Chapter 6
On the Harmonic Balance Method Augmented with Nonsmooth Basis Functions for Contact/Impact Problems

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Abstract In this work, we evaluate the usefulness of nonsmooth basis functions for representing the periodic response of a nonlinear system subject to contact/impact behavior. As with sine and cosine basis functions for classical Fourier series, which have C^{∞} smoothness, nonsmooth counterparts with C^{0} smoothness are defined to develop a nonsmooth functional representation of the solution. Some properties of these basis functions are outlined, such as periodicity, derivatives, and orthogonality, which are useful for functional series applied via the Galerkin method. Least-squares fits of the classical Fourier series and nonsmooth basis functions are presented and compared using goodness-of-fit metrics for time histories from vibro-impact systems with varying contact stiffnesses. This formulation has the potential to significantly reduce the computational cost of harmonic balance solvers for nonsmooth dynamical systems. Rather than requiring many harmonics to capture a system response using classical, smooth Fourier terms, the frequency domain discretization could be captured by a combination of a finite Fourier series supplemented with nonsmooth basis functions to improve convergence of the solution for contact-impact problems.

Keywords Harmonic balance · Contact/impact · Nonsmooth · Basis functions · Galerkin method

6.1 Introduction

The harmonic balance method (HBM) is now widely used in recent years to obtain periodic responses to both weakly and strongly nonlinear dynamical systems. Recent developments have allowed HBM to be efficiently combined with numerical continuation for both small- and large-scale systems and for stability analysis, bifurcation detection and tracking, and nonlinear modal analysis [1]. Some researchers have worked to expand HBM to nonsmooth nonlinearities such as friction and contact [2]. A major difficulty with nonsmooth nonlinearities is the Gibbs phenomenon, which leads to inaccuracies near nonsmooth points with discontinuous derivatives. Also, nonsmooth solutions only converge polynomially unlike exponentially for smooth solutions [3]. To resolve this problem, some researchers used Lanczos filtering to improve the values of the Fourier coefficients [2]. Some work has been done on appending additional, nonsmooth, or discontinuous terms to a system's solution [4], as well as replacing some or all the terms in classical Fourier series with nonsmooth terms [5]. A priori knowledge of the state transition times may be required. Other works utilized event-driven schemes to find and integrate between the state transition times to compute nonlinear forces [6]. Nonsmooth temporal and spatial transformations have also been studied in detail [7].

In this work, we evaluate the usefulness of nonsmooth basis functions for efficiently representing the periodic response of a nonlinear system subject to contact/impact behavior. The objective is to combine desirable traits from previous works and eventually create an improved harmonic balance formulation that supplements the Fourier series with additional nonsmooth terms. Nonsmooth functions with C^{0} smoothness are defined to develop a nonsmooth functional representation of a solution. Rather than using Fourier series with many harmonics to capture a nonsmooth system response, it may instead be obtainable

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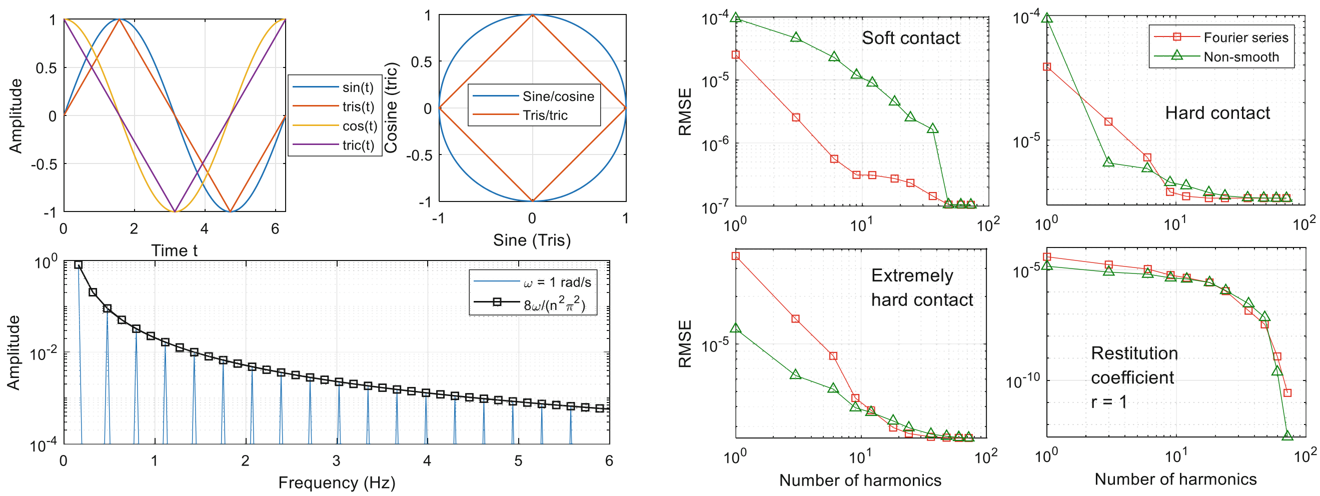

(a) (b)

Fig. 6.1 (**a**) Time histories, phase portraits, and FFT spectra comparing the sine/cosine functions to their nonsmooth counterparts. (**b**) RMSE of Fourier series and nonsmooth series curve fits to time data of dynamical systems with three levels of penalty stiffness, and with a perfectly elastic, restitution-based contact law

by combining a truncated Fourier series with a number of nonsmooth basis functions. This highlights a potential to significantly reduce the computational cost of harmonic balance solvers for nonsmooth dynamical systems. Properties, such as periodicity, derivatives, and orthogonality are outlined. Goodness-of-fit metrics are used to evaluate the classical Fourier series and the nonsmooth basis functions, using time data from different contact systems, to provide a proof of concept for the formulation.

6.2 Nonsmooth Galerkin Formulation

The nonsmooth basis functions investigated here are periodic triangle waves with the same periodicity, maxima, minima, and roots as sine and cosine waves. The basis functions are denoted as "triangle sine" tris(ωt) and "triangle cosine" tric(ωt), respectively, for frequency ω and time t. The basis functions can be mathematically defined and used in a nonsmooth series:

$$\text{tris}(\omega t) = \begin{cases} \frac{4}{T} t_m, & t_m < \frac{T}{4} \\ \frac{-4}{T} t_m + 2, & \frac{T}{4} \le t_m \le \frac{3T}{4} \\ \frac{4}{T} t_m - 4, & t_m > \frac{3T}{4} \end{cases}, \quad \text{tric}\,(\omega t) = \begin{cases} \frac{-4}{T} t_m + 1, & t_m \le \frac{T}{2} \\ \frac{4}{T} t_m - 3, & t_m > \frac{T}{2} \end{cases}, \qquad (6.1\text{a, b})$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \text{tric}\,(n\omega t) + b_n \text{tris}\,(n\omega t)\,, \qquad (6.2)$$

where $t_m = \omega t(\text{mod}T)$ ensures periodicity of the function over a time period $T = 2\pi$, and the values a_0, a_n, b_n denote weighting coefficients that scale each function. Given a time history, a nonsmooth series can be curve-fitted to the data using least-squares regression. Figure 6.1a shows the time histories, phase portraits, and FFT spectra comparing the sine and cosine functions to their nonsmooth counterparts tris and tric. The nonsmooth functions parametrize the unit diamond, and the FFT peaks follow a known pattern, which is mathematically derived to decay polynomially.

6.3 Least-Squares Regression Analysis

Figure 6.1b presents the root mean square error (RMSE) of the Fourier series and nonsmooth series curve fits to different sets of contact/impact time data for a periodic orbit, as the number of harmonics increases. The time histories were obtained from a forced Duffing oscillator with freeplay using two different contact laws, shown in Eq. (6.3). Three responses use a penalty stiffnesses (soft, hard, and extremely hard), namely linear stiffness K_c added when the system crosses contact boundaries $x = -j_1$ or $x = j_2$. A fourth case uses an elastic coefficient of restitution ($r = 1$) to relate the velocities right before and after an impact, x^-, x^+. The third stiffness case produces a response, which resembles the restitution-based response.

$$m\ddot{x} + c\dot{x} + kx + \alpha x^3 + F_c(x) = p\cos(\omega t)\,, \tag{6.3a}$$

$$F_c(x) = \begin{cases} K_c\,(x + j_1)\,, & x < -j_1 \\ 0, & -j_1 \le x \le j_2 \\ K_c\,(x - j_2)\,, & x > j_2 \end{cases}\,,\quad x^+ = -rx^-, r = 1 \tag{6.3b, c}$$

For soft contact, the system response is smooth, and classical Fourier series converges much faster than the nonsmooth series. Hard contact produces a less-smooth response, and both curve fits tend to swap accuracy. Extremely hard stiffness leads to a strong nonsmooth response, and the nonsmooth series converges faster than Fourier series until approximately 12 harmonics are used. Elastic rigid impact leads to a perfectly nonsmooth response, and the nonsmooth series produces closer results if either few or many harmonics are used.

6.4 Conclusion

In this work, we evaluated the usefulness of nonsmooth basis functions for efficiently obtaining the response of a nonlinear system subject to contact/impact behavior. The nonsmooth functions were defined, their similarities to sine and cosine functions were highlighted, and different cases of contact/impact time histories were curve-fitted with Fourier series and the nonsmooth series. Results show Fourier series is superior for smooth responses, but the nonsmooth series becomes superior for increasingly nonsmooth responses. However, Fourier series tends to become more accurate again when many harmonics are used. This behavior will be explored in the final conference presentation, along with other methods for defining the nonsmooth series to represent periodic orbits from vibro-impact systems.

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