A Novel Difference-Integral Equation Satisfied Asymptotically by the Riemann Zeta Function

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Abstract In 2009 Fokas began a program of study of the investigation of the large t-asymptotics of the Riemann zeta function, $\zeta(\sigma + it)$. In the current work we present a novel difference-integral equation which is satisfied asymptotically by $\zeta(1/2 + it)$. This equation is obtained starting with a singular integral equation presented for the first time in 2019 and using a finite Fourier transform representation of the Riemann zeta function. The relevant analysis involves a plethora of tools and techniques developed by Fokas and collaborators during the last decade.

1 Introduction

In 2009, one of the authors, motivated by the understanding of the importance of complex analysis in the investigation of asymptotics, begun a program of study of the investigation of the asymptotics of the Riemann zeta function $\zeta(s)$, $s \in \mathbb{C}$.

It is well known that the leading asymptotics of $\zeta(s)$ as $t = \text{Im}s \to \infty$, is expressed in terms of two transcendental sums whose ranges of summation are from 0 to x and from 0 to y, where x and y satisfy the constraint $xy = t/2\pi$. Siegel, in his classical paper [8] presented the asymptotics of $\zeta(s)$ to all orders in the important particular case of $x = y = \sqrt{t/2\pi}$. In a recent publication in the Memoirs of the

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American Mathematical Society [5], two of the authors presented analogous results for $\zeta(s)$, as well as for a novel two parameter generalization of $\zeta(s)$, for any x and y to all orders.

Fokas pioneered a new approach to the asymptotics of $\zeta(s)$ based on the derivation of a novel integral equation satisfied by $|\zeta(s)|^2$, see Eq. (2.1). The large t analysis of this equation led to an interesting asymptotic result, namely, it provided the analogue of the famous Lindelöf hypothesis for a certain variation of $\zeta(s)$ [3]. Additional results were derived in [6, 7].

The analysis of the novel integral equation mentioned above is based on the following: the interval of integration of the associated integral is decomposed into four subintervals. For the first two of the resulting integrals it is possible to obtain explicit estimates, whereas for the remaining two integrals one needs to use an appropriate representation for $\zeta(s)$. In all our earlier works, we replaced $\zeta(s)$ by its leading asymptotics. This has two limitations. First, it makes it very difficult to control the relevant errors, and second, it introduces sums, for which it is difficult to obtain rigorous estimates.

Here we introduce a new idea: we express $\zeta(s)$ in terms of its the Fourier transform representation. It is worth noting that this development was motivated by the so-called unified transform, also known as the Fokas method [2, 4]. Indeed, if a function is defined on the full line it is well known that it can be represented in terms of the Fourier transform, whereas if it is defined on the half line it is often represented in terms of the Laplace transform, which is equivalent to the Fourier transform defined on the half-line. If a function is defined on a finite interval, traditionally, it is expressed in terms of a Fourier series. However, the unified transform suggests a paradigm shift: such a function should be expressed in terms of the Fourier transform defined on a finite domain. Using this idea and employing some earlier results of [3] we are led to the following difference-integral equation satisfied by the Riemann zeta function of $\sigma = 1/2$:

$$
\left| \zeta \left(\frac{1}{2} + it \right) \right|^2 \sim \mathrm{Re} \left\{ \sqrt{\frac{2}{\pi}} \left(c - e^{-i\frac{\pi}{4}} e^{-i} \right) t^{-i} \right\} \left| \zeta \left(\frac{1}{2} + i(t-1) \right) \right|^2 \left(1 + O \left(\frac{1}{\ln t} \right) \right)
$$
$$
+ \mathrm{Re} \left\{ \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} t^{-it} \int_{t^{\delta_2}}^{t-1} \frac{e^{i(t-\rho)\ln(t-\rho) + i\rho \ln \rho}}{\sqrt{t-\rho}} \left| \zeta \left(\frac{1}{2} + i\rho \right) \right|^2 d\rho \right\}
$$
$$
+ \ln t + 2\gamma - \ln 2\pi, \qquad t \to \infty, \tag{1.1}
$$

where c is a complex constant given by

$$
c = \int_1^{+\infty} \frac{e^{-ix}}{x^{1-i}} dx \approx -0.0713 - 1.0417i. \tag{1.2}
$$

This paper is organised as follows. In Sect. 2 we review some of the basic results of [3], which includes decomposing the integral appearing in (2.1) into 4 integrals, I_j, $j = 1, 2, 3, 4$. In Sect. 3 we express the leading asymptotic behaviour of I_3 and

I_4 in terms of the finite Fourier transform of $\zeta(s)$. In Sect. 4 we sketch the derivation of (1.1). In Sect. 5 we present numerical evidence of the validity of (1.1).

2 Review of Some of the Results of [3]

In this section we review the singular integral equation for the Riemann zeta function, as well as associated results which were derived in [3].

We start with the singular integral equation for all $t > 0$:

$$\frac{t}{\pi} \oint_{-\infty}^{\infty} \mathrm{Re}\left\{ \frac{\Gamma(it - i\tau t)}{\Gamma\left(\frac{1}{2} + it\right)} \Gamma\left(\frac{1}{2} + i\tau t\right) \right\} \left| \zeta\left(\frac{1}{2} + i\tau t\right) \right|^2 d\tau + \mathcal{G}(t) = 0, \quad (2.1)$$

where the principal value integral is defined with respect to $\tau = 1$, and the function $\mathcal{G}(t)$ is defined by the formula

$$\mathcal{G}(t) = \mathrm{Re}\left\{ \Psi\left(\frac{1}{2} + it\right) \right\} + 2\gamma - \ln 2\pi + \frac{2}{1 + 4t^2}, \quad (2.2)$$

with $\Psi(z)$ denoting the digamma function, i.e.,

$$\Psi(z) = \frac{\frac{d}{dz}\Gamma(z)}{\Gamma(z)}, \quad z \in \mathbb{C},$$

and γ denoting the Euler constant.

It is shown in [3] that for $\delta_1 > 0$, $\delta_4 > 0$, $\delta_{14} = \min(\delta_1, \delta_4)$, Eq. (2.1) simplifies to the equation

$$\frac{t}{\pi} \oint_{-t^{\delta_1 - 1}}^{1 + t^{\delta_4 - 1}} \mathrm{Re}\left\{ \frac{\Gamma(it - i\tau t)}{\Gamma\left(\frac{1}{2} + it\right)} \Gamma\left(\frac{1}{2} + i\tau t\right) \right\} \left| \zeta\left(\frac{1}{2} + i\tau t\right) \right|^2 d\tau + \mathcal{G}(t)$$
$$+ O\left(e^{-\pi t^{\delta_{14}}}\right) = 0, \quad t \to \infty, \quad (2.3)$$

where the principal value integral is defined with respect to $\tau = 1$.

We split the above interval of integration into the following subintervals:

$$L_1 = [-t^{\delta_1 - 1}, t^{-1}], \quad L_2 = [t^{-1}, t^{\delta_2 - 1}], \quad L_3 = [t^{\delta_2 - 1}, 1 - t^{\delta_3 - 1}],$$
$$L_4 = [1 - t^{\delta_3 - 1}, 1 + t^{\delta_4 - 1}], \quad \delta_2 > 0, \ \delta_3 > 0. \quad (2.4)$$

Denote by I_j the integrals along the intervals L_j. It is shown in [3] that

$$I_1(t, \delta_1) = O\left(t^{-\frac{1}{2}+\frac{4}{3}\delta_1}\right), \qquad I_2(t, \delta_2) = O\left(t^{-\frac{1}{2}+\delta_2} \ln t\right),$$
$$\mathcal{G}(t) = \ln t + 2\gamma - \ln 2\pi = O(\ln t), \qquad t \to \infty. \tag{2.5}$$

Thus, if $\delta_1 \leq \frac{3}{8}$ and $\delta_2 < \frac{1}{2}$, Eq. (2.3) becomes

$$I_3 + I_4 = -\ln t - 2\gamma + \ln 2\pi + o(1) \qquad t \to \infty. \tag{2.6}$$

Let $\check{I}_3$ and $\check{I}_4$ denote the leading order terms as $t \to \infty$ of I_3 and I_4, respectively. Then, Eq. (2.6) is

$$\check{I}_3 + \check{I}_4 = O(\ln t) - \left(I_3 - \check{I}_3\right) - \left(I_4 - \check{I}_4\right), \qquad t \to \infty. \tag{2.7}$$

A rigorous treatment of the RHS of (2.7), which will be presented in forthcoming publication, yields

$$\left(I_3 - \check{I}_3\right) + \left(I_4 - \check{I}_4\right) = o(1), \qquad t \to \infty. \tag{2.8}$$

Employing (2.8) into (2.6) yields

$$\check{I}_3 + \check{I}_4 = -\ln t - 2\gamma + \ln 2\pi + o(1) \qquad t \to \infty. \tag{2.9}$$

In what follows we will present arguments suggesting that (2.9) leads to the difference-integral Eq. (1.1) for $\left|\zeta\left(\frac{1}{2} + it\right)\right|^2$. The rigorous derivation of (1.1) will be presented in forthcoming publication.

3 Computation of $\check{I}_3$ and $\check{I}_4$

Preliminaries for I_3

Letting $\tau = \frac{\rho}{t}$ in the definition of I_3, we find

$$I_3 = \frac{1}{\pi} \int_{t^{\delta_2}}^{t-t^{\delta_3}} \text{Re}\left\{\frac{\Gamma(it - i\rho)}{\Gamma\left(\frac{1}{2} + it\right)} \Gamma\left(\frac{1}{2} + i\rho\right)\right\} \left|\zeta\left(\frac{1}{2} + i\rho\right)\right|^2 d\rho. \tag{3.1}$$

Using

$$\frac{\Gamma(it - i\rho)}{\Gamma\left(\frac{1}{2} + it\right)} \Gamma\left(\frac{1}{2} + i\rho\right) \sim \frac{\sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{itF\left(\frac{\rho}{t}\right)}}{\sqrt{t - \rho}}, \qquad t \to \infty, \tag{3.2}$$

where $F(x)$ is defined by

$$F(x) = (1 - x) \ln(1 - x) + x \ln x, \tag{3.3}$$

we find that the leading contribution of I_3 is given by

$$\check{I}_3 := \sqrt{\frac{2}{\pi}} \int_{t^{\delta_2}}^{t - t^{\delta_3}} \frac{\mathrm{Re}\left\{ e^{-i\frac{\pi}{4}} e^{it F\left(\frac{\rho}{t}\right)} \right\}}{\sqrt{t - \rho}} \left| \zeta\left(\frac{1}{2} + i\rho\right) \right|^2 d\rho. \tag{3.4}$$

Preliminaries for I₄

Letting $\tau = \frac{\rho}{t}$ in the definition of I_4, we find

$$I_4 = \frac{1}{\pi} \oint_{t - t^{\delta_3}}^{t + t^{\delta_4}} \mathrm{Re}\left\{ \frac{\Gamma(it - i\rho)}{\Gamma\left(\frac{1}{2} + it\right)} \Gamma\left(\frac{1}{2} + i\rho\right) \right\} \left| \zeta\left(\frac{1}{2} + i\rho\right) \right|^2 d\rho, \tag{3.5}$$

where the principal value integral is defined with respect to $\rho = t$. Letting $x = t - \rho$, we obtain

$$I_4 = \frac{1}{\pi} \oint_{-t^{\delta_4}}^{t^{\delta_3}} \mathrm{Re}\left\{ \Gamma(ix) \frac{\Gamma\left(\frac{1}{2} + it - ix\right)}{\Gamma\left(\frac{1}{2} + it\right)} \right\} \left| \zeta\left(\frac{1}{2} + it - ix\right) \right|^2 dx, \tag{3.6}$$

where the principal value integral is defined with respect to $x = 0$. It is well-known that the Gamma function admits the integral representation

$$\Gamma(ix) = \frac{1}{e^{-\pi x} - e^{\pi x}} \int_{H_1} \frac{e^z}{z} z^{ix} dz, \tag{3.7}$$
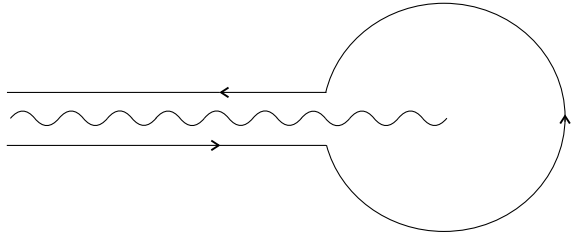
with H_1 denoting the Hankel contour with a branch cut along the negative real axis, see Fig. 1, defined by

$$H_1 = \{ re^{-i\pi} | 1 < r < \infty \} \cup \{ e^{i\theta} | -\pi < \theta < \pi \} \cup \{ re^{i\pi} | 1 < r < \infty \}. \tag{3.8}$$

Using the asymptotic formula

$$\frac{\Gamma\left(\frac{1}{2} + it - ix\right)}{\Gamma\left(\frac{1}{2} + it\right)} = e^{\frac{\pi x}{2}} t^{-ix} e^{ix} \left(1 - \frac{x}{t}\right)^{i(t - x)} \left[1 + O\left(\frac{1}{t}\right)\right], \qquad t \to \infty, \tag{3.9}$$

Fig. 1 The Hankel contour H_1

as well as the estimate

$$e^{ix}\left(1 - \frac{x}{t}\right)^{i(t-x)} = 1 + O\left(\frac{x^2}{t}\right), \qquad \frac{x}{t} \to 0,$$

we find that the leading contribution of I_4 is given by

$$\check{I}_4 = \text{Re}\left\{\frac{1}{\pi}\int_{H_1}\frac{e^z}{z}\oint_{-t^{\delta_4}}^{t^{\delta_3}}\frac{e^{\frac{\pi x}{2}}}{e^{-\pi x} - e^{\pi x}}\left(\frac{z}{t}\right)^{ix}\left|\zeta\left(\frac{1}{2} + it - ix\right)\right|^2 dx dz\right\}, \quad (3.10)$$

with the principal value integral defined with respect to $x = 0$.

The Finite Fourier Transform

In order to compute the large t asymptotics of the RHS of (3.4) and (3.10) we will employ the finite Fourier transform, where it turns out that it will be more convenient to integrate from $t = 1$:

$$\Phi(\nu) := \int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 e^{i\nu t} dt, \qquad \nu \in \mathbb{C}. \quad (3.11)$$

Then,

$$\left|\zeta\left(\frac{1}{2} + it\right)\right|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}d\nu, \qquad t \in [1, T]. \quad (3.12)$$

Remark 1 Equations (3.11) and (3.12) imply

$$\int_{-\infty}^{\infty}\Phi(\nu)\left(\frac{1}{2\pi}\int_1^T e^{i\tau(k-\nu)}d\tau\right)d\nu = \Phi(k). \quad (3.13)$$

Note that

$$\frac{1}{2\pi}\int_1^T e^{i\tau(k-\nu)}d\tau = \frac{1}{2i\pi}\frac{e^{iT(k-\nu)} - e^{i(k-\nu)}}{k-\nu}. \quad (3.14)$$

In the case of the Fourier transform on the full line the analogue of the LHS in (3.14) equals $\delta(k - \nu)$. In the case of the finite Fourier transform, Eq. (3.13) is a direct consequence of analyticity: $\Phi(\nu)$ is an entire function for which

$$\Phi(\nu) \sim \frac{1}{i\nu} \left[e^{i\nu T} \left| \zeta\left(\frac{1}{2} + iT\right) \right|^2 - e^{i\nu} \left| \zeta\left(\frac{1}{2} + i\right) \right|^2 \right], \qquad \nu \to \infty. \quad (3.15)$$

Thus, $e^{-i\nu T} \Phi(\nu)$ is an entire function for which

$$e^{-i\nu T} \Phi(\nu) \sim \frac{1}{i\nu} \left[\left| \zeta\left(\frac{1}{2} + iT\right) \right|^2 - e^{-i\nu(T-1)} \left| \zeta\left(\frac{1}{2}\right) \right|^2 \right].$$

Hence we rewrite (3.13) as

$$\frac{1}{2i\pi} \int_{\tilde{R}} \Phi(\nu) \frac{e^{iT(k-\nu)}}{k - \nu} d\nu - \frac{1}{2i\pi} \int_{\tilde{R}} \Phi(\nu) \frac{e^{i(k-\nu)}}{k - \nu} d\nu,$$

where $\tilde{R}$ is the real line slightly deformed at the point $\nu = k$, with a small semicircle of radius $\epsilon \to 0$ contained in the lower half complex plane. Thus, the first integral vanishes, by closing at $\mathbb{C}^-$, whereas the second integral gives $\Phi(k)$, by closing at $\mathbb{C}^+$.

Remark 2 Using in (3.11) the fact that $\left| \zeta\left(\frac{1}{2} + it\right) \right|^2$ is real yields the condition $\overline{\Phi(\nu)} = \Phi(-\nu)$.

The Derivation of $\check{I}_4$

Equation (3.12) yields

$$\left| \zeta\left(\frac{1}{2} + i(t - x)\right) \right|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\nu) e^{-i\nu t} e^{i\nu x} d\nu, \qquad t \in [1, T]. \quad (3.16)$$

Using the above equation into (3.10), we obtain

$$\check{I}_4 = \text{Re} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\nu) e^{-i\nu t} \tilde{I}_4 d\nu \right\}, \quad (3.17)$$

with

$$\tilde{I}_4 = \frac{1}{\pi} \int_{H_1} \frac{e^z}{z} \oint_{-t^{\delta_4}}^{t^{\delta_3}} \frac{e^{\frac{\pi x}{2}}}{e^{-\pi x} - e^{\pi x}} A^{ix} dx dz, \qquad A := \frac{z}{t} e^\nu. \quad (3.18)$$

Using the result of Proposition 6.3 of [3], namely the estimate

$$\frac{1}{\pi}\int_{H_1}\frac{e^z}{z}\oint_{-t^{\delta_4}}^{t^{\delta_3}}\frac{e^{\frac{\pi x}{2}}}{e^{-\pi x}-e^{\pi x}}A^{ix}dxdz \sim -1+\frac{1}{\pi}\int_{H_1}\frac{e^z}{z}\frac{e^{it^{\delta_3}\ln A-\frac{\pi t^{\delta_3}}{2}}}{\frac{\pi}{2}-i\ln A}dz, \qquad (3.19)$$

we obtain

$$\tilde{I}_4 \sim -1+\frac{1}{\pi}\int_{H_1}\frac{e^z}{z}\frac{e^{it^{\delta_3}\ln(Mz)-\frac{\pi t^{\delta_3}}{2}}}{\frac{\pi}{2}-i\ln(Mz)}dz, \qquad M=\frac{e^\nu}{t}, \qquad t\to\infty. \qquad (3.20)$$

Furthermore, Proposition 6.4 of [3] yields

$$\frac{1}{\pi}\int_{H_1}\frac{e^z}{z}\frac{e^{it^{\delta_3}\ln(Mz)-\frac{\pi t^{\delta_3}}{2}}}{\frac{\pi}{2}-i\ln(Mz)}dz = 2e^{-\frac{i}{M}}+E_4^{SD}(\nu,t), \qquad (3.21)$$

where the first term occurs iff $e^\nu\in\left(t^{1-\delta_3},t\right)$ and

$$E_4^{SD}(\nu,t)=\frac{1}{\pi}\int_{H_1}\frac{e^{\lambda[w+i\ln\left(\frac{iw}{\alpha}\right)]}}{-iw\ln\left(\frac{iw}{\alpha}\right)}dw, \qquad \lambda=t^{\delta_3}, \quad \alpha=\frac{t^{1-\delta_3}}{e^\nu}. \qquad (3.22)$$

Hence,

$$\tilde{I}_4 \sim -1+2e^{-ite^{-\nu}}+E_4^{SD}, \qquad t\to\infty, \qquad (3.23)$$

where the second term occurs iff $e^\nu\in\left(t^{1-\delta_3},t\right)$.

Thus, $\check{I}_4$ takes the form

$$\check{I}_4 = \text{Re}\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}d\nu+\frac{1}{\pi}\int_{(1-\delta_3)\ln t}^{\ln t}\Phi(\nu)e^{-it(\nu+e^{-\nu})}d\nu\right.$$
$$\left.+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}E_4^{SD}(\nu,t)d\nu\right\}.$$

Using the fact that $\overline{\Phi(\nu)}=\Phi(-\nu)$, see Remark 2, the term $\check{I}_4$ can also be written in the following form:

$$\check{I}_4(t) = \frac{1}{2\pi}\int_{(1-\delta_3)\ln t}^{\ln t}\Phi(\nu)e^{-it(\nu+e^{-\nu})}d\nu+\frac{1}{2\pi}\int_{-\ln t}^{(\delta_3-1)\ln t}\Phi(\nu)e^{-it(\nu-e^\nu)}d\nu$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}E_4^{SD}(\nu,t)d\nu+\frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}\overline{E_4^{SD}(-\nu,t)}d\nu$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}d\nu. \qquad (3.24)$$

Proposition 1 *Let $\check{I}_4$ be defined by* (3.24). *Then*

$$\check{I}_4 = -\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \tag{3.25}$$
$$+ Re\left\{\frac{1}{\pi}\int_{(1-\delta_3)\ln t}^{\ln t}\Phi(\nu)e^{-it(\nu+e^{-\nu})}d\nu + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}E_4^{SD}(\nu,t)d\nu\right\},$$

where

$$E_4^{SD}(\nu,t) = \frac{1}{\pi}\int_{H_1}\frac{e^{\lambda[w+i\ln\left(\frac{iw}{\alpha}\right)]}}{-iw\ln\left(\frac{iw}{\alpha}\right)}dw, \qquad \lambda = t^{\delta_3}, \;\; \alpha = \frac{t^{1-\delta_3}}{e^{\nu}}. \tag{3.26}$$

Proof Employing (3.12) in the last term of the RHS of Eq. (3.24) yields (3.25).

The term $2e^{-i/M}$ appearing in the RHS of (3.21) arises from the evaluation of the contribution of the pole $z_P = -i/M$ and gives rise to the second term in the RHS of (3.25). Thus, we will use the notation $\check{I}_4^P(t)$ for this term. Similarly, we denote the last term (3.25) as $\check{I}_4^{SD}$.

Hence, (3.25) takes the form

$$\check{I}_4 = -\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 + Re\left\{\check{I}_4^P + \check{I}_4^{SD}\right\}, \tag{3.27}$$

where

$$\check{I}_4^P(t) = \frac{1}{\pi}\int_{(1-\delta_3)\ln t}^{\ln t}\Phi(\nu)e^{-it(\nu+e^{-\nu})}d\nu \tag{3.28}$$

and

$$\check{I}_4^{SD}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\nu)e^{-i\nu t}E_4^{SD}(\nu,t)d\nu. \tag{3.29}$$

4 Sketch of the Derivation of (1.1)

It can be shown that $\check{I}_4^{SD}$ is negligible, compared to $\check{I}_4^P$; the rigorous derivation will be presented in forthcoming publication. Thus, in what follows we analyse the contribution of $\check{I}_4^P$.

Using (3.11) in (3.28) yields

$$\check{I}_4^P(t) = \frac{1}{\pi}\int_{(1-\delta_3)\ln t}^{\ln t}\int_1^T\left|\zeta\left(\frac{1}{2}+i\tau\right)\right|^2 e^{-it(\nu+e^{-\nu}-\frac{\tau}{t}\nu)}d\tau d\nu$$
$$= \frac{1}{\pi}\int_1^T J\left(\tau;t,\delta_3\right)\left|\zeta\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau, \tag{4.1}$$

where

$$J(\tau; t, \delta_3) = \int_{(1-\delta_3)\ln t}^{\ln t} e^{-itg\left(\nu, \frac{\tau}{t}\right)} d\nu, \tag{4.2}$$

with

$$g\left(\nu, \frac{\tau}{t}\right) = \nu + e^{-\nu} - \frac{\tau}{t}\nu. \tag{4.3}$$

The stationary phase method yields the estimate

$$\int_{(1-\delta_3)\ln t}^{\ln t} e^{-itg\left(\nu, \frac{\tau}{t}\right)} d\nu \sim \sqrt{2\pi} e^{-i\frac{\pi}{4}} \frac{e^{-i(t-\tau)} e^{i(t-\tau)\ln(t-\tau)} e^{-i(t-\tau)\ln t}}{\sqrt{t-\tau}},$$
$$\text{if } \tau \in \left(t - t^{\delta_3}, t - 1\right). \tag{4.4}$$

Indeed, the stationary point is given by solving $g_\nu = 0$, which yields $\nu^* = -\ln\left(1 - \frac{\tau}{t}\right)$. Hence,

$$\nu^* \in \left((1 - \delta_3)\ln t, \ln t\right) \iff \tau \in \left(t - t^{\delta_3}, t - 1\right).$$

It is interesting to note that the RHS of (4.4) can be rewritten in the form

$$\frac{1}{\pi} \int_{(1-\delta_3)\ln t}^{\ln t} e^{-itg\left(\nu, \frac{\tau}{t}\right)} d\nu \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} \frac{e^{itF\left(\frac{\tau}{t}\right)}}{\sqrt{t-\tau}}, \quad \text{if } \tau \in \left(t - t^{\delta_3}, t - 1\right), \tag{4.5}$$

with F defined in (3.3). This can be derived by using

$$e^{-i(t-\tau)} e^{i(t-\tau)\ln(t-\tau)} e^{-i(t-\tau)\ln t} = e^{itF\left(\frac{\tau}{t}\right)} \left[1 + O\left(t^{2\delta_3 - 1}\right)\right], \quad t \to \infty, \tag{4.6}$$

into (4.4). In order to prove (4.6) we observe that

$$e^{-i(t-\tau)} e^{i(t-\tau)\ln(t-\tau)} e^{-i(t-\tau)\ln t} = e^{itF\left(\frac{\tau}{t}\right)} \exp\left\{-i\left[t - \tau + \tau \ln\left(\frac{\tau}{t}\right)\right]\right\}.$$

Making the change of variables $\tau = t - x$, $x \in \left(1, t^{\delta_3}\right)$, we find

$$t - \tau + \tau \ln\left(\frac{\tau}{t}\right) = x + (t - x) \ln\left(1 - \frac{x}{t}\right) = x + (t - x)\left(-\frac{x}{t} + O\left(\frac{x}{t}\right)^2\right) \sim \frac{x^2}{t},$$

hence

$$\exp\left\{-i\left[t - \tau + \tau \ln\left(\frac{\tau}{t}\right)\right]\right\} = 1 + O\left(\frac{x^2}{t}\right).$$

The fact that $x \in \left(1, t^{\delta_3}\right)$ yields (4.6).

The stationary point of J coincides with the endpoint $\ln t$ if $\tau = t - 1$. Evaluating the RHS of (4.4) at $\tau = t - 1$ we find $\sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{-i} t^{-i}$. Thus, as $\nu \to \ln t$ and $\tau \to t - 1$, we obtain the following contribution:

$$\frac{1}{\pi} \left[\hat{J}(t; \delta_3) - \sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{-i} t^{-i} \right] \left| \zeta \left(\frac{1}{2} + i(t-1) \right) \right|^2$$

where $\hat{J}(t; \delta_3)$ denotes the contribution of $J(\tau, t; \delta_3)$ in the neighbourhood of $\tau = t - 1$. It turns out that

$$\hat{J}(t; \delta_3) = \sqrt{2\pi} J(t-1, t; \delta_3) = \sqrt{2\pi} \int_{(1-\delta_3)\ln t}^{\ln t} e^{-i(\nu + t e^{-\nu})} d\nu.$$

In order to evaluate the above integral we let $\nu = \ln t - \ln x$, and find

$$\int_{(1-\delta_3)\ln t}^{\ln t} e^{-i(\nu + t e^{-\nu})} d\nu = t^{-i} \int_1^{t^{\delta_3}} \frac{e^{-ix}}{x^{1-i}} dx = t^{-i} \int_1^\infty \frac{e^{-ix}}{x^{1-i}} dx - t^{-i} \int_{t^{\delta_3}}^\infty \frac{e^{-ix}}{x^{1-i}} dx. \tag{4.7}$$

Using integration by parts we find that the second integral in the RHS of (4.7) is $O\left(t^{-\delta_3} \right)$.

Similar considerations apply to the case that $\tau = t - t^{\delta_3}$, where the stationary point approaches the other endpoint, $(1 - \delta_3) \ln t$, but now the relevant contribution is $O\left(t^{-\frac{\delta_3}{2}} \right)$. Hence we find,

$$\check{I}_4^P \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} \int_{t-t^{\delta_3}}^{t-1} \frac{e^{it F\left(\frac{\tau}{t}\right)}}{\sqrt{t - \tau}} \left| \zeta \left(\frac{1}{2} + i\tau \right) \right|^2 d\tau \tag{4.8}$$

$$+ \sqrt{\frac{2}{\pi}} \left(c - e^{-i\frac{\pi}{4}} e^{-i} \right) t^{-i} \left| \zeta \left(\frac{1}{2} + i(t-1) \right) \right|^2 \left(1 + O\left(\frac{1}{\ln t} \right) \right), \qquad t \to \infty,$$

with F and c defined in (3.3) and (1.2), respectively.

Simplifying $t F\left(\frac{\rho}{t} \right)$, we find,

$$(t - \rho) \ln \left(1 - \frac{\rho}{t} \right) + \rho \ln \left(\frac{\rho}{t} \right) = (t - \rho) \ln (t - \rho) - (t - \rho) \ln t + \rho \ln \rho - \rho \ln t$$

$$= (t - \rho) \ln (t - \rho) + \rho \ln \rho - t \ln t.$$

Hence, (1.1) follows by employing (3.4), (3.27) and (4.8) in (2.9).

5 Numerical Evidence

In this section we check numerically the validity of the difference-integral Eq. (1.1), namely

$$
\left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \sim \mathrm{Re}\left\{ \sqrt{\frac{2}{\pi}} \left(c - e^{-i\frac{\pi}{4}} e^{-i}\right) t^{-i} \right\} \left| \zeta\left(\frac{1}{2} + i(t-1)\right) \right|^2
$$

$$
+ \sqrt{\frac{2}{\pi}} \int_{t^{\delta_2}}^{t-1} \frac{\mathrm{Re}\left\{ e^{-i\frac{\pi}{4}} e^{it F\left(\frac{\rho}{t}\right)} \right\}}{\sqrt{t-\rho}} \left| \zeta\left(\frac{1}{2} + i\rho\right) \right|^2 d\rho + \ln t + 2\gamma - \ln 2\pi
$$

$$
+ O\left(\frac{\left| \zeta\left(\frac{1}{2} + i(t-1)\right) \right|^2}{\ln t} \right), \qquad t \to \infty, \tag{5.1}
$$

with $F(x)$ and c defined in (3.3) and (1.2), respectively.

In Fig. 2 we depict the LHS by the blue curve, and the RHS (ignoring the error term) by the red dashed line, for the range $t \in (357, 440)$. In Fig. 3 we depict the difference of LHS minus the RHS, for the same range of t. In Fig. 4 we observe that this difference is dominated by the error term $O\left(\frac{|\zeta(\frac{1}{2}+i(t-1))|^2}{\ln t} \right)$; we plot the absolute value of the above-mentioned difference in green, and the $\frac{|\zeta(\frac{1}{2}+i(t-1))|^2}{\ln t}$ in black. We
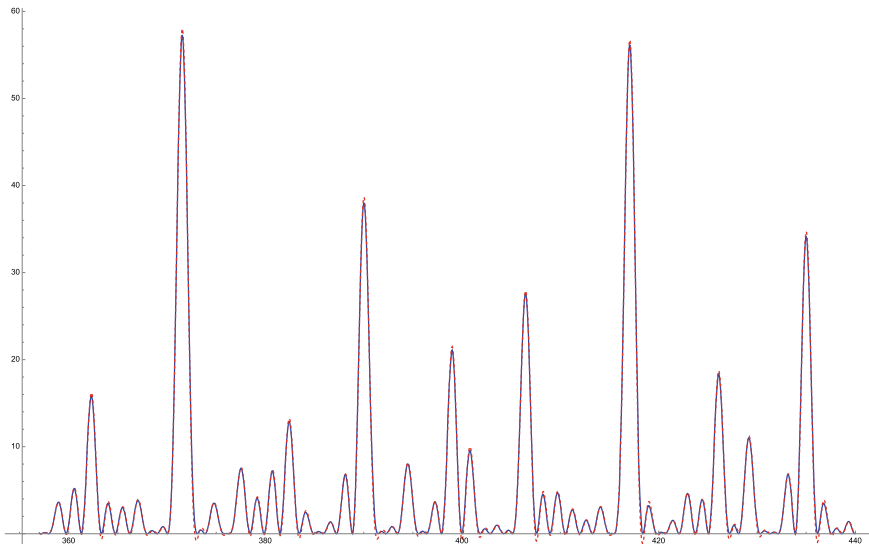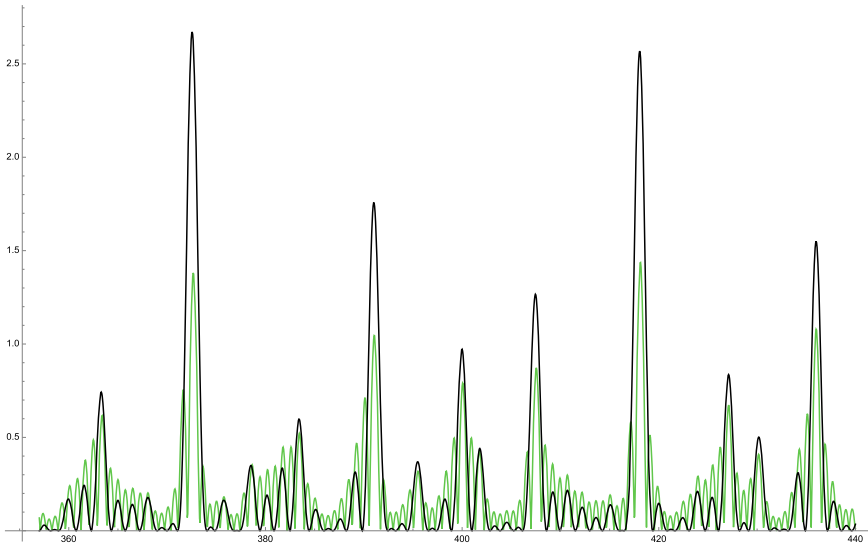

Fig. 2 The LHS (blue) and the RHS (red dashed), for the range $t \in (357, 440)$
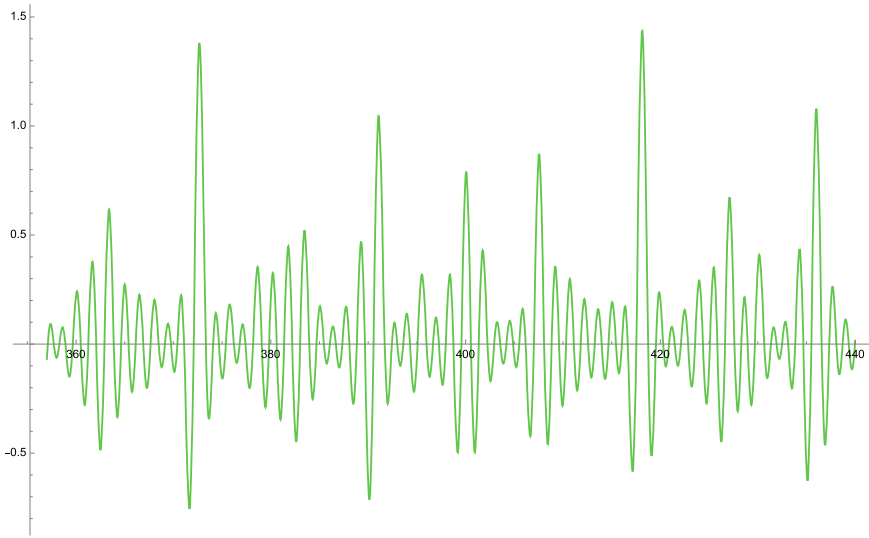
Fig. 3 The difference of LHS minus the RHS
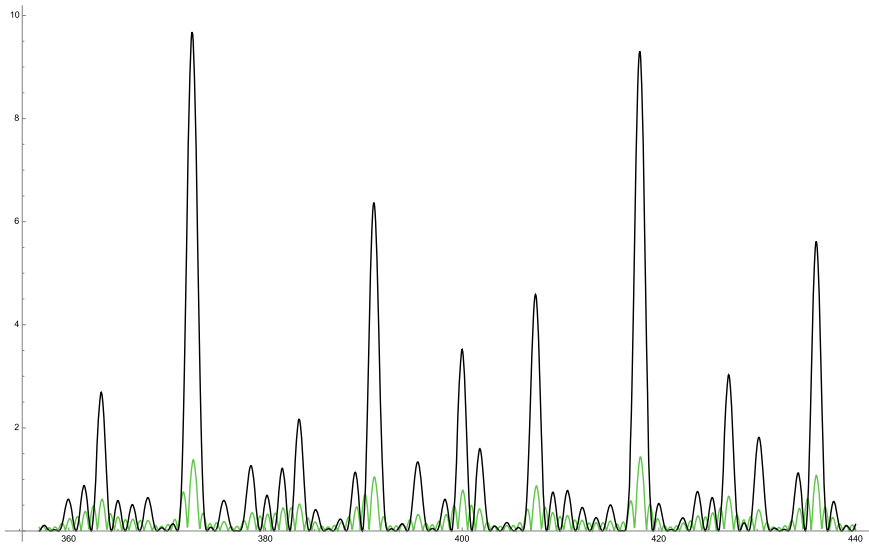
Fig. 4 The absolute difference of LHS minus the RHS (green), versus $\dfrac{\left|\zeta\left(\frac{1}{2}+i(t-1)\right)\right|^2}{\ln t}$ (black)

Fig. 5 The absolute difference of LHS minus the RHS (green), versus $\frac{\sqrt{\pi}\left|\left(c - e^{-i\frac{\pi}{4}}e^{-i}\right)\right|}{\ln t}$ $\left|\zeta\left(\frac{1}{2} + i(t-1)\right)\right|^2$ (black)

find interesting that if we scale $\frac{\left|\zeta\left(\frac{1}{2} + i(t-1)\right)\right|^2}{\ln t}$ by $\sqrt{\pi}\left|\left(c - e^{-i\frac{\pi}{4}}e^{-i}\right)\right| \approx 0.276$, Fig. 5 illustrate clearer that the error term $O\left(\frac{\left|\zeta\left(\frac{1}{2} + i(t-1)\right)\right|^2}{\ln t}\right)$ 'captures the peaks' of the explicit difference LHS minus RHS.

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