Chapter 15
Resonant Characterization of Nonlinear Structures in the Co-existence of Multiple Resonant Components

Nidish Narayanaa Balaji, Matthew R. W. Brake, D. Dane Quinn, and Malte Krack

Abstract The study of nonlinear normal modes has become a very popular subdiscipline in the structural dynamics community. This is principally due to the fact that they allow for a practical generalization of the concept of spectral invariant manifolds in linear dynamics. There have been a lot of analytical successes achieved in this area through the application of the method of multiple scales which decomposes the response of a nonlinear oscillator into fast-changing dynamics occurring on a slowly varying manifold. Numerical calculations have also been carried out using a periodic ansatz (with resonance-based constraints). Most of such investigations in the past have focused on cases where a single nonlinear mode is studied in isolation. Although this alone provides very interesting results, such as invariant manifold characterization and internal resonance detection, these are usually not sufficient to study the coupling between multiple nonlinear modes. Recent experimental studies have shown that such coupling can lead to very nontrivial trends in the resonant characteristics, making it difficult to correlate computational realizations with experimental measurements. The present paper takes a computational approach frequency-domain numerical simulations conducted using quasi-periodic harmonic balance to obtain numerical insights into the steady-state multi-resonant behavior. These results are compared with signal processing conducted on transient ringdown responses to determine the ramifications of commonly employed signal processing techniques like frequency-domain filtering for mode isolation. Analytical results are also computed using multiple scales for comparisons. Results are presented for a simplified system with two kinds of nonlinearities: geometric and dry friction.

Keywords Nonlinear resonance · Quasi-periodic oscillations · Modal coupling · Method of multiple scales · Nonlinear structural dynamics

15.1 Introduction

Resonance of nonlinear structures is an area that has been undergoing increasing interest in the structural dynamics community over the last few decades [1]. Nonlinear normal modes (NNMs) generalize the concept of resonance in linear dynamics to nonlinear systems. Unlike linear systems, in which the resonance is an invariant feature of the dynamics irrespective of the amplitude of response (or excitation), the presence of nonlinearity implies a strong amplitude dependence of the resonance characteristics (resonant frequency, damping, mode shapes, etc.). Consequently, there have been several computational as well as analytical techniques that have been successfully applied for the estimation of these NNMs in terms of amplitude-dependent resonant characterization (see, for instance, [2–4]).

Apart from the theoretical interest in the NNMs, the practical interest stems from the fact that, like modal representations in linear vibrations, these NNMs allow for the construction of reduced-order nonlinear modal models. Such models provide a computationally very efficient representation of a potentially much larger system. Stemming from the single nonlinear mode

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theory [5], this idea has been successfully applied for the synthesis of both the free and the forced responses of nonlinear structures [6, 7]. In these studies, after the relevant NNM is computed, the complexification-averaging (CX-A) technique is employed to derive the governing equations for the so-called slow dynamics, representing the instantaneous amplitude and phase of the response. Since the frequencies are known, the "fast dynamics" is reconstructed as a harmonic response enveloped by the slow dynamics. The interested reader is directed to [6, 7] for details.

Most of the abovementioned studies employ the singe nonlinear mode theory [5], i.e., resonant characteristics of a single nonlinear mode taken in isolation is employed for the syntheses (other modes, if included, are only taken as linear contributors to the response). While being a very powerful and widely used approach, this limits the domain of accuracy to scenarios where the excitation/response is close to the chosen mode (in frequency). Multiple modes may interact in a real structure through internal resonances (occurring at amplitude ranges where modal frequencies are commensurate) and through mode coupling (coupling that does not require commensurate frequencies) (see [8] for an analytical treatment covering both these aspects for a nonlinear structure).

The current paper proposes a nonlinear modal analysis approach which allows for the consideration of multiple modes together. The first complication this introduces is that the solutions for which the equations of motion are solved for are not periodic but are quasi-periodic since the frequencies corresponding to the different modes are not necessarily commensurate. Multi-frequency harmonic balance is therefore used as the computational technique of choice. A two-degree-of-freedom (DOF) model is used for demonstrating the technique numerically, with amplitude ranges chosen in such a manner so as to avoid any internal resonances.

15.2 Methodology

This section first describes the quasi-periodic formalism proposed for resonant characterization of multiple modes at the same time. Throughout the rest of the paper, the terminology "multi-resonant" will be adopted to denote response cases where the response consists of multiple modes that are resonant (and at a specified amplitude).

Given a nonlinear system of equations of the form

$$\underline{\underline{M}}\,\ddot{\underline{u}} + \underline{\underline{C}}\,\dot{\underline{u}} + \underline{\underline{K}}\,\underline{u} + \underline{f}_{nl}(\underline{u}, \dots) = \underline{0} \tag{15.1}$$

the extended periodic motion concept [2] (EPMC) constrains the response to be periodic by introducing a periodic ansatz. Since the system could have dissipative components, the formulation also introduces a negative damping term of the form $\xi\underline{\underline{M}}\,\dot{\underline{u}}$ (with an unknown coefficient ξ) that is meant to cancel out the dissipation by supplying sufficient energy over a cycle of oscillation. The equations of motion for EPMC is

$$\underline{\underline{M}}\,\ddot{\underline{u}} + \underline{\underline{C}}\,\dot{\underline{u}} + \underline{\underline{K}}\,\underline{u} + \underline{f}_{nl}(\underline{u}, \dots) - \xi\underline{\underline{M}}\,\dot{\underline{u}} = \underline{0}\,. \tag{15.2}$$

The "modal amplitude" is defined as the cyclic amplitude q such that $\oint \underline{u}^T\underline{\underline{M}}\,\underline{u}dt = q^2$ (other amplitude definitions also exist), which is user-specified. It is to be noted that a practical implementation will involve a phase constraint, whereby all of the unknowns can then be parameterized by the modal amplitude q and the phase θ. Solving this system of equations will give a complex multi-harmonic mode shape $\underline{\Psi}$, a natural frequency ω, and the effective dissipation coefficient ξ, each of which is parameterized by the modal amplitude q.

Considering the case where a multi-resonant response is desired that has more than one resonant frequency component, the response will have to be quasi-periodic in general. So opting for a quasi-periodic multi-harmonic ansatz must provide sufficient flexibility for such a study. Unlike the previous case, there will have to be at least as many independent damping terms as the number of resonant components under consideration. We use the so-called Caughey series of matrices to generate sets of matrices to serve as coefficient matrices for the damping terms. It must be stressed at this point that, in general, the exact choice of the sequence of matrix should not change the results, i.e., this should not constrain the system in any manner. This claim will have to be verified for the current choice, but is deferred for future investigations and this is taken to be a fact for the current investigation. The negative damping terms are hereby

$$\underline{f}_{nd} = \left(\xi_1\underline{\underline{M}} + \xi_2\underline{\underline{K}} + \xi_3\underline{\underline{K}}\,\underline{\underline{M}}^{-1}\underline{\underline{K}} + \dots\right)\dot{\underline{u}}$$

$$= \sum_{i=1} \xi_i \underbrace{\underline{\underline{M}} \left(\underline{\underline{M}}^{-1} \underline{\underline{K}} \right)^{i-1}} \underline{\dot{u}}$$

$$= \sum_{i=1} \xi_i \underline{\underline{C}}_i \underline{\dot{u}} \,, \tag{15.3}$$

where $\underline{\underline{C}}_i$ are the Caughey matrices. Similar to EPMC, the idea here is to use proportional negative damping to cancel out the dissipative components in a hyper-cyclic (cyclic in hyper-time) sense.

The equations of motion for this case will be

$$\underline{\underline{M}}\,\underline{\ddot{u}} + \underline{\underline{C}}\,\underline{\dot{u}} + \underline{\underline{K}}\,\underline{u} + \underline{f}_{nl}(\underline{u}, \dots) - \sum_i \xi_i \underline{\underline{C}}_i \underline{\dot{u}} = \underline{0}\,. \tag{15.4}$$

In this case, the definition of the modal amplitudes of the individual resonant components come from the first harmonic coefficients of each frequency component. Solving this system will result in a quasi-periodic deflection shape with multiple harmonic components: the resulting resonant components $\omega_1, \omega_2, \dots$; and the damping coefficients $\xi_1, \xi_2, \dots$, all parameterized by modal amplitudes $q_1, q_2, \dots$.

15.3 Numerical Illustrations on a Two-DOF System

The above-developed formulation is applied to a two-DOF nonlinear oscillator that has previously been considered in the literature (see [2], for instance). The governing equations for this are taken to be

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.02 & -0.01 \\ -0.01 & 0.02 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} x_1^3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{15.5}$$

Two modes are studied concurrently; therefore, two independent frequency components need to be considered. Figure 15.1 shows, in graphical form, the list of harmonics selected for the quasi-periodic ansatz. The harmonic (n, m) corresponds to an ansatz of the form $\cos(n\omega_1 t + m\omega_2 t) + j\sin(n\omega_1 t + m\omega_2 t)$. Truncation is carried out in such a manner that $|n| + |m| \leq H_{max}$ with $H_{max} = 7$.

Figure 15.2 shows the natural frequencies of the two modes of this system as a function of the two modal amplitudes Q_1 and Q_2. These simulations are conducted by setting H_{max} to 7. It must be noted here that since there are two control parameters Q_1, Q_2, traditional numerical continuation is not applicable (designed for single parameter continuation). For the purposes of the current study, an additional constraint is enforced of the form $(Q_1, Q_2) = Q(\cos\theta, \sin\theta)$ and single parameter continuation is carried out in the parameter Q. This is repeated for different θ values so as to traverse the solution

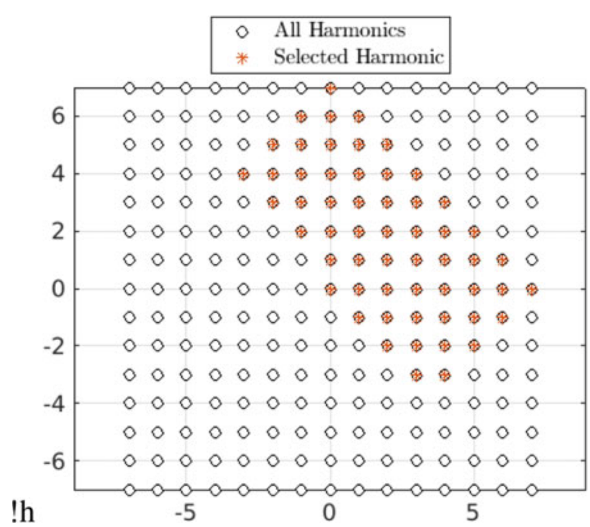

Fig. 15.1 Selected harmonics for two frequency components and $H_{max} = 7$ (Redundant indices removed)

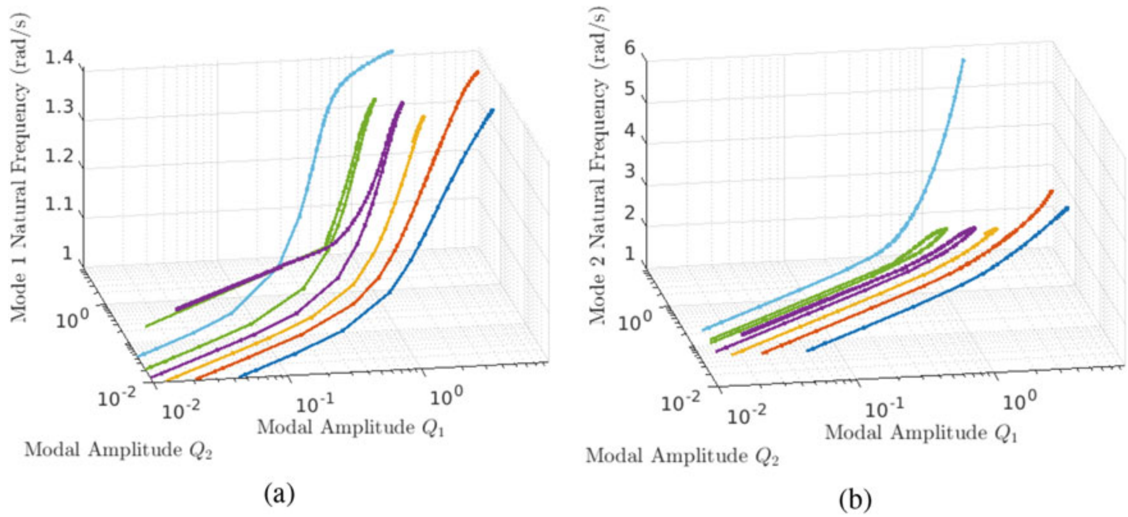

Fig. 15.2 Results from the seven-harmonic QP simulation, plotting the (**a**) first and (**b**) second natural frequencies as functions of the modal amplitudes Q_1, Q_2

domain (which is a planar region) through single lines. The results shown in Fig. 15.2 demonstrate that beyond a certain amplitude, the frequency backbone seems to change direction, indicating strong presence of modal interactions/couplings. Little can be said about such phenomena with the continuation scheme employed presently, since the constraint enforced above for the continuation may end up artificially constraining the system, rendering these features on the backbone plot to have limited physical meaning. One will have to resort to multi-parameter continuation techniques in order to fully explore such features.

For the purposes of this study, the following analyses will be restricted to single harmonic simulations. Although this has limited physical significance, it will, at the minimum, enable a better understanding of modal interactions that are not related to internal resonances (which necessarily require a multi-harmonic ansatz).

15.3.1 Two-Component Single Harmonic Balance Results

For the system at hand, it must be noted that one may employ the method of multiple scales (MMS, as in [8]) in order to obtain analytical expressions for the nonlinear natural frequency (and damping) characteristics. For the system at hand, the expressions for the natural frequencies are

$$\omega^{(1)} = \omega_0^{(1)} + \epsilon \frac{3\alpha}{8}(Q_1^2 + 2Q_2^2), \text{ and} \tag{15.6a}$$

$$\omega^{(2)} = \omega_0^{(2)} + \epsilon \frac{3\alpha}{8}(2Q_1^2 + Q_2^2). \tag{15.6b}$$

Note that these expressions are derived using MMS considering just a single harmonic, thereby enabling direct comparison to the numerical results generated by the quasi-periodic harmonic balance (QP-HB) approach developed above. ϵ represents the book-keeping parameter used in the perturbation analysis. The above expressions correspond to the analysis carried up to an $\mathcal{O}(\epsilon)$ accuracy. Figure 15.3 shows a comparison of the resonant frequency backbone surfaces (since these are functions of both the modal amplitudes) computed using the numerical approach (QP-HB) and the analytical approach (MMS). A close match can be observed, especially for small amplitudes.

Another aspect can be highlighted by constructing vertical contour sets on the above modal surfaces. These are contour levels drawn for different values of Q_1 and Q_2, as shown in Fig. 15.4. These highlight the fact that even when the mode 1 amplitude is fixed (and nonzero), the mode 1 frequency can vary purely due to the changes in the mode 2 amplitude (vice versa for the mode 2 frequency). This fact is also reflected by the analytical dependence in Eq. 15.6. This seems to suggest that in a practical setting, the employment of data from hammer impact tests (which provides a broad spectrum and therefore

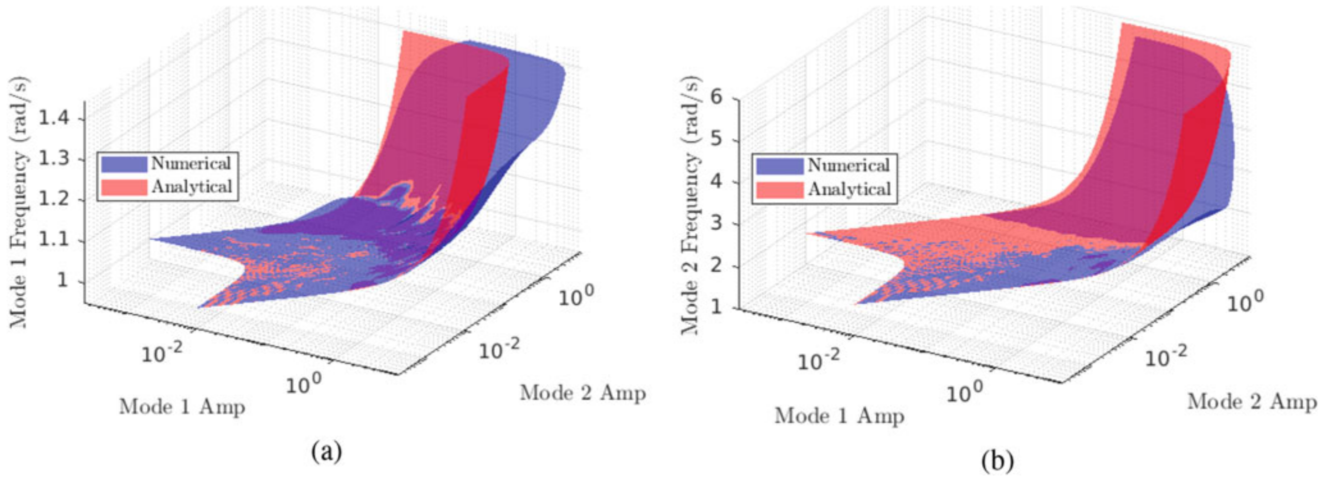

Fig. 15.3 Surfaces of the (**a**) first and (**b**) second mode natural frequencies as a function of the two modal amplitudes, numerical and analytical predictions

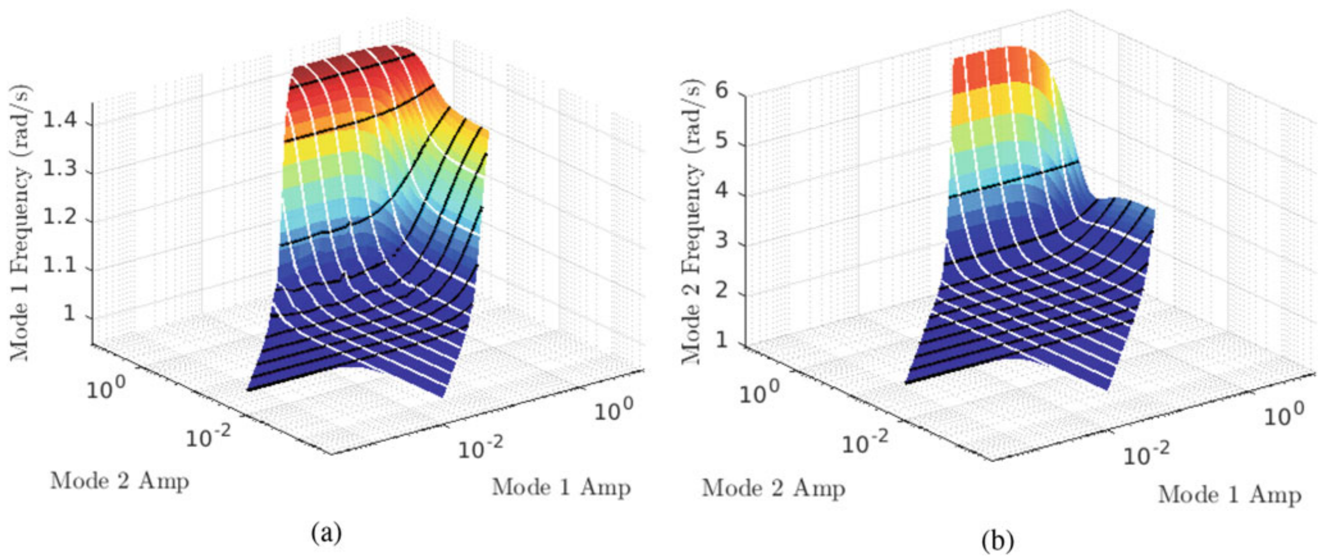

Fig. 15.4 Surfaces of the (**a**) first and (**b**) second mode natural frequencies as a function of the two modal amplitudes, showing vertical contours

multimodal excitation) could possibly be of questionable reliability. In the literature, similar observations have been made experimentally in [9].

As a first step in assessing the acuteness of this issue, a transient simulation is conducted wherein the initial condition is provided as a mix of the two mode shapes at a sufficiently high amplitude level so as to exhibit nonlinear free decay (but not too high so as to avoid internal resonance phenomena). The PFF approach [10] is used to postprocess the ringdown data into modal backbones. Figure 15.5a presents the transient ringdown data in time as well as frequency domains, showing the raw acceleration data extracted from DOF 2 (x_2). In Fig. 15.5b, the characteristics of the mode 1 frequency are plotted against the DOF 2 acceleration amplitude along contour levels corresponding to different values of Q_2 (the other mode in the system). The first observation that can be made is that at higher amplitudes of Q_2, there seems to be a very nontrivial relationship between the second mode amplitude and the first mode frequency (this is also verified for mode 2, but not shown). Here, the MMS results corresponding to the same amplitude levels as the contours are plotted as dash-dotted lines—the match is once again very good at low amplitude levels, but becomes inaccurate as either Q_1 or Q_2 becomes large. Also plotted (in black) is the frequency backbone post-processed from the transient ringdown data using PFF. There seems to be some disagreement between the transient result and any of the contour levels in the nonlinear regimes.

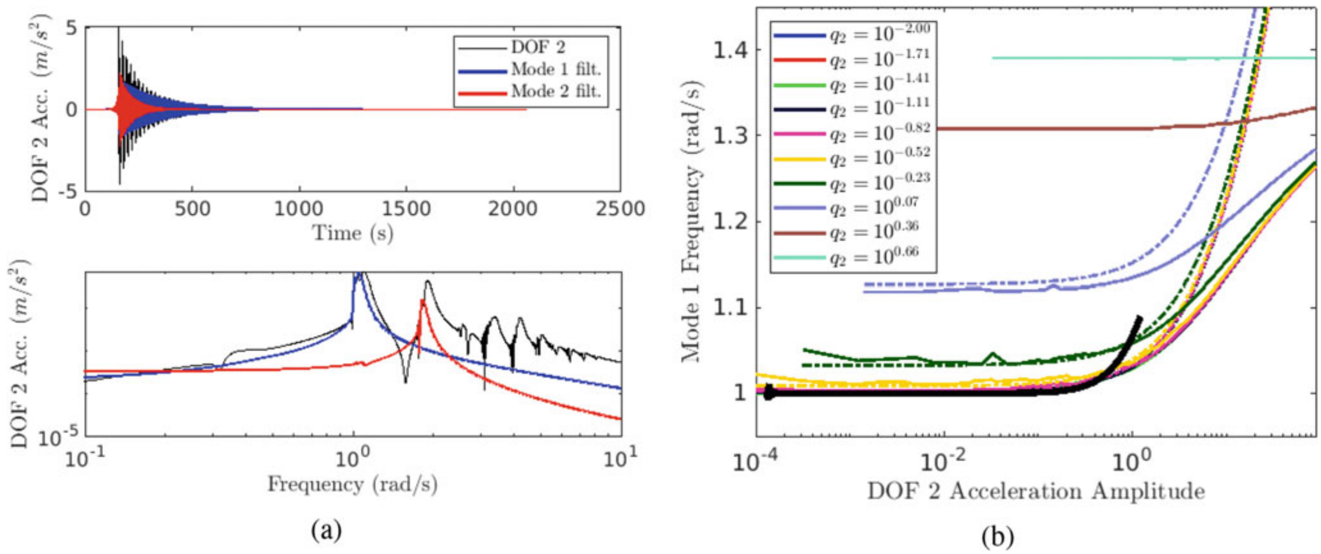

Fig. 15.5 (**a**) Time and frequency-domain representations of the ringdown data (before and after filtering and **b**) contour curves from the mode 1 backbone surface (Fig. 15.4a), shown as solid lines, along with their analytical counterparts (using MMS), shown as dash-dotted lines. Also plotted in (**b**) is the mode 1 backbone post-processed from the transient data in (**a**)

15.4 Conclusions

A new computational approach based on the extended periodic motion concept (EPMC) has been proposed for the study of nonlinear multi-resonant dynamics. Resonant backbones have been computed for a two-DOF oscillator with a cubic nonlinearity. The results show that the proposed approach is capable of diagnosing both modal interactions as well as modal coupling. Some areas that have been identified for future research are:

- The viability of multi-parameter continuation for the investigation of the resonant backbone manifolds in the presence of modal interactions must be explored.
- The generizability of the assumed negative damping form needs to be understood better.
- Modal coupling (away from the internal resonance regimes) needs to be studied in more detail, as a way of determining how much inaccuracies are incurred while employing hammer impact test-based data in an experimental setting.

Apart from these, it is remarked that the presented approach can potentially allow the direct detection of modal interactions through bifurcation tracking. This is possible due to the fact that when the two frequencies become commensurate, the Jacobian matrix of the system of equations being solved for the QP-HB approach becomes singular, i.e., indicating a bifurcation.

Furthermore, the practical utility of this approach will be greatly enhanced if it becomes possible to derive slow flow equations of motions for the amplitude and phase coordinates of multiple modes concurrently (such as the developments in [6, 7]). This will enable the construction of nonlinear modal reduced-order models, leading to potentially faster computations for large systems.

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