Chapter 3
Creating Data-Driven Reduced-Order Models for Nonlinear Vibration via Physics-Informed Neural Networks

Alex J. Elliott

Abstract Modern engineering solutions often aim to improve the energy efficiency of their structures by including lightweight, flexible, and slender designs. In practice, it is not always possible to maximise these characteristics and hence the efficiency of the structures, as they exhibit complex, nonlinear structural dynamics. Unless this behaviour is accurately predicted or controlled, the system may encounter extremely destructive behaviour, which can lead to catastrophic mechanical failure. Non-intrusive reduced-order models (NIROMs)—which project the system dynamics onto a reduced set of modes and approximate the nonlinear components of the behaviour—have, therefore, been of great interest and have the potential to greatly increase the industrial uptake of high-efficiency, nonlinear structures. Existing methodologies for NIROM generation apply linear regression to static force and displacement cases, but this approach has previously been demonstrated to be overly dependent on the scale of these characteristics, a point that has prevented wider application. In this work, initial steps are taken to utilise physics-informed recurrent neural networks (RNNs) in place of the static step, allowing the dynamic behaviour to be more accurately captured in the NIROM. First, the use of random and periodic data series is applied in the training stage, with low-pass filtered white noise shown to provide the more reliable model. Following this, long short-term memory RNNs are developed both with and without a physics-informed loss function, with the former demonstrating faster convergence and more accurate predictions. This study represents the first steps taken in a wider project that aims to improve the accuracy and reliability of nonlinear NIROMs, so that they may be more readily applied in a real-world setting.

Keywords Nonlinear vibrations · Long short-term memory · Physics-informed neural networks · Reduced-order model

3.1 Introduction

The global climate effort is increasingly dependent on lightweight, flexible designs to provide engineering solutions capable of meeting ambitious emissions targets. Examples of these designs include high-aspect ratio wings [1, 2], which are capable of achieving extended flight times using significantly less energy, but their complexity introduces geometric nonlinearity to the system, leading to a substantial increase in complexity [3]. Both the structural behaviour and the mathematical techniques required to capture it provide a much greater modelling challenge. The potential benefits offered by these next-generation designs necessitate accurate and responsive modelling techniques, leading to a marked interest in reliable, non-intrusive reduced-order models (NIROMs) in recent years.

The application of NIROMs to efficiently calculate nonlinear structural behaviour is well-established, with the associated literature including examples of their use to capture local [4] and global nonlinearities [5]. For the designs mentioned above, the increased flexibility and associated strains lead to nonlinearity on the global scale, so this is the focus of the present work. As outlined in [5], NIROM generation traditionally applies nonlinear regression to a series of static cases to estimate the nonlinear coefficients in the force-displacement relationship. This is particularly important for models developed using commercial finite element software, in which the black-box nature of the source code prevents NIROMs from being generated directly.

Naturally, there are two major strategies for generating the aforementioned static cases. In the first of these, often referred to as the implicit condensation [6] (and expansion [7]), a static *force* is applied and the resulting displacements are recorded. Alternatively, it is possible to apply a series of static displacements to the system and record the resultant forces [8].

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Given their similarity, the comparison of these methodologies has been relatively common [9], with the authors of [10] investigating the impact that their differences have on the accuracy of the associated NIROMs. A key finding of this work was that accurately capturing the cross-coupling of the vibration modes was fundamental to achieving an accurate NIROM, an attribute that was not consistently observed for either method.

In light of this underperformance, the recent literature has started to explore the use of machine learning, and particularly recurrent neural networks (RNNs), to overcome this barrier [11–14]. Simpson et al. [11] employ several toy models to investigate the use of long short-term memory (LSTM) RNNs [15], utilising an autoencoder that reduces the order of the model. In contrast, Cenedese et al. [12, 13] expand the spectral submanifold approach to nonlinear normal modes by incorporating a data-driven learning of the dynamics on the submanifold. Across these examples, the methodologies show exceptional promise, though their relative complexity restricts their applicability by a non-expert in a real-world setting.

This paper explores the initial steps in the development of an analogous strategy that aims to directly address this need. A simple, five-degree-of-freedom (5DOF) mass-spring model, including nonlinear springs, is introduced to explore the fundamental accuracy and practicability of the proposed methodology. The application of physics-informed (PI) RNNs, in which verifiable knowledge of the physics of the system is used to guarantee convergence to a physically interpretable solution [16], is identified as a key strategy in achieving this overall aim.

3.2 Methodology

The general equations of motion for a nonlinear mechanical system are given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_\mathbf{x}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{F}_{\mathrm{NL,x}}(\mathbf{x}) = \mathbf{F}_\mathbf{x}, \tag{3.1}$$

where $\mathbf{x}$ denotes the physical coordinates and $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ denote the mass, damping, and stiffness matrices, respectively. $\mathbf{F}_{\mathrm{NL}}(\mathbf{x})$ and $\mathbf{F}_x$ represent the vectors of nonlinear and external forces; note that the subscript x on the forces and damping matrix denotes the fact that this vector is in physical coordinates, rather than modal, as this will simplify the notation below.

It is common practice to consider this system in terms of the basis of modal coordinates—denoted by $\mathbf{q}$—which can be found through the application of the mass-normalised eigenvector matrix, Φ, with $\mathbf{x} = \Phi\mathbf{q}$. Through this transformation, Eq. (3.1) can be rewritten as [3]

$$\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \Lambda\mathbf{q} + \mathbf{F}_{\mathrm{NL}}(\mathbf{q}) = \mathbf{F}, \tag{3.2}$$

where Λ is the diagonal matrix of the squared natural frequencies (i.e. the nth diagonal entry is ω_n^2) and $\mathbf{F}_{\mathrm{NL}}(\mathbf{q}) = \Phi^{\mathrm{T}}\mathbf{F}_{\mathrm{NL,x}}(\Phi\mathbf{q})$.

The NIROM methodologies discussed in the Introduction are designed to preserve the structure of Eq. (3.2), but reducing the number of modes retained in the model. Specifically, the NIROM is defined in terms of some subset $\{\hat{q}\} \subset \{q\}$, with associated reduced eigenvector matrix $\hat{\Phi}$. The number of modes retained in $\{\hat{q}\}$ will be denoted by R. The reduced equations of motion can now be written in the form

$$\ddot{\hat{\mathbf{q}}} + \hat{\mathbf{C}}\dot{\hat{\mathbf{q}}} + \hat{\Lambda}\hat{\mathbf{q}} + \hat{\mathbf{F}}_{\mathrm{NL}}(\hat{\mathbf{q}}) = \hat{\mathbf{F}}. \tag{3.3}$$

Both the linear components of Eq. (3.1) and $\hat{\Phi}$ can be readily obtained, either analytically or from the finite element software, so the linear elements of Eq. (3.3) can be easily calculated. In contrast, the nonlinear function is typically complicated and is often approximated through a numerical methodology that is not made available to the user. Thus, the main aim of the NIROM generation strategy is to analytically approximate this behaviour in the function $\hat{\mathbf{F}}_{\mathrm{NL}}$.

Since the motivation for developing NIROMs is frequently driven by a need for significant improvements in response time, $\hat{\mathbf{F}}_{\mathrm{NL}}$ is usually written as a cubic polynomial in terms of the elements of $\hat{\mathbf{q}}$. The m^{th} element of this vector is given by

$$\hat{\mathbf{F}}_{\mathrm{NL},m} = \sum_{i=1}^{R}\sum_{j=1}^{R}\sum_{k=1}^{R} A_{i,j,k}^{(m)}\hat{q}_i\hat{q}_j\hat{q}_k + \sum_{i=1}^{R}\sum_{j=1}^{R} B_{i,j}^{(m)}\hat{q}_i\hat{q}_j, \tag{3.4}$$

where the coefficients, $A_{i,j,k}$ and $B_{i,j}$, are the terms estimated by the reduction steps. With the nonlinear terms approximated as in Eq. (3.4), the static cases applied to the system take the form

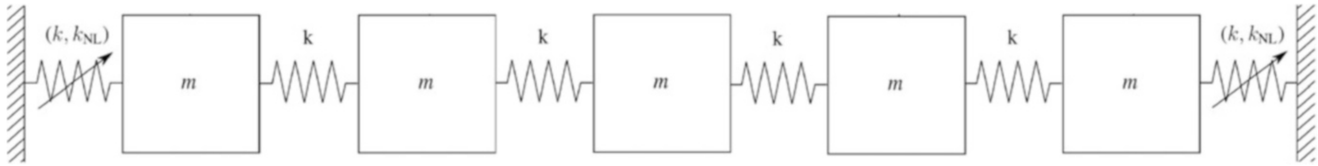

Fig. 3.1 Schematic for a 5DOF mass-spring system with nonlinear grounding springs

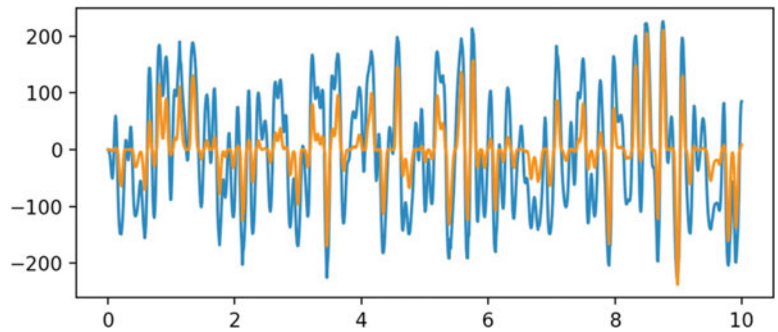

Fig. 3.2 Relative magnitudes of the linear (blue) and nonlinear (orange) forces observed for the 5DOF mass-spring system

$$\omega_m^2 \hat{q}_m + \sum_{i=1}^{R}\sum_{j=1}^{R}\sum_{k=1}^{R} A_{i,j,k}^{(m)} \hat{q}_i \hat{q}_j \hat{q}_k + \sum_{i=1}^{R}\sum_{j=1}^{R} B_{i,j}^{(m)} \hat{q}_i \hat{q}_j = \hat{\mathbf{F}}_m. \tag{3.5}$$

In Eq. (3.5), it is only the nonlinear coefficients that are unknown. Therefore, by generating a sufficient number of these cases, it will be possible to apply linear regression to estimate the value for $A_{i,j,k}$ and $B_{i,j}$. The details for both methodologies are outlined in [10], so they are not provided here.

3.2.1 Example: Five-Degree-of-Freedom, Nonlinear Mass-Spring Model

To allow the accuracy and usability of the methods to be compared, a simple toy model is introduced and presented in Fig. 3.1. This consists of five masses, of mass $m = 0.1$ kg, arranged in parallel and connected by linear springs with stiffness coefficient $k = 100$ N/m and an associated damping coefficient of $c = 0.1$ kg/s. The end masses are grounded by springs that include an additional cubic component with stiffness, $k_{\text{NL}} = 25$ N/m^3, so that its force is defined by the equation $F(x_n) = kx_n + k_{\text{NL}}x_n^3$, where x_n denotes the horizontal displacement of the nth mass.
The forcing of the system is scaled by the amplitude vector $\mathbf{A} = [75, 0, 0, 0, 75]^{\text{T}}$. As can be observed in Fig. 3.2, this forcing level results in the linear and nonlinear forces observed in the system time histories being of a similar magnitude. This has been selected to mirror the typical approach for the static methodologies, which utilise a user-defined scaling factor to ensure that the modal cross-couplings have been sufficiently activated [5, 10].

3.2.2 Data Generation

The longer-term vision for this approach is that knowledge of the system can be used to strategically force the system and generate data that leads to an optimal NIROM, though this is beyond the scope of the present work. Instead, two types of forcing are considered:

- *Sinusoidal:* A simple sinusoidal function with amplitude vector $\mathbf{A}$ and frequency Ω is applied to one or multiple masses, i.e., $\hat{\mathbf{F}} = \mathbf{A}\sin(\Omega t)$.
- *Random:* Low-pass filtered white noise, with cut-off frequency of 7.5 Hz is applied to one or multiple masses.

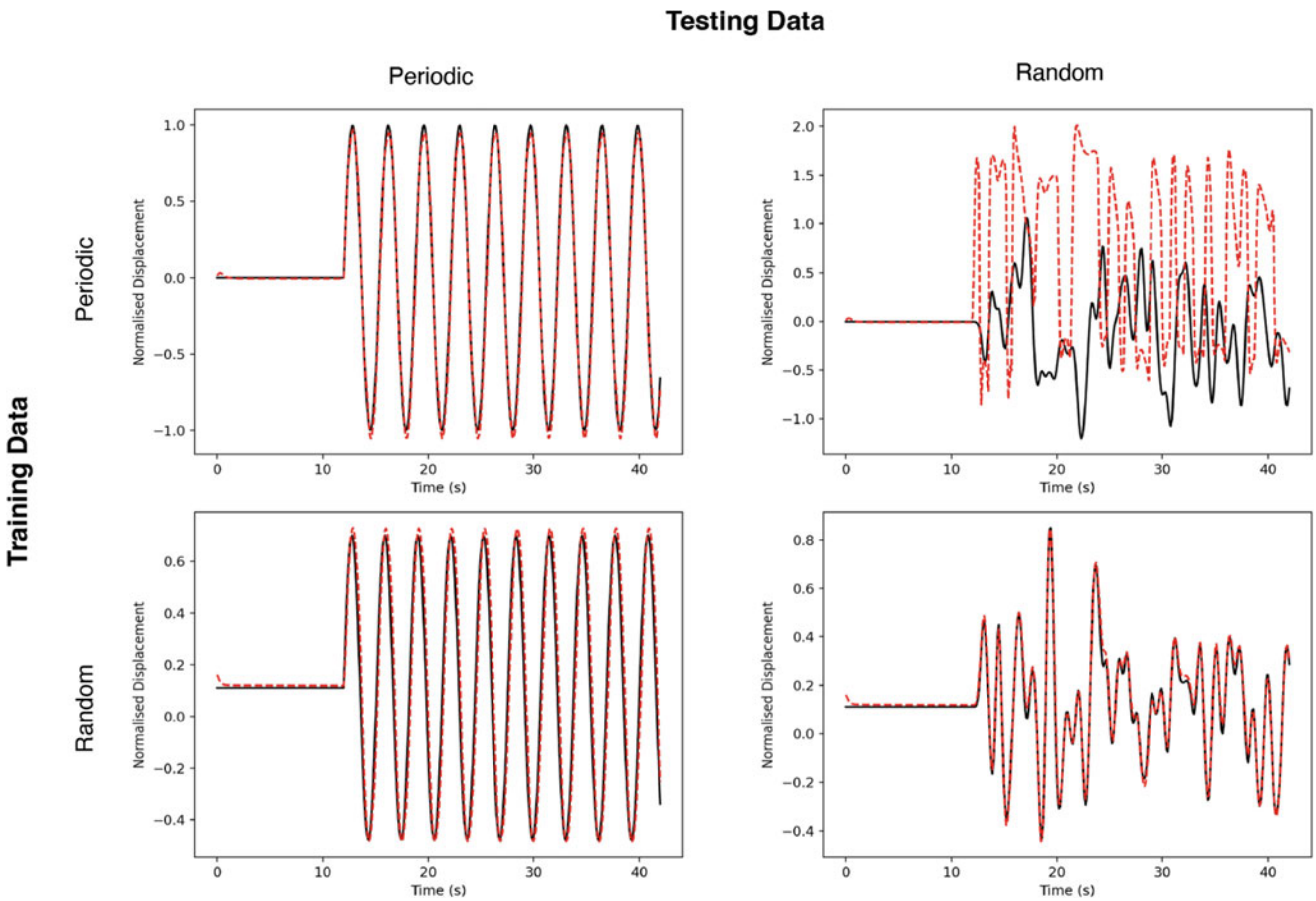

Fig. 3.3 Representative model performance for models trained using periodic and random data

In both cases, each time history is preceded by a period of zero forcing to improve the prediction if the system is forced from a resting position, as recommended in [11].

The relative performance of models generated with periodic and random forcing data is presented in Fig. 3.3. For both cases, the prediction of periodic data is observed to be accurate. It can be noted that there are minor discrepancies in the predicted amplitude, with slight improvements in the periodic model, though these are negligible in relation to the full vibration magnitude. In contrast, there are significant differences in the accuracy of the prediction of randomly forced data. As expected, the model that is also trained with random data predicts the system response very accurately, with no real change in performance between the periodic and random test cases. However, the periodic model performs very poorly, failing to predict the structural dynamics as soon as the force is applied. These initial results highlight the importance of representative data in the NIROM training phase. This point is particularly important for the aforementioned longer-term data generation strategy, which must incorporate the full range of system behaviour if it is to be reliably used in the real world.

3.2.3 Long Short-Term Memory

In this work, we investigate the application of LSTM [15] as a strategy for incorporating the dynamical behaviour of the system into the NIROMs generated. LSTM networks are a specific (and popular) subcategory of RNNs, which use gated activation functions to up the "cell state". This cell has the ability to remember the system values over a pre-selected number of steps, allowing longer-term behaviour to be captured; this is of particular interest for nonlinear structures, for which the behaviour is often hysteretic in nature. As the exact details of the LSTM mechanism are of little relevance to the generated NIROMs—and with the wider project aiming to remove the need for the user to have in-depth technical knowledge of machine learning—further these details are not provided here, and the reader is instead directed to [15] for this information.

3.2.4 Physics-Informed Loss Function

While LSTM is a proven strategy for developing surrogate models and NIROMs, there is no explicit guarantee that the model developed is consistent with physical laws that define the system behaviour. To address this concern, this section investigates the application of PI RNNs—as introduced in [16]—to allow the existing knowledge of the system to be incorporated into the process.

As has been discussed throughout the previous sections, the underlying linear system (which can be found by simply removing the nonlinear force function from Eq. (3.1) or (3.2)) has been extensively studied and defines a large amount of the structural dynamics. By incorporating this knowledge into the loss function utilised in the LSTM network, it is possible to create a PI loss function that can aid the accuracy, physical interpretability, and convergence of the NIROM. In the vanilla LSTM approach, the loss function consists solely of the standard mean squared error (MSE) applied to displacement and velocity:

$$L(\hat{\mathbf{q}}, \dot{\hat{\mathbf{q}}}) = \|\mathbf{z}_1 - \hat{\mathbf{q}}\|_2^2 + \|\mathbf{z}_2 - \dot{\hat{\mathbf{q}}}\|_2^2, \tag{3.6}$$

where $\mathbf{z}_1$ and $\mathbf{z}_2$ denote the true values of the modal displacements and velocities, respectively, and $\|\bullet\|_2$ denotes the standard ℓ^2-norm. In practical terms, the loss function in Eq. (3.6) will aid the convergence to a surrogate model that accurately predicts behaviour similar to what it has been trained with, but there is no guarantee that this will be sufficient to capture unseen physical phenomena. This is something that a PI loss function should address, with the added benefit that convergence may occur more quickly.

The changes required to update Eq. (3.6) are relatively minimal and take advantage of the fact that the unknown nonlinear function, $\hat{F}_{\mathrm{NL}}$, can be expressed in terms of the known linear terms by rearranging Eq. (3.3). Therefore, Eq. (3.6) can be adapted to also assess the error in the prediction of the nonlinear term, as follows:

$$L(\hat{\mathbf{q}}, \dot{\hat{\mathbf{q}}}) = \|\mathbf{z}_1 - \hat{\mathbf{q}}\|_2^2 + \|\mathbf{z}_2 - \dot{\hat{\mathbf{q}}}\|_2^2 + \|\hat{\mathbf{F}}_{\mathrm{NL}}(\mathbf{z}_1) - \hat{\mathbf{F}}_{\mathrm{NL}}(\hat{\mathbf{q}})\|_2^2. \tag{3.7}$$

The impact of this additional term can be observed in Fig. 3.4, which compares the convergence and performance of NIROMs generated using the MSE and PI loss functions. The performance of both models is strong, though the vanilla NIROM slightly underperforms in both amplitude and phase, which could prove problematic for systems with very high-frequency components of the response. However, there is a much clearer difference in the convergence of the model. While the first 1000 epochs of training show lower losses for the MSE model, it eventually takes over 4000 iterations to show signs of real convergence. In contrast, the PI loss values are much more volatile for the initial 1500 epochs, but converge by 2000 epochs. This suggests that the use of a PI loss function can more than halve the computation time.

3.3 Conclusions

This paper presents initial insights into the use of physics-informed neural networks in the development of NIROMs for nonlinear mechanical systems. It is envisaged that this approach will eventually provide a more accurate and reliable methodology than current approaches that utilise static forces and displacements to approximate the dynamic nonlinear behaviour. A 5DOF mass-spring model with nonlinear grounding springs is used to explore the generation of data that leads to robust NIROMs that are capable of capturing complex, nonlinear behaviour, with random data providing clear advantages over simpler periodic time histories. This model is further applied to investigate the use of an LSTM-based machine learning methodology. This approach has been demonstrated to accurately capture the system dynamics, with additional accuracy and savings in computation being achieved through the inclusion of a PI loss function. Future work on this topic will focus on developing this strategy to be able to capture the dynamics of much larger systems, including those built with commercial finite element packages, as well as incorporating uncertainty quantification and optimisation.

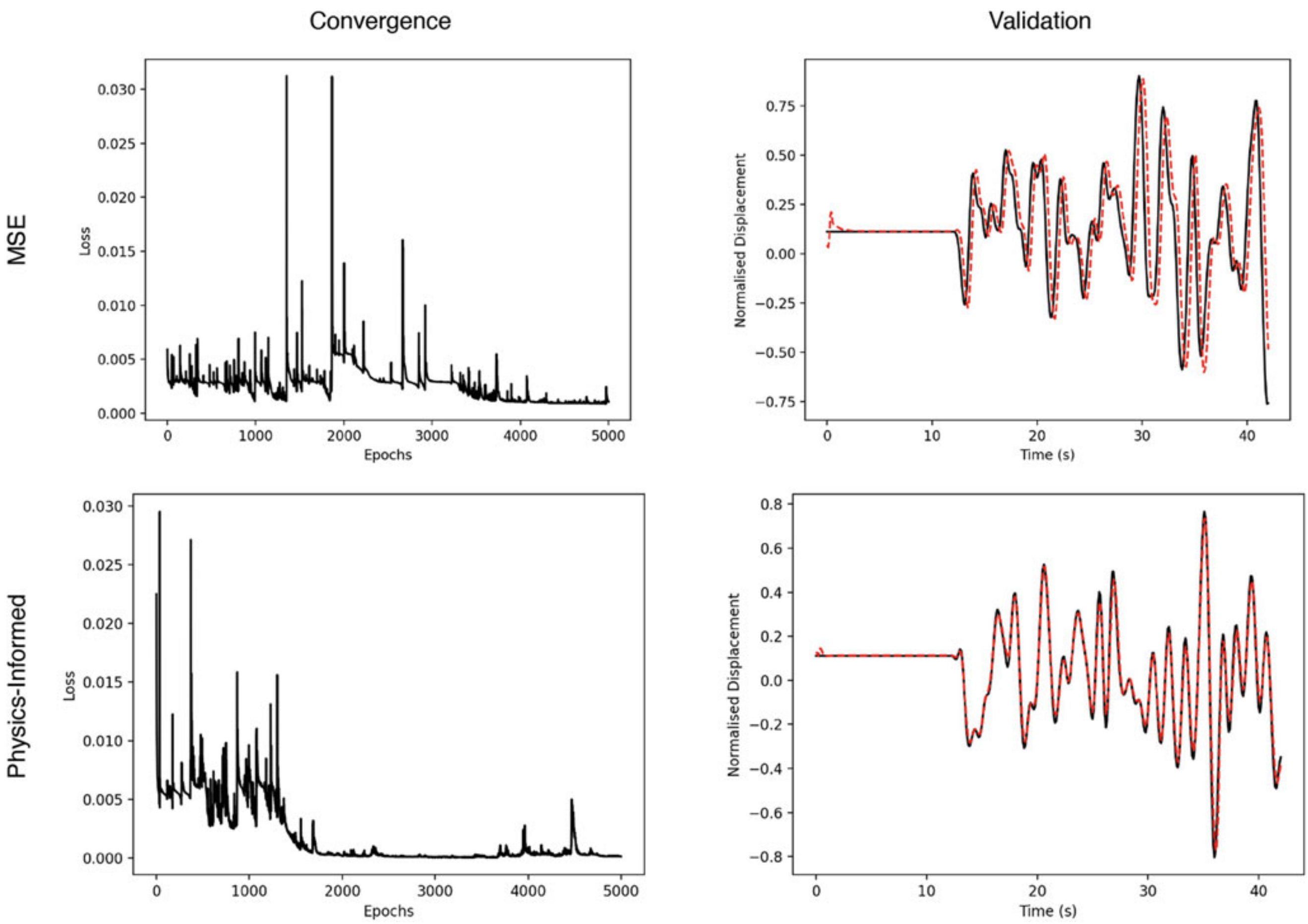

Fig. 3.4 Convergence and performance data for NIROMs generated using MSE and PI loss functions

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