), in which the statements are first interpreted individually in the order in which they occur, to then be able to understand their relationships to each other following conversation analysis (Sidnell & Stivers, 2013). This makes it possible to work out both the negotiated mathematical meaning and the characteristics of the turn-taking system. The construct of responsiveness is used as a "sensitizing concept" (Blumer, 1954, p. 7) to describe the extent to which mathematical situations create opportunities for all children to participate.

Analysis of Two Mathematical Situations of Play and Exploration

Snail Shell – Mathematical Situation Called "Ropes 01"

In our first empirical case, we analyze the mathematical situation "ropes", in which four children between the ages of 4 and 5 and a guiding person are dealing with ropes of different lengths (short, medium, long) and colors (blue, green, red and yellow). The group tries to sort the ropes by size. Therefore, they are doing a side-by-side comparison by holding two ropes side by side. This measurement process is quite challenging because the arm span of the children is shorter than the length of the ropes.

001	René:	but I can do something with these ropes.[a]
002		*holds a yellow rope of a medium length in his hand raises a long green rope*
003	Marie:	*holds a coiled-up blue rope in her left hand*
004	Chris:	*ties a short blue rope around his head*
005	B:	**what can you do? show it to me.**
006	René:	*thrusts aside a long red rope in Marie's direction*
007		**snail shell**.
008		*gets on his knees and puts the rope on the floor in the shape of a spiral* (Fig. 1)
009	Levent:	*swings a green rope above his head*
010	B:	**a snail shell**.
011		*looks towards Marie*
012		great. Look. can you do this too.
013		*looks at Levent*

Notes: [a]In the transcribed sequence actions are printed in italics, stressed words are coded in bold letters. All specialities of the spoken language (mistakes, grammar etc.) are mentioned in the translation of the transcribed sequence.

In the following 30 seconds Chris, Levent and Marie also try to make a snail shell out of the ropes. These seconds are skipped.

Fig. 1 René developing snail shell

088	René:	so **snail shell** is ready.
089	B:	**g r e a t.** I'm going to do one as well
090	Marie:	*takes a short blue rope*
091	Chris:	*puts the yellow rope away and looks at B's rope*
092	B:	let's see if my **snail shell** gets **bigger.**
093	Marie:	*positions herself behind René and lets the rope hang down to the ground*
094	René:	*pulls a piece of the outer edge of the spiral apart* (Fig. 2)
095		**snail** is built.
096	Chris:	*looks at René's snail*
097	B:	**great!**
098	Levent:	if we got a **bigger** one. I will do it even **bigger.**
099	B:	see if this is bigger.
100		*puts away a short red rope, which lies between him and René, and puts his spiral in front of René*
101	René:	this is smaller.

While comparing the ropes, René discovers that he can lay snail shells (#001-#008), which mathematically represent Archimedean spirals (Figs. 1 and 2). In Mehan's three steps (1979), René's 'spiral idea' can be interpreted as a form of initiation that does not emanate from the person who is considered competent, but directly from the child. The guiding person (B) responds to this initiation by asking René to show the snail shells (#005) and asking Marie, Levent and Chris to lay their ropes in snail shells too (#012). B takes up the child's terminology and encourages the children to localize the concept of the snail shell in the material. This stimulates a cooperative negotiation process in which B involves the other children (#012, #092, #099). In addition, B seems to identify the mathematical potential in René's idea, which becomes increasingly visible as the situation progresses. Triggered by the 'spiral idea', an interactive space opens up for all participants, in which they can deal with the length of the ropes from an extended, geometric perspective (#092, #099). They also take a step forward in solving their problem of comparing the lengths of the ropes by replacing their impractical approach of directly comparing two uncoiled ropes with the 'snail shell' idea. This becomes clear from the second scene. During this process, B also places a snail shell and makes assumptions about its size (#092). He also praises René when he builds his snail (#012). Linking almost immediately to this scene, Levent also reflects on the length of a rope and the size

Fig. 2 Created snails

of a snail shell (#098). Finally, René can compare the two snail shells that he and B have laid in terms of their surface area or their diameter (#101). Since he argues with the term “smaller” (and not “shorter”), the comparison in terms of the surface area seems more likely. However, it should be noted that children of this age often use the adjective “small” synonymously with “short”, which is why both possibilities of interpretation are given at this point. There is no explicit evaluation of René’s initiation by B, but a ‘polyadic evaluation’ is implicit in the activities described, in which the group implements René’s idea and uses it to compare the ropes. In addition to René and B, Levent and later Chris are also actively involved in this, while Marie participates more receptively in laying and comparing the snail shells. Thus, all of children are given the opportunity to deal with mathematically demanding concepts such as size (area), length (ropes) and even invariance in a way that was initiated by them (Beck, 2022; Vogler & Beck, 2020).

Crosswalk – Mathematical Situation Called “Wooden Sticks 01”

In the present play and exploration situation, four children (almost 4 years old) interact with the guiding person (B). First there is a free building phase in which the children place different figures (squares, triangles) and construct buildings from sticks by arranging the sticks alternately in squares (Fig. 3). After some time, B confronts the children with the concept of the *mathematical design pattern* and evokes the investigation of a linearly repeating pattern sequence with the two-element basic unit of green and blue sticks. Initially, this pattern is (determined) by B by alternating five sticks in the orientation green-blue-green-blue-green. Following on from this, she asks the children to continue the pattern (Fig. 4). Jonas then places a yellow stick next to the last green one (Vogler & Beck, 2020).

Fig. 3 Children interacting with the adult

Fig. 4 Linearly repeating pattern sequence

001	B:	Now I don't even know which one to take (...)
002		*Scratches her head, then takes a blue chopstick in her hand and shows the chopstick to the children*
003		Leonard what do you think?
004		*looks at the boy Leonard*
005	Leo:	blue.
006	B:	**blue?**
007		*raises the hand with the blue chopstick again*
008		and the yellow?
009	Leo:	Dropping (unintelligible) that doesn't belong to blue
010	B:	*looks at him and whispers*
011		why?
012	Leo:	Because (.) because otherwise, **it wouldn't change anymore**.
013	B:	Then **it wouldn't change anymore**? then come here, put the blue stick along.
014		*hands Leonard the blue chopstick from her hand*
015		Put it where it has to go. Here.
016		*taps her finger on the carpet, next to the sequence of patterns*
017		look, you can put it there.
018		*points with her finger to the pattern*
019		you have explained it to me, otherwise it wouldn't change anymore
020		*looks at Leonard*
021	Leo:	*stretches out his arm, but doesn't place the blue double sticks next to the blue-green sequence, but on a pile of different double sticks*
022	B:	There? Look. I'll put it there again.
023		*takes the blue stick that Leonard put on the pile and places it at a certain distance from the already existing pattern sequence*
024		Do we have to remove it?
025		*points to the yellow stick in the pattern sequence* (Fig. 4)
026		what do you think? yes or no?
027		*looks at Jonas and Leonard*
028		what do you think?

029		*looks at Jeremy*
030	Jeremy:	Yes. Removing.
031	B:	Why?
032	Jeremy:	Because (...) (unintelligible) (...) because **otherwise, that would be weird**.
033		*looks at his hands, in which he is holding a chopstick*
034	B:	**Otherwise it's weird**?
035		*takes the yellow stick out of the pattern sequence*
036	Jonas:	*picks up a stick from the ground and breaks it with both hands*
037		Oh
038		*looks down at his hands holding the broken pieces*
039	B:	Oh that's bad. Put it aside.
040		*pushes the blue stick to the green stick* (...)
041		Look, I'll take the blue one over here and now comes purple. Who has to come now.
042		*rests her head on her hand looks at Jeremy, who is holding a broken stick and says out loud*
043		Oh I think that's nasty.

In this situation, the pattern sequence with the basic unit blue-green (Fig. 3) is addressed again and again by B (from line #001), probably to draw the attention of the children to the pattern topic. She takes up Leo's idea of placing a blue stick next in the form of a question (#006), implicitly asking for a possible reason. She also asks him to help her continue the pattern sequence (#002). B thus implicitly evaluates the boy's answer positively. Then she asks directly for the yellow stick, which interrupts the blue-green pattern sequence (#008). Leo explains that this is not one of them (#009). This time B explicitly asks in line #011 for a reason and Leo answers that the change (between blue and green) would otherwise be interrupted (#012). B then uses Leo's formulation and repeats his statement in the form of a question, "Then that wouldn't change anymore?" (#013), and at the same time motivates him to put down the blue stick instead of the yellow one. At this point, Leo seems to step out of the interaction, instead placing the blue stick on top of a stack of different sticks. Then B turns first to Jonas and Leo (#027) and then to Jeremy (#029) and integrates both boys by asking them if they should put the yellow stick away (#026 and #028). Jeremy affirms this, because otherwise, "it would be weird" (#032). However, the boy only actively participates in the interaction for a brief moment before he, like Jonas, grabs a wooden stick and breaks it (#036). It becomes clear how the children actively and productively participate in the interaction and provide reasons for the statement that the yellow stick cannot belong to the pattern, but how a long-term and cooperative negotiation process fails to materialize. In addition, the polyadic interaction structure that can be reconstructed at the beginning of the situation, in which the children build and discuss their structures, 'disintegrates' into changing child-companion dyads in which the children, who are not actively productive, 'flee' and turn to other things again. Although the guiding person (B) repeatedly opens up opportunities for participation for the children and tries to tie in with the children's contributions by adopting their formulation and terminology (lines #006, #013 and #034), it can be assumed that their persistent insistence on the two-element pattern sequence, which obviously deviates from the ideas of the children, the children gradually withdraw from the negotiation process. In addition, their strategy of repeating or taking up the children's formulations in the form of questions (#006, #0013, #034) seems to have an unfavorable effect on the course of the interaction. The children may understand the question as implicit criticism and not as an extended request to justify or explain their statements. Breaking the chopsticks towards the end of the situation proves to be much more exciting than dealing with the pattern sequence, which can possibly also be attributed to the fact that B uses the children's mathematically creative ideas to lay out figures and erect tall buildings, thus switching to the mathematical area of quantities and measurements is not taken up here.

Empirical Findings Concerning Responsiveness in Mathematical Children-Adult-Interactions Through the Use of Mathematical Design Patterns

In the comparison of the information from the pattern itself and the analysis of the two situations, it can be assumed that the accompanying persons in both situations are familiar with the mathematically comprehensively prepared situation pattern. This is evident from the fact that in both examples, mathematically demanding topics are negotiated with the support of the guiding adult (B), such as the invariance of the length of the ropes, which are highlighted in the mathematical design patterns. In the second example, crosswalk, this becomes clear from the fact that B, although the children build buildings and place figures, introduces the patterns' sequence shown in the design *pattern* and obstinately pursues it as a theme. In the first example, the use of the mathematical design pattern obviously supports the guiding person (B) in interacting supportively by highlighting René's idea and integrating/developing it in the group for an easy way to compare different rope sizes. B creates linguistically interactive and thematically adaptive opportunities for participation with the children and thus opens up interactional potential (Vogler, 2019) for the children. This leads to a high degree of *content-related responsiveness* as well as to a high degree of *interactive-discursive responsiveness* (Vogler & Beck, 2020; Beck & Vogler, 2021). In the second example, the use of the mathematical design pattern seems to lead the guiding person (B) into an attitude of inertia. B strictly adheres to the focus of the mathematical design pattern and tries to enforce the theme of pattern laying, even though the children are engaged in building and gaining their own mathematical experiences. The sudden change from the building phase into the pattern sequences, which is initiated by the adult, does not pick up on the mathematical ideas of the children and may generate disinterest on their part. As a consequence of this interpretation, we reconstructed a low degree of content-related responsiveness in the second example. While B persistently builds the sequence of patterns in the zebra crossing scene and seems to take over all the actions here, the children are often only passively active as silent listeners and show little interest. Despite the efforts of the accompanying person to take up the children's moves on an interactive-discursive level, e.g. in repeating their central utterances, this results in a very one-sided negotiation. We have reconstructed a high degree of interactive-discursive responsiveness but the children are only less engaged in the mathematical negotiation process that is initiated by the accompanying person.

Conclusion and Outlook

The paper aimed to investigate empirically how the use of design patterns influence the behavior of adults in mathematical situations in kindergarten and thus lead to a high level of responsiveness in interactions. In our analyses, it could be shown how the patterns promote responsiveness: one of the prepared accompanying persons react adaptively to creative mathematical ideas of a child while working on a mathematical problem. It can be traced that the design pattern plays a major role in recognizing the child's mathematical idea and in the subsequent interactive and co-construction-promoting implementation. Especially through the actions of the guiding person - as can be seen in the first analysis - an interactive space in the group emerges, in which the children can collectively deal with mathematically demanding concepts. In the context of the analyses, it becomes apparent, in line with the research findings of Bishop et al. (2016) and Robertson et al. (2015), how helpful the construct of responsiveness is for reconstructing the supportiveness of interaction systems.

The paradigmatic examples in this paper cannot be used to clarify how great the influence of the preparation of the guiding person is. The causal connection can also and especially be suspected through the comparative example of the second analysis: here it was possible to understand that the design patterns do not always have a positive effect on responsiveness in a concrete situation if they are interpreted as rigid guidelines for action that must be implemented in any case and in full. However, it can be assumed that this is due less to the structure of the mathematical design pattern and more to its situational implementation. In this respect, the pattern is a very promising form of preparation for mathematically rich and responsive child-adult and children-preschool teacher interactions. Due to the comprehensive mathematical description and diverse multimodal action impulses, the patterns offer a basis for being able to foster the mathematical development of children of preschool age. In our paper, we were also able to show how fruitful the construct of responsiveness is for analysing the interactive quality of early negotiation processes in kindergarten, and the assessment of supportive structures. With the help of the construct responsiveness, it was possible to work out in the context of the analyses presented here to what extent the support of the adults (accompanying persons) is responsive in terms of language, but above all also in terms of content.

Further analyses should provide detailed information about the interactional facets in which responsiveness emerges in a mathematical negotiation process between children and a guiding adult. On a theoretical level, the concept of responsiveness applies further differentiation. As the two examples show, a distinction between content-related and interactive-discursive responsiveness proves to be a viable basis for an extended, empirically-based theory genesis. Within the framework of the theoretical examination of the adaptivity of early childhood support, responsiveness can be viewed as a central, situational element of adaptive actions by elementary education professionals, which can be reconstructed in children-preschool teacher discourses and is therefore empirically identifiable and measurable.

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Kindergarten Professionals' Perspectives on Observing Children's Mathematical Competencies

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Introduction

Historically, German kindergarten has been characterized by a holistic approach rather than a focus on content specific teaching and learning processes. Now, even after recent changes in curricula and educational policy concerning early mathematics education, in Germany there is still a need for further development in in-service education. Kindergarten professionals' training routes can differ widely and for many of the professionals currently working in kindergarten or pre-school, early mathematics education has not been part of their own pre-service education. The majority of kindergarten professionals in Germany do not have a university background but graduated from vocational schools.

The long-term in-service project "Children and Adults Explore Mathematics together" acts as one answer to this demand. One major part of the long-term project is a visit of kindergarten professionals together with a group of children to a join-in-studio at university twice a year. The visit is embedded in different project components like a workshop for the professionals before they visit the join-in-studio and a reflection meeting afterwards (Benz, 2016).

Due to the pandemic situation, in 2020 and 2021 many of the activities could not be conducted as usual. However, it was not only in-service training that could not be implemented as it was before the pandemic: mathematics education in kindergarten was also affected over a long period of time, because kindergartens were closed. Even after they re-opened, mathematics was not necessarily in the focus of the daily kindergarten routine.

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Therefore, we planned a short in-service offer from May to July 2021 with the focus on enabling kindergarten professionals to support especially those children who would enter primary school in September 2021 and who had not had many chances to acquire mathematical basic skills. In order to enable mathematical individual-adaptive learning for these children, we offered the professionals a short in-service training. The focus of the in-service offer was on the mathematical content of counting and using structures for seeing the quantity of sets (Björklund et al., 2019; Sprenger & Benz, 2020). The short-term in-service project started with the component of a virtual training, providing content knowledge about children's mathematical basic skills in the area of numbers and operations. The main focus in the workshop was on presenting different possibilities and tools for observing and diagnosing these skills as well as materials for supporting counting and using structures. In this paper, we focus on the professionals' use and assessment of the different diagnostic tools.

Theoretical and Empirical Background – Importance of Observing and Diagnosing Children's Competencies

Observing children's mathematical skills is seen as a cornerstone for supporting children's mathematical learning (e.g. Bruns et al., 2020; Fosse et al., 2018) not only in early mathematics education. Teacher adaptability as a widely accepted important aspect of effective instruction and support (Parsons et al., 2018) is based on observing and diagnosing children's mathematical competencies. Wullschlegel (2017, p. 141) proposes that diagnosing is the first step for an individual-adaptive support. It is important to note that in the educational context, the term *diagnosing* is increasingly used in a non-pathologizing way, signifying "various practices of continuously gathering and evaluating knowledge about students" (Hoppe et al., 2020, p. 3). In this paper, we use the term diagnosing in the sense of one aspect of teachers' competencies: "These diagnostic processes and activities include, for example, teachers choosing an appropriate question to learn more about a student's conceptions or teachers evaluating the information given by a student in order to gain an understanding of this student's conceptions. Growing research interest in those diagnostic processes and activities is one reason the term 'diagnosing' has become increasingly prevalent in the educational field" (Hoppe et al., 2020, p. 3).

Diagnostic activities help to get an idea of what mathematical skills a specific child still needs to acquire. Therefore, observing and diagnosing children's mathematical skills can be found as a facet or aspect in many different models of professional knowledge (Gasteiger & Benz, 2018), and also in studies investigating professional competencies (Bruns et al., 2014; Yang et al., 2019). In the competence model of professional knowledge for early mathematics education (Gasteiger & Benz, 2018, p. 85) it is part of the facet *situative observing and perceiving* (see

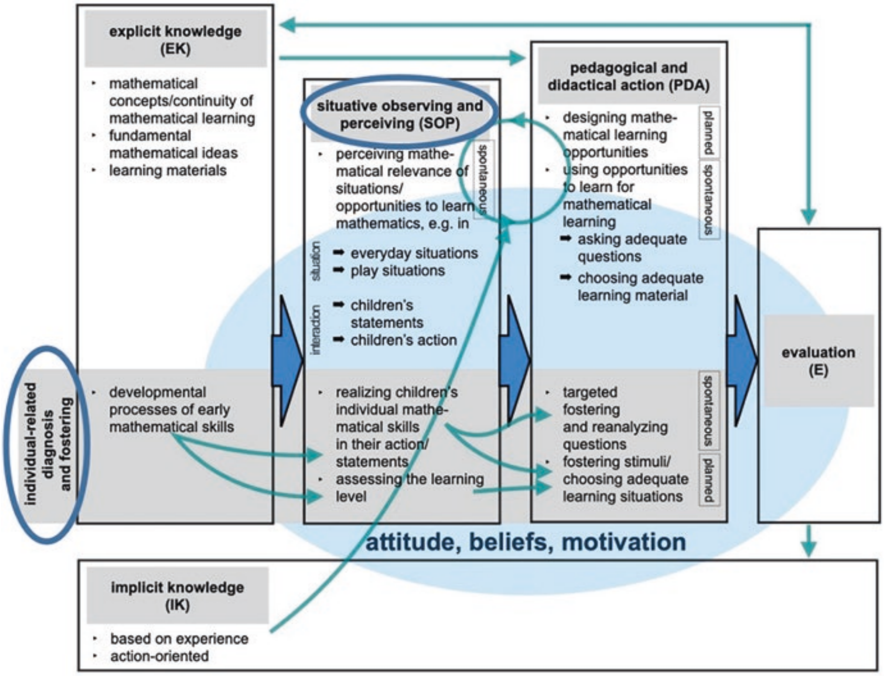

Fig. 1 Competence model of professional knowledge for early mathematics education. (Adapted from Gasteiger & Benz, 2018, p. 85)

Fig. 1) and also part of the perspective *individual-related diagnosis and fostering* when assessing the learning levels.

Perceiving mathematically relevant aspects in children's actions as well as in their statements is necessary in order to ask adequate questions and to support children's mathematical learning. This perspective shows the importance of observing for acting and responding adaptively in a situation of concrete interaction. The situational observing and perceiving of individual abilities (diagnostic aspect SOP) represents the prerequisite for knowing how to foster individual children. The ability to diagnose and foster as part of the pedagogical didactical action can manifest itself on the one hand in spontaneous purposeful interventional-diagnostic questions and stimulations (Steinweg, 2009) and on the other hand in the deliberate choice of learning stimulations, games or materials that adequately foster the mathematical learning process.

Diagnosing as one main facet of teachers' professional competencies is described as generating information about children's understanding, eliciting students' cognition (Kron et al., 2021) or as the goal-directed accumulation and integration of information to reduce uncertainty when making educational decisions (Heitzmann et al., 2019). Diagnosing implies obtaining a diagnosis.

Different Ways of Diagnosing – Generating and Recording Information About Children's Mathematical Competencies

Diagnostic competencies are required in very different situations with different aims. In each diagnostic situation the professional aims to gain information about children's skills either for future learning (initial) or for adapting instructional choices during the learning process (formative) or for getting information about learning results (summative). Moreover, the way of generating information can also be differentiated: information can be generated by a diagnostic interview or by conducting a test with individual children about the chosen mathematical content (e.g. Clarke et al., 2006). Both are artificial, planned one-to-one-situations and a child could easily realize that someone aims to interrogate them. According to the level of structuring and standardizing, this is the most structured way to generate information.

An everyday observation during times of free play in kindergarten is the least structured way and demands a high level of competence from early childhood professionals (Lembrer et al., 2018; Fosse et al., 2018; Bruns et al., 2020). Also, research on teacher noticing underlines the importance of situative observing. Mason points out that "[e]very act of teaching depends on noticing: noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be said or done next" (Mason, 2002, p. 7). Schoenfeld (2011, p. 228) also describes the importance of noticing for consequential actions: "What you see and don't see shapes what you do and don't do." For the broad spectrum of perceiving children's mathematical skills, different terms with slightly different notions are already used in this paper. Situative observing and perceiving as well as noticing have a strong situative aspect and focus on the interaction between professionals and children. These terms cover a broader spectrum where not only the (mathematical) skills of children are observed, seen or noticed but also other aspects of a teaching situation are identified. In the current discussion about the construct of noticing, not only the identification of significant interactions (concerning children's thinking) or the attention to students' ideas but also the interpretation and the decision how to respond are included (Sherin et al., 2011).

Between these two poles, different situations can be designed with direct tasks, board or card games, and guided play, in order to diagnose children's mathematical competencies.

Sometimes, the way of recording the information is linked to the way of generating it. In a standardized test and in some diagnostic interviews the answers are scored or labeled as wrong or right (Peter Koop & Grüßing, 2011; Wollring, 2004). A semi structured way of collecting data for example can be found in the *Documentation of Learning* (Lerndokumentation) introduced by Steinweg (2009). Different mathematical competencies are presented in tables and the professionals can fill in their observations. For each given competence there are different columns labeled "with assistance", "sometimes independently", "often independently", and "absolutely independently". Thus, development in each competence can be documented as well. The least structured way of recording would be an essay or free notes. Between these different levels many nuances of pre-structured recording/collecting are possible.

Concerning adaptive learning support, the evaluation of the generated and recorded information does not focus on summative scores or on labels based on statistic data. For adaptive learning support, detailed information about the child's specific mathematical skills and understanding are important.

Design of the Study

Different diagnostic and observation tools were offered to get detailed information about the children's mathematical competencies and moreover, to evaluate the use of different diagnostic tools in terms of professional development.

Diagnostic Possibilities in an In-Service Offer

1. *Direct observation*
   A diagnostic one-to-one interview with a detailed description of mathematical tasks as well as a recording sheet on which the child's approach could be recorded.
2. *Situative observation in everyday or planned learning situations*
   (a) A recording sheet with a table presenting concrete actions for the diagnosis of different mathematical competencies, e.g. "identifying quantity of fingers of two hands without counting".
   (b) A recording sheet without structured suggestions (open questions).

Figure 2 shows an example of one of the recording tools, a table for situative observation.

After introducing different diagnostic tools in the workshop, games, materials, picture books and tasks supporting mathematical number skills were illustrated. Every participant got the above mentioned different diagnostic tools and a box with games, materials and picture books for their institution. For every game, material, or picture book, the game idea and the learning possibilities were explained in a booklet, as well as possible impulses and questions to support and diagnose children's competencies.

Altogether, 25 professionals from 20 different kindergartens participated in the workshop. Thus, 20 boxes were needed. In order to introduce the materials in the box, a student teacher came to each kindergarten, presented the box and played with the children at least one or two times. Thus, the kindergarten professionals were not only introduced to the games, but also had the opportunity to observe their children while they were interacting with the student teacher.

Minis und Erwachsene entdecken Mathematik
MiniMa

Counting and Seeing
Daily Observation - Table

Page 2

Child: Emma

Note the date and further observation in the particular field.

	What quantity	With help	Partly with help	autonomously
13) Identifying quantity of fingers of two hands – without counting	8, 10, 7			11.03.22
14) Taking a given quantity of objects at once – without counting	4, 3, 2	16.02.22 (comment: you don't have to count)		13.04.22
15) Identifying quantities based on structuring	1,2,3, 4,5,6		24.03.22 Number on a dice	
16) Identifying differences of quantities based on structuring	2, 3	28.04.22 Could you determine where are more?		

Fig. 2 Recording situative observation with a table

Two-Part Evaluation

In order to evaluate the in-service offer, we asked the professionals to take part in an online-survey. Here, they were asked to assess the practicability of the different tools. 11 kindergarten professionals took part in the online-survey. Three professionals recorded different agreements to the different diagnostic tools, showing highest agreement on the practicability of the direct observation, followed by the situative observation with the given table and lastly by the situative observation without the possibility of structured recording. The answers to the open questions concerning the practicability of the different tools led to more insights: "For every colleague, especially the 1:1 interview was interesting, because the guidelines were clear and within a short time we got meaningful information." "…I experienced that the children know 'more', the knowledge is wider, because the children could concentrate better."

The agreement and the answers to the open question led us to the following research question for a subsequent qualitative evaluation:

What do professionals report about the use of different diagnostic tools for initial and formative diagnosis of basic mathematical skills in the area of number and operation?

To better understand professionals' assessment of the different tools, a second, qualitative research approach was conducted. Two synchronous virtual focus-group interviews (Stewart & Shamdesani, 2015) with respectively two participants were conducted. The two structured guideline interviews were conducted by the same interviewer and lasted 72 and 60 min. The group interviews were transcribed and categorized by qualitative content analysis in accordance to Kuckartz (2018). Parts of the data were analyzed by three different researchers in order to validate the categories via intercoder reliability. As a result, five main categories were defined:

- Organisation
- Perceived Quality
- Arrangement
- Professional Development
- Suggested Improvements

For each category, subcategories were defined and for both diagnosing situations it was recorded whether the statements given had positive or negative connotations.

Results

1. Organisation

Organisation of the observation: As in the case of situative observation the situation itself is not pre-organized by the professionals, only statements for the 1:1 interview were assigned to this category. The statements mentioned place and time as relevant factors that are specifically required for the 1:1 interview. Solutions were named though, like conducting the interviews in the morning when children are arriving, and using "a table in the hall."

Organisation of the documentation situation: Organisation of documentation in the 1:1 interview is reported to be easy, whereas for the situative observation it often seems to be complicated and challenging because the "sheet is not always at hand"

and the professionals "forget" the situation easily or "cannot remember it at the end of the day".

2. Perceived Quality

Accuracy: The 1:1 interview is perceived as being more accurate than the situative observation. As an explanation, the professionals name on the one hand the explicitly different tasks that approach different aspects of basic mathematical skills, and on the other hand the suggestions given for inquiring about the children's thinking, or for requesting further explanations from them about how they are solving the problems: "My observations of the children were more focused [...] with the subitizing and I realized that really many children counted very much". Also, the professionals reported that the 1:1 situation helped them to alter as well as to complete their evaluation of children's competencies they had generated from everyday situations.

Mathematical focus: As everyday situations are very complex, the professionals reported difficulties in focusing on mathematical aspects here: „in the free observation, it is still sometimes the case that I lay the focus on other things, but then, one still perceives how the children interact with each other and then I am not focused anymore on what I wanted to observe… "Another aspect of the mentioned complexity is that professionals often do not have sufficient time and/or space to follow through with their observation because there are demands of other children as well. The professionals describe it as an advantage that in the 1:1 interview they have the possibility to focus on different "important" mathematical aspects at once. They also perceive their observation in the 1:1 interview as more detailed: „Still, with your interview guideline one gets – of course – a much more differentiated picture of the mathematical competencies of the child."

Usability of documentation: The documentation of the 1:1 interview is also reported to be very helpful in conversations with parents or teachers about the specific mathematical development of each child. This is due to the detailed description of tasks and the feeling of having a diagnosis "in black and white" at hand. A similar use of the documentation of the situative observation is not mentioned.

3. Arrangement

Learning situation: The 1:1 interview was described as an artificial situation compared to natural situations. Besides reduction of complexity and a need for extra place and time (see categories above), different impacts of this artificial situation on the children were perceived.

Motivation of children: Most of the professionals reported that the children were very eager to join the 1:1 interview and to get undivided attention as well as to show the professionals their capabilities: "I was at the table, the other children also walked by – children who usually were not easy to capture also wanted to take part in the interview".

Emotional stress: One professional reported that some children perceived the artificial 1:1 interview as a "test-situation": "they immediately have the feeling of a test-situation" (…) "I also have the reporting sheet next to me, filling it in and then they ask 'what are you writing there?'." The professional reported that mainly insecure children felt uncomfortable in the situation. However, it should be mentioned that this professional called the 1:1 interview a test herself, whereas the other professionals rather described it as "a game", indicating an awareness of possible emotional stress. This aspect was never linked to the situative observation.

Interaction-feedback: In situative observation*s*, the professionals did not perceive a need to give feedback to children's actions, but one reported about the challenge of appropriate reactions in the 1:1 situation if children could not solve the tasks. Here they also perceived that children felt sometimes discouraged.

4. Professional Development

Especially the 1:1 interview was mentioned concerning perceived professional development for themselves as well as for their colleagues.

"Door-opener" for mathematics education: The 1:1 interview was reported to represent an access to mathematics education for professionals who are not affiliated with it: „It helped my colleagues to gain access to the subject at all. Because, I think for many it is just really difficult, somehow. Mathematics – where do I begin? It's counting, somewhat... And I think it was helpful for many to conduct a direct observation in order to get a sense of important aspects."

The last part of the statement also refers to the professional competence facet *explicit knowledge* (Gasteiger & Benz, 2018).

The data also provides insights into the professionals' self-perceived development.

Perceived acquisition of explicit knowledge: The participants reported that by using the detailed guidelines of the 1:1 interview and the pre-structured table – they acquired *explicit knowledge.* Especially knowledge about perceiving and using structures was mentioned.

Perceived acquisition of situative observing and perceiving skills: All professionals reported that conducting the 1:1 interview led to an enhanced view for children's competencies as well as for mathematical potential in learning situations: "because it also simply trains the view on the child's competencies and because then, it is also easier to observe in everyday situations".

Perceived development of pedagogical and didactical actions: The participants also reported a higher awareness of mathematical content in observations and the deliberate initiation of situations to promote mathematical competencies.

Conducting the 1:1 interviews promoted the professionals' process oriented view on children's mathematical competencies, especially concerning the determination of quantities by counting or using structure. Adaptive learning support was reportedly implemented as a result of insights gained from the interviews.

5. Suggested Changes for designing observation situations and documentation

Despite the 1:1 interview being their preferred diagnosing tool, the participants suggested some modulation concerning the role of the interviewer. As stated above, a perceived "test situation" may induce emotional stress in children. Thus the professionals expressed a need for proposed wording of feedback that encourages a child's participation in the interview whilst remaining neutral towards the accuracy of their answer.

A suggestion relating to the documentation of the situative observation was to integrate it into their existing documentation tools.

Discussion

The professionals reported about many positive aspects of the 1:1 interview, not only concerning the interview situation itself in terms of accuracy, mathematical focus, organization and usability of the documentation, and the children's motivation, but also concerning self-perceived positive effects on their implementation of adaptive learning support. They felt they could train their diagnosing competencies in a prescribed situation reduced of complexity, and thus had the possibility to acquire knowledge not only about relevant mathematical contents but also about what (mathematical) thought processes are important and how they can be assessed. They reported about transferring acquired competencies from the artificial diagnosing situation to everyday situations in which they were able to discover mathematical aspects and act and react adaptively.

They also mentioned negative aspects of the artificial, strongly focused situation which may be perceived as a test situation. Those professionals who called the interview a "test" also reported about children's emotional stress and problems with getting feedback. This shows the danger of using direct diagnostic tools as assessment in an artificial "test-situation" in pre-school settings. 1:1 interviews can create negative experiences for children if they get negative feedback like "this is an incorrect answer" or "you are not able to solve the task". Also, one consequence of 1:1 interviews could be that due to summative scores, children are labelled as "weak" or as "children at risk" instead of using the differentiated results for further adaptive support (Gasteiger, 2010; Meisels & Atkins-Burnett, 2000; Meisels, 2007).

Therefore, materials which can be used as tests should not be accessible without training for the professionals.

Some professionals stated that they started to integrate parts of the 1:1 interview into suitable everyday situations – here it can be interpreted that their aim is to apply those parts in natural situations and thus reduce the artificial, test-like atmosphere. The previous use of the 1:1 interview could be interpreted as an exercise for observing and a possibility for getting to know "what and how" mathematical competencies can be observed in natural situations.

The situative observation was reported to be quite demanding because of the complexity of everyday situations, especially the organization of documentation. Still, regarding the above mentioned perceived acquisition of facets of professional competencies, the professionals reported transferring knowledge from the 1:1 interview to natural situations, so it can be assumed that they do see a need and utility of situative observation for supporting mathematics education.

Therefore, the following conclusion can be drawn with regard to the design and development of diagnosing and observation tools for early mathematical education: In order to compensate for the negative aspects of the 1:1 interview but also to keep the accuracy, mathematical focus, organization, and usability of the documentation, a prescribed diagnostic guided play and corresponding documentation tool could be developed. Thus, professionals would be able to train their diagnostic competencies in a situation reduced of complexity with time for documentation scheduled in advance, but without any negative effects on the children. Such structured diagnosis situation would also aim to sharpen the professionals' view of children's mathematical competencies and situational mathematical potential, which is needed for adaptive learning support.

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Play Responsive Assessment of Children's Mathematical Proficiencies in Norwegian Kindergarten

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Introduction

According to The Norwegian Framework Plan for the Content and Tasks of Kindergartens, "Staff shall create opportunities for mathematical experiences by enriching the children's play and day-to-day lives with mathematical ideas and in-depth conversations" (Ministry of Education and Research, 2017, p. 54). Enrichment of play has been theorized in Sweden as play-responsive teaching (Pramling et al., 2019). The Norwegian and Swedish traditions are similar, except that the word "teaching" may be more controversial in Norway. Situations of daily life can be seen as cultural activities and be seen through the lenses of cultural historical activity theory (CHAT). Within this approach van Oers (2010, 2013a, b, 2014), theorizes play as a format of cultural activity in which children are highly involved, follow implicitly or explicitly some shared rules and have some degrees of freedom with regard to how the activity should be carried out (Van Oers & Duijkers, 2013). CHAT as a theoretical framework is thus suitable for analyzing whether children are observed in adherence to the ideals of the Norwegian kindergarten context. CHAT has been used in developmental education and play-responsive teaching, two approaches which both acknowledge play and the important role of adults in young children's development (Pramling et al., 2019; Van Oers & Duijkers, 2013).

Assessment of children's mathematical competency or proficiencies in Norwegian kindergarten has been investigated from the perspective of mathematical learning difficulties and early intervention. In Norway, 'kindergarten' is used for

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children aged 1–6 years. Reikerås et al. (2012) and Reikerås (2016) have studied quantitatively the results of assessment of competence or proficiency in mathematics, language, social skills and movement in Norwegian kindergartens. The kindergarten teachers in their study assessed every child at the ages of 2.5 and 4.5 years old and were told to do this through observation in daily life and play situations. Reikerås et al. (2012) and Reikerås (2016) used the observation scheme (MIO), which is also available for assessment purposes in Norwegian kindergartens (Davidsen et al., 2008). The referred study and the availability of assessment schemes make it a relevant research question to ask whether kindergarten teachers using that kind of observation scheme really observe children in play and daily life activity, and how the observations interfere with ongoing play and activity. An observation scheme restricted to proficiencies related to numbers and counting was developed for the study this article is based on, in place of the MIO scheme. This is a simplification compared to MIO, but the developed observation scheme adds more detail to the area numbers and counting and makes it possible to avoid a ceiling effect for children who are 4 and 5 years old. The chosen mathematical area had largest variance among toddlers in Reikerås et al. (2012), and it is also most relevant in order to identify children at risk of developing mathematical learning difficulties (Geary et al., 2018). This was a motivation in the study, even though learning difficulties are not discussed in this article.

The concepts responsiveness to cultural meaning and play-responsive assessment will be developed, and the research question will be refined by using these concepts, CHAT and van Oers' theory of play as a format of cultural activity. The first refined question asks how observations were made, and the second and third ask about responsiveness of observation to cultural meaning and play respectively. Thus, in this article we will investigate the following three research questions:

– What are the main types of observation of mathematical proficiencies used by the kindergarten teachers?
– How do observations of mathematical proficiencies relate to the cultural meaning of the activity in which observation was done?
– How was children's involvement and freedom during observation related to cultural and personal meanings of the cultural activities in which observation was done?

Theory

Cultural historical activity theory, originating with Leont'ev and Vygotsky, conceives human development as a process based on interactions between humans with the help of cultural tools in the context of historically produced practices (Van Oers, 2013b, p. 189). A central idea is that human activity is part of human culture and develops through history. Learning takes place when children participate in cultural activity, and development is dependent on participation together with adults or more capable peers. Mathematics is a human activity, an abstract one with scientific concepts. Mathematical concepts and actions have to be put into words, and this is crucial for young children's development of mathematical meaning (Van Oers, 2010, p. 29).

Bearing on El'konin, cultural historical activity theory claims that before school age, the best way of learning is when children take part in playful activity (El'konin, 1999). Play and learning are unified by seeing learning as a format of cultural activity in which children are highly involved, follow implicitly or explicitly some shared rules and have some degrees of freedom with regard to how the activity should be carried out (Van Oers & Duijkers, 2013). Since daily life activities are cultural activities, children's participation in them may become play when the above definition is satisfied. The three characteristics involvement, rules and freedom of change apply to all cultural activity, and thus can be used also for potential non-play. A sociocultural approach makes it possible to overcome the assessment-teaching dualism. In the Vygotsky-based dynamic assessment theory, mediated interaction is necessary to understand the range of a child's functioning, but this interaction also guides further development of these abilities (Poehner, 2008, p. 24). From this point of view, assessment may involve proficiencies and activities that children can only approach together with adults or more capable peers. Assessment may start in cultural activity which is non-play, but which may be turned into play. The concept play-responsive teaching includes both adult interference responsive to children's initiatives and interests, and interference which enables children to have experiences they otherwise would not have had (Pramling et al., 2019, p. 180). Similarities to this will be discussed for the concept play-responsive assessment introduced in this article.

Another kind of responsiveness in assessment is responsiveness to cultural meaning. A cultural activity has a motive which is collective or cultural, and the actions of a participant in the activity will have personal meanings, similar or sometimes different from the cultural motive of the activity. Leontyev (2009) identifies personal meaning with subjective meaning and links this to the German 'Sinn' as opposed to objective meaning 'Bedeutung'. We will follow van Oers (2010, p. 26) and use the phrase 'cultural meaning' in place of 'cultural motive'. An adult intervention in a cultural activity for the purpose of assessment will be called responsive to cultural meaning if the intervention is compatible or faithful to the cultural meaning.

Meaningful learning happens when cultural and personal meanings merge. Play can be seen as imitative participation in meaningful cultural practices (Van Oers, 2010, p. 29). The child's personal meaning normally is not quite the same as the cultural meaning that the child's play is based on. Adults' intervention in children's play may change the cultural meaning and also be at odds with children's personal meanings. Responsiveness to cultural meaning must not be confused with the concepts culturally responsive teaching and culturally responsive assessment, which emphasize the multicultural perspective (Gay, 2018).

The concept of teaching is clearly distinguished from the related concept of instruction when play-responsive teaching is conceptualized. Instruction takes place regardless of response to the instruction, but teaching presumes responsiveness to response (Pramling et al., 2019, p. 176). According to Pramling et al. (2019, p. 176), "*instruction is an action* while *teaching is an activity*". A parallel distinction between testing and assessment for the concept play-responsive assessment will be discussed.

Method

This study was planned as a pilot for future studies. The observations involve 47 children aged 2:10–3:1, 3 children aged 3:10–4:1 and 11 children aged 4:10–5:1. The observations were carried out by the kindergarten teachers in 2019 using an observation scheme in the twelve kindergartens that participated in the project. The observation scheme has been developed by the authors, based on Stock et al. (2010) and inspired by the project described in Reikerås (2016), but limited to solely focusing on the children's proficiencies related to numbers and counting. This article will focus on the following five of a total of eight proficiencies in the observation scheme: 'recitation of the counting sequence', 'counting a set of objects' (arranged in order and by random placement), 'fetching N objects by a given number word', 'estimation by separating most from fewest' and 'seriation and classification'. The observations were carried out during normal practice in the kindergarten.

Four researchers, pairwise, conducted interviews with the kindergarten teachers in groups of one to four, but typically two, in each of the twelve kindergartens. The main content of the interviews was related to how the kindergarten staff carried out the observations and how they themselves and the children experienced these observations. The interviews lasted about 45 minutes and were audio recorded. We analyzed the data using thematic analysis, in which an initial thematic coding was done during transcription. Then as a result of repeated readings of the coded transcriptions, we identified about ten themes of research interest. In the third phase of analysis we both searched for a theoretical framework and tried to restrict attention to a coherent subset of the identified themes. This resulted in four themes defined by how the kindergarten teachers observed children's mathematical proficiencies and the choice of van Oers' use of CHAT to theorize play and learning. One of the four themes was excluded because non-participant observation was not found to be central, and daily life situations and play situations were united as one theme that was called *natural activity*. The fourth theme was initially given the name *adult initiated activity*, but was renamed *organized activity.*

The fact that four different researchers conducted semi-structured interviews may have influenced the reliability of the data. For example, the amount of specific follow-up questions versus more general questions could affect the level of details in the answers from the informants. The fact that the project leader participated in ten of the twelve interviews together with one of the other three, has greatly reduced this bias in the collected data material. The reliability is strengthened by the fact that the two authors, largely independently of each other, have read and analyzed the data over time and from different perspectives.

The excerpts from the interviews are labelled with a K, and a number between 1 and 12, which refers to the kindergarten number in our data, followed by the quote's time point in the interview. The age of the children, except for one excerpt, is 2:10–3:1. Written consent for participation has been given by the parents or guardians of all the children who have been observed and by all the kindergarten teachers who have been interviewed in this project. The project has been approved by the Norwegian Centre for Research Data (NSD).

Results

The thematic analysis combined with the cultural historical activity theory approach resulted in these main types of observation of mathematical proficiencies:

- Extension of natural activities
- Organized activities

In a few cases, observation was done without organization or adults extending a natural activity, but this third alternative is beyond the scope of this article. The kindergarten teachers themselves used phrases like 'natural situations', in the sense of both daily life activities and play initiated by children.

Extension of natural activities means that the kindergarten teachers for the purpose of observing one or several proficiencies in the observation scheme, take some action which adds to and potentially modifies or changes a cultural activity that has a cultural meaning independent of the need of observation. In the case of children's free play, the activity is ongoing when the adult takes part or influences it. For daily life activities the adult may initiate an activity like a real meal, but for another purpose than observation.

The term 'organized activity' was used by the kindergarten teachers and will be used in the sense of initiating a cultural mathematical activity for the sole purpose of observing a proficiency in the observation scheme. It is not ruled out that an organized activity could turn into a natural activity. Sometimes a natural activity is extended in a way that neither responds to its cultural meaning nor children's meaning, and such cases may more appropriately be called organized activities.

Responsiveness to Cultural and Personal Meanings

Natural activity was regarded as the typical kindergarten approach by the kindergarten teachers. This means that their preferred way of observation was by extension of such activity, as long as this was regarded as a possible choice. The typical way the kindergarten teachers add to a natural activity is to ask questions with mathematical content, dependent on the proficiency being observed.

> *When dressing, to ask how many shoes do you need to wear, how many mittens do you need?* (K5: 7.40)

Putting on shoes or mittens is an example of a daily life activity which regularly happens in the kindergarten. The questions to the children are consistent with the cultural meaning of the activity and put a mathematical concept into words. Thus, the questions are responsive to the cultural meaning and make the activity mathematical for children.

> *when we came back from shopping and there were these little tins of liver pate […] instead of lining up something arbitrary and saying that they should count. […] There it was natural, we were unpacking the box of groceries, and they were lying higgledy-piggledy, so then it was just a completely natural activity.* (K12: 4.17)

The kindergarten had ordered tins of liver pate, and the kindergarten teacher asked the children (aged 4–5) to count the tins which were randomly placed, probably on a table. For the adult it makes sense to count the tins, in order to be sure that what is brought to the kindergarten is in accordance with what they planned to buy, and so her question to the children was consistent with the cultural meaning of the activity. Introducing counting into the activity makes it mathematical, but in this case children's personal meanings do not necessarily include the cultural meaning.

> *A: We were in the woods, playing with leaves and sticks. Where are there most / fewest. M: Was it planned? A: No, then I had someone with me to be observed. So, I thought it was a good idea to use things from nature and include it in the play.* (K4: 11.00)

To go for a walk in the woods is a daily life activity in Norway. This is an activity with some degrees of freedom, and for instance, talking about what you see in the woods can be seen as part of the activity. One of the items in the observation scheme was to decide which of two sets is most numerous. This extension of the activity would probably not have occurred without the observation needing to be done and is as such an organized activity. However, it has similarities to rule-based competitive game activities common among children which possibly may be seen as part of a walk in the woods.

Like separating most from fewest, to fetch N objects also was regarded by some of the kindergarten teachers as difficult to observe through natural activity. One example from the data is that children were playing with toy building bricks, and that the kindergarten teacher asked a child to fetch five bricks. Typically, this question was not related to the cultural meaning of building. However, in one situation with building bricks the kindergarten teacher said "We need another two. Can you find two more and build upon this?" This extension of the activity is responsive to cultural meaning.

Children's Involvement and Freedom During Observation

Observation in extensions of natural activities is described generally in positive terms. Specific examples of situations from the data where the kindergarten teachers mention positively involved children in extensions not changing the cultural or children's personal meaning are sparse, but sometimes the kindergarten teachers said that they contributed to children's involvement. Some of them also mentioned learning, even though the study focused on assessment. An expression used by some kindergarten teachers was 'learning by playing'.

> *We must be the driving force in free play, make it interesting and 'play it in'. Even more than we do. We see that what we are involved in, in 'learning by playing', shows up in other contexts. That's what kindergarten is all about.* (K1: 39.40)

This kindergarten teacher has an overall belief that play is of utmost importance in the kindergarten, and that adults must contribute to playful learning. Both in this excerpt and in other cases it is unclear whether "free play" is restricted to play initiated by children, or whether they have in mind also other kinds of cultural activities with a playful approach.

Organized activities could also lead to positive involvement, even if the border between play and organized activity sometimes may be discussed.

> *LB: What kinds of activities did you use when she sorted them from smallest to largest, for example? Then it was families that were the thing. Then it was domestic animals. Mom-horse and dad-horse and the baby(-horse). Sheep and pigs. Then she was eager, then she could line them up by size. But she also wanted to arrange them so that the piglet stood between mom and dad, then she focused on that.* (K6: 31.55)

The kindergarten teachers had handed out toy animals and asked children to sort the animals according to size. This girl enjoyed role play and families, and her engagement in that may have influenced her involvement in the extended or organized activity. She was given freedom to change the rules, and she turned the activity into play with another cultural meaning. By placing the baby-animal between its parents she is focusing on social relations rather that ordering by size.

Often organized activity was done with a group of children, a math-group, in a separate room. These organized activities with a playful attitude were popular with many and said to have similarities with engaging activities children were used to doing in the kindergarten.

> *T1: Then we took some things into a separate room. We were there together so that we could both observe and instruct.* […] *R: So it was more like a test? T1: Yes. Yes, but I don't think the kids experienced it like that. T2: No, more like play maybe. T1: It is not unusual for them that we are doing it.* […] *T2: I found that they were very eager. Everyone wanted to join in. Because it's like playing.* […] *It was rather a bit like what I experienced when we were indoors in the kindergarten and played with the teddy bears or the cars and made them count.* (K1: 20.52)

Children's freedom of change was restricted when they were observed in a separate room, and the words 'instruct' and 'like a test' indicate restrictions in children's freedom of change. Even so, the kindergarten teachers thought that the activities were a positive experience for the children. The kindergarten teachers argued for this by saying that the children were used to restrictions in their freedom, and the adults managed to maintain a playful attitude and enthusiasm towards the activity.

Children's negative involvement related to organized activities was mostly described by words like 'artificial', 'like a test' and 'schoolish'. The word 'artificial' could point to difficulties in relating both to cultural and personal meaning, and 'like a test' and 'schoolish' express activities with restricted degrees of freedom. One kindergarten teacher related the word 'artificial' to a feeling among children that this was not what they wanted to play, and said that the adults took control in order to observe what they needed to see. The adults in these cases sometimes were uninvolved, and the children sometimes expressed bad feelings towards the kindergarten teachers. Children for instance said that they did not want to do this. Some children disliked being observed or understood that something was expected of them. Other times children withdrew from the activity, preferring to do something else, but not necessarily with bad feelings.

Discussion

The research questions will be discussed using cultural historical activity theory, theory of play in the version of van Oers, and the theory of play-responsive teaching (Pramling et al., 2019). Van Oers (2010, 2013a, b, 2014), theorizes play as a format of cultural activity in which children are highly involved, follow implicitly or explicitly some shared rules and have some degrees of freedom with regard to how the activity should be carried out (Van Oers & Duijkers, 2013). A central construct is the idea of cultural activity and that children's involvement, and children's freedom of change is what distinguishes play from non-play. A new contribution in this article is the introduction of the concepts responsiveness to cultural meaning and play-responsive assessment, inspired by the concept play-responsive teaching (Pramling et al., 2019).

– What are the main types of observation of mathematical proficiencies used by the kindergarten teachers?

The main distinction found is between organized activity and extension of natural activity. Natural activity is either play initiated by children or daily life activities which are initiated by adults for other reasons than observation of mathematical proficiencies. Only in a few cases were the kindergarten teachers able to observe mathematical proficiencies passively without any participation. The following two remaining research questions will be discussed together.

– How do observations of mathematical proficiencies relate to the cultural meaning of the activity in which observation was done?
– How was children's involvement and freedom during observation related to cultural and personal meanings of the cultural activities in which observation was done?

Both play and daily life activities have cultural meaning as well as a personal meaning for the participating children. In natural activity the kindergarten teachers in many cases contributed by introducing mathematical vocabulary or procedures in a way that preserved or was consistent with the cultural meaning of the activity. By doing so, adults contributed to the child's process of gaining mathematical meaning (Van Oers, 2010, p. 29). In the case of ongoing play, such extensions often preserved children's involvement and so were play-responsive. Responsiveness to play failed in cases when adults were not attentive to children's meanings or did not give children enough freedom.

A central finding is that children could be engaged also when they were observed in organized activity. Both explanations related to children and explanations related to adults may explain children's involvement in such activity. Adults' involvement and the way organized activity is presented are explanations related to kindergarten teachers. Some of the kindergarten teachers said that they have to make it interesting and 'play it in', or that involvement is contagious. One child-related explanation was that children are used to similar activities in the kindergarten with restrictions in their freedom. Restrictions in freedom as a reason for involvement is strange, and it is more plausible that enough freedom was still present. If so, it makes sense that similarity to usual activities in the kindergarten explains the involvement. Activities with the sole purpose of observing proficiencies in the observation scheme often were similar to engaging activities like games or rule-based play. Another child-related explanation is that some of them liked to play with the same kind of objects or material as used in organized observational activities. This did not always apply, as children sometimes turned to their preferred way of using the material. Bad feelings in such situations were avoided when children were given freedom to withdraw from the intended adult extension. That children are members of a math-group is another kind of explanation. Such a group is said to be popular, but the data do not contribute to answering why this is so. A final reason related to children is that movement and physical action are engaging. Relay race and collecting sticks in the forest are examples from the interviews. When children later compared the lengths of the sticks, their involvement from the collecting process may be prolonged into the mathematical activity.

The concepts of responsiveness to cultural meaning and play-responsive assessment are inspired by the concept of play-responsive teaching (Pramling et al., 2019), but are about assessment rather than teaching. Especially in the first part of the observation period, kindergarten teachers thought assessment to be clearly distinct from teaching and learning, and had ideas similar to not 'teaching to the test'. Gradually they began to use observation activities that also led to children's learning. In principle, assessment may be possible in daily life situations in which children neither are involved nor have freedom of influencing what is going on. The possibility of that kind of assessment could be a question of research, but whether this can be done in an ethically justifiable way is questionable. Even so, the concept of responsiveness to cultural meaning can be defended as a supplement to play-responsive assessment. In language learning, words and concepts are used before children have an ownership to the words, and in the same way mathematical language and procedures may be introduced into daily life situations. When this happens in children's zone of proximal development, both assessment and learning are possible. However, play-responsive teaching is said not necessarily to start in play, but has to be responsive to play, if it starts in play (Pramling et al., 2019, p. 180). This could be interpreted as if play-responsive teaching also includes teaching that starts in, for instance, daily life situations. Play-responsive didaktik (Pramling et al., 2019) subsumes both play-responsive teaching and play-responsive assessment.

One finding is that the word 'test' is used in organized activities which impose restrictions on children's freedom of change. Kindergarten teachers from one kindergarten also used the word 'instruction' for this kind of activities. We argue that play-responsive assessment in play and daily life situations is just as distinct from testing as play-responsive teaching is from instruction according to (Pramling et al., 2019, p. 176). It was found that children could be involved and even have some kind of freedom to influence organized activities, but that sometimes involvement was low and freedom almost absent. When some freedom of change is present, this is, however, not freedom regarding aspects central to the mathematical proficiencies to be assessed in the activity. In play-responsive assessment, as in play-responsive teaching, assessment may start with adult-initiated activity, but must be responsive to how children react.

A possible limitation in the study is that the analysis and discussion use a theory of play, but that the kindergarten teachers' use of the word 'play' is not necessarily always consistent with the theoretical definition. For instance, some talk about play could be interpreted as related to some kind of material or to physical movement, but also to involvement, which is central in the theory of van Oers. In the study such utterances are taken as reasons for children's involvement. The way kindergarten teachers talk about play also in some cases makes it unclear whether they are talking about child-initiated play, extensions of daily life activities or organized activity. One solution to this can be the term play-responsive didaktik and a playful attitude to all interaction with children, but also the finding that the different kinds of activities transform into each other, and that the borders between them sometimes are unclear. Another limitation is that in some of the interviews, kindergarten teachers who have assessed different children independently, share their experiences, but their individual stories are difficult to distinguish in the transcriptions. Care has been taken to avoid mixing up such stories, but a possible danger is to fit together unrelated individual voices into a false story.

Questions for Further Research

One finding was that extensions of daily life activities can respond to cultural meaning and contribute to mathematical meaning, even if this does not necessarily respond to children's meaning. An interesting question to investigate is how children's involvement is in that kind of extensions. A plausible hypothesis is that responsiveness to cultural meaning makes the adults become involved, which may then be contagious. Taking part together with adults in that kind of activity may also in itself be attractive to children, in much the same way as taking part in a math-group.

Some mathematical proficiencies, like to fetch a given number of items or deciding which of two sets is most numerous, were found to be more challenging to assess through natural activity than proficiencies related to counting a given set of items. A question for further research is to investigate why this is so from the perspective of mathematical teacher competency or tasks of teaching. An alternative approach is didactical phenomenology in the sense of Freudenthal (1983), by searching for suitable cultural activities for the mathematical proficiencies that it is most relevant to assess in kindergarten.

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Toddlers' Mathematics and Language – Two Sides of the Same Coin?

Elin Reikerås

Introduction

Through the toddler period (from 1 to 3 years of age), children develop many central mathematical skills (Reikerås et al., 2012; Reikerås, 2016). At the same period, they have an extensive development of their language (Slot et al., 2020). Although relations between development in mathematics and language were noted several decades ago (e.g., Durkin & Shire, 1991), there is still a lack of research on the topic, especially regarding children as young as toddlers. The present study aims to examine how the relations are between toddlers' skills in mathematics and language. This includes the relations between different parts of mathematics and language, as well as the relations between mathematics and different aspects of language. It will also be explored how the relations between mathematical skills and language skills are for children with different levels of mathematical skills.

Mathematics and Language

In early childhood and care institutions (ECECs), toddlers are surrounded by staff and children who use language to explain sizes, shapes, numbers, relations, directions, quantities, etc. Such social use of language forms an interactive and linguistic context that is important for children's learning of mathematics (Björklund, 2008; Durkin et al., 1986).

Toddlers express their mathematics mainly through action since they are not fully able to express themselves verbally (Björklund, 2007). They show that they

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have an understanding of mathematical content related to, e.g., size, shapes and numbers before they start to use language to express their understanding in the later part of the toddler period (Reikerås et al., 2012; Solem & Reikerås, 2017). Language comprehension is found to be the best predictor for variations in number tasks in 7–9-year-olds (Cowan et al., 2005), and Bower et al. (2020) found spatial language comprehension in three-year-olds to be strongly associated with mathematical skills. The importance of children's expressive language skills for mathematical development was investigated in a study of 3–5-year-olds by Purpura et al. (2011) and in a study of 5–6-year-olds by Praet et al. (2013). Both studies found that vocabulary and other expressive language skills were central factors in explaining the variance in mathematical skills; therefore, these language skills are crucial for mathematical development.

The largest challenge when studying the relation between the development of mathematical skills and language skills is related to how mathematics is defined and whether mathematical language is seen as a part of mathematical skills or not. Mathematical language is classified in some taxonomies in the problem-solving area together with logical reasoning (Magne et al., 2019), even though this language develops through interaction with the acquisition of skills in most other mathematical areas. Verbal counting is, for instance, central in quantitative development (Mix et al., 2002), and relational language is found to be a powerful contribution to spatial learning (Gentner, 2008). In the same way, quantitative development affects geometrical development and not only arithmetical development (Mix et al., 2002). Geometrical development also supports the learning of numbers and later arithmetic (Arcavi, 2003). However, some researchers find language skills of lesser importance for mathematical development based on neuroscientific research (e.g., Gelman & Butterworth, 2005).

The differences in the findings may also be due to what kind of language the children's mathematical skills are compared with. Purpura and Reid (2016) found that although general language performance was initially a significant predictor of numeracy performance, when both mathematical language and general language were included in the model, only mathematical language was a significant predictor of numeracy performance.

In a study by Reikerås and Salomonsen (2019) comparing skills in children with different levels of mathematics, the group of toddlers with weak mathematics had the largest variation of results within the mathematical area of mathematical language. Kleemans et al. (2011), who conducted research on children with language difficulties, found that these difficulties were not restricted to language acquisition but also had serious consequences for the development of key concepts in early numeracy skills. Durkin et al. (2013) showed that the weaker the language skills children have, the more difficult it is to master number skills. However, Arvedson (2002) found that children with language difficulties differ from children with typical development only in verbal numeral skills but not in nonverbal number skills.

The Present Study

Earlier studies about the relations between mathematics and language, as presented above, underline the need for more research, both on the relation between the developmental areas' mathematics and language for all children and for children with different skill levels. Most earlier studies are based on research with few children, and few of the studies included toddlers. The present study has a large group of toddlers as participants, and the data were collected in social interactions in the ECEC by their teachers. The study aims to answer the research questions

1. How are the relations between mathematical skills and language skills in toddler age?
2. If and in which way does the relationship differ between language and mathematics for children at different skill levels?

Method

The present study is part of *The Stavanger Project – The Learning Child* following children's development from 2½ to 10 years of age, and the 1086 participating toddlers (534 girls, 552 boys) were recruited through the project (for a description see Reikerås et al., 2012). The parents gave written consent to let their children participate, and the project was approved by the Norwegian Social Science Data Services.

Data on the toddler's mathematical skills and language skills were collected by the staff in the ECECs using structured observation of the toddlers in play and everyday situations over a three-month period (from the toddlers being 2 years and 6 months to they turning 2 years and 9 months). This method is based on authentic assessment, a recommended and accepted practice, and a nonintrusive way of assessing children's skills in ECECs (Bagnato et al., 2014).

The assessment material used for mathematics was MIO - The Mathematics, the Individual and the Environment (Davidsen et al., 2008). The structure in MIO is based on Olof Magnes taxonomy (Magne & Thørn, 1987; Magne et al., 2019), and the research grounds for choosing the included items can be found in the handbook (Davidsen et al., 2008). The MIO scheme consisted of six sections (mathematical areas), each including six items (36 items in total): *Mathematical language*, *Logical reasoning, Shape and space*, *Pattern and order, Counting and series of numbers* and *Enumeration.* The children's *language skills* were assessed using the observation material TRAS - Early registration of language development (Espenakk, 2003). The material consists of a total of 72 items divided into three main language areas *Social Language* (with the sections Communication, Interaction, and Attention), *Language Comprehension* (with the sections Language comprehension and Linguistic awareness), *and Expressive Language (*with the sections Sentence production, Word production, and Pronunciation). These assessment materials were developed for use in Norwegian ECECs. The sections in the materials were selected on a theoretical basis. Both MIO and TRAS had been repeatedly piloted in ECECs, with observation of several hundred children to find items suitable for finding valid measures of math and language skills for the age range 2–5 years in ECECs. The data from the pilot were unable to be validated by comparison with data from other assessment materials because at the time, there was no other assessment material for the age groups available in Norwegian that mapped the same fields. However, the calculation of Cronbach's alpha showed scores ≥0.90 for both materials, indicating good internal consistency. To ensure the reliability of the materials, 90 children were observed by two different ECEC teachers. The Wilcoxon signed rank test showed good interrater reliability for both TRAS and MIO (for more details, see Espenakk, 2003; Reikerås et al. 2012).

TRAS and MIO are both based on a social cultural approach to learning, as it assesses the children's functional skills: the skills they show in play and interaction with peers and staff in the ECECs and provides ecologically valid data (Keilty et al., 2009). In addition, both materials included the assessment of skills in the child's proximal zone, in line with Vygotsky (e.g., Vygotskij & Kozulin, 2001). Several studies have documented the advantages of authentic assessment compared with standardised assessments, and authentic assessment has become a recommended and accepted practice (Bagnato et al., 2014). The staff was trained in scoring, and in addition, detailed descriptions of each item in the registration form and directions for scoring were available to facilitate data collection and increase the reliability of the data collection procedure (Helvig & Løge, 2006; Davidsen et al., 2008). Both MIO and TRAS have three difficulty levels, with level 1 as the easiest and level 3 as the most difficult. An example of an item in MIO: *Have started with pointing when saying the number sequence.* If the child points and says the number sequence (not necessarily the correct link between numbers and things), the staff should mark with show competence (two points). For example, Karine plays alone in the playroom. She mutters the number sequence as she points to the teddy bears and dolls. "I need to know how many you are for us to have enough food!" she says. If the child has begun to show interest in other people's pointing when they count, the staff should mark with partial competence (one point). For example, Per is closely monitoring when Jasmine points at the cars and says the number sequence. Not yet showing competence (zero point): If the child does not show interest in the relation between pointing and the number sequence. When summing up for each child, the results were normally distributed.

The first analysis is correlation analysis since such analysis is suitable when describing relations between variables. Next, a graphic presentation between the sum scores in mathematics and language is presented to see if there is a linear relation between these variables. If the graph is not linear, it is necessary to divide the children into skill groups and look at the relation between MIO and TRAS for each group. Dividing into quartiles, four groups with approximately the same size, will in the present study be suitable to maintain a certain group size appropriate for statistical analysis (Tabachnick & Fidell, 2014).

Results

To answer the research question about how the relations between mathematical skills and language skills in toddler age are, correlation analyses were performed between the MIO-total and TRAS-total, as well as analyses between the different MIO-sections and the TRAS-sections.

As seen from Table 1, all sections in MIO and all sections in TRAS correlate, and in line with Cohen (1988), all the effect sizes are large. This indicates that across all mathematical areas included in MIO, the mathematical skills in toddlers are related to their language skills.

The relationship between language and mathematics varies for children at different skill levels and was first examined by a graph with the sum score in mathematics on one axis and the sum score of language on the other for all the participants (see Fig. 1).

The graph indicates a more complex picture than a linear relationship, and to be able to answer the second research question, the toddlers' results on MIO were divided into quartiles, which give us four groups with approximately the same size (see Table 2).

When examining the correlations between MIO-total and TRAS total for each quartile group of MIO, all are significant, but smaller than when all children were included as in Table 1, now with medium effect size. One cause for this can be that the variance in the results in the quartiles is much smaller than the variance in the whole sample, as seen in the SDs in the table. The children with the lowest mathematical scores (MIO quartile 1) had language scores mostly in TRAS quartile 1 (58.5%) and TRAS-quartile 2 (32.9%). Only 8.6% of toddlers with results in MIO quartile 1 had language scores in one of the TRAS-quartiles 3 and 4. For children with mathematical scores within quartiles 2 and 3, only approximately 50% of the children were in the corresponding language quartile (51.0% and 48.0%, respectively), whereas most of the rest were in their neighbouring quartiles. Of the toddlers with scores in MIO quartile 2, 21.5% had TRAS scores within TRAS quartile 1 and 24.1% in TRAS quartile 3. For MIO quartile 3, the children's TRAS scores within TRAS quartiles 2 and 4 were 22.3% and 23.8%, respectively. For the children with mathematical scores in the highest quartile, 71.0% also had language scores within the highest language quartile (TRAS-quartile 4), and 21.9% had language scores within TRAS-quartile 3. Only 2 of the children within MIO quartile 4 (under 1%) are in TRAS quartile 1.

Discussion

The present study found strong correlations between toddlers' different math skills and the aspects of language skills all over when the whole group results were analysed together. The nuances are very small between the size of the correlation

Table 1 Relations between mathematical skills and language skills in toddler age

N = 1086	*Mean (SD)*	MIO-total	aMathematical language	b Logical reasoning	c Shape and space	d Pattern and order	e Counting and series of numbers	f Enumeration	1	2	3	TRAS-total
Mean (SD)		30.80 (10.89)	5.57 (2.18)	5.14 (2.11)	5.94 (2.01)	5.13 (1.88)	4.31 (2.60)	4.71 (2.23)				
TRAS-total	69.21 (23.31)	.78**	.70**	.68**	.69**	.63**	.62**	.59**				
1. Social language	28.01 (9.61)	.69**	.63**	.60**	.65**	.58**	.51**	.50**	1	.80**	.80**	.94**
2. Language comprehension	16.00 (5.18)	.77**	.69**	.66**	.67**	.64**	.61**	.61**		1	.84**	.92**
3. Expressive language	25.21 (10.07)	.75**	.70**	.68**	.69**	.63**	.62**	.59**			1	.95**
a			1									
b			.72**	1								
c			.69**	.69**	1							
d			.62**	.67**	.67**	1						
e			.66**	.64**	.57**	.56**	1					
f			.64**	.65**	.60**	.58**	.65**	1				
MIO-total			.86**	.87**	.83**	.80**	.83**	.83**				

**Correlations (Pearson's *r*) were significant at the.01 level (2-tailed)
[1)]Effect size: ≥.1 = small, ≥.3 = medium, and ≥.5 = large (Cohen, 1988)

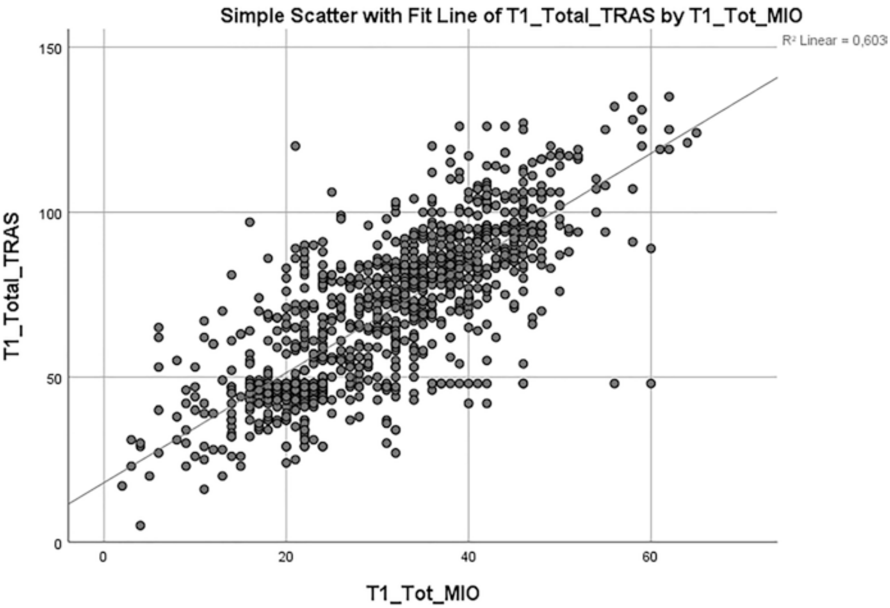

Fig. 1 Plot of the participants' results in mathematics (MIO total) and language (TRAS total)

Table 2 The toddlers result on MIO and TRAS divided into quartiles. N, mean and SD for the quartiles, correlations between TRAS-total and MIO-quarttiles, and the proportion of toddlers in the TRAS-quartiles in the MIO-quartiles

MIO-total			MIO-quartile 1	MIO-quartile 2	MIO-quartile 3	MIO-quartile 4
n			277	261	265	283
MIO Mean (SD)			17,95 (4,61)	25,61(2,33)	34,34 (2,26)	44,85 (5,32)
Correlations with TRAS-total			.31**	.45**	.36**	.43**
	n	TRAS Mean (SD)				
TRAS-quartile 1	236	40.63 (7.38)	162	56	16	2
TRAS-quartile 2	301	55.84 (7.55)	91	133	59	18
TRAS-quartile 3	268	78.05 (4.63)	16	63	127	62
TRAS-quartile 4	281	99.12 (11.19)	8	9	63	201

**Correlations (Pearson's *r*) were significant at the .01 level (2-tailed)
[1)]Effect size: ≥.1 = small, ≥.3 = medium, and ≥.5 = large (Cohen, 1988)

coefficients, which makes a meaningful discussion of which is larger than another difficult. However, it may be slightly surprising that the correlations between the section Mathematical language and the three language sections in TRAS do not stand out to a particular extent, although the relation to *Language comprehension* and *Expressive language* are among the largest correlations. Another thing worth noting is that the lowest correlations were between the language sections and two mathematical areas related to the numbers *Counting and series of numbers* and *Enumeration*. However, the correlations are still due to Cohen (1988) being large.

The findings of these strong relationships between the different kinds of mathematical skills and types of language skills in toddlers are in line with research presented earlier in the paper that supports the close relations between development in mathematics and language for older children (e.g., Cowan et al., 2005; Purpura et al., 2011; Praet et al., 2013).

From the correlation in Table 1, it may be considered that the strong connections between the toddler's mathematical skills and language skills should give a relatively linear graphic presentation with the sum score in mathematics on one axis and the sum score of language on the other for all the participants. As shown in Fig. 1, this is the case for many of the children. However, the figure shows a messier picture. There are children with high MIO results and much lower language than expected, and there are children with low results in mathematics and high language scores.

When divided into quartiles as displayed in Table 2, this becomes even more visible. There is no 1-1 correspondence between skill level in mathematics and skill level in language. Perhaps the most surprising is that under 60% of the children with MIO results in the lowest quartile are also in TRAS-quartile 1. This stands in contrast to Kleemans et al.'s (2011) research, which found language difficulties to have a high impact on mathematical development. However, if TRAS quartile 2 is included, over 90% of the toddlers in the present study with results in MIO quartile 1 are in the two lowest TRAS quartiles. Nevertheless, there are some toddlers with results in the lowest quartile in mathematics and with language scores above the mean score of TRAS. It should be noted that for children in the next lowest MIO quartile, most of the language scores were within TRAS quartiles 1, 2 and 3, and only 9 of 261 children were in TRAS quartile 4. This implies that it is not common to have under the mean math skills and very good language skills, but for some children, this is the case. When looking at MIO quartile 3, most TRAS scores were within TRAS quartiles 3 and 4, and much fewer were in TRAS quartile 1. For the children with mathematical scores in the highest quartile, 93% also had language scores within the two highest language quartiles. The present study cannot say anything about causality, still, it appears to be children within the highest MIO-quartile are most likely to have over medium language scores. However, there are a few toddlers in MIO-quartile 4 with low to very low language skills.

The variation that appears when dividing a large sample into skill level groups as in the present study shows that the relations between the development of mathematical skills and language skills are much more complex than previously stated. Our results stand in some contrast to earlier research claiming a more 1-1 correspondence between language level and math skills, especially in regard to children with low skills (e.g., Durkin et al., 2013; Kleemans et al., 2011).

In the present study, we used MIO and TRAS to assess mathematics and language respectively. The materials include items based on theoretical terms and concepts central to the two developmental areas. They are not completely separated, e.g., mathematical language is part of MIO, knowing the counting sequence a part of *Counting and series of numbers* and so on. There are items in TRAS, including relational words and words for the classification of shapes. However, in the language context of TRAS, these items are a part of *Language comprehension*. This overlap between the assessment materials may be seen as a limitation of the study, but on the other hand, it would be difficult, maybe impossible, to find language-independent assessment material. The present study can therefore only describe how the skills in mathematics and language seem to be related as assessed by MIO and TRAS but cannot say anything about causality.

Although there are relations between the learning of mathematics and language, the results of the present study question whether the two developmental areas can be treated as two sides of the same coin. In further research, children's skill levels need to be taken into consideration since the relations between mathematics and language may be different related to this. In the ECECs, the staff need to create a rich learning environment where the children have many experiences with play and exploring with a focus on learning mathematics both with an emphasis on language and more language-independent activities to meet children with different language levels.

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Mathematics in Actions and Gestures – A Young Learner's Diagrammatic Reasoning

Lara Billion and Melanie Huth

Introduction

It is well-investigated, that learners use different modes in mathematical interactions which are understood as a central space of mathematical learning (Huth, 2022; Krummheuer, 1992). The use of different modes is often framed as *multimodality* in mathematics education (Arzarello, 2006; Radford, 2009). It refers to an integrated use of modes to construct the interactionally negotiated mathematical content. Radford (2009) describes "[…] that mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a *genuine sense*, by actions, gestures, and other types of signs." (p. 112). Additionally, in line with C. S. Peirce' concept of signs and diagrams mathematical learning is described as a socially grounded and visible activity on and with signs and diagrams (Dörfler, 2006). This assumption picks up Peirce' description of *diagrammatic reasoning* as the core of doing math *with others*, even in a very early stage of mathematical development. In this paper, especially actions and gestures of a kindergartner are considered in relation to the simultaneous speech while participating in an open-designed mathematical interaction. In the course, the focused child Rigon develops different mathematical diagrams: First, he places blue and green wooden dogs in a disorderly crowd in front of him, then he creates a patterned row out of these dogs. His gestures, actions, and speech address different mathematical ideas sometimes in an interwoven multimodal way, but sometimes rather in a parallel process. In the following, mathematical learning is considered diagrammatically, the research focus and the used method

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are described, and different sequences of Rigon's diagrammatic work are analytically considered and summarized in the conclusions.

Theoretical Background – Mathematical Learning as Diagrammatic Reasoning

In a semiotic view, mathematical reasoning can be characterized as the use of *signs* and *diagrams* (Hoffmann, 2003). Both concepts are understood in the sense of Peirce. A sign after Peirce is a triadic relation of a perceived sign (*representamen*), the referred meant (*object*), and the effect of the sign in the mind of a sign reader (*interpretant*) (Peirce, 1932, CP 2.228). Thus, anything can become a sign when it is perceived as such. The *representamen* can be understood as the perceivable sign, e.g., an action or a gesture, even though Peirce also refers to exclusively mind-based sign processes. The *object* is describable as what is assumed by the sign reader to be meant by the sign creator (Schreiber, 2013). It is not necessarily a thing or something materialized, but can also be an idea or a statement. The *interpretant* is a kind of impact on the mind of the sign reader. It is not a person, but an effect of the sign stimulated by perceiving something as a sign. It can be expressed as a new representamen, interpreted again and so on, which refers to the infinity of the sign process. *Diagrams*, as complex signs, can be seen multifaceted: rule-based written, arranged with material (Peirce, 1933; Billion, 2021; Dörfler, 2006; Schreiber, 2013), or even gestured as a fleeting diagram (Huth, 2022). They can evolve out of rule-related *inscriptions*, a concept based on Latour and Woolgar (1986). A line, as a first inscription, can be diagrammatically interpreted as a side of a square, a part of a tally sheet of data collection, or as a straight line, etc. Following Peirce, the construction of a diagram, the observation of diagram relations and the manipulation of that diagram to prove if the socially established rules "will hold for all such diagrams" (Peirce, 1931, CP 1.54) are the core of doing mathematics. Diagrammatic reasoning needs to be constantly done with others, especially in early education. Thus, interactionally negotiated and shared meanings and the use of mathematical diagrams are the results of mathematical interactions, offer learning opportunities, and potentially new insights into mathematical relations.

Diagrammatic Reasoning with Gestures and Actions

Previous research on the diagrammatic reasoning of learners shows, that their actions and gestures can be used to reconstruct their diagrammatic interpretations (Billion, 2021; Huth, 2022). With actions, materials can be arranged rule-based and mathematical relations can come to the fore. These are based on that materialized construction and the reading and use by learners *as a* diagram. Gestures show different functions in mathematical learners' interactions, e.g., they can link different material parts, show diagram manipulations or even be used as diagrams themselves (Huth, 2022). Vogel and Huth (2020) investigate interfaces of gestures and actions in function, meaning and chronology which refer to the special interconnectedness of gestures and actions in mathematical learning. These results prove that gestures and actions are of great importance in learning mathematics and open up new opportunities to understand the learning process as a multimodal emergence. Gesture researchers agree that the interplay of gestures and actions is complex in daily interaction and that these modes show a special relation to the speech used (e.g. Harrison, 2018). A clear demarcation between action and gesture is difficult and probably rather a matter of theoretical perspective. Some gesture definitions seem to provide a clear distinction at first glance, e.g., that of Goldin-Meadow (2003) who distinguishes a movement to communicate from a functional act on objects. But this turns out not to be tenable, because the material can be integrated into gestures and actions can show communicative intention (Huth, 2022). Andrén (2010) describes two perspectives to be found in the literature, where the first ascribes nearly every body movement to gestures, like hands, arms, and head movements, but also actions, gaze or mimic expressions. The second perspective is more narrowed and in line with Kendon's (2004) definition (Andrén, 2010, p. 11): Kendon (2004) defines gestures to be interpreted from a counterpart as movements with "features of manifest deliberate expressiveness" (p. 15). He describes gestures as "visible actions" (p. 7). Following Kendon (2004), actions and gestures can rather be distinguished by their function and usage due to the mathematical context and sign interpretation. And sometimes it will still remain uncertain. Harrison (2018) focuses on the commonalities of actions and gestures more than distinguishing features, so he clarifies that across an interaction, actions can become gestures, which however always refer to the action made at the outset. Gestures are performed according to the material world, and "may require elements in the physical surround as an integral part of their semiotic structure […]" (Harrison, 2018, p. 161). E.g., a pointing can be made at a material arrangement and by using an object like a pencil to broaden the pointing itself. As discussed in Huth (2022), the differentiation of gestures and actions cannot be purely made by the question of material use or not. It is more to be seen on a kind of continuum where action and gestures meet at the pole of material use to change an arrangement (action), or material is used while performing a gestural utterance. Actions cannot be ascribed purely functional features but also a potentially communicative intention when made in interaction. It always depends on the interactional interpretation and use of these expressions.

For the present paper and in line with the above discussed perspectives, it can be assumed, that actions and gestures show comparable structures, features and forms. Furthermore, they show comparable relations to the simultaneously uttered speech (Andrén, 2010; Harrison, 2018).The definition as and the meaning of a gesture or an action depends on the overall activity of which they are a part and thus also focuses on usage in attempting to reconstruct the meaning and function of an action or gesture in interaction (Andrén, 2010). Describing gestures and actions in that way fits in line with the thoughts of Wittgenstein (1984), who states that words only gain their meaning in usage. Transferred to actions and gestures, their meaning is reconstructed in their use in the mathematical occupation.

For the analysis in this paper, a distinction between actions and gestures of the learner is operationalized as following: Actions on material arrangements generate new material arrangements which then exist independently of these generating actions (Billion & Huth, 2023). These new material arrangements can potentially be interpreted as a diagram in a semiotic sense. Gestures can be generated on these material arrangements and can indicate manipulations of the material (Huth, 2022). However, the display of the manipulation is not fixed in a new material arrangement and is thus ephemeral (Billion & Huth, 2023). Therefore, actions and gestures have a certain proximity, because a gesture can indicate manipulations, whereas an action can perform the manipulations and fix them in a new material arrangement (Billion & Huth, 2023).The *multimodality* in mathematics learning is seen as an integrated view of gestures, actions, and speech (and potentially other expressive modes) which show a specific relation to each other and which are interpreted as potentially meaningful for the mathematical occupation based on their interactional usage by the participants.

Research Focus

In relation to the discussed view on gestures and actions and their interplay in mathematical learning understood as diagrammatic reasoning of learners, this paper aims to identify how a young learner (Rigon) uses these expressive modes in relation to his speech while doing mathematics. As a first analytical approach, the interaction in the group is considered, building on this, Rigon's gestures and actions are focused in relation to his speech based on a semiotic perspective on mathematics learning. Rigon's usage of actions, gestures, and speech, while he is obviously following his own mathematical interpretations in a complex networking with the surrounding situation and his counterparts in interaction, is of main interest to clarify his multimodal diagrammatic reasoning in a very early stage of learning. From the theoretical considerations, the following research question arises: How are especially actions and gestures in their interplay concerning speech used by a young learner Rigon to create a mathematical idea of determining the number of countable things in the interaction with others?

Methodology

The considered example is taken from the longitudinal study erStMaL (early Steps in Mathematical Learning), which investigated the mathematical development of learners from kindergarten to the second year of primary school in different situations with the potential for mathematical discoveries (Brandt & Vogel, 2017).

Another part of the chosen example is analyzed in Billion et al. (2020) and in Brandt and Krummheuer (2013).

Data Generation and Method of Data Analysis

The situation analyzed in parts can be assigned to *numbers and operations*. For the implementation of the situation with four kids in a German kindergarten, the accompanying person was trained with the help of "mathematical situation patterns" (Vogel, 2014, p. 232) to support uniformity in implementation. The considered situation was videotaped and transcribed using a special notation of all gestures and actions uttered simultaneously with speech (Huth & Schreiber, 2017). The method of analysis combines the interactionist and the semiotic perspective on mathematical learning. Based on the transcript, an interaction analysis in the sense of the reconstructive social research approach was conducted in which the focus is on gestures and actions in relation to speech to generate an interpretation of the ongoing interaction that proves to be plausible (Krummheuer, 1992; Huth & Schreiber, 2017). In the second step, semiotic process cards are created based on the results of the interaction analysis (Huth & Schreiber, 2017). In these cards, the above-described triadic sign concept of Peirce is used to reconstruct the related sign process of actions, gestures, and speech. Due to space constraints, only simplified sections of the semiotic process cards are shown in the following in which the analytical results of the interaction analysis are integrated.

Empirical Example – The Mathematical Exploration Situation

At the beginning of the mathematical exploration situation, four young learners are offered a large number of different wooden animals. The learners investigate the question of whether the number of various animal species differs. All participants sort the unordered quantity of wooden animals according to the different types of animals. To determine the number of wooden animals of a species, various mathematical ideas can be identified, which the learners express in a multimodal way. For the analysis, the actions, gestures, and speech of Rigon (4.7 years) are focused to identify which mathematical ideas he expresses about number determination in these modes. Rigon has chosen the wooden dogs, which are in a disorderly crowd in front of him. In total the 19 dogs consist of green and blue dogs of the same size and shape. First, a sequence is considered, where Rigon wants to find out the number of dogs by counting. Second, he places the dogs next to each other in a gapless line so that he may be able to determine the number by the length of the line. While setting up the series, Rigon has another mathematical idea that deals with pattern sequences. He places the green and blue dogs alternately next to each other, creating a continuous pattern of green and blue, possibly to count them with help of the pattern. In the last sequence, Rigon manipulates the pattern. In the following, these ideas of number determination, including mathematical areas of numbers (counting), measurement (comparing length), and patterns and structures (patterns of blue and green dogs), are analyzed with a focus on the interplay of actions, gestures, and speech.

The Idea of Number Determination by Counting

Rigon first expresses verbally the idea of counting the dogs addressed to the accompanying person. She paraphrases Rigon's statement and asks all learners to count their animals. Following the request, Rigon says "one" aloud and pushes a green dog a few centimeters to the front left (see Fig. 1, line 1).

With this action, he assigns his phonetic number to a green dog and separates it from the other dogs. It can be assumed that he marks this dog as counted and wants to separate it from the rest of the dogs. He then moves another green dog in the same direction (see Fig. 1, line 2). After this action, he makes a pointing gesture to the latest moved dog and utters "two" at the same time (see Fig. 1, line 3). It can be assumed that Rigon wants to separate the second green dog from the rest of the dogs with his action. However, he marks this one as counted only after the separation from the other dogs, using a gesture. The marking and separation of the first dog is expressed beforehand by an action, that of the second dog successively by actions

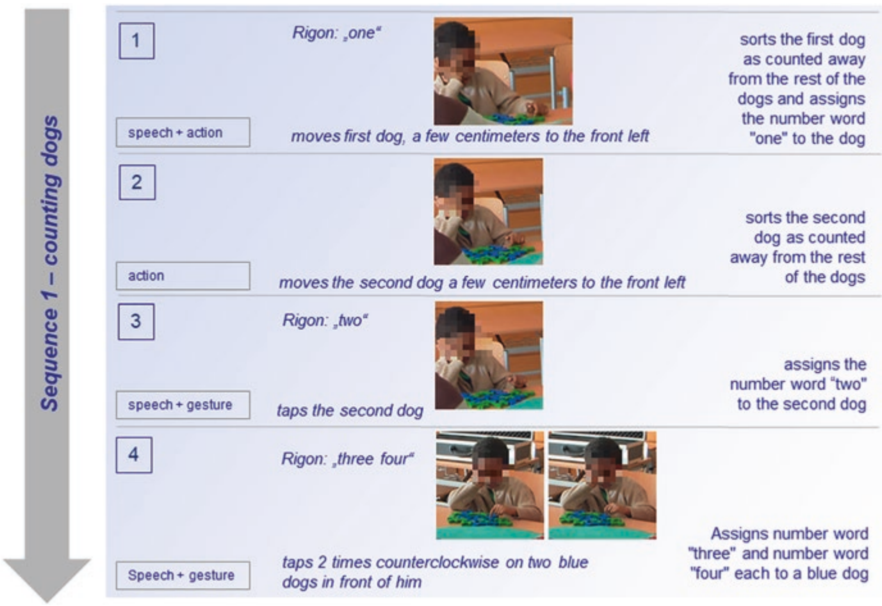

Fig. 1 Counting process of the first four dogs

and gestures. The marking of the dog is now done by a gesture, the separation by an action. In the speech, the counting process is expressed. Through analysis, it is likely that the mathematical idea of a one-to-one assignment and separation of the quantity is expressed first exclusively in action and then with action and gestures. Looking at the further counting process, Rigon abbreviates the interaction of the modes and exclusively uses gestures to mark the dogs and to assign each number word to one of them. He utters "three four" and assigns each of the number words to blue dogs with gestures (see Fig. 1, line 4). For the number word five, he does not make this one-to-one assignment of the word to one dog in the quantity. While he is uttering "five", he is simultaneously marking two green dogs gesturally. Rigon seems to get confused in counting as he makes more gestural marks than phonetically named ones. Maybe this emerges out of the abbreviated interplay of gesture and action. By pushing the dog away and separating it, he might have assigned only one dog to each number word. Rigon makes a similar assignment with the number word seven as he did earlier with five. He gesturally assigns two dogs to the number word, too, maybe because of the given rhythm with two syllables "sie-ben" in German. Otherwise, he gesturally assigns a blue or green dog to each number word up to 12. After the number word 12, he continues counting with the number word 21 and continues to establish a one-to-one assignment between the dogs in front of him and the number words 21, 22, and 23. Subsequently, his speech becomes slurred and he does not make purposeful pointing gestures to individual dogs, but a fluid movement over the crowd of dogs. He then emphasizes in his speech that he has 12 dogs in front of him. In this sequence, actions, gestures, and speech express the same mathematical idea for determining an unknown number of dogs, even not always successfully. The characteristic color does not seem to play a crucial role. In the sequence gestures and actions are in a close interplay. Rigon creates his trial of a one-to-one assignment multimodally. There is a transition from the usage of actions, to actions and gestures, to gestures, whereas the matching speech is always present.

The Idea of Number Determination by Comparing Length and Pattern Structures

In the second sequence, Rigon seems to determine the number of dogs based on the length of a gapless row. He could be inspired by the interactors who have already placed animals next to each other in a gapless species-wise row. To determine the number of dogs, Rigon first places two green dogs next to each other and then alternates their color. The result is a pattern with the basic unit green-blue. Maybe he changes his idea of determining the number of dogs by the length of the row. The pattern should perhaps help him count them. It remains open which mathematical idea he follows. With a closer look at setting up the dogs, it becomes clear how he expresses the idea of the gapless pattern. In his action, Rigon chooses a green dog and places it to the right of a green dog in the row (see Fig. 2, line 1, right). Maybe

Fig. 2 Setting up the pattern of dogs

he thinks this dog is suitable to be the next one. During his action, the accompanying person asks if another wooden animal is a chicken or a rooster (see Fig. 2, line 1, left). Rigon replies "chickens" (see Fig. 2, line, 2 left). During his speech, he selects a green dog, leads it to the row of lined-up dogs, leads the green dog back to the disorderly crowd of dogs, and lets it go (see Fig. 2, line 2, right). Possibly, Rigon feels this dog is not suitable to continue the pattern. Rigon then selects a blue dog and places it to the right of the green dog. Maybe he thinks this dog fits the pattern. He lines it up without any gaps. Across this excerpt, it can be reconstructed that Rigon's mathematical idea of setting up a row with a certain pattern is expressed in his actions. How he uses this row to determine numbers cannot be figured out. Nevertheless, he participates in the interaction with his speech, which is unrelated to the mathematical idea he pursues in his actions. He uses the modes of speech and action separately to follow both his mathematical diagrammatic idea of a pattern and the interaction theme.

A few minutes later, after Rigon has setup the line of dogs he leads his two hands to the ends of his set up row respectively (see Fig. 3, line 1). Maybe Rigon wants to measure the length of the row with his gesture which is followed by an action. He grasps the far-left dog with his left hand and transfers this dog to his right hand. He places it to the right of the blue dog on the far right of the row (see Fig. 3, line 2). The row that started earlier with two green dogs now starts with a green dog followed by a blue dog. It now ends with a green dog after a blue dog. Maybe Rigon's gesture shows a mathematical idea to measure the length of the row to deduce the

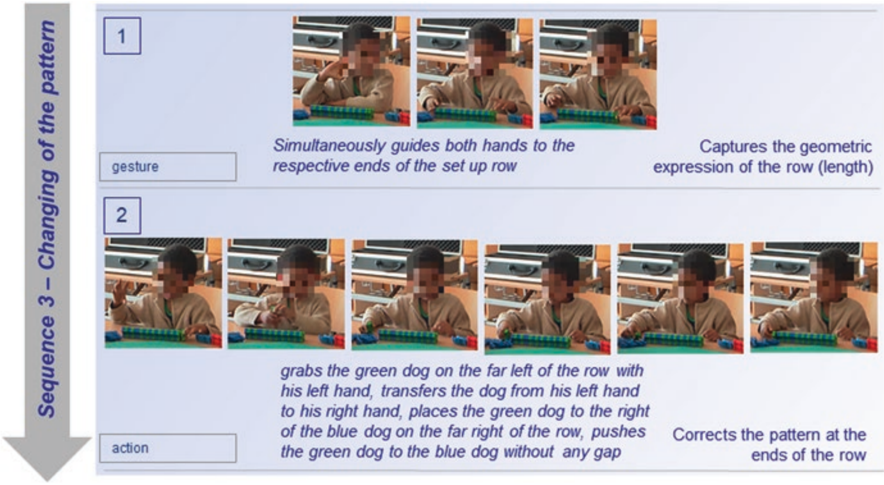

Fig. 3 Changing of the pattern

number of dogs. This fits the created rows of the other kids. Possibly Rigon marks the row as finished and now measures it as a whole.

While he marks the length of the row with his hands, he recognizes the ends of the row. He notices that the pattern is carried on differently at the beginning of the row, which he then changes with an action. The mathematical idea of creating a pattern with the basic unit green-blue comes to the fore. A gesture changes into an action, and in the transition, Rigon's mathematical idea also changes.

Conclusion

The paper aimed to investigate how Rigon's actions and gestures interplay with speech to create the diagrammatic work of a kindergartener while he constructs mathematical ideas for determining the number of an unknown crowd. The results show that Rigon is likely to focus on different mathematical areas in different modes, implements the same mathematical ideas in different modes, or addresses mathematical and non-mathematical content in different modes.

First, the analysis shows that his focus is on numbers and operations with a counting process. He probably interprets mathematical relations in an already constructed flat arranged crowd. In the semiotic sense, he is likely to create with an interplay of gestures, actions and speech a countable diagram out of a disordered crowd. For this diagram of determining the number of the crowd, the color of the wooden dogs is not essential. The analysis shows that he then changes this diagram in his actions. In the new diagram, he probably establishes two mathematical relations in different modes: In his actions he, first, makes a gapless row to map the length of all dogs. Second, he places the dogs next to each other in a pattern where the color alternates. In his gesture he embraces the row of dogs as a whole, marking the length, Rigon becomes aware of an irregularity in the pattern and changes it with an action. The first sequence shows that gestures, actions and speech can express the same mathematical idea (counting a crowd) and that an action can be replaced by a gesture in the counting process, not necessarily working. This result fits the description by Harrison (2018), who uses everyday examples to show that actions can become gestures in interaction. His results can be extended in this paper because in the case study described here it can be shown that gestures can also replace actions in mathematical interactions. In the second sequence, the analysis shows that the modes are not necessarily interwoven but rather proceed in phases of mathematical interaction parallel to each other. Rigon participates in the interaction involving "chickens" in speech and simultaneously sets up a row of dogs with a specific pattern in his action. Regarding Radford's (2009) quote at the beginning of this paper "[...] that mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a genuine sense, by actions, gestures, and other types of signs." (p. 112), it becomes clear concerning the second sequence that actions, gestures and speech interplay, but not necessarily in relation to the same (mathematical) topic. Thus, multimodality in mathematical learning is not always to be understood in a 'genuine sense'. In the third analyzed sequence, Rigon probably expresses different mathematical relationships in different modes. He is likely to focus on the pattern sequence in action and on the row as a whole in gesture (geometric length).

Theoretically, these findings show that multimodality in mathematics learning is not only characterizable by an interwoven construction of mathematical expressions, but used to participate in an ongoing interaction, potentially not always about mathematics, and simultaneously to pursue own mathematical ideas, construct diagrams and focus on different mathematical relations.

Acknowledgments The preparation of this paper was funded by the LOEWE initiative of the state government of Hessen.

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The Real World of Toddler Mathematics

Hanna Palmér and Camilla Björklund

Introduction

Early mathematics education has received increased attention in both politics and research during the last decades. Even so, there are large differences when it comes to perceptions of what preschool mathematics is, how it should be designed and what constitutes appropriate content (Palmér & Björklund, 2016). Despite these differences, there is a consensus that early mathematics matters, and a large number of studies have shown that mathematical competencies acquired in early childhood have positive effects on later school achievement (e.g., Aunio & Niemivirta, 2010; Duncan et al., 2007). Mathematics thus concerns even the youngest learners, and there is thus a need to conceptualize what early mathematics might be. Some notions, such as "mathematizing", are introduced to the education and learning of young children (e.g., Björklund et al., 2018; Gejard, 2018; Reis, 2011). These notions are important for communicating and developing ideas that are to be implemented in educational practices, but they also call attention to what notions are used, the meaning they mediate and what implications for practice they might have.

As a contribution to the development of the field of early childhood mathematics education, the focus of this paper is on the notion of *mathematizing*, an expression introduced by the famous Dutch mathematician Hans Freudenthal. He pointed out several challenges in mathematics education, one of which was "How to create suitable contexts in order to teach mathematizing" (Freudenthal, 1981, p.145). Even though Freudenthal's studies were not conducted in a preschool environment, the

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notion of mathematizing is often used in relation to preschool mathematics. However, quite often the expression is used as equivalent to "everyday mathematics" or "mathematics in everyday life" without any reference to Freudenthal's original writings. In this paper, we elaborate on what mathematizing may imply in a context of preschool mathematics, taking Freudenthal's original writings as the starting point. The aim is not to evaluate or rate how mathematics in preschool is or ought to be taught, but only to elaborate on the meaning of the notion of mathematizing in relation to empirical examples from mathematics education in preschool. The context for these empirical examples is Swedish preschool, thus a pedagogical practice with a play-oriented approach.

Mathematizing

The starting point for Freudenthal's work on mathematizing was at that time the common teaching in school mathematics. According to Freudenthal, mathematics teaching commonly took its departure from the sophisticated knowledge and strategies of experts resulting in a series of learning objectives that made sense from the perspective of the experts but not necessarily from the perspective of the learners. Based on this, Freudenthal suggested a change in instructional approach, so that instead of decomposing ready-made expert knowledge, students would elaborate, refine and adjust their current ways of knowing (Gravemeijer, 2004). According to Freudenthal (1981), mathematics is always both form and content and therefore it should not be taught as isolated form or as isolated content, but always with regard to the interplay that exists between the two. For example, even though children can learn to perform mathematical procedures (e.g., read the numbers on a ruler) and memorize facts (e.g., a square is a plane figure with four equal straight sides and four right angles), such skills and abilities are in themselves of little value if the child does not understand the purpose of the procedure or the memorized fact, or how and why a procedure works or a statement is true. Based on this, Freudenthal's starting point was that mathematics should be taught so that the knowledge becomes useful for the learner, which is why all mathematics teaching should be based on the learner's world and experiences (Freudenthal, 1968).

Mathematizing means, in brief, the process of making use of mathematical thinking and skills in problem solving where there is an actual *need* for mathematics in order to complete a task:

> Sets will not be formed, unless there is some need that they should be. In the laboratory experiment the child is expected to view some hotch-potch as a set, but why should it? What could be the genuine need to form sets? (Freudenthal, 1978, p. 217).

Freudenthal had an idea of mathematics as a human activity where students should be given the opportunity to reinvent mathematics by mathematizing in the sense of "mathematizing subject matter from reality and mathematizing mathematical subject matter" (Gravemeijer, 2004, p. 109). For education, he distinguished between horizontal and vertical mathematization. Horizontal mathematization is the use of mathematics as it appears in the "real world" based on the learner's life and experiences, while vertical mathematization refers to a process where symbols are shaped, reshaped and manipulated to make a problem solvable by reducing "noise" that the "real world" usually induces. To learn and develop mathematical skills, both are needed. However, in both cases, the subject matter that is to be mathematized should be experientially real for the learners. Freudenthal pointed out that "real world" implies different things to different individuals; for example, mathematical objects are part of the "real world" for mathematicians in a different way than they are for students in school. Thus, in school mathematics, he emphasized first the real world and then mathematizing and clarified that "the real world" in school mathematics implied a context that includes a mathematical problem that is relevant to the learners and where mathematics is needed to solve the problem (Freudenthal, 1981).

Based on Freudenthal's ideas, an approach to mathematics education called *Realistic Mathematics Education* (RME) was established. The basic idea of RME was that mathematics is a human activity where students should mathematize real situations in a context that made sense to them. RME implied designed support materials conjectured learning paths along which students, through instructional activities, could reinvent conventional mathematics. Within these learning paths, however, there is a tension between the openness toward the students' own constructions and the obligation to work toward certain given endpoints (Gravemeijer, 2004).

It is not evident how Freudenthal's writing can be understood in a preschool context. In preschool, designed support materials conjectured learning paths are not that common. Also, young children may perform actions that we recognize as mathematical even though they may not be mathematical for the child. As expressed by Van Oers (2010, p. 28), 'we actually cannot maintain that very young children (1–3 years old) perform mathematical actions, even when they may carry out actions that we, as encultured adults, may recognize as mathematical. As long as these actions are not intentionally and reflectively carried out, we cannot say that children perform mathematical actions." Based on this, in this paper we will elaborate on what "real world" and thus mathematizing may imply in the context of preschool mathematics if taking Freudenthal's original writings as the starting point.

The Play-Oriented Context of Swedish Preschools

The context of the empirical examples in this paper is mathematics education in Swedish preschools. In Sweden, preschool is available to all children aged 1–5 years, with a national curriculum that clearly states that teaching is to be conducted. Further, play is described as the basis for children's development, learning and well-being, hence preschool activities should be organized so that children can play and learn together. Based on a play-oriented approach, Swedish preschools should include teaching and opportunities for children to explore different knowledge areas, including mathematics (National Agency for Education, 2019).

There are many ways to describe what play is, but regardless of how play is understood or explained, it is characterized by an openness of the narrative where the direction of the play is not predetermined. Even if play is always about "something", the direction of the play activity is constantly renegotiated in metacommunication between the players. In play, there is also a constant shift between *as is* and *as if*, thus there is movement both towards and away from reality (Pramling et al., 2019). According to Fleer (2011), imagination is the bridge between play and learning. In moving across this bridge in an iterative movement, the child directs his or her attention to the material world – not the physical objects per se but their meaning. Imagination then makes it possible to change meanings depending on the conditions or needs of the activities (e.g., pretending that a banana is a car). Yet the child is perfectly aware of the object's meaning in "reality" (as is) and in his or her imagination (as if). Thus, reality and imagination are not separated but give meaning to each other and are dialectically related. We see this argument already in Vygotsky's (1987) writings, where he writes that intervention and creativity build on realistic thinking and imagination working in unison.

In the play-oriented approach of Swedish preschools, there is a tension between the openness of play and the goal orientation of teaching (Björklund & Palmér, 2019). This tension is similar to the tension within RME described above, a tension between the obligation to work toward certain given endpoints and the openness toward students' own constructions (Gravemeijer, 2004). Previous studies in Swedish preschools have shown, however, that goal-oriented processes can be integrated into play without changing the intentions of play, but depend on the teachers' awareness and responsiveness to children's intentions (Björklund & Palmér, 2019).

The Study

The data used in this paper was generated in a combined research-development project conducted in close collaboration between researchers and three preschool teachers for 2 years. The selection of the three preschools was based on the teachers' interest in participating. The three teachers have a university (bachelor) level preschool teacher exam and have worked in preschool for several years. Twenty-seven toddlers (at the start of the study, the children were between 12 and 27 months old) from three preschools were involved in designed teaching activities and their learning was followed through task-based interviews.[1] Each activity and task was designed to be engaging for the toddlers, based on experiences familiar to them and aiming to broaden their understanding of mathematical ideas embedded in the

[1] The study has been approved by the Swedish Ethical Review Authority (Dnr: 2019-01037), and written consent was given by the toddlers' legal guardians.

play-oriented activities. The play-oriented activities and the task-based interviews were designed based on the following design principles (for thorough description see Palmér & Björklund, 2022):

- The context of the activities ought to be based on children's experiences, needs, and interests, being familiar so that they can participate, relate to, and reason about the content based on their previous social and cultural experiences.
- The activities ought to make it possible for the children to discern essential aspects of numbers (representations, cardinality, ordinality, and part-whole relations).
- The activities ought to allow the children to express different ways of understanding and allow a variety of experiences and expressions both between the children and for the same child over a prolonged period.

During the project, a large body of video documentations of play-oriented activities as well as task-based interviews was collected. These have been analysed in detail regarding different issues (e.g., critical conditions for learning numbers' meaning, the toddlers' development of number knowledge and methodological issues in the designed activities and task-based interviews). In this particular study, we take a holistic view on the whole data set, now focusing on the practice that was developed in collaboration between teachers and researchers. The aim of this paper is to elaborate on Freudenthal's notions of "real world" and mathematizing in relation to preschool mathematics. That is, the toddlers' numerical development is not the primary focus. Instead, we direct our attention to their ways of experiencing numbers in situations they are involved in. We take the toddlers' perspective as an outset and focus our analysis on how the mathematical content embedded in the activities appears to the toddlers. Their expressed intentions (in words and actions) are the unit of analysis and are described in terms of different ways of experiencing mathematical content and form. This analysis is in line with a phenomenographical approach to interpreting the meaning of lived experiences among certain groups of people (Marton & Booth, 1997) – in this case toddlers. The examples presented below are chosen to illustrate how mathematizing is realized, or not realized, based on our reading and interpretation of Freudenthal and analysis of toddlers' expressions. Of particular interest is how "real world" can be conceptualized in play-oriented activities. The consolidation of content and form, which Freudenthal held essential for mathematizing, works as a guideline in the elaboration of what "real world" entails. How the toddlers experience the mathematics in different ways thereby becomes the key to interpreting what "real world" entails and how mathematizing may be realized in the preschool activities.

"Real World" and Mathematizing in Preschool

Below we present three examples that illustrate how mathematizing is realized or not. The first two examples are from pre-planned singing activities where materials are used to illustrate the numerical elements of the song. The third example is from one of the interview situations. Thus, the first two examples are teaching-learning situations while the third is a research situation. However, from the perspective of the involved children, the different purposes of the activities are likely not evident.

Mathematics But Not Mathematizing

In the project, the teachers together with researchers designed play-oriented activities where numbers were to be possible to elaborate on with the toddlers. However, the content "numbers" were not always connected to a "problem" that, from the children's perspective, needed to be solved.

In the following example, one teacher and three children (2 and 3 years old) are involved in a singing activity. The children sit at a table, facing each other. The teacher also sits at the table holding a box containing five plastic elephants:

Teacher: We're going to sing about the elephants. *Puts one elephant on the table*. How many elephants is this?
Holger: One. *Takes the elephant.*
Teacher: One elephant. *Holds up one finger.* Can you show one elephant with your fingers? How many fingers is one?

Holger starts to play with the elephant. One of the other children starts to sing another song about spiders, with movements. The third child starts to do the movements to the spider song. Then, the teacher turns to Holger who is still playing with the elephant.

Teacher: Can you show one with your fingers?
Holger: One. *Holds up the elephant. Takes it down and points at the elephant's trunk.*
Teacher: One trunk. *The children start to talk about the elephant's legs. The teacher asks them to count the legs. One child counts three legs, another five legs.* Five feet. *Then the teacher starts to sing the elephant song.*

In this example, the teacher is trying to teach mathematics within the play-oriented approach. The teaching starts in the "real world" in the sense of singing a familiar song and using props that the children are used to playing with. Before they are about to sing, the teacher asks the children to express the same numerical meaning of *one* through different representations (words, fingers). In this way, the teacher tries to bridge the "real" lived experiences with the symbolic representations, connecting content and form. However, the children apparently do not experience this as necessary for singing the song. To the children, this activity does not seem to contain any problem that would need to be solved by showing one with their fingers – showing one with one's fingers serves no purpose. Thus, even though the question of showing "one with your fingers" is a mathematical question based on an activity in the "real world", this example cannot be seen as an example of mathematizing as the mathematical problem given by the teacher is not relevant for the children in this context.

Mathematics in the Sense of Mathematizing

As shown in the example above, the toddlers' perspective and directed attention in a situation has great influence on how the mathematical content can be consolidated with form. Even when the starting point is within the toddlers' "real world", it becomes a challenge to accomplish mathematizing where symbolic representations, such as counting words and the counting sequence, become necessary tools for solving a "problem" that, from the children's perspective, needs to be solved.

The following example illustrates the same elephant song as in the previous example, to be sung by one teacher and one child (2 years old) with the exception that they use small toy bears instead of elephants. The teacher has taped a line on a table – the spiderweb – where the bears are to balance. She then takes out a box containing several bears (Fig. 1):

Teacher: Can you take three bears? *Holding up three fingers.* Three.

The child puts one bear on the line on the table.

Teacher: How many have you taken now?
Sander: One.
Teacher: One. And you were to take three.

Fig. 1 Teacher asking child to take three bears out of the box

The child nods and puts one more bear on the line on the table.

Teacher: How many do you have now?
Sander: *The child takes another bear from the box while the teacher asks the question, and says "three" while pointing at the third bear.*
Teacher: Are there three now?
Sander: Yes.

In this example, the teacher asks the child to put forth the number of bears they are to sing about. Based on the child's actions, he is apparently experiencing it as necessary for singing the song. The episode takes place before the playing and singing is about to start, thus the "real world" can be understood as the setting in which preparations are made for the play. This can be seen as an example of mathematizing as the mathematical task introduced by the teacher starts in the "real world" and connects content and form. Thus, the child appears to consider the problem of enumerating as relevant to solve, and he has knowledge of both form (enumerating) and relevant content (items to sing about).

Mathematizing in Play

As the focus of this paper is early childhood mathematics education, the essential play-orientation in this context cannot be overlooked. Therefore, the following example further elaborates, based on the insights described above, how play may affect how "real world" and "mathematizing" are interpreted.

In this third example, one teacher and one child (soon to be 3 years old) are in an interview situation where the child is asked to set the table for a toy cat having a birthday party.

Teacher: Look, now the kitty was to have a birthday party. Because it was the kitty's birthday. Now you are to help the kitty to set the plates.
Gustav: Then we first must bring the cake.
Teacher: Yes, also cake.
Gustav: And some muffins.
Teacher: And some muffins.
Gustav: All his friend should be.
Teacher: All his friends are coming.

While talking, the teacher puts forward one toy kitty, two plastic plates and twelve cookies (small circular pieces of wood). Then, she knocks with her hand under the table and says "here comes a friend" as she puts forth another toy kitty. She places the kitties next to one plate each.

Gustav: I will set the table for them.
Teacher: Yes please, serve them the cookies.
Gustav: One for you. *Puts one cookie on one plate.* And one for you. *Puts one cookie on the other plate.* This is not fair! (Fig. 2)
Teacher: Is it not?

Fig. 2 The child starting to divide the 12 cookies putting one cookie on each plate

Fig. 3 Holds up four fingers and says that the kitties must have four cookies

Gustav: No. They must have only, they must have four cookies. *Holds up four fingers at the same time as he says four.* One for you. *Puts another cookie on the first plate.* And one for you. *Puts another cookie on the second plate.* How many is it? One, two, three, four. *Points at one cookie at a time as he says each number word.* But they must have four cookies! (Fig. 3)

Gustav continues handing out cookies, one at a time on each plate. Then he says, "How many are there?" and counts the cookies on one plate at a time. When there are four cookies on each plate he says:

Gustav: I think it was both their birthdays.
Teacher: Did both have a birthday?
Gustav: Yes. Now they are to chew on their cookies.

Gustav holds the cookies in front of the kitties making a chewing sound. He says the kitties say that the cookies taste like strawberry cake. Then Gustav says, "What do you do when the cookie is finished?" He puts one cookie behind the kitties saying, "I put the cookie, I mean the stick here." When all the cookies are eaten, a third kitty joins the birthday party. The teacher puts forth a third plate and collects the twelve cookies in the middle of the table. Gustav says that they have forgotten to sing the birthday song. The teacher says that they can do this after he has handed out the cookies once again. Like the first time, he hands out the cookies one at a time and counts the cookies on the plates. However, he has now decided that the kitty is turning 5 years old and therefore he wants the kitties to have five cookies each. He asks for more cookies but the teacher says that there are no more cookies. He looks under the table for possibly dropped cookies. Then the teacher asks if they now are to sing the birthday song. The boy starts to sing and the teacher claps her hands.

Gustav: Now we must go get the balloons.

The child collects small gadgets from a box in the room. He divides these between the kitties.

Gustav: They only got three. They were to have four.

This is an example of an activity with a problem that, from the perspective of the child, needs to be solved with mathematics. It is also an example of an activity with clear indicators of play: there is a narrative (birthday party), meta conversation (e.g., what to do with eaten cookies, the forgotten birthday song) and a continuous shift between as is and as if. Both the child and the teacher act as is and as if, thus the "real world" is sometimes as is and sometimes as if. "Real world" may therefore imply the world of fantasy where the problem to be solved is an imagined problem within the narrative of the play. While the child is free to take the play activity in new directions (e.g., collect balloons), the teacher stays with the intended mathematical content (partitioning of twelve cookies). When all the cookies are distributed and the child wants more, it is an example of the tension between openness of play and the goal orientation of teaching in preschool (Björklund & Palmér, 2019) as well as between the openness toward students' own constructions and the obligation to work toward certain given endpoints that, for example, is highlighted within RME (Gravemeijer, 2004).

Discussion

The focus of this paper is on what "real world" and thus mathematizing may imply in the context of preschool mathematics if taking Freudenthal's original writings as the starting point. The three empirical examples in this paper illustrate that not all activities in preschool that involve mathematics can be labelled as mathematizing. The first example illustrates that even if an activity can be seen as taking children's "real world" as an outset, there may not be a problem involved that, from the children's perspective, needs to be solved by using mathematics – content and form are not consolidated. On the other hand, the second example illustrates a problem that, from the child's perspective, needs to be solved by using mathematics, thus it is an example of mathematizing. In both these examples, "real world" implies a situation where the children are to sing a song. The third example illustrates that "real world" can also include the world of play and fantasy. This means that "real world" may imply the world of fantasy where the problem to be solved is an imagined problem within the narrative of the play. "Real world" may in this respect, in the context of early childhood education, imply different kinds of activities, but it only becomes mathematization if there is a problem that, from the children's perspective, needs to be solved.

According to Van Oers (2010, p. 28), we cannot maintain that very young children perform mathematical actions as long as these actions are not intentionally and reflectively carried out. However, when children mathematize in the sense of using mathematics to solve a problem that according them needs to be solved, this is no longer an issue. Thus, when children mathematize, even when the "real world" is the world of play and fantasy, we can maintain that they are performing mathematical actions. As illustrated in the third example, the problem that needs to be solved can emerge from both teachers and children. The key is to find and consolidate form and content that constitute a relevant asset for *the child's* problem solving.

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The Appearance of Playfulness in Swedish Preschool Class Mathematics Teaching

Camilla Björklund and Jessica Elofsson

Introduction

In this paper, we elaborate on play orientation in mathematics teaching, an issue originating from curricula and policy guidelines stating that play should be an integrated part of the education for six-year-olds in Sweden ("preschool class", a mandatory form of education the year before Grade 1). When the preschool class was introduced as a school form in 1998, it was said to integrate the best from the play-oriented preschool practice and the knowledge-oriented primary school (Govt Bill 1997/98:6, 2009/10:165). The preschool class has its own section in the national curriculum (Swedish National Agency for Education, 2018) with goals referred to as *knowledge to strive for*. As mentioned above, play should be an integrated part of the practice in preschool class at the same time as, for example, mathematical development is to be given a salient position. In an inquiry about ten-year compulsory school in Sweden (SOU, 2021:33), it is repeatedly emphasized that play is an important tool for exploring and understanding the surrounding world, which implies that a play-oriented and explorative approach should be an essential part of education for six-year-olds. However, what "play-oriented teaching" means in terms of teaching practices and the contextualization of learning content is not elaborated on in the preschool class education policy documents. This is also reflected in an evaluation of the preschool class practices, which revealed that child-initiated play, or "free play", is the most common way of including play in the education. However, it is also pointed out that this kind of play most often lacks support from teachers and thereby includes no guided direction toward national curriculum goals (Swedish Schools Inspectorate, 2015).

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Based on the above-described expectations and shortcomings in implementing play as an aspect of early childhood education for Swedish six-year-olds, there seems to be a need for a deeper inquiry into how play appears in the teaching practice. As a basis for intervention programs and further research on early mathematics education, here we present an observation study of mathematics teaching in preschool classes in Sweden, aiming to give an overview of how play, and in particular the notion playfulness, and mathematics as a content for learning are expressed in teaching activities. The specific research question to be answered is thereby how playfulness and mathematics as a content for learning appear in mathematics teaching about numbers with six-year-olds.

Making Sense and Making Meaning

Recent educational research (see Pramling et al., 2019) considers learning in the context of play to be instances in which the meaning of objects and actions may (and usually does) change in the creation of imaginary situations. It is conjectured that this is where new ways of understanding emerge, because the children move between the actual and imaginary experiences, "as if" something were in a certain way and "as is", which means the way objects or occurrences are actually perceived in the situation. This is an iterative process that keeps the children's play activity going and thus constitutes the motor for *meaningful* learning. Van Oers (2010) describes mathematical development as an emerging process in the context of children's own activities in which actions and operations with numbers can be carried out, for example in the context of children's play that makes *sense* to these children.

It is often stated that mathematics education should take its starting point in the lives, experiences, and needs of the learner (see Freudenthal, 1968), to which symbols may be introduced to shape and manipulate the experiences such that a problem is made possible to solve (thereby making mathematics "useful" to the learner). This process, known as "mathematization", reduces the noise that the experiences in the real world induce. A precondition, however, is that the problem to be solved must be found in a context that is relevant to the learner and that mathematics is needed in order to solve the problem (Freudenthal, 1981). By this, one can draw the conclusion that the content to be learned should be considered meaningful in terms of practical use to the learner, as a tool for solving problems that the learner encounters. Following Leont'ev (1978), however, there are two dimensions of "meaning". The first is the cultural meaning that is found in actions, objects, goals, or tools as well as the actions that are culturally attributed to them, a sort of standard way of understanding, which is mediated through, for example, books or by others in the same community. The second dimension is the personal meaning, or rather "sense", that relates to the personal values that someone attributes to these objects, goals, tools, or actions. Thus, to aim for meaningful learning, teaching should include both a cultural and a personal dimension of meaning at the same time. That is, the cultural dimension relates to providing relevant cultural tools to the learner while the personal dimension relates to the involvement of the learner in practices that make sense to him/her.

Play Orientation in Early Childhood Education

Considering young learners, many scholars describe play as a context for putting abstract or academic concepts into a familiar setting in attempts to illustrate their culturally developed meaning, or how to use the concept in a way that is accepted by the main community. However, play orientation in educational settings could also be seen as "both culturally framed and unframed activities that are subsumed under the umbrella of 'playfulness'" (Roopnarine, 2011, p. 20). In this way, play – or rather playfulness – is more of an approach that goes beyond certain activities. Whether a situation is regarded as "playful" in this sense depends on how those involved in an activity experience a joint focus of attention, goal, and boundaries (frames of the activity, in Roopnarine's words) for what is possible to do in the situation. The playful approach thereby makes all participants agents in how the activity develops, regardless of whether the activity is scheduled, preplanned, or spontaneous.

Theories emanating from Vygotsky maintain that until the age of seven children learn to their best potential when the learning is embedded in playful activities (El'konin, 1999). Nevertheless, there seems to be a debate concerning how to conduct play-oriented teaching. Burghardt (2011) even experiences that the label "play" might better be avoided in trying to integrate playful activities into school curricula. On the other hand, there are research-based education programs aiming to further the idea of play as young children's primary activity for learning. One such program is Developmental Education (Van Oers, 2014), which takes its starting point in the notion that any activity can be interpreted as more or less playfully formatted. This implies that play is not considered a separate kind of activity but rather part of a continuum, in line with how Roopnarine (2011) conceptualizes play (see quotation above). In accordance with this, the relationship between play and mathematics teaching can be seen both as "mathematics made playful", for instance through games in which counting and mathematical operations are used, and as "mathematizing elements of play" in which the primary act is play and the teacher actively introduces mathematical concepts or operations to the playing child. Regardless of whether a play activity is framed in a context of manipulative or role play, it is usually characterized by children having a high degree of freedom in how they carry out the rule-governed activity. In such a context, the children can encounter tasks that are solved with tools that may look mathematics-like, but can also often be solved through intuition. It is then the actions of the teacher and how s/he articulates mathematical relationships that extend the children's encounters with mathematics and that may induce learning. This means that children's initiatives and explorations are important, but it is not enough that mathematical representations and concepts are present in a play activity – teachers must also provide new content and altering perspectives that extend the children's experiences (Van Oers, 2010).

To support mathematical learning, it is not enough to merely confirm what the child him/herself initiates, as this will not contribute to extending the child's knowledge or skills. Offering strategies for completing a task or a different perspective on how to interpret a problem is more a goal-oriented act that may direct the child's attention toward skills or tools that help him/her complete a task in a more advanced way. The teacher's way of responding to children's initiatives and possible mathematical content in play activities thereby leads to different learning opportunities. Particularly, maintaining a shared focus and handling the balance between foregrounding play or the content for learning is a core issue and is not easily operationalized in educational settings (Björklund et al., 2018). In reality, teaching most likely moves across this continuum, sometimes starting from problems initiated by the children and sometimes from curricular goals determining what mathematics they are expected to learn about, but most often moving in between these. When this occurs, the two dimensions of meaning – the cultural meaning and the personal sense – are likely to be connected (Leont'ev, 1978).

The Study

In this paper, we present an analysis of teaching in Swedish preschool classes. It is not our ambition to classify teaching activities as play or not play; rather, we focus on the term *playfulness* as it comes through in the teaching of numbers. While playfulness does not have a clear-cut definition in the literature, we nevertheless understand the notion as shared attention and responsiveness to the other's (the child's) perspective and experiences, including imaginary creations, and particularly an openness in the direction of the activity whereby any participant may introduce alternatives and renegotiate the rules of a game, task, or play activity (Pramling et al., 2019; Roopnarine, 2011).

The data for analysis consists of fieldnotes and documentation in a protocol (originally developed by Venkat & Askew, 2018) focusing on teachers' talk, gestures, use of artefacts, and notations in bringing forth numbers as the object of learning. Researchers (including the authors of this paper) made observations of teaching that the teachers themselves considered to be about numbers, the features of numbers, and how to make use of numbers in problem-solving. The data was collected during fall 2021. From the large data set, 81 episodes (from 46 individual teachers), observed and documented by the two authors of this paper, were used for analysis.

The object of analysis in this study is mathematics teaching about numbers, conducted in Swedish preschool class. Two features of this phenomenon are of specific interest: expressions of playfulness that come through in the mathematics teaching, and how mathematics appears as a content for learning. In this sense the analysis is phenomenological, aiming to reduce, describe, and search for essence in the observed phenomenon (Giorgi, 1997). Thus, the analysis started with identifying and describing the context in which the mathematics is being taught, as it appeared to the participants of the activity based on how the mathematics was framed as well as the teachers' and students' actions, freedom to pose suggestions and alternatives, and the nature of their engagement. Several qualitative differences emerged, in which similarities and differences in the intentions of the contexts shaped in the teaching activity came through. In this identification process, it was also possible to recognize a pattern in how the mathematical content appeared. In the following process, we condensed the contexts and connected meanings of the mathematical content into thick descriptions that bear a common meaning. Playfulness being the main object or phenomenon of inquiry is the guiding notion in these descriptions. The result of the analysis is thereby a condensed description of playfulness as it appeared in the observed lessons, and of how the content for learning was made discernable for the learners.

Results

In the Results section we present a synthesis of playfulness and the mathematical content for learning as it appeared during the teaching activities in the observed preschool classes. Three different appearances of how mathematics appears as content for learning *in regard to* the playfulness expressed in the teaching situation can be found in the data set. The observations are not exclusively classified to one or another as there are overlapping observations, but in this presentation, we describe the characteristic pattern of contextualization that was found in the data.

The Mathematical Content as the Primary Target

Firstly, a good many observations have the common characteristic of "completing a task", in which the mathematical content appears as the primary target. The tasks are not contextualized based in the children's experiences or lives. If manipulatives are used for visualizing number relations and operations, it seems irrelevant what kind of objects are used. The mathematical content constitutes the task, and is thereby the center of attention. There is rarely any playfulness observed, such as negotiating about rules or offering imaginary suggestions, and when there is some sense of playfulness this occurs when the teacher is building up tension in activities in which the outcome is not known and the children are involved in making guesses. The following example describes a teaching activity in which the mathematical content appears as the primary task:

> The teacher hands out cards with numbers on them (1-20) and asks the children to arrange them in order from smallest to largest. The children take on the task and try to place the cards in the right order. During the activity the teacher supports the children, asking "Do you remember, 17, what number comes before?". Child: "18". Teacher: "What number is before?". Child: "16". Teacher: "Then this is your place" (points at a position on the number track). When all the numbers have been placed on the number track, teacher and children check together that the numbers have been correctly ordered by counting out loud together while the teacher points to one number at a time.

The goal for this group activity is clearly stated and does not allow for alternative solutions. The children are expected to participate in a certain way, and a specific content is in focus. There is no room for spontaneity and the focus is on completing the task, i.e., placing the number cards in the correct order.

Exploring the Mathematical Content in a Relevant Context

Secondly, playfulness is observed in cases in which teacher and children explore the mathematical content in a relevant context; that is, numbers are a central feature of the activity and exploring mathematical possibilities is necessary for completing the task. Here, the children are involved in activities in which they interact and solve tasks together with their peers and the teacher. Typically, the teacher directs the activity and asks questions like "Why…?" and "How come…?". Mathematics is used as a tool for understanding the outcome of some investigation or suggestion, to help structure the children's experiences. In these observations we can see that a common approach in the teaching takes its starting point in the children's experiences and the teacher extends these by pointing out surprising results and hidden questions, and making them objects of inquiry. For instance, the teacher may introduce "conflicts" to highlight issues that can only be resolved through mathematical reasoning. In such cases, the goal for the activity is predetermined and known by the teacher, but the exact direction to take within the activity in order to arrive at the goal is not determined beforehand. This can be seen as a criterion for playfulness, as the approach allows for alternative routes and exploration. By structuring the mathematical content, the children make sense of their experiences and take part in the mediated cultural meaning in the process. Furthermore, this way of teaching mathematics, with playfulness and keeping content foregrounded, is based on either the children's own lived experiences or a collectively created context shaped in the ongoing situation. Both seem to function as facilitators for engaging the children in the activity and connecting cultural and personal mathematical meaning in joint exploration. In the following example, the teacher and the children examine what fruits the children have brought to school as a starting point for a mathematically informed exploration:

> Teacher: "Let's try to find out which fruit is the most common one today!". The teacher makes a horizontal axis on the whiteboard and asks the children what fruits they have brought with them today, writing their answers under the axis. The teacher then systematically asks for each fruit: "How many of you have apples with you today?", and the children raise their hands. The teacher documents each answer with an X in separate stacks on the board for each fruit, and a stacked bar chart emerges. After this survey, the teacher uses the stacked bar chart to ask questions, helping the children answer them by interpreting the data on the chart: "Which fruit is the most common one today?" "How can we know that without counting?" and "And if we compare pears and bananas?". In the end, the teacher describes the use of diagrams in everyday life and says it is very useful to have the ability to interpret data presented in this way.

In this activity, the children's experiences are taken as the starting point for mathematizing. The teacher helps reduce the "noise" and provides mathematical explanations for the children's experiences. The engagement is strong among the children during the activity. The teacher points out surprising results on the chart, and challenges the children to figure out answers to different questions and make them objects of joint inquiry. In this activity, mathematics is necessary in structuring the mathematical content and reasoning about the results. Playfulness is also a central part of the activity, in terms of the explorative and curious approach that both teacher and students engage in.

Parallel Activities

Thirdly, the children are involved in activities framed in an imaginary narrative, usually participating by helping a protagonist to complete a task. The mathematical task is then situated in a playful setting, but the narrative and the mathematical content are rather parallel activities. Children are invited to an imaginary setting, within which they are engaged in solving tasks through their own means and suggestions; that is, with a high degree of freedom. Mathematics may become part of the activity, but as the attention is not necessarily directed at exploring the mathematical content in a mathematically relevant context it is possible to participate without extending the children's view on the present mathematical content. The teaching appears to constitute a combination of creating a playful setting in order to (re)gain interest among the children and of completing a task in which the setting does not support the *mathematical* inquiry. In the following example, the teacher invites the children into an activity that is shaped as an imaginary narrative:

> The teacher tells the children about Findus (a well known cat in children's literature) who wants to play a joke on Pettson (his owner) by hiding eggs in his boots. The teacher places two boots in front of the children and shows them five tennis balls to represent the eggs. The children are told that Findus needs their help to figure out how the five eggs can be divided between the two boots. The teacher captures each suggestion the children offer, and processes them together with the children by writing the solutions in triads on tablets. At the end of the activity, teacher and children state that they have found all the possible ways that the five eggs can be divided between the two boots.

This example is a teaching activity in which the mathematical task is situated in a playful setting. The task itself is well structured to facilitate an exploration of part-whole relationships in numbers, and its playfulness in playing a joke on a familiar fictional character seems to engage the children in participating in the activity. Nevertheless, the narrative and the mathematical content are parallel activities in the teaching.

Discussion

In this paper we set out to describe how playfulness and mathematics as a content for learning appear in preschool class teaching activities. The analysis is therefore focused on both the playfulness and the content for learning, as a contribution to the discussion on how play (or rather playfulness) may become an informed part of preschool class mathematics teaching. The appearance of these two features of preschool class teaching forms the phenomenon in our inquiry, and the analysis of the 81 observations reveals that the way in which these features emerge shapes the mathematical learning opportunities differently. Thus, we do not intend to make any quantitative comparisons of frequency of observations, as the data can best be described as "touchdowns in time" rather than necessarily being representative of classroom teaching from a broader perspective.

When we first identified expressions of playfulness, a continuum of the extent to which play was given space in the teaching activities appeared. At one end of this continuum, we find teaching characterized by an orientation toward completing a (mathematical) task with highly limited expressions of playful exploration or open-ended inquiry. At the other end, we find teaching framed in narratives and a use of props with the intention to engage the children in interaction whereby the teacher, through the playful narrative, guides their communication about some mathematical content. In between, we find examples of interaction that centers around exploring a specific content, characterized by active involvement (from both teachers and children), that induces an explorative approach with a high degree of freedom. These inquiries are often (but not exclusively) guided by the teacher's open-ended questions that take their starting point in the children's own experiences or a familiar setting and embrace alternative suggestions and imaginary proposals.

The common content in all our observations is numbers, but the mathematics appear in different ways in the observed teaching. In some observations the mathematics becomes the central task to engage in, often through the mediation of a standard solution to symbolically presented problems (e.g., numerals written on the whiteboard). However, the content can sometimes be presented very well in terms of visualizing mathematical structures and procedures but not connect to contexts outside the mathematical. The mathematical content is thereby heavily foregrounded. In these cases, there is also often a closed solution or expected way to complete the task at hand. Meanwhile, in other observations, there is a more open approach in which the children are invited to offer suggestions for how to complete a task. The teacher still has a clear emphasis on the mathematical content and goal of the activity, but encourages different solutions for reaching the goal.

Numbers are the central feature in the observed teaching acts, and do appear in all the observations. However, the analysis of how the mathematical content is foregrounded, seen through the lens of playfulness, reveals differences in the opportunities for learning mathematics as useful and relevant (and thus meaningful) to the children. The first and third categories exhibit teaching practices in which the mathematical task to be completed is central and the goal is clearly determined, as are the methods and tools to be used. In the first category, in which the mathematics constitute the context, the cultural meaning of mathematical tools and concepts comes through. However, if one focuses on merely the cultural meaning of tools and goals, the teaching risks being reduced to the training of specific operations (Leont'ev, 1978). This stands out in comparison with the second category, which exemplifies open-ended inquiries that to a greater extent involve the children's experiences and suggestions as an outset for the teaching and are characterized by an openness in the direction of the activity, whereby any participant may introduce alternatives and renegotiate the rules of the game, task, or play activity. In other words, the children are invited to explore mathematical content in a relevant context, with their experiences taken as the starting point for mathematizing (see Freudenthal, 1981). The teacher helps reduce the "noise" and provides mathematical explanations for the children's experiences. In this way, the interplay between mediating (cultural) meaning and (personal) sense (see Leont'ev, 1978; Van Oers, 2010, 2014) becomes operationalized.

There is no doubt as to the benefits of linking educational goals to play, as play can be seen as a motivating factor for children and make them perceive the learning as meaningful, enjoyable, satisfying, and thereby help to arouse their interest in further learning (Simeonsdotter Svensson, 2009). However, our study may contribute through problematizing how play and playfulness can be implemented in teaching in ways that facilitate the learning of a specific content, a task that is known to be challenging (Björklund et al., 2018). We claim that it is not enough to embed the content for learning in a playful context, as in the third category. Instead, we argue that there are greater opportunities for learning when the setting facilitates structuring the mathematical content to be comprehensible and useful to the learner. We observed this in the second category. Play then becomes more than simply having fun; it is a valuable educational tool that includes children's experiences and meanings, creating opportunities for their deeper engagement.

The openness and high degree of freedom that playfulness offers can be characterized by a "what if" type of thinking (see Vaihinger, 1924/2001) that arises as children are challenged to go past their current level of understanding. This allows them to realize that changes to or variations in a specific task, as well as their consequences, can be anticipated and calculated. In this way, what-if thinking provides a kind of prospective thinking whereby the essential direction is forward (not looking back), which is in stark contrast to the commonly used notion of reflective thinking that instead presents a meta-perspective on an occurrence. This makes playfulness an asset in mathematics teaching, and our study has provided empirical observations of how this can take place in the Swedish preschool class. We suggest that this is an important insight to consider when developing teaching practice and policy guidelines for how education for six-year-olds should be conducted; particularly in the Swedish context, where the role of the preschool class in the education system is under review. For mathematics to remain meaningful to learners, and for education to provide culturally mediated tools that support individuals' development of mathematical knowledge and skills, it is essential to gain a deeper understanding of the significance of play and playfulness in regard to mathematics teaching and learning in early childhood education.

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Methodological Choices in Research on Early Mathematics Education: Elicitation of Parents' Views

Dorota Lembrér

Introduction

In this paper, I discuss two data collection methods: an online survey and photo-elicitation interviews (PEIs) used in my PhD project (Lembrér, 2021). These different ways of gathering data were used to investigate parents' views on mathematics education for young children at home and in Early Childhood Education and Care (ECEC) institutions in Sweden and Norway. The data included parents' stories about mathematics education and provided insights into pedagogical and mathematical aspects of home and ECEC that parents value.

Curricula documents for ECEC in Sweden and Norway highlight the importance of collaboration between teachers and parents for children's learning and development (Norwegian Ministry of Education, 2017; Swedish National Agency for Education, 2018). Yet, parents' views have received little attention in Scandinavian early childhood research. International research highlights the importance of understanding children's learning as embedded in the social, cultural and family contexts in which it occurs (Goodall & Montgomery, 2014; Phillipson et al., 2017). Most studies about parents and mathematics education in ECEC have focused on informing parents about good activities to do at home and how teachers in ECEC could support parents to better understand mathematics-related learning opportunities for young children (e.g. Blevins-Knabe et al., 2000). Some studies have shown that the involvement of parents in their children's mathematics education is considered important in ensuring that they achieve an appropriate academic outcome. For example, Mapp (2003) and Missall et al. (2015) found that informal home mathematics activities, such as when parents highlight number symbols or counting with children, seem to contribute to children's early mathematical skills, which

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contribute to their achievement at school. Other researchers, such as Anderson and Anderson (2018), have queried the one-way communication from ECEC to home and have stressed the need to understand the mathematics that adults introduce to young children at home. Previous research suggests that a lack of collaboration between ECEC and parents may be linked to the different roles parents and teachers play in children's lives (see for example, Green et al., 2007; Sonnenschein et al., 2012; Van Laere & Vandenbroeck, 2017). Parents may have limited opportunities to share what they already do without awareness of how these differences may interfere with developing opportunities for richer collaborations. Opportunities for collaboration often depend on the context, such as the parents' familiarity with the educational institutions (Murray et al., 2015; Norheim & Moser, 2020). Against this background, research on mathematics education for young children in terms of parents' views can be divided into three themes. The first theme is about how research findings should be translated into the home. The second theme identifies research concerned with how parents engage children in mathematical ideas through everyday experience. The third indicates the different roles that parents play in children's lives and the contextual considerations that may interfere with opportunities for such collaboration. Nevertheless, research that values parents' views requires appropriate methodologies that reveal how parents see, think about, and value their children's mathematics at home and in ECEC.

Vázquez Campos and Liz Gutiérrez (2015) stated that points of view are identified by highlighting of specific content, related not only to what is experienced but also to what is possible to experience. Therefore, in this paper, I define a view as a way of seeing the world that comes from the life experiences that people carry with them. A point of view is not only a place from which people view things and events but also how those things and events can be viewed from a certain kind of situation or position. The events that parents share provide insights into their experiences of mathematics education for young children at home and ECEC.

What kinds of parents' views can be identified in the research is related to methodological choices as these shows how the data collection contexts affect how they are interpreted. In this paper, I discuss the methodological choices in research on early mathematics education framed by two data collection methods: an online survey and photo-elicited interviews (PEIs), with a particular interpretation of insights, gained about parents' views on mathematics education for young children. I argue for the need for more discussion about the methodological choices for investigating parents' views on mathematics education for young children.

In the following section, I present Bruner's theory of narrative construction (1991, 2004), which links people's views and the cultures created and negotiated within a society (Bruner, 2009). Bruner' theory provides the foundation for how investigating parents' views is undertaken. I also briefly describe previous research into parents' views on mathematics education for young children and describe the data collection methods used in my studies (e.g., Lange et al., 2022; Lembrér, 2020). In the last sections, I discuss these methods with particular understanding of how the data collection has affected the research insights from investigating parents' views on mathematics education for young children.

Theoretical Perspective About Parents' Views

Bruner (1991) stressed that narrative construction should be understood as universal, where the personal view is constructed and reconstructed through social interactions and cultural activities. In his work, Bruner used stories to talk about people's experiences, he stated that: "to narrate" derives from both "telling" (narrate) and "knowing in some particular way" (gnarus), the two tangled beyond sorting" (Bruner, 2002, p. 27). He highlighted the power of narrative or storytelling for imposing order on life's uncertainties and one's expectations of life. Therefore, narrative construction shows the importance of contexts and provides opportunities for exploring views on particular topics, such as children and the family culture in which mathematics education occurs.

Bruner (2002) highlighted the importance of meaning as a central process of the individual mind and social interaction. Bruner stated:

> There is no such thing as an intuitively obvious and essential self to know, one that just sits there ready to be portrayed in words. Rather, we constantly construct and reconstruct ourselves to meet the needs of the situations we encounter. (Bruner, 2002, p. 64)

From this understanding, parents' views become identifiable because narratives do not just recall memories but indicate the values connected to others' expectations and culture as well as their own experiences, ideas, and opinions. Parents' views appear in the problems, dilemmas, or contradictions in the narratives that connect past, present, and future events with mathematics education for young children. Past experiences are connected with what may be yet to come through the values developed from past experiences, as these are likely to be used to interpret future events. Contexts affect what comes to be views or valuable knowledge about mathematics education for young children. For example, institutional views often determine events with children at home. Takeuchi (2018) showed that in Filipino immigrant parents' interactions with children about multiplication methods, the methods used in the children's Japanese school were valued more highly. The parents considered their past experiences with an informal finger method, commonly used in Filipino culture, to be a counterproductive activity for doing multiplication, even if they knew that it provided correct answers.

For Bruner, a narrative is about "the desire to communicate meaning" (1990, p. 8) and that people use narratives to construct and make sense of their views of the world. As such, he emphasised the importance of language as a tool for understanding the world. Through narratives, people build up a view of themselves and their place in the world. A narrative is, thus, situated in the context of its time and provides a sequence of events with an interrelated, meaningful connection, which allows for the reasons behind these events to be interpreted. Bruner's narrative construction provides opportunities to better understand the meanings that people create from their experiences. This is because narratives can reveal or confirm cultural norms, values, rules, and regulations (Bruner, 1991, 2004). These narratives include people recalling events using particular knowledge or understanding. For example, in a study by Wager and Whyte (2013), parents provided information to ECEC teachers about mathematics activities done at home. The teachers incorporated this information into their planning of mathematics activities at ECEC in two ways. The first way involved including only activities presented by parents that teachers were familiar with. These teachers disregarded the activities they did not recognise from their own experiences or practices. The second way involved integrating home experiences that teachers were unfamiliar with. In this way, teachers adopted some of the parents' experiences of home activities, but it was a much less common approach. In both cases, the teachers made choices about what mathematics activities should be included based on how the teacher interpreted the narratives told to them by the parents.

Bruner's narrative construction can be used to understand the meanings that parents created from their experiences. The sequences of events described in the narratives may also include what Clandinin and Connelly (2004) describe as the three-dimensional narrative research space. These dimensional spaces are: the personal and social (the interaction); the past, present, and future (continuity); and the place (situation). Narratives are constructed in a specific place and situation and in a way that the narrative modes of thought generally do not highlight or make clear to those trying to understand them. The sequence of events may include information about the setting or context of participants' experiences. Einarsdottir and Jónsdóttir's (2017) study of collaborations in Iceland between parents and preschool teachers identified tensions when the parents became more interested in early childhood policy and pedagogical practices. The results indicated that while teachers sought to keep their professional status as educators when talking about children's early years' education, the parents were viewed as providing teachers with informal knowledge about their children. The views of parents were valued only to the extent that it was individualised, so it was only about their particular children. As a result, the teachers did not seek collaboration in implementing the institutional goals and organisation of pedagogical practices at ECEC. Bruner (1991) stated that in interactions, people navigate between their previous experiences and knowledge of the world around them. This kind of navigation could be seen in the teachers' views about what should happen in the Icelandic ECEC institutions affected their interpretation of what the parents told them.

People share their views when interacting by telling narratives about events and experiences. Narratives are social or personal stories which refer to cultural values and traditions as they offer meaningful connections to events. Bruner (1991, 2004) stated that narratives address the meanings people create from their experiences. These experiences include layers of understanding about representations of time, interpretations, what feels right to say or do, cultural norms and contextual background knowledge. Narratives about events to do with mathematics activities for your children appeared in the two data collection methods. The stories told through these methods, either explicitly or implicitly, provided insights into parents' lived experiences in society. Therefore, the connection between parents' narratives and views became apparent in how they constructed their narratives in the individual survey responses and the interactions between parents in a PEI.

In the next section, I describe both methods and then discuss how Bruner's narrative construction was used to identify how the parents' views were constructed by merging their individual and social understandings of the world.

Data Collection Strategies

The two data collection methods were an online survey and photo-elicited interviews (PEIs). Earlier studies indicated that photo-elicitation interviews yielded narratives jointly constructed by the participants and offered a sharing of attention because photos have communicative features when viewed together (Lapenta, 2011). The online survey was useful in describing a population's characteristics (Braun et al., 2021). For my PhD project, Polish immigrant parents and their responses provided opportunities to gain insights into their experiences in their home countries and their new countries.

The first data collection phase focused on immigrant parents' views on mathematics education for young children; therefore, an online survey was designed to capture some of these views from Polish immigrant parents living in Sweden and Norway. The focus on Polish parents was because there are more Polish immigrants yearly in Norway than any other nationality (Østby, 2016), and Polish citizens were also the fourth-largest immigrant group in Sweden in 2015 (Statistics Sweden, 2016).

The use of surveys for data collection is a common technique for focusing on a specific population sample. Surveys can be conducted within a limited period of time and are cost-effective for collecting data (Cohen et al., 2000; Trost, 2012). This method allowed participants to remain anonymous. The survey consisted of 16 questions. The majority were open-ended, and two were multiple-choice. These questions are described in detail in Lembrér (2021, pp. 30–34). The survey questions aimed to find out about parents' individual views and understand how these views could inform ECEC in both countries. The data from the first phase were responses to a survey on an online platform (SurveyPlanet), answered between June 2016 and closed in November 2016 (41 Polish parents resident in Sweden completed the survey), and May and September 2017 (54 Polish parents resident in Norway completed the survey). The links to both surveys were made available on the websites of the Polish organisation in Sweden (Polonia info), and an organisation in Norway (Moja Norwegia/My Norway). All the parents that responded to the survey made an explicit or implicit reference to early childhood education in Sweden/Norway and had to have at least one child in ECEC in Sweden/Norway.

The photo-elicitation focus group interviews (PEIs) comprised the second phase of the data collection. Photo elicitation is a method that involves participants taking photos that are later used as stimuli during interviews. Basing the interviews on their own photos helps participants to articulate their interpretations (Hurworth, 2004). This provides insights into how the parents see the relationship between individual views and the wider societal context. Photos as stimuli provide familiarity (Harper, 2002), and the user-generated image is a term often used in research

Table 1 Overview of data collected

Method	Data material	Specification of participants
Online survey	Participant responses to 16 survey questions	41 Polish immigrant parents living in Sweden
Online survey	Participant responses to 16 survey questions	54 Polish immigrant parents living in Norway
Photo-elicitation group interview	2 × 60–75 min interviews	Total of 9 Norwegian parents in 2 groups: Group 1: 5 parents Group 2: 4 parents

when participants take photos (Epstein et al., 2006). The participants' choice of photos is an initial consideration when investigating specific groups' views on certain experiences from particular environments. The parents were asked to take photos of their children engaged in mathematical activities, with no information provided about what mathematical activities could be. The data collection for PEIs began in May 2017 and ended in November 2017.

An overview of the data is given in Table 1 and includes the method, data material, and participants' descriptions.

The narratives that were produced from these two data collections were about events and experiences of mathematics education for young children. The initial analysis began with identifying what insights appeared in the surveys, about events to do with young children's mathematics activities, and the transcripts of the photo-elicited interviews. Reflecting on the data, I describe how the narratives produced from these two data collections provided different insights into parents' views on mathematics education for young children. These narratives were investigated to identify the relationship between parents' individual views and wider societal views. In the next section, I describe the two types of narratives constructed when gathering data from parents.

Results and Discussion

The types of narratives that appeared were produced as a result of the way the survey questions were asked and how the interactions between the participants in PEIs developed. The context in which these narratives were produced provided insights into why various aspects of individual and societal views appeared. The context also gave insight into what can or should be described in the narratives.

The parents' narratives were shaped by the settings in which they were collected and connected to the three-dimensional space narrative enquiry framework (Clandinin & Connelly, 2004). In the narratives about parents' lived experiences, I found patterns, descriptions of what mathematics education is, and evidence of the social influence that affected parents' views from specific cultural standpoints (see Chap. 5 in Lembrér, 2021). The narratives brought up the personal and social and were dominated by the autonomy of a parent to be original when telling stories, their commitment to the group, and the values they might share. For example, the interactions show links between personal and social factors of experiences involving sharing life experiences through which a teller looks inside into their feelings, hope, and outside to external environments. The narrative reveals not only the mathematical events but also the settings in which the narratives were collected.

The Individual Context of the Online Survey

In the survey, the questions were designed to elicit narratives, albeit short ones. The questions asked about individual parents' experiences related to the environments of home and ECEC where children might be involved in mathematics education. When designing the survey questions, particular mathematics activities at home and ECEC identified in earlier research (e.g. Aubrey et al., 2003; Bottle, 1999) were presented, and the parents were asked to identify the mathematics activities their children did at home and ECEC.

The survey results gave a broader view of a particular group of parents on a specific set of questions about parents' experiences of mathematics education for young children at home, in ECEC in Poland and Sweden, and between Poland and Norway. Overall, the answers to the survey questions were relatively short, making it difficult to know if more details would have changed the understanding of the values connected.

Although the questions were designed to encourage parents to describe mathematics activities, the shortness of the responses limited the kinds of narratives parents could share, which led to the identification of particular kinds of views. The collected set of views that emerged from the survey about participants' experiences of mathematics education for young children indicated an implicit recognition of the influence of everyday life in the given society (Bruner, 2002).

The narratives allowed for practical and situated knowledge to be identified. This knowledge suggested a similar view from this group of immigrant parents about what was valuable for the mathematics education of young children. For example, the following two narratives illustrate Polish immigrant parents' dissatisfaction with the ECEC institutions that their children attended in Sweden and Norway.

Survey response (Sweden):	I do not think that playing in the sandbox has a greater impact on learning mathematics – unless they count sand molds or distribute a group of toys in equal parts among the children.
Survey response (Norway):	There is a tragic level of education in Norwegian preschools compared to any preschool in Poland. The Norwegian preschool is a children's storage room until parents take them home. I am very disappointed; I plan to return to Poland because I see that children do not learn anything here, and only at school from the first Grade, do they start learning anything.

In the first narrative, the parent valued counting. Other aspects of mathematics, such as volume, measurement or understanding of shapes, which can occur when playing with sand, were not recognised and not valued. The second narrative is from a parent who, influenced by expectations from the Polish approach to early childhood mathematics education, considered that Norwegian preschool did not provide appropriate experiences to their child. From these two narratives, it is possible to see how an online survey could elicit parents' critical points of view. The complexity connected to the construction of these narratives was apparent in that there were different sources that parents drew on when developing their views, which came from wider societal and institutional expectations and their own previous experience of mathematics education.

Although the survey required individual responses, it was assumed that there would be societal influences on the narratives through the parents' expectations about the kinds of answers that the survey developer were expecting. There was a range of views presented, from the critical perspective provided earlier, to those which valued the approaches used in the Scandinavian countries. Thus, parents' narrative could include contextual background knowledge from interpreting the pedagogical practices that they were aware of in the ECEC:

Survey response (Sweden):	Children learn to count in play activities. I think that play is a good approach to learning mathematics.
Survey response (Norway):	I am very happy with the way children are taught mathematics in our (Norwegian) kindergarten. Children learn it casually, on specific examples. The road to abstract thinking goes gradually, starting with things that children know that they can touch. Thanks to this, they get used to mathematics as a natural part of life. I am very happy about this approach.

Narratives such as these provided insights into the relationship between specific individual experiences and societal norms in that they emphasised the particularity of mathematics education for young children at ECEC from a particular cultural experience. This can be seen in the description of the valued pedagogical norm, "learning through play", as being a "good approach". The societal influence appears in the similarity between how play is described by the parent and how it is described in the Swedish preschool curriculum, highlighting that children learn through play (Swedish National Agency for Education, 2018). This narrative indicates that, potentially, this Polish parent had knowledge of and was influenced by the curriculum. In the second example, Polish parents living in Norway indicated an appreciation of how the Norwegian ECEC supported children to learn in everyday situation and a view of the Norwegian ECEC providing appropriate mathematics education for their children.

The societal impact on parents' general view of the institutional values to do with mathematics education was evident from viewing the whole set of responses to the questionnaires. They provided narratives that included their view, both as being individual and showing the influence of societal expectations, which had been adopted by the narrator about ways of engaging with mathematics. For example, in response to a specific survey question, when a series of individual narratives present a similar view about a range of examples of mathematical learning situations, from playing with sand and water, to measuring or jigsaw puzzles, the societal impact on that view becomes more evident.

The Context of Interactions in the Photo-Elicited Group Interviews

The narratives produced in the PEIs reflected the specific context of their construction, including the contributions of other participants about selected photos and related experiences and the impact of broader society on sharing personal memories of events within a group. The PEIs provided a more nuanced understanding of small groups, but were more specific to the experiences of those groups. In the PEIs, participants elaborated on each other's contributions, contributing to the jointly constructed narratives. As such, the parents' interpretations of their individual events were less apparent, and as the interaction developed, a joint narrative about a set of events was often produced.

Bruner (2002) argued that group interactions grow on the interplay of narratives, on the sharing of common ideas that can be negotiated for their consequences producing an outcome or resolution. In the PEIs, the Norwegian parents described several events in which they played games with their children, often as examples of adapting mathematics in everyday situations. Yahtzee was raised as a general experience of activity that parents engaged with their children at home. Here the emphasis on the social processes that shaped their understanding of people's positions was found. Bruner (2002) described narratives as providing shared meanings and symbolic modes for maintaining original or existing genres that contributed to creating and communicating the world. The narratives produced were not always familiar to those who were listening. Several narratives provided details about how the game involved mathematics and indicated that parents could interpret the features of playing a board game with their children and value playing Yahtzee as it provided mathematical learning possibilities in the home environment. From this discussion, the view seemed to be developed that this was the right or valuable activity to do with young children. For example, parents said:

PEI Group 1: Yahtzee, for example, involved "gathering all the sixes", and "We count how many dots there are on the die."

PEI Group 2: We play with a die with the numbers 1 to 6, but with the younger child we use the die with pictures. It is a little easier then.

PEI Group 1: We are playing Yahtzee with him (their younger son) in order to collect all the sixes, but it's also worth gathering all the sixes

Fig. 1 A photo taken by a parent illustrating playing the Yahtzee game

In these examples, the parents focused on encouraging children to count with them and recognise specific amounts through pattern shapes or from the numerals (Fig. 1). Yathzee was mentioned in both PEIs groups, and this further suggests that parents have valued learning to count into their general view about mathematics education. Talking about the Yahtzee game made at least one parent reflect on whether they participated in what was presented as a societal norm and reflected on their commitment to the group and the values they might share. This parent said:

PEI Group 2: I have got a little guilty conscience because we almost never played Yahtzee with our other child. So we must go home and do that.

Although PEIs were designed to elicit individual views, the specific context of the PEIs revealed that there would be societal influences on the narratives, including the contributions of other parents and the impact of wider society on sharing individual memories within a group.

The narratives produced in the interactions in PEIs were both individual and social and provided insights into what influenced participants' views beyond what was evident in the responses to the survey questions. Analysis of these narratives aims to understand why or how something happened and the participants' motivation in the events. Parents' views are linked to their intentionality of actions regarding what they value in their roles and the kinds of engagement they described themselves as having with their children. For example, a parent said:

PEI Group 1: I think that there is a strong connection between home and barnehage [ECEC] because children may get [mathematical] ideas in barnehage [ECEC] and can engage with it at home too and learn a little more about it.

This example indicates an appreciation of the Norwegian ECEC and how they might have been influenced by what was done at the ECEC. This suggests that a parent had subsumed valuing Norwegian ECEC providing appropriate mathematics education, situating the work with children being shared undertaking.

Birkeland (2013) stated that, in PEIs, telling stories about specific contexts included participants confronting each other with different ideas. The co-constructed nature of the narratives was evident in the transcripts, as the participants drew upon input from each other's narratives when they brought up different points. Therefore, the jointly constructed narratives contained the negotiation of ideas about mathematics education for young children from a set of experiences, using the pictures of their children's engagement in mathematics activities at home as stimuli. The interactions between the participants prompted broader discussions about mathematics education than the individual parents' interpretations of their own experiences.

Conclusion

In this paper, I have discussed two data collection methods, an online survey and photo-elicited interviews (PEIs), for gathering data from parents to identify their views on mathematics education. The different types of narratives produced were affected by the ways the data were collected and provided insights into some of the parents' views on mathematics education for young children. The kinds of narratives that arose in the responses to survey questions brought attention to parents' individual experiences against the backdrop of societal expectations. In the PEIs, as the interactions developed, parents could reconstruct, unpack and contextualise their views, which gave insights into what influenced those views beyond what had been evident in the survey responses.

In mathematics education research, it is acknowledged that parents contribute to their children's educational outcomes. Many researchers have studied these contributions using different methodologies (see for example, Anders et al., 2012; Colliver & Arguel, 2018; Hoover-Dempsey et al., 2005; Sheri-Lynn et al., 2014). In this paper, I have illustrated how parents' identifiable views seem to be situated simultaneously within the individual and social expectations that appeared differently in the data collected through these two methods. Thus, in the support for collaboration stated in curricula documents for ECEC in Sweden and Norway (Norwegian Ministry of Education, 2017; Swedish National Agency for Education, 2018), there are no formal directions about how this is to be achieved. For the research related to parents' views on mathematics education for young children, the study has shown that further reflection is needed on the influence of data collection on what can be said about parents' knowledge, experiences and views. The cultural nature of mathematics education for young children influences parents' views. The way parents are asked about their views might emphasise particular aspects of their knowledge about differences or similarities in mathematics education, such as between home and ECEC institutions.

The focus on the social also indicates the role of the political context in social interactions. In the case of the Polish parent in Norway, it seemed that they lacked the power to have the Norwegian preschool adopt their view on mathematics education for their children, which led them to consider taking their children back to Poland (Lange et al., 2022). The issue of who had the power to affect the mathematics learning opportunities offered to children came up in different ways when considering what might influence parents' views. Further reflection on the cultural and political nature of data collection is needed.

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