Maximal Solution of Tropical Linear Systems by Normalization Method

B. Amutha and R. Perumal

Keywords Semiring · Tropical semiring · Solution of linear system · Maximal solution

1 Introduction

Imre Simon [2], a Brazilian mathematician and computer scientist, was the first who brought tropical geometry into the literature. French mathematicians coined the term "tropical" to recognize Simon's efforts in applying min-plus algebra to optimization theory. In tropical geometry, tropical semirings play a significant role. Semirings with an underlying carrier set, that is, a subset of the set of real numbers and a binary operation of addition as maximum or minimum, product as addition, have been devised and reinvented numerous times in diverse fields of research since the late 1950s [3]. There are two tropical semirings, depending on the operation. One is the minimum tropical semiring, while the other one is maximum. In the minimum tropical semiring, an addition of two elements will be a minimum of that two elements and multiplication of two elements obtained by adding them. Min-plus semiring is another name for this algebraic structure. Similarly in maximum tropical semiring, addition of two elements will be the maximum of two elements, and the tropical product is a sum of the elements. It is also called as max-plus semiring [4]. Examples of max-plus semirings are $(\mathbb{R} \cup (-\infty), \bigoplus, \bigodot)$, $(\mathbb{Z}^+ \cup (-\infty), \bigoplus, \bigodot)$. The tropical semiring $(\mathbb{Z}^+ \cup (-\infty), \bigoplus, \bigodot)$ was introduced by Simons. Max-plus semiring is isomorphic to a min-plus semiring, and both are idempotent semirings [5]. Working with tropical semirings is appealing because of its simplicity and resemblance to algebraic geometry [9]. As a result, the ease of use and applicability might be inspiring. The tropical semiring structure is used in a variety of fields, including computer science, linear algebra, number theory, automata theory, etc.

B. Amutha · R. Perumal (✉)
Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu, India
e-mail: ab5552@srmist.edu.in; perumalr@srmist.edu.in

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[3, 8, 12]. Tropical semirings are also used in language theory, control theory, and operation research [3]. Tropical semirings are playing an important role in linear algebra, especially in solving the linear systems [10, 11, 13]. We intend to decide the behavior of some matrices over the tropical semiring. Tropical addition is denoted as $\bigoplus$ and the tropical product as $\bigodot$. In this chapter, we are concentrating on the maximum tropical semiring [8].

2 Preliminaries

A semiring S is a non-empty set with two binary operations, say addition and multiplication, that guarantees the conditions that, $(S, +)$ has the identity element 0 and it is commutative monoid; $(S, \cdot)$ is a monoid that has a single identity element which is 1; multiplication distributes over addition, that is, $a\ (b + c) = ab + ac$ and $(b + c)\ a = ba + ca$, $\forall\ \ a, b, c \in S$, $a.0 = 0.a = 0\ \forall\ a \in S$; and an element 1 is not equal to zero [1, 10]. A semiring S is said to be an idempotent semiring if $\forall\ a \in S, a + a = a$ [5]. A semiring is said to be zero-sum-free if $a + b = 0 \implies a = b = 0$. The maximum tropical semiring is the semiring $R = (S \cup (-\infty), \bigoplus, \bigodot)$, since the operations $\bigoplus$ and $\bigodot$ denoted the maximum tropical addition and maximum tropical multiplication, respectively, since S is a semiring and R should satisfy the following properties that commutative under the tropical addition, i.e., $a \bigoplus b = b \bigoplus a\ \forall\ a, b \in R$. It satisfies the associative property under the tropical addition and tropical multiplication i.e., $(a \bigoplus b) \bigoplus c = a \bigoplus (b \bigoplus c)$ and $(a \bigodot b) \bigodot c = a \bigodot (b \bigodot c)\ \forall\ a, b, c \in R$, It satisfies the property that multliplication distributes over addition $i.e.$ $a \bigodot (b \bigoplus c) = (a \bigodot b) \bigoplus (a \bigodot c)$ $\forall\ a, b, c \in R$, property of existence of additive identity $i.e.$ $\exists\ e \in R, \forall\ \ a \in R$ such that $e \bigoplus a = a \bigoplus e = a$ (since the additive identity is $-\infty$), and it never has an additive inverse [10, 20]. Similarly in minimum tropical semiring, instead of maximum we have to choose minimum [11, 19]. Maximum tropical semiring is a idempotent semiring, and all idempotent semirings are zero-sum-free [5]. Suppose that there is a semiring, say S; we denote the set of all $m \times n$ matrices over the semiring as $M_{m\times n}(S)$ and we denoting every ijth element of $P \in M_{m\times n}(S)$ matrix as p_{ij}; transpose of the matrix P is denoted as P^T. Let $P = (p_{ij}) \in M_{m\times n}(S)$, $Q = (q_{ij}) \in M_{m\times n}(S)$, $T = (t_{ij}) \in M_{n\times l}(S)$ and $\alpha \in S$. Addition of two matrices generally calculated by $P + Q = ((p_{ij}) + (q_{ij}))_{m\times n}$ and similarly product of two matrices PT can be calculated by,

$$\sum_{i=1}^{n} ((p_{ik})(t_{kj}))_{m\times l}$$

and $\alpha P = (\alpha(p_{ij}))_{m\times n}$.

Similarly in the max-plus semiring, addition of two tropical matrices, $P \bigoplus Q$, can be calculated by $(max((p_{ij}), (q_{ij})))_{m\times l}$, and the multiplication of two tropical matrices P$\bigodot$T is calculated by

$$max((p_{ik}) + (t_{kj}))_{m\times l}$$

and $\alpha \bigodot P = (\alpha + (p_{ij}))_{m\times n}$. A system $P \bigodot x = q$ is said to be a tropical system if all the entries of the system from the tropical semiring $R = (S \cup (\pm\infty), \bigoplus, \bigodot)$ [16, 17]. A matrix $P \in M_{m\times n}(S)$ is said to be a tropical matrix if all the elements of a matrix from the tropical semiring $R = (S \cup (\pm\infty), \bigoplus, \bigodot)$[5]. A matrix P is said to be maximum tropical matrix if all the elements of the matrix from the maximum tropical semiring $R = (S \cup (-\infty), \bigoplus, \bigodot)$. A matrix P is said to be minimum tropical matrix if all the elements of the matrix from the minimum tropical semiring $R = (S \cup (\infty), \bigoplus, \bigodot)$. Let $S = \mathbb{R}$ be the extended real number system under the max-plus algebra, and let P and Q be $m \times n$ matrices over the extended real numbers under the operation of maximum tropical semirings, where $P = (p_{ij})_{m\times n}$ and $Q = (q_{ij})_{m\times n}$ and (p_{ij}), (q_{ij}) are the ij^{th} entries of P and Q, respectively, $P \leq Q \leftrightarrow (p_{ij}) \leq (q_{ij})\ \forall\ i, j$ [11, 18]. A matrix $P = (p_{ij})$ is said to be regular if $(p_{ij}) \neq \pm\infty$. A vector $b \in S^m$ is said to be a normal vector or regular vector if $b_j \neq -\infty\ \forall\ j \in m$ [10]. Since we have considered max-plus semiring, if we consider the min-plus algebra, then in the regular vector, each entries $b_j \neq \infty$ $\forall\ j \in m$ [11]. A solution x^* of the tropical system $P \bigodot x = q$ is called as the maximal solution if $x \leq x^*$ for any other solution x [10, 11]. A linear system $P \bigodot x = q$ is said to be a tropical linear system if the elements of the linear system are all from any one of the tropical semirings [14, 15].

3 Main Results

A linear system $P \bigodot x = q$ is said to be a maximum if the coefficients of the linear systems from the maximum tropical semirings [7]. We know that there are different methods to solving the linear equations [6]. In this chapter, we have used the method of normalization [10, 11]. Consider the system of equation $P \bigodot x = q$. $P = (p_{ij}) \in M_{m\times n}(S/(-\infty))$, $Q = (q_{ij}) \in M_{m\times n}(S/(-\infty))$ since $(S/(x))$ denote all the values of R except x, $q = (q_j)$ is a regular vector $1 \leq j \leq m$, and j^{th} column of P matrix denoted as P_j. We begin this section with some basic definitions, and then we discuss the general maximal solution of the particular matrices. Let us assume the tropical semirings $\mathbf{T} = (\mathbb{Z}^+ \cup (-\infty), \bigoplus, \bigodot)$ where $\mathbb{Z}^+$ denoting the set of all natural numbers, **V**=$(\mathbb{R} \cup (-\infty), \bigoplus, \bigodot)$ where $\mathbb{R}$ is a set of all real numbers, **W**=$(\mathbb{Z} \cup (-\infty), \bigoplus, \bigodot)$ where $\mathbb{Z}$ is a set of all integers.

Theorem 1 *The linear system $P \bigodot x = q$ has solution if and only if every row of associated normalized matrix U contains at least one element, which is column minimum.*

3.1 Analyzing the Maximal Solution of the Tropical Linear Systems with Natural Matrix

Definition 1 A matrix $P \in M_{m\times n}(\mathbf{T})$ is said to be a natural matrix if the entries of the P matrix are continuously written with the natural numbers in the way followed by the row or column. Types of natural matrix are:

- Row natural matrix
- Column natural matrix

Definition 2 A matrix $P \in M_{m\times n}(\mathbf{T})$ is said be a row natural matrix if it is in the form of

$$\begin{bmatrix} 1 & 2 & 3 & \ldots.. & n \\ n+1 & n+2 & n+3 & \ldots.. & 2n \\ 2n+1 & 2n+2 & 2n+3 & \ldots.. & 3n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ .. & .. & .. & .. & m.n \end{bmatrix}$$

Definition 3 A matrix $P \in M_{m\times n}(\mathbf{T})$ is said be column natural matrix, if it is in the below form,

$$\begin{bmatrix} 1 & m+1 & 2m+1 & \ldots & .. \\ 2 & m+2 & 2m+2 & \ldots & .. \\ 3 & m+3 & 2m+3 & \ldots & .. \\ \vdots & \vdots & \vdots & \vdots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \\ m & 2m & 3m & .. & n.m \end{bmatrix}$$

Theorem 2 *Let $P \in M_{m\times m}(\mathbf{T}/(-\infty))$ be a column natural matrix and $P \bigodot x = q$ the linear system over the tropical semiring $(\mathbf{T}/(-\infty))$. If the $m \times 1$ regular vector q is of the form $q_i = m^2 + i, 1 \le i \le m$, then the linear system $P \bigodot x = q$ has a solution with the maximal solution*

$$x^* = \begin{bmatrix} m^2 \\ m^2 - m \\ m^2 - 2m \\ m^2 - 3m \\ \vdots \\ m \end{bmatrix}$$

Proof Given P is a column natural matrix over the tropical semiring $\mathbf{T}/(-\infty)$

$$\begin{bmatrix} 1 & m+1 & 2m+1 & 3m+1 & .. & .. & m(m-1)+1 \\ 2 & m+2 & 2m+2 & 3m+2 & .. & .. & m(m-1)+2 \\ 3 & m+3 & 2m+3 & 3m+3 & .. & .. & .. \\ \vdots & \vdots & \vdots & \vdots & .. & .. & .. \\ \vdots & \vdots & \vdots & \vdots & .. & .. & .. \\ m & 2m & 3m & 4m & .. & m(m-1) & m(m-1)+m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} m^2+1 \\ m^2+2 \\ m^2+3 \\ \vdots \\ \vdots \\ m^2+m \end{bmatrix}$$

Since $\hat{P}_1 = \frac{m+1}{2}$, $\hat{P}_2 = \frac{3m+1}{2}, \ldots\ldots, \hat{P}_m = \frac{2m^2-m+1}{2}$, $\hat{q} = \frac{2m^2+m+1}{2}$ finally U matrix is a zero matrix. Clearly every row of U matrix has at least one element, which is the minimum element in any one of the columns. By Theorem 1, the given system has a solution. The maximal solution of given system obtained by ${x_j}^* = {y_j}^* - \hat{P}_j + \hat{q}$

$$x^* = \begin{bmatrix} m^2 \\ m^2 - m \\ m^2 - 2m \\ m^2 - 3m \\ \vdots \\ m \end{bmatrix}$$

Theorem 3 *Let $P \in M_{m\times m}(\mathbf{T}/(-\infty))$ be a row natural matrix and $P \bigodot x = q$ linear system over the tropical semiring $(\mathbf{T}/(-\infty))$. Since $\mathbf{T} = (\mathbb{Z}^+ \cup (-\infty), \bigoplus, \bigodot)$. If the $m\times 1$ regular vector q is of the form $q_i = m^2+im$, $1 \le i \le m$ then the linear system $P \bigodot x = q$ has a solution with the maximal solution $x_i^* = m^2 + m - i$, for $1 \le i \le m$.*

Proof Given a row natural matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & .. \; .. & m \\ m+1 & m+2 & m+3 & m+4 & .. \; .. & 2m \\ 2m+1 & 2m+2 & 2m+3 & 2m+4 & .. \; .. & 3m \\ \vdots & \vdots & \vdots & \vdots & .. \; .. & .. \\ \vdots & \vdots & \vdots & \vdots & .. \; .. & .. \\ (m-1)m+1 & (m-1)m+2 & (m-1)m+3 & .. & .. \; .. & m(m-1)+m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} m^2+m \\ m^2+2m \\ m^2+3m \\ \vdots \\ \vdots \\ 2m^2 \end{bmatrix}$$

Since $\hat{P}_1 = (\frac{m^2-m+2}{2})$, $\hat{P}_2 = (\frac{m^2-m+4}{2})$, $\hat{P}_3 = (\frac{m^2-m+6}{2}), \ldots\ldots, \hat{P}_m = (\frac{m^2-m+2m}{2}))$, $\hat{q} = \frac{3m^2+m}{2}$, now the matrix U has all of its entries zero $\Longrightarrow$ All the rows contain at least one column minimum element. By Theorem 1, given system has a solution. The general form of the maximal solution is

$$x^* = \begin{bmatrix} m^2 + m - 1 \\ m^2 + m - 2 \\ m^2 + m - 3 \\ m^2 + m - 4 \\ \vdots \\ m^2 + m - m \end{bmatrix}$$

3.2 Analysis of the Maximal Solution of the Tropical Linear Systems with J-Matrix

Definition 4 Let P be a $m \times n$ matrix, and it is named as **J**-matrix if all the entries of the P matrix are only j.

$$\begin{bmatrix} j & j & j & .. & j \\ j & j & j & .. & j \\ j & j & j & .. & j \\ \vdots & \vdots & \vdots & \vdots & j \\ j & j & j & .. & j \end{bmatrix}$$

Theorem 4 *Let $P \in M_{m \times m}$ be a* **J***-matrix and $P \bigodot x = q$ a linear system over the tropical semiring $\mathbf{V}/(-\infty)$ where $\mathbf{V}$=($\mathbb{R} \cup (-\infty), \bigoplus, \bigodot$) with the $m \times 1$ normal vector q of the form*

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ \vdots \\ q_m \end{bmatrix}$$

then

1. *$q_i = q_j$ for all $1 \leq i, j \leq m$ if and only if the system has solution.*
2. *$q_i \neq q_j$ for some $1 \leq i, j \leq m$ if and only if the system has no solution.*

Proof Given matrix is a **J**-matrix over the tropical semiring $\mathbf{V}/(-\infty)$

$$\begin{bmatrix} j & j & j & .. & j \\ j & j & j & .. & j \\ j & j & j & .. & j \\ : & : & : & : & j \\ j & j & j & .. & j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ : \\ x_m \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ : \\ : \\ q_m \end{bmatrix}$$

Since we have $\hat{P}_1 = j$, $\hat{P}_2 = j$, $\hat{P}_3 = j$, ….., $\hat{P}_m = j$, $\hat{q} = (\frac{q_1+q_2+\ldots+q_m}{m}) = k$. In U matrix, all the entries are equal in first row, all the entries are equal in second row, and similarly, this condition holds till for the last row.

1. Assume that $q_i = q_j = k$, for all $1 \leq i, j \leq m$; then

$$q_1 - (\frac{q_1 + q_2 + \ldots + q_m}{m}) = \ldots.. = q_m - (\frac{q_1 + q_2 + \ldots + q_m}{m}) = 0$$

   now clearly verify that all the elements are column minimum elements. Since every row of U matrix has a column minimum element as 0. By Theorem 1, the system has a solution. To prove the converse part, assume that the system has a solution. By the method of contradiction, suppose that $q_i \neq q_j$ for some $1 \leq i, j \leq m$; then clearly we know that minimum element among $q_k{}^s$ where $1 \leq k \leq m$ can be either q_i or q_j for some $1 \leq i, j \leq m$; then that minimum element will be placed in the same row. All other rows have no column minimum element. By Theorem 1, the system has no solution, which is the contradiction to our assumption that system has a solution. So q_k^s should be equal for every $1 \leq k \leq m$. The general form of the maximal solution for this case will be $x_i^* = -j + k, \forall\, 1 \leq i \leq m$.
2. Assume $q_i \neq q_j$ for some $1 \leq i, j \leq m$; then minimum element can be one of the values of q_k^s where $1 \leq k \leq m$. The column minimum element will be placed in any one of the rows of U matrix. Other rows cannot have the column minimum element. By Theorem 1, that implies system has no solution. Conversely, let us assume that system has no solution. We can say that some row of the U matrix does not contain any column minimum element. Suppose $q_i = q_j$ for all i and j; then, by first part of Theorem 4, the system has a solution, which is a contradiction.

3.3 Analysis of the Maximal Solution of the Tropical Linear Systems with γ-Diagonal Matrix

Theorem 5 *Let $P \in M_{m\times m}$ be a γ-diagonal matrix and $P \bigodot x = q$ a linear systems over the tropical semiring* $\mathbf{V}/(-\infty)$ *where* $\mathbf{V}$=$(\mathbb{R} \cup (-\infty), \bigoplus, \bigodot)$ *with the normal vector $q_i = \gamma, \forall 1 \leq i \leq m$ then $U = -\tilde{P}$ and the system has a solution.*

Proof Given is a γ-diagonal matrix,

$$\begin{bmatrix} \gamma & 0 & 0 & .. & 0 \\ 0 & \gamma & 0 & .. & 0 \\ 0 & 0 & \gamma & .. & 0 \\ : & : & : & : & : \\ 0 & 0 & 0 & .. & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ : \\ x_m \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \\ \gamma \\ : \\ : \\ \gamma \end{bmatrix}$$

when we compare the normalized matrix $\tilde{P}$ and associated normalized matrix $\implies$ $U = -\tilde{P}$. Now we want to prove that the system has a solution. Associated normalized matrix has only two elements $(\frac{\gamma}{m})$ and $-(\gamma - \frac{\gamma}{m})$.

Case 1:

If $(\frac{\gamma}{m}) < -(\gamma - \frac{\gamma}{m})$, then $(\frac{\gamma}{m})$ is the column minimum element in every column. Also we know that every row and every column has an entry $(\frac{\gamma}{m})$, so every row has atleast one column minimum element. Hence the system always has a solution. Now the maximal solution of this system is

$$x^* = \begin{bmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \\ : \\ \gamma \end{bmatrix}$$

Case 2:

If $-(\gamma - \frac{\gamma}{m}) < (\frac{\gamma}{m})$ then $-(\gamma - \frac{\gamma}{m})$ be the column minimum element in every column. Also we know that every row and every column has an entry $-(\gamma - \frac{\gamma}{m})$. So that every row has at least one column minimum element $\implies$ System has a solution. In this case the maximal solution is

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ : \\ 0 \end{bmatrix}$$

3.4 Analysis of the Maximal Solution of the Tropical Linear Systems with Circulant Matrix

Theorem 6 *Let $P \in M_{m\times m}(\mathbf{V}(-\infty))$ be a circulant matrix and $P \bigodot x = q$ a linear systems over the tropical semiring $\mathbf{V}/(-\infty)$ where $\mathbf{V}$=$(\mathbb{R} \cup (-\infty), \bigoplus, \bigodot)$ with the $m \times 1$ normal vector q of the form $q = C_j$, where C_j is a j^{th} column of the circulant matrix; then the following conditions hold:*

1. *$\hat{P_i} = \hat{P_j} = \hat{q}$, $\forall\, 1 \leq i, j \leq m$*
2. *$\tilde{P}$ is also circulant matrix.*
3. *System has a solution.*
4. *$x^* = y^*$*

Proof

1. Given $P \in M_{m\times m}$ is a circulant matrix over the tropical semiring $\mathbf{V}/(-\infty)$. We know that $\hat{P_j} = (\frac{p_{1j}+p_{2j}+\ldots+p_{mj}}{m})$, $\forall j \in m$. Clearly every row of circulant matrix has every element from ${c_i}^s$, $\forall\, 0 \leq i \leq m-1$ exactly once and every column of the circulant matrix has every element from ${c_i}^s$, $\forall\, 0 \leq i \leq m-1$ exactly once. Sum of the entries in every columns is equal. Let the column sum of the circulant matrix be r. When calculating the $\hat{P_j}$, the $\hat{P_j} = \frac{r}{m} = k\ \forall\ j \in 1, 2, \ldots m$. So we conclude that $\hat{P_i} = \hat{P_j} = \hat{q} = k$, $\forall\, 1 \leq i, j \leq m$.
2. For the given system $P \bigodot x = q$, the normalized system is $\tilde{P} \bigodot y = \tilde{q}$
   We know that by the first part of Theorem 6, we know $\hat{P_i} = \hat{P_j} = \hat{q}$. Assume that $\hat{P_i} = \hat{P_j} = \hat{q} = k$

$$\begin{bmatrix} c_0-k & c_{m-1}-k & c_{m-2}-k & .. & c_1-k \\ c_1-k & c_0-k & c_{m-1}-k & .. & c_2-k \\ c_2-k & c_1-k & c_0-k & .. & c_3-k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & c_{m-1}-k \\ c_{m-1}-k & c_{m-2}-k & c_2-k & .. & c_0-k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_0-\hat{q} \\ c_1-\hat{q} \\ c_2-\hat{q} \\ \vdots \\ \vdots \\ c_{m-1}-\hat{q} \end{bmatrix}$$

   This normalized matrix satisfies all the conditions of a circulant matrix. We can conclude that $\tilde{P}$ is also a circulant matrix.
3. After finding the normalized matrix when we are finding the associated normalized matrix, we are getting U matrix as,

$$U = \begin{bmatrix} \tilde{q_1}-(c_0-k) & \tilde{q_1}-(c_{m-1}-k) & \tilde{q_1}-(c_{m-2}-k) & .. & \tilde{q_1}-(c_1-k) \\ \tilde{q_2}-(c_1-k) & \tilde{q_2}-(c_0-k) & \tilde{q_2}-(c_{m-1}-k) & .. & \tilde{q_2}-(c_2-k) \\ \tilde{q_3}-(c_2-k) & \tilde{q_3}-(c_1-k) & \tilde{q_3}-(c_0-k) & .. & \tilde{q_3}-(c_3-k) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \tilde{q_{m-1}}-(c_{m-1}-k) \\ \tilde{q_m}-(c_{m-1}-k) & \tilde{q_m}-(c_{m-2}-k) & \tilde{q_m}-(c_2-k) & .. & \tilde{q_m}-(c_0-k) \end{bmatrix}$$

   if $q = C_j$, j^{th} column of U matrix is zero, and in the j^{th} column, all elements are column minimum elements. We have at least one column minimum element in every row of the associated normalized matrix U. By Theorem 1, we can conclude that the system has a solution. Maximal solution of this system depending upon the y^*. For each value of y^*, we can find different maximal solution.

4. We know that

$$x^* = \begin{bmatrix} {y_1}^* - \hat{P}_1 + \hat{q} \\ {y_2}^* - \hat{P}_2 + \hat{q} \\ {y_3}^* - \hat{P}_3 + \hat{q} \\ \vdots \\ {y_n}^* - \hat{P}_n + \hat{q} \end{bmatrix}$$

By the first part of Theorem 6, we have $\hat{P}_i = \hat{P}_j = \hat{q}, \forall\ 1 \leq i, j \leq m$

$$x^* = \begin{bmatrix} {y_1}^* \\ {y_2}^* \\ {y_3}^* \\ \vdots \\ {y_n}^* \end{bmatrix} = y^*$$

hence $x^* = y^*$.

Notes and Comments To determine the solutions of tropical linear systems, we employed the normalization method in this article. We talked about the conditions in tropical systems and came up with a unique solution, many solutions, and no solution. We used normalized method to determine the maximal solution of the linear equations over the tropical semirings. We worked on some special matrices and studied the general form of the maximal solution of that special matrices. We have also given several theorems about the general maximal solutions of specific linear systems over the tropical semirings.

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